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The Term Structure of Defaultable Bond Prices

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Abstract

In this paper we present a new methodology for modelling the development of the prices of defaultable zero coupon bonds that is inspired by the Heath-Jarrow-Morton (HJM) [19] approach to risk-free interest rate modelling. Instead of precisely specifying the mechanism that triggers the default we concentrate on modelling the development of the term structure of the defaultable bonds and give conditions under which these dynamics are arbitrage-free. These conditions are a drift restriction that is closely related to the HJM drift restriction for risk-free bonds, and the restriction that the defaultable short rate must always be not below the risk-free short rate.
The same restrictions apply for the extended versions of the model that allow for restructuring of defaulted debt and multiple defaults, and loss quotas that are not predictable. In its most general version the model is set in a marked point process framework, to allow for jumps in the defaultable rates at times of default.

JEL Classification: G 13

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Contents

1 Introduction

The aim of this paper is to provide a new approach to the modelling of the term structure of defaultable bonds that is inspired by the Heath-Jarrow-Morton [19] approach.

There are two approaches to the valuation of defaultable securities: In the older approach starting with the initial proposal of Black and Scholes [7] a credit risky security is regarded as a contingent claim on the value of the issuing firm and is valuated according to option pricing theory. Here the firm's value is assumed to follow a diffusion process and default is modeled as the first time the firm's value hits a pre-specified boundary. The models of Merton [29], Black and Cox [5] Longstaff and Schwartz [27], Das [11] and Geske [18] are representatives of this approach. The second, more recent approach, leaves the direct reference to the firm's value and models the time of default directly as the time of the first jump of a Poisson process with random intensity (a Cox process), or – more generally – as a totally inaccessible stopping time with an intensity. This approach is followed by Madan and Unal [28], a group around Duffie (Duffie and Singleton [15], Duffie, Schroder and Skiadas [14], Duffie and Huang [13] and Duffie [12]), Lando [25] [26], Flesaker et.al. [16], Artzner and Delbaen [1] [2] and Jarrow and Turnbull [24].

Most of these approaches first specify a mechanism that triggers the default (e.g. the firm's value process that hits a barrier or the first jump of a Cox process) and then go on to derive a valuation formula for a defaultable zero coupon bond from this specification. Especially in the second group of models (the intensity models) a striking similarity between these models and risk-free interest rate modelling is found.

In this paper we invert the classical modelling process: Instead of specifying the dynamics of a process that triggers the default, we specify the dynamics of the defaultable bond prices and forward rates and look for necessary and sufficient conditions for the absence of arbitrage opportunities. Of course we also have to define the time of default as a stopping time, but the emphasis lies on the modelling of a whole term structure of defaultable bonds in this setup. We specify the term structure of defaultable bonds in terms of the defaultable forward rates which allows us to use many of the results of the Heath, Jarrow and Morton (HJM) [19] model. Using this approach we explore the connection between the dynamics of defaultable interest rates and default-free interest rates and give necessary and sufficient conditions for the absence of arbitrage.

We start with a model in which the value of a defaultable bond drops to zero upon default. After deriving the interconnections between the dynamics of the risky interest rates and the defaultable bond prices, we derive the key relationship that *under the martingale measure the difference between the*

defaultable short rate and the default-free short rate is the intensity of the default process. This result drives the conditions for the absence of arbitrage that are derived subsequently, and the arbitrage-free dynamics of the defaultable bond prices. We find a very strong similarity between the defaultable and the default free interest rate dynamics and drift restrictions.
Next we explore how a model of the spread of the defaultable interest rates over the default-free interest rates may be used to add a default-risk module to an existing model of default-free interest rates.
In the following sections we extend the model to include positive recovery rates, reorganisations of the defaulted firms with the possibility of multiple defaults and uncertainty about the magnitude of the default. Even though it may seem that this will make the model far more complicated the restrictions for absence of arbitrage and the price dynamics remain unchanged.
The connection to the modelling of the short rates (instead of the forward rates) is shown in the next section. We find that instead of modelling the forward rates one can model the defaultable short rate alone. Here the only restriction is to ensure a positive spread between risk-free and defaultable short rate.
The paper is concluded by allowing the defaultable forward rates to change discontinously at default times if there are multiple defaults. Here we use the methods of Björk, Kabanov and Runggaldier (BKR) [4] and give the most general version of the model presented.
In the conclusion we mention some of the many potential extensions of this model.

2 Setup and Notation

For ease of exposition we first introduce the simplest setup which will be generalised in the following sections to include positive recovery rates, multiple defaults and jumps in the defaultable term structure.

The model is set in a filtered probability space $(\Omega, (\mathcal{F}_t)_{(t\geq 0)}, P)$ where P is some subjective probability measure. We assume the filtration $(\mathcal{F}_t)_{(t\geq 0)}$ satisfies the usual conditions[2].

The time of default is defined as follows:

Definition 1
The time of default *is a stopping time* τ. *We denote with* $N(t) := \mathbf{1}_{\{\tau\leq t\}}$ *the* default indicator function *and* $A(t)$ *the predictable compensator of* $N(t)$*, thus*

$$M(t) := N(t) - A(t)$$

[2]See Jacod and Shiryaev [22]. We also assume a large but finite time horizon.

is a (purely discontinuous) martingale. A is nondecreasing (because N is), predictable and of finite variation. Frequently we will assume that A has an intensity, i.e.

$$A(t) = \int_0^t h(s)\,ds. \tag{1}$$

The filtration $(\mathcal{F}_t)_{(t\geq 0)}$ is generated[3] by n Brownian motions $W^i,\ i = 1,\dots,n$ and the default indicator $N(t)$.

For the default risk-free bond markets we use the HJM setup:

DEFINITION 2

1. *At any time t there are* default-risk free zero coupon bonds *of all maturities $T > t$. The time-t price of the bond with maturity T is denoted by $B(t,T)$.*

2. *The* risk-free forward rate *over the period $[T_1, T_2]$ contracted at time t is defined (for $t \leq T_1 < T_2$)*

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left(\ln B(t, T_1) - \ln B(t, T_2)\right). \tag{2}$$

3. *If the T-derivative of $B(t,T)$ exists, the* instantaneous risk-free forward rate *at time t for date $T > t$ is defined as*

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T). \tag{3}$$

4. *The* instantaneous risk-free short rate $r(t)$,
the risk-free discount factor $\beta(t)$
and the risk-free bank account $b(t)$ *are defined by*

$$r(t) := f(t,t), \qquad \beta(t) := \exp\{-\int_0^t r(s)ds\}, \quad b(t) := 1/\beta(t). \tag{4}$$

We use similar notation to describe the term structure of the defaultable bonds.

DEFINITION 3

1. *At any time t there are* defaultable zero coupon bonds *of all maturities $T > t$. The time-t price of the bond with maturity T is denoted by $C(t,T)$. The payoff at time T of this bond is $\mathbf{1}_{\{\tau > T\}} = 1 - N(t)$: one unit of account if the default has not occurred until T, and nothing otherwise.*

[3] This assumption will be relaxed later on to include a marked point process in the case of multiple defaults.

2. *The* defaultable forward rate *over the period* $[T_1, T_2]$ *contracted at time* t *is defined (for* $t \leq T_1 < T_2$*)*

$$g(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left(\ln C(t, T_1) - \ln C(t, T_2) \right). \tag{5}$$

3. *If the* T*-derivative of* $C(t, T)$ *exists, the* instantaneous defaultable forward rate *at time* t *for date* $T > t$ *is defined as*

$$g(t, T) = -\frac{\partial}{\partial T} \ln C(t, T). \tag{6}$$

4. *The* instantaneous defaultable short rate $r^d(t)$,
the defaultable discount factor $\gamma(t)$
and the defaultable bank account $c(t)$ *are defined by*

$$r^d(t) := g(t, t), \quad \gamma(t) := \exp\{-\int_0^t r^d(s) ds\}, \quad c(t) := \mathbf{1}_{\{t<\tau\}} \frac{1}{\gamma(t)}. \tag{7}$$

All definitions of defaultable interest rates are only valid for times $t < \tau$ *before default.*

The defaultable forward rate $g(t, T_1, T_2)$ as it is defined above is *not* the value of a T_1-forward contract on a defaultable bond with maturity T_2, but the *promised* yield of the following portfolio:

short	one	defaultable bond	$C(t, T_1)$
long	$C(t, T_1)/C(t, T_2)$	defaultable bonds	$C(t, T_2)$.

A forward contract on the defaultable bond T_2 would involve a short position in the default free bond $B(t, T_1)$. See also section 4.2 for some consequences of this definition.

The defaultable bank account $c(t)$ is the value of \$ 1 invested at $t = 0$ in a defaultable zero coupon bond of very short maturity and rolled over until t, given there has been no default until t. It will play a similar role to the default-free bank account $b(t)$ in default-free interest rate modelling.

In the definition of the defaultable forward rates — to avoid taking logarithms of defaultable bond prices that are zero — we assume that a future default cannot be predicted with certainty. At any time $t < \tau$ strictly before default, and for every finite predicition-horizon T ($t < T < \infty$) the probability of a default until T is not one: $P[\tau \leq T | \mathcal{F}_t] < 1$. This can be achieved by setting the default time to be the first time at which a future default can be predicted with certainty: $\tau' := \inf\{t \geq 0 | \exists\, T < \infty \text{ s.t. } P[\tau \leq T | \mathcal{F}_t] = 1\}$.

We assume τ has been defined as above. This assumption is in keeping with the real-world legal provisions that a bankruptcy must be filed as soon as the fact of the bankruptcy is known. Furthermore it does not change any of the qualitative features of the model. In addition to this we assume that all (forward) interest rates have continuous paths and that the instantaneous forward rates are well-defined.

3 Pricing with Zero Recovery

3.1 Dynamics: The Risky Forward Rates

Given the above definitions we can start to explore the connections between the dynamics of the risky bond prices and the risky forward rates. We assume the following representation as stochastic integrals for the dynamics of the risky forward rates $g(t,T)$ and the risky bonds $C(t,T)$:

Assumption 1
1. The dynamics of the risky forward rates are given by

$$dg(t,T) = \alpha(t,T)\, dt + \sum_{i=1}^{n} \sigma_i(t,T)\, dW^i(t). \tag{8}$$

2. The dynamics of the risky bond prices are [4]

$$\frac{dC(t,T)}{C(t-,T)} = \mu(t,T)dt + \sum_{i=1}^{n} \eta_i(t,T)\, dW^i(t) - dN(t). \tag{9}$$

3. The integrands $\alpha(t,T), \sigma_i(t,T), \mu(t,T)$ and $\eta_i(t,T)$ are predictable processes that are regular enough to allow
– differentiation under the integral sign
– interchange of the order of integration
– partial derivatives with respect to the T-variable
– bounded prices $C(t,\cdot)$ for almost all $\omega \in \Omega$.

We start by analysing the consequences of the specification (8) of the risky forward rates. The *dynamics of the risky spot rate process* are [5]

$$\begin{aligned} r^d(t) &= g(t,t) = g(0,t) + \int_0^t \alpha(s,t)ds \\ &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)dW^i(s). \end{aligned} \tag{10}$$

[4] The notation $dY(t)/Y(t-) = dX(t)$ is a shorthand for $dY(t)/Y(t-) = dX(t)$ for $Y(t-) > 0$ and $dY(t) = 0$ for $Y(t-) = 0$.

[5] It is understood that dynamics of defaultable interest rates are always the dynamics before default $t < \tau$.

From definition (6) of the defaultable forward rates and definition 3 of the defaultable bonds the price of a risky zero coupon bond is given by

$$C(t,T) = (1 - N(t)) \exp\left\{ - \int_t^T g(t,s)\, ds \right\}. \tag{11}$$

The factor of $(1 - N(t))$ follows from the default condition $C(t,T) = 0$ for $t \geq \tau$. Writing $G(t,T) := \int_t^T g(t,s)\, ds$ this yields for $t \leq \tau$ using Itô's lemma

$$dC(t,T)/C(t-,T) = -dG(t,T) + d < G, G > -dN, \tag{12}$$

where we have used that G is continuous. For the process $G(t,T)$ we have

$$\begin{aligned}
G(t,T) - G(0,T) &= \int_t^T [g(t,s) - g(0,s)]\, ds - \int_0^t g(0,s)\, ds \\
&= \int_t^T\int_0^t \alpha(u,s)\, du\, ds + \sum_{i=1}^n \int_t^T\int_0^t \sigma_i(u,s)\, dW^i(u)\, ds \\
&\quad - \int_0^t g(0,s)\, ds \\
&= \int_0^t\int_t^T \alpha(u,s)\, ds\, du + \sum_{i=1}^n \int_0^t\int_t^T \sigma_i(u,s)\, ds\, dW^i(u) \\
&\quad - \int_0^t g(0,s)\, ds \\
&= \int_0^t\int_u^T \alpha(u,s)\, ds\, du + \sum_{i=1}^n \int_0^t\int_u^T \sigma_i(u,s)\, ds\, dW^i(u) \\
&\quad - \int_0^t g(0,s)\, ds - \int_0^t\int_u^t \alpha(u,s)\, ds\, du \\
&\quad - \sum_{i=1}^n \int_0^t\int_u^t \sigma_i(u,s)\, ds\, dW^i(u) \\
&= \int_0^t \hat{b}(u,T)\, du + \sum_{i=1}^n \int_0^t a_i(u,T)\, dW^i(u) \\
&\quad - \int_0^t g(0,s)\, ds - \int_0^t\int_0^s \alpha(u,s)\, du\, ds \\
&\quad - \sum_{i=1}^n \int_0^t\int_0^s \sigma_i(u,s)\, dW^i(u)\, ds \\
&= \int_0^t \hat{b}(u,T) - r^d(u)\, du + \sum_{i=1}^n \int_0^t a_i(u,T)\, dW^i(u)
\end{aligned}$$

where

$$a_i(t,T) := \int_t^T \sigma_i(t,v)\, dv \tag{13}$$

$$\hat{b}(t,T) := \int_t^T \alpha(t,v)\, dv. \tag{14}$$

The main tool in the equations above is Fubini's theorem and Fubini's theorem for stochastic integrals (see e.g. HJM [19] and Protter [31]).

With this result we reach the dynamics of the defaultable zero coupon bond prices

$$\begin{aligned}\frac{dC(t,T)}{C(t-,T)} &= \left[-\hat{b}(t,T) + r^d(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\right] dt \\ &\quad + \sum_{i=1}^{n} a_i(t,T)\, dW^i(t) - dN(t). \end{aligned} \tag{15}$$

The final condition $C(T,T) = 0$ for $\tau < T$ is automatically satisfied by the functional specification of C.

The above derivation of the dynamics of $G(t,T)$ follows the derivation of the dynamics of the risk-free bond prices in HJM [19]. Here the only addition is the jump term $-dN(t)$ which is introduced by the default process. Summing up:

PROPOSITION 1
1. Given the dynamics of the risky forward rates (8)
(i) the dynamics of the risky bond prices are given by

$$\begin{aligned}\frac{dC(t,T)}{C(t-,T)} &= \left[-\hat{b}(t,T) + r^d(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\right] dt \\ &\quad + \sum_{i=1}^{n} a_i(t,T)\, dW^i(t) - dN(t). \end{aligned} \tag{16}$$

where $a_i(t,T)$ and $\hat{b}(t,T)$ are defined by (13) and (14) resp..
(ii) the dynamics of the risky short rate are given by

$$\begin{aligned} r^d(t) &= g(t,t) = g(0,t) + \int_0^t \alpha(s,t)ds \\ &+ \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)dW^i(s). \end{aligned} \tag{17}$$

2. Given the dynamics (9) of the risky bond prices the dynamics of the risky forward rates are (for $t \leq \tau$) given by (8) with

$$\alpha(t,T) = \sum_{i=1}^{n} \eta_i(t,T)\frac{\partial}{\partial T}\eta_i(t,T) - \frac{\partial}{\partial T}\mu(t,T) \tag{18}$$

$$\sigma_i(t,T) = -\frac{\partial}{\partial T}\eta_i(t,T). \tag{19}$$

Proof: 1.) has been derived above, 2.) follows from Itô's lemma on $\ln C(t,T)$ and taking the partial derivative w.r.t. T.

These relationships are well-known in the case of the default-risk free term structure. Assume the following dynamics of the risk-free forward rates $f(t,T)$ and the risk-free bond prices $B(t,T)$ (the risk-free dynamics are marked with a dash '):

Assumption 2

1. The dynamics of the default risk free forward rates are given by

$$df(t,T) = \alpha'(t,T)\, dt + \sum_{i=1}^{n} \sigma_i'(t,T)\, dW^i(t). \tag{20}$$

2. The dynamics of the default risk free bond prices are

$$\frac{dB(t,T)}{B(t-,T)} = \mu'(t,T)dt + \sum_{i=1}^{n} \eta_i'(t,T)\, dW^i(t). \tag{21}$$

3. The integrands $\alpha'(t,T), \sigma_i'(t,T), \mu'(t,T)$ and $\eta_i'(t,T)$ are predictable processes that are regular enough to allow

– differentiation under the integral sign

– interchange of the order of integration

– partial derivatives with respect to the T-variable

– bounded prices $B(t,\cdot)$ for almost all $\omega \in \Omega$.

The dynamics of the risk-free term structure do not contain any jumps at τ. Volatilities and drifts may change at τ but the direct impact of the default is only on the risky bonds.

Given these dynamics the following proposition is a well-known result by Heath, Jarrow and Morton [19].

Proposition 2

1. Given the dynamics of the risk free forward rates (20)

(i) the dynamics of the risk free bond prices are given by

$$\begin{aligned} \frac{dB(t,T)}{B(t-,T)} &= \left[-\hat{b}'(t,T) + r(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i'^2(t,T)\right] dt \\ &\quad + \sum_{i=1}^{n} a_i'(t,T)\, dW^i(t). \end{aligned} \tag{22}$$

where $a_i'(t,T)$ and $\hat{b}'(t,T)$ are defined by

$$a_i'(t,T) := \int_t^T \sigma_i'(t,v)\, dv \tag{23}$$

$$\hat{b}'(t,T) := \int_t^T \alpha'(t,v)\, dv. \tag{24}$$

(ii) the dynamics of the risk free short rate are given by

$$\begin{aligned} r(t) &= f(t,t) = f(0,t) + \int_0^t \alpha'(s,t)ds \\ &+ \sum_{i=1}^n \int_0^t \sigma_i'(s,t)dW^i(s). \end{aligned} \tag{25}$$

2. Given the dynamics (21) of the risk free bond prices the dynamics of the risky forward rates are given by (20) with

$$\alpha'(t,T) = \sum_{i=1}^n \eta_i'(t,T)\frac{\partial}{\partial T}\eta_i'(t,T) - \frac{\partial}{\partial T}\mu'(t,T) \tag{26}$$

$$\sigma_i'(t,T) = -\frac{\partial}{\partial T}\eta_i'(t,T). \tag{27}$$

3.2 Change of Measure

Now that the connections between the dynamics of the risky zero coupon bonds and the forward rates are clarified, we can start analysing the conditions for absence of arbitrage opportunities in this model. We use the following standard definition:

DEFINITION 4
There are no arbitrage opportunities if and only if there is a probability measure Q equivalent to P under which the discounted security price processes become local martingales. This measure Q is called the martingale measure, *and for any security price process $X(t)$ the discounted price process is defined as $X(t)/b(t)$.*

The main tool to classify all to P equivalent probability measures is the following version of Girsanov's Theorem (see Jacod and Shiryaev [22] III.3 and III.5 and BKR [4]):

THEOREM 1
Assume that the default process has an intensity. Let λ be a n-dimensional predictable processes $\lambda_1(t), \ldots, \lambda_n(t)$ and $\phi(t)$ a strictly positive predictable process with

$$\int_0^t \|\lambda(s)\|^2 ds < \infty, \qquad \int_0^t |\phi(s) - 1| h(s) ds < \infty$$

for finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \sum_{i=1}^n \lambda_i(t)dW^i(t) + (\phi(t) - 1)dM(t).$$

Assume that $\mathrm{E}[\, L(t)\,] < \infty$ *for finite* t.
Then there is a probability measure Q *equivalent to* P *with*

$$dQ_t = L_t dP_t \tag{28}$$

such that

$$dW(t) - \lambda(t)dt = d\tilde{W}(t) \tag{29}$$

defines $\tilde{W}$ *as* Q*-Brownian motion and*

$$h_Q(t) = \phi(t)h(t) \tag{30}$$

is the intensity of the default indicator process under Q.
Furthermore every probability measure that is equivalent to P *can be represented in the way given above.*

In the financial context here the processes λ_i are the *market prices of diffusion risk*, and the process ϕ represents the *market price of jump risk* (per unit of jump intensity). To ensure absence of arbitrage the financial requirement of a well-defined set of market prices of risk with validity for all securities translates into the mathematical requirement of having a well-defined intensity process for the change of measure.

Given the risky bond price dynamics (9) the change of measure to the martingale measure leaves the volatilities of the risky bond prices unaffected, the same is true of the integral with respect to dN (the *compensator* of this integral has changed, though), the only effect is a change of drift in the defaultable bond price process.

From now on we will assume that the change of measure to the martingale measure has already been performed. The results of the preceding section on the dynamics remain valid if the underlying measure is the martingale measure. Therefore we simplify notation such that all specifications in section 3.1 are already with respect to Q. [6]

3.3 Absence of Arbitrage

By Itô's lemma we require under the martingale measure for absence of arbitrage that for all $t > T$

$$\mathrm{E}\left[\, \tfrac{dC(t,T)}{C(t-,T)}\,\right] = r(t)\, dt. \tag{31}$$

This means using (16)

$$r(t)\, dt \quad = \quad \mathrm{E}\left[\, \tfrac{dC(t,T)}{C(t-,T)}\,\right]$$

[6] If $P = Q$ then $\lambda \equiv 0$ and $\phi \equiv 1$.

$$
\begin{aligned}
&= \mathrm{E}\left[\left[-\hat{b}(t,T) + r^d(t) + \tfrac{1}{2}\textstyle\sum_{i=1}^n a_i^2(t,T)\right] dt \right] \\
&\quad + \mathrm{E}\left[\textstyle\sum_{i=1}^n a_i(t,T)\, dW^i(t) - dN(t) \right] \\
r(t) &= -\hat{b}(t,T) + r^d(t) + \tfrac{1}{2}\sum_{i=1}^n a_i^2(t,T) - h(t) \qquad (32)
\end{aligned}
$$

Now we have to take a closer look at the compensator $A(t)$ of the default indicator process $N(t)$. We assumed that A is continuous and therefore has an intensity $dA(t) = h(t)dt$. Then we note that

$$
-M(t) = -N(t) + A(t) = -N(t) + \int_0^t h(s)\, ds \qquad (33)
$$

is a martingale by the definition of the predictable compensator. The Doleans-Dande exponential of $-M$ is a martingale, too:

$$
\exp\{\int_0^t h(s)ds\} \prod_{s \le t}(1 - \Delta M(s)) = \mathbf{1}_{\{\tau > t\}} \exp\{\int_0^t h(s)ds\}. \qquad (34)
$$

Here we used that the only jump in $-M$ is a jump of -1 at $t = \tau$, which moves the Doleans-Dade exponential down to zero.

Now consider the value process of the *risky* bank account $c(t)$, i.e. the development of \$ 1 invested at time 0 at the risky short rate and rolled over from then on. By definition its value at time t is

$$
c(t) = \mathbf{1}_{\{\tau > t\}} \exp\{\int_0^t r^d(s)ds\}. \qquad (35)
$$

The similarity to equation (34) above will now be used. Under the martingale measure the discounted (discounting with the *risk-free* interest rate) value process of c

$$
\bar{c}(t) := \frac{c(t)}{b(t)} = \mathbf{1}_{\{\tau > t\}} \exp\{\int_0^t r^d(s) - r(s)\; ds\} \qquad (36)
$$

must be a martingale. But this is also a Doleans-Dade exponential, the Doleans-Dade exponential of

$$
\hat{M}(t) := -N(t) + \int_0^{t\wedge\tau} r^d(s) - r(s)\; ds, \qquad (37)
$$

which in turn must also be a martingale. (The martingale property can also be seen from $\hat{M}(t) = \int_0^t \frac{1}{\bar{c}(s-)} d\bar{c}(s)$ and the uniqueness of the Doleans-Dade exponential up to τ.) We use the freedom we had in the specification of $r^d(t)$ for $t \ge \tau$ and set $r^d(t) := r(t)$ for $t \ge \tau$.

Taking the difference of (33) and (37)

$$
M(t) - \hat{M}(t) = \int_0^t h(s) - r^d(s) + r(s)\; ds \qquad (38)
$$

one sees that – while the l.h.s. is a martingale – the r.h.s. is predictable, the only predictable martingales are constant, thus we have for all s

$$h(s) = r^d(s) - r(s). \tag{39}$$

The hazard rate $h(s)$ of the default is exactly the short interest rate spread. Note that this relationship can also be inverted to define the risky short rate as $r^d(s) := r(s) + h(s)$.

Equation (39) is the key relation that yields, substituted in (32), as necessary condition for the absence of arbitrage:

$$\hat{b}(t,T) = \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T). \tag{40}$$

Substituting in this condition the definition of $\hat{b}$ yields the results of the following theorem.

THEOREM 2
The following are equivalent:
1. The measure under which the dynamics are specified is a martingale measure.
2. (i) The short interest rate spread is the intensity of the default process. It is positive.

$$h(t) = r^d(t) - r(t) > 0. \tag{41}$$

(ii) The drift coefficients of the defaultable forward rates satisfy for all $t \le T$, $t < \tau$

$$\int_t^T \alpha(t,v)dv = \tfrac{1}{2}\sum_{i=1}^{n}\left(\int_t^T \sigma_i(t,v)dv\right)^2 \tag{42}$$

or, differentiated,

$$\alpha(t,T) = \sum_{i=1}^{n} \sigma_i(t,T)\int_t^T \sigma_i(t,v)dv. \tag{43}$$

(iii) The drift coefficients of the risk-free forward rates satisfy for all $t \le T$

$$\alpha'(t,T) = \sum_{i=1}^{n} \sigma_i'(t,T)\int_t^T \sigma_i'(t,v)dv. \tag{44}$$

3. (i) $r^d(t) - r(t) = h(t) > 0$.
(ii) The dynamics of the risky bond prices are given by

$$\frac{dC(t,T)}{C(t-,T)} = r^d(t)dt + \sum_{i=1}^{n} a_i(t,T)dW^i(t) - dN(t) \tag{45}$$

or, solving the s.d.e.

$$\begin{aligned} C(t,T) &= \mathbf{1}_{\{\tau>t\}} C(0,T) \exp\left\{ \int_0^t r^d(s)\,ds - \tfrac{1}{2}\sum_{i=1}^n \int_0^t a_i^2(s,T)\,ds \right. \\ &\left. + \sum_{i=1}^n \int_0^t a_i(s,T)\,dW^i(s) \right\}. \end{aligned} \tag{46}$$

(iii) The dynamics of the risk-free bonds satisfy under the martingale measure

$$\frac{dB(t,T)}{B(t-,T)} = r(t)dt + \sum_{i=1}^n a_i'(t,T)dW^i(t). \tag{47}$$

Proof:
1.) $\Rightarrow$ 2.) : (i) and (ii) have been derived above, (iii) has been shown in HJM [19].
2.) $\Rightarrow$ 3.) : 2.(i) and 3.(i) coincide, 3.(ii) follows from 2.(ii) and (i) by substituting in proposition 1, again (iii) is by HJM [19].
3.) $\Rightarrow$ 1.): follows from the definition of the martingale measure.

q.e.d.

The most important result of this section is equation (43), the defaultable-bond equivalent of the well-known Heath-Jarrow-Morton drift-restriction. This restriction has been derived for default risk-free bonds in HJM [19], and, as we see here, it is also an important part of the modelling of the defaultable bonds' dynamics.

Another important insight is that precise knowledge of the nature of the default process N and its compensator A is not necessary for setting up an arbitrage-free model of the term structure of defaultable bonds. With the restricitions 2.) of theorem 2 one can set up a model of defaultable bonds that uses readily observable market data (the term structure of the risky forward rates) as input, without having to try and find out about the precise nature of N.

We assumed that the default process has an intensity: $dA(t) = h(t)dt$. This implies that the time of default is a totally inaccessible stopping time. Dropping this assumption (to allow discontinuities in A) one sees readily from the derivation of equation (39) that the risky spot rate r^d cannot be finite at jumps of A. One would have to specify the risky term structure in a more general way by defining a process $R^d(t) := \int_0^t r^d(s)ds$ which will be well-defined and can account for the jumps in A. Similar definitions will be needed for the forward rates. Then (39) translates into $dR^d(t) = r(t)dt + dA(t)$. With this specification we can also drop the initial assumption that a default cannot be predicted with certainty.

It is important to note that the default risk-free term structure and the defaultable term structure must satisfy the conditions simultaneously. This will become clearer in the following version of theorem 2 that is set under the *subjective* measure P:

THEOREM 3
If the dynamics are given under a subjective probability measure P the following are equivalent:
1.) The dynamics are arbitrage-free.
2.) There are predictable processes $\lambda_1(t), \ldots, \lambda_n(t)$ and a strictly positive predictable process $\phi(t)$ that satisfy the regularity conditions of theorem 1 such that for all $t < T$:
(i) The difference between risk-free and defaultable short rate is ϕ times the hazard rate:

$$r^d(t) - r(t) = \phi(t)h(t). \tag{48}$$

(ii) The defaultable and the default-free forward rates satisfy

$$-\alpha(t,T) + \sum_{i=1}^{n} \sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv = \sum_{i=1}^{n} \sigma_i(t,T)\lambda_i(t) \tag{49}$$

$$-\alpha'(t,T) + \sum_{i=1}^{n} \sigma'_i(t,T) \int_t^T \sigma'_i(t,v)\,dv = \sum_{i=1}^{n} \sigma'_i(t,T)\lambda_i(t) \tag{50}$$

a.s. for all $t < T$.

Proof:
1.) $\Leftrightarrow$ There is an equivalent martingale measure Q
$\Leftrightarrow$ (Theorem 1) There are predictable processes $\lambda_1(t), \ldots, \lambda_n(t)$ and a strictly positive predictable process $\phi(t)$ that satisfy the regularity conditions of theorem 1 such that (using theorem 2):
(i) $h^Q = \phi h$ and $h^Q(t) = r^d(t) - r(t) = \phi(t)h(t)$.
(ii) $dW_i^Q = dW_i - \lambda_i dt$ and

$$\begin{aligned} dg(t,T) &= \sum_{i=1}^{n} \left(\sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv \right) dt + \sum_{i=1}^{n} \sigma_i(t,T) dW_i^Q(t) \\ df(t,T) &= \sum_{i=1}^{n} \left(\sigma'_i(t,T) \int_t^T \sigma'_i(t,v)\,dv \right) dt + \sum_{i=1}^{n} \sigma'_i(t,T) dW_i^Q(t). \end{aligned}$$

By substituting the P-dynamics of $g(t,T)$ and $f(t,T)$ and equating coefficients the proof is concluded.

q.e.d.

From theorem 3 one sees directly that there is only one set of market prices of risk for both the defaultable and the risk-free term structure. This follows from the fact that there is only one set of underlying Brownian motions that drive the market. The market price of jump risk ϕ is uniquely determined by equation (48) which can be used as defining relationship for ϕ. Even with defaultable zero coupon bonds one can set up portfolios that are hedged against default risk (by making sure that the sum of the portfolio weights is zero). These portfolios must be related to the risk-free term structure in their dynamics, and this relation is given in the two theorems above.

4 Modelling the Spread between the Forward Rates

When trying to connect the dynamics of the defaultable term structure and the default-free term-structure the most important relationship is (41):

$$0 < h(t) = r^d(t) - r(t).$$

In many cases a model of the risk-free interest rates and forward rates will already be in place and the task is to find a specification of a model of the defaultable term structure that does not violate (41). If one directly estimated and implemented a model for the defaultable term structure $g(t,T)$ without reference to the existing model of the default-free term structure, situations where $r^d < r$ are bound to arise and the (combined) model will not be arbitrage-free.

A way around this problem is not to model the forward rates but the difference between these:

DEFINITION 5
The forward rate spread $h(t,T)$ *is defined as the difference between the defaultable forward rate and the default-free forward rate:*

$$h(t,T) = g(t,T) - f(t,T). \tag{51}$$

Under the martingale measure we have

$$h(t) = h(t,t)$$

which justifies the slight abuse of notation by doubly defining h.

Now one has to find a model for $h(t,T)$ which is compatible with theorem 2. The advantage of modelling $h(t,T)$ instead of $g(t,T)$ is that (41) reduces to the well known problem of ensuring that $h(t,t) > 0$. and we can hope to use some of the extensive literature on interest rate models with positive short rates. We use the following dynamics for h:

Assumption 3

The dynamics of h are given by

$$h(t,T) - h(0,T) = \int_0^t \alpha^h(v,T)\,dv + \sum_{i=1}^n \sigma_i^h(v,T)dW^i(v). \tag{52}$$

Then

$$\begin{aligned}\alpha(v,t) &= \alpha'(v,t) + \alpha^h(v,t) && (53)\\ \sigma_i(v,t) &= \sigma_i'(v,t) + \sigma_i^h(v,t). && (54)\end{aligned}$$

In place of the drift restriction (43) we reach:

Corollary 1

Let the risk-free interest rates satisfy the HJM drift restriction (44). A model for the risky forward rates based on the forward rate spread $h(t,T)$ must imply under the martingale measure

$$\begin{aligned}\alpha^h(t,T) &= \sum_{i=1}^n \left[\sigma_i'(t,T)\int_t^T \sigma_i^h(t,v)\,dv\right.\\ &\quad +\sigma_i^h(t,T)\int_t^T \sigma_i'(t,v)\,dv\\ &\quad \left.+\,\sigma_i^h(t,T)\int_t^T \sigma_i^h(t,v)\,dv\right]. \end{aligned} \tag{55}$$

Proof: Substitute (53) and (54) in (43).

Again – as in the original HJM model – the drift of the spread is given in terms of the volatilities of the interest rates and spreads. Given these drift specifications one has to require that the process $h(t,t)$ is nonnegative. This will enable us to add a defaultable interest rate model to a given model of the default-free interest rate in a modular fashion.

If one chooses a specification of the dynamics of h that has nonnegative $h(t,t)$ a.s. under the subjective measure, this will ensure that $h(t,t)$ will be nonnegative a.s. under the martingale measure, too.

It will be interesting to analyse some possible specifications and the problems that may arise when modeling the spread structure.

4.1 Independence of Spreads and Risk-Free Rates

The easiest way to specify the spreads is to avoid the cross-variation terms with the risk-free term-structure in (55). Assume that every factor W_i either influences f or h but never both. Then $\forall\, i = 1, \ldots, n$

$$\begin{aligned} \sigma_i^h(t,T) \neq 0 \quad &\Rightarrow \quad \int_t^T \sigma_i^\prime(t,v)\, dv = 0 \\ \sigma_i^\prime(t,T) \neq 0 \quad &\Rightarrow \quad \int_t^T \sigma_i^h(t,v)\, dv = 0 \end{aligned} \tag{56}$$

and the drift restriction for the spreads becomes the usual HJM restriction:

Corollary 2
(i) If σ_i^h and $\sigma_i^\prime$ satisfy (56) then (under the martingale measure)

$$\alpha^h(t,T) = \sum_{i=1}^n \sigma_i^h(t,T) \int_t^T \sigma_i^h(t,v)dv \tag{57}$$

and $h(t,t) > 0$ a.s. are necessary and sufficient for absence of arbitrage.
(ii) Equation (56) is satisfied if $h(t,T_1)$ and $g(t,T_2)$ are independent for all $t \leq T_1,\ t \leq T_2$, i.e. the term structure of the spreads and the term structure of the risk-free forward rates are independent.

Proof: The first part follows directly by substituting the assumptions (56) in (55). For the last part observe that independence of the term structures of spreads and risk-free rates implies that

$$\sigma_i^\prime(t,T_1)\sigma_i^h(t,T_2) = 0$$

for (almost) all $t \leq T_1, T_2$ which in turn implies (56) directly. q.e.d.

Note that strict stochastic independence of h and g is not needed. One might imagine a model where the term structure of the spreads is driven by an additional Brownian motion alone (this will ensure (56)), but the volatility of the spread might still depend on the level of the interest rates.

Satisfying the positivity requirement (41) on $h(t,t)$ becomes very easy in the setup of corollary 2: One can use any interest rate model for $h(t,T)$ that is known to generate positive short rates, e.g. the square root model of Cox, Ingersoll and Ross [9][10] or the model with lognormal interest rates by Sandmann and Sondermann [32].

4.2 Negative Forward Spreads

There is one additional caveat when using the nonnegative rate model for the forward rate spreads: Even though we require that the 'short' spread $h(t,t) > 0$ is greater than zero, we might still have that a 'forward' spread $h(t,T)$ $(T > t)$ becomes negative.

As an example how this can arise we consider the two-period economy with points in time $t = 0, 1, 2$ from figure 1. There are three states $\omega_1, \omega_2, \omega_3$

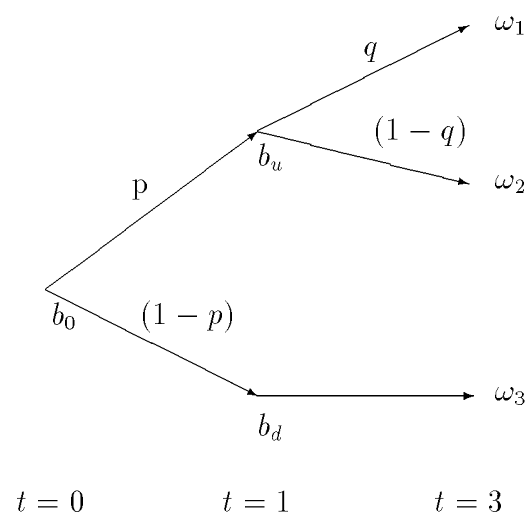

Figure 1: Example for negative forward spreads

and the filtration is $\mathcal{F}_0 = \{\{\omega_1, \omega_2, \omega_3\}, \emptyset\}$; $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \omega_3)$; $\mathcal{F}_2 = \sigma(\omega_1, \omega_2, \omega_3)$. The states have risk-neutral probabilites $P(\omega_1) = pq$; $P(\omega_2) = p(1-q)$; $P(\omega_3) = (1-p)$. There are risk-free bonds $B(0,1), B(0,2)$ and risky bonds $C(0,1), C(0,2)$. In state ω_1 both risky bonds survive, in ω_2 only the risky bond with maturity 2 defaults and in ω_3 both bonds default (with zero recovery).

The initial risk-free term structure is given by $B(0,1) = \beta_0$ and $B(0,2) = \beta_0(p\beta_u + (1-p)\beta_d)$. Thus the bond prices are

$$
\begin{aligned}
B(0,1) &= \beta_0 \\
B(0,2) &= \beta_0(p\beta_u + (1-p)\beta_d) \\
C(0,1) &= p\,\beta_0 \\
C(0,2) &= p\,q\,\beta_0\beta_u \\
\hat{C}(0,1) &= p \\
\hat{C}(0,2) &= \frac{pq}{p\,\beta_u + (1-p)\,\beta_d}.
\end{aligned}
$$

The forward rates are

$$
\begin{aligned}
f(0,1,2) &= -\ln(p\,\beta_u + (1-p)\,\beta_d) \\
g(0,1,2) &= -\ln(q\,\beta_u).
\end{aligned}
$$

We have a negative forward spread $h(0,1,2) < 0$ or $g(0,1,2) < f(0,1,2)$ if

$$p\,\beta_u + (1-p)\,\beta_d < q\,\beta_u \tag{58}$$

holds. This is equivalent to

$$q > p + (1-p)\frac{\beta_d}{\beta_u}, \tag{59}$$

so a necessary condition for negative forward spreads is that $\beta_d < \beta_u$. Choose for instance $p = 0.9, \quad \beta_d/\beta_u = 0.9, \quad q = 0.995$.

This example can be regarded as a 'snapshot' from a continuous-time model in which the relevant prices and probabilities have been aggregated to the two-period example.

The condition $\beta_d < \beta_u$ means that $r_d > r_u$. For negative forward spreads to arise we need
– either q and p are of the same order of magnitude, then the ratio β_d/β_u must be very small, $r_u << r_d$,
– or q is much larger than p, then β_u can be of the same order as β_d. In practice $q >> p$ only occurs if $T_1 >> T_2 - T_1$. But then $\beta_d/\beta_u \sim 1$ because of the short horizon $T_2 - T_1$ which is very far in the future, as well.

The occurrence of negative forward spreads is due to the special way in which we defined the risky forward rates. It is not possible to exploit the negative forward spread as an arbitrage-opportunity because the portfolio one would typically use for that will be destroyed by an early default:

In the risk-free bonds one can set up a portfolio that replicates the payoff of a default free forward contract, but set up in defaultable bonds this portfolio disappears in the case of an early default. If one had gone long a risk-free forward contract and short a *replicating portfolio* of a *defaultable* forward contract, an early default (which eliminates the replicating portfolio for the risky forward contract) leaves one with the default-risk free half of the portfolio, which now is exposed to changes in the risk-free term structure. If the subsequent risk-free interest rates are high the remainder of the portfolio will generate a loss.

Summarizing, negative forward spreads can only occur if there is a strong correlation between early default (event ω_3) and high interest rates (β_d small), and a strong correlation between early survivial (events u) and low interest rates (β_u large). It has to investigated empirically whether negative forward spreads actually occur in the markets.

5 Positive Recovery and Restructuring

In the preceding sections we assumed that a defaultable zero coupon bond has a payoff of zero upon default. This assumption is unnecessarily restrictive and does not agree with market experience.

In actual markets a default of a bonds does not mean that this bond becomes worthless. Usually there is a positive recovery; in their empirical study Franks and Torous [17] find recovery rates between 40 and 80 percent for distressed exchanges and Chapter 11 reorganisations.
A second observation from actual bankruptcy procedures is that the majority of firms in financial distress are reorganised and re-floated, they are not liquidated.
Thirdly, even though on average about one third of the compensation to the holders of defaulted bonds is in cash , most of the compensation payments (about two thirds) are in terms of new securities of the defaulted and restructured firm.
Fourth, if a firm is reorganized and the payoff to the defaulted debtors is in terms of securities of the new firm, a second default of this firm on their (new) debt is possible. We have the possibility of multiple defaults with in principle any number of defaults (each with subsequent restructuring of the defaulted firm's debt).

The main results of the preceding sections are still valid if the recovery of the bond is positive and not zero. We choose the following setup including the possibility of multiple defaults:

If a default occurs a *restructuring* of the debts occurs. Holders of the old debt loose a fraction of q of their claims, where $q \in [0,1]$ is possibly unpredictable, but known at default.

> A pre-default claim of \$ 1 face value becomes a claim of \$ $(1-q)$ face value after the default. The maturity of the claim remains unchanged.

This model mimicks the effect of a rescue plan as it is described in many bankruptcy codes: The old claimants have to give up some of their claims in order to allow for rescue capital to be invested in the defaulted firm. They are *not* paid out in cash [7] (this would drain the defaulted firm of valuable liquidity) but in 'new' defaultable bonds of the same maturity. As the vast majority of defaulted debtors continue to operate after default, a representation of the loss of a defaulted bond in terms of a reduction in face value is possible even if the actual payoff procedure is different.

[7] The holder of a defaulted bond is free to sell this bond on the market, though.

For ease of modelling we use the convention that the *defaultable forward rates are quoted with respect to a bond of face value 1 $.*

The reduction in the face value of a bond in default is *not* reflected in the forward rates. This convention enables us to separate the effects of changes in interest rates (representing expectations on future defaults) and the direct effect of the default.

Mathematically the setup is as follows:

ASSUMPTION 4
(i) Defaults occur at the stopping times $\tau_1 < \tau_2 < \dots$.
(ii) At each time τ_i *of default a loss quota* $q_i \in E$ *is drawn from a measurable space* $(E, \mathcal{E})$, $E \subset I\!R$, *the* mark space. *(Usually* $E = [0, 1]$ *with the Borel sets.)*
(iii) The double sequence $(\tau_i, q_i), i \in I\!N_+$ *defines a marked point process* [8] *with defining measure*

$$\mu(t, \omega;\, dq, dt) \tag{60}$$

and predictable compensator

$$\nu(\omega, t)(dt, dq) = K(\omega, t)(dq)\; h(t)\, dt \tag{61}$$

(iv) Consider the defaultable zero coupon bond $C(0, T)$. *At time* T, *the maturity of this bond, it pays out*

$$Q(T) := \prod_{\tau_i \leq T} (1 - q_i), \tag{62}$$

the remainders after all fractional default losses. [9] $Q(t)$ *can be represented as a Doleans-Dade exponential:* $Q(0) = 1$ *and*

$$\frac{dQ(t)}{Q(t-)} = -\int_0^1 q\, \mu(dq, dt). \tag{63}$$

(v) The filtration is generated by the Brownian motions W^i *and the marked point process* μ.
(vi) We assume sufficient regularity on the marked point process μ *to justify all subsequent manipulations.*
– The sequence of default times is nonexplosive.
– μ *is a multivariate point process (see [22]).*
– $\int_0^t \int_E K(dq) h(s)\, ds < \infty$ *for all* $t < \infty$.
– The processes introduced in definition 6 are square-integrable.
– The resulting bond prices are bounded.

---

[8] For a general reference on marked point processes see Jacod and Shiryaev [22] and Bremaud [8].

[9] It will be clear from the context whether Q denotes the martingale measure or the accumulated fractional default losses. The latter will usually be the case from now on.

For the subsequent analysis we need to define the following processes:

Definition 6
The default counter function $N'(t)$, the instantaneous expected loss rate $q(t)$, the default compensator $A'(t)$ and the default martingale $M'(t)$ are defined as:

$$N'(t) := \int_0^t \int_0^1 q\,\mu(dq, ds) \tag{64}$$

$$q(t) := \int_0^1 q\,K(dq) \tag{65}$$

$$A'(t) := \int_0^t \int_0^1 q\,K(dq)h(s)ds = \int_0^t q(s)h(s)\,ds \tag{66}$$

$$M'(t) := N'(t) - A'(t). \tag{67}$$

Note that A' is the predictable compensator of N', and M' is a martingale, and $dQ(t)/Q(t-) = -dN(t)$.

This modelling approach has much in common with the *fractional recovery* introduced by Duffie et.al. in [15] [12] [13] [14].

Duffie specifies the payoff in default to be a predictable fraction $(1-q)$ of the value of a 'non-defaulted but otherwise equivalent security'. This is inspired by the default procedures in swap contracts. In Duffie's mathematical model the value V_τ of the defaulted security directly after default (i.e. the payoff in default) is specified as $V_\tau := (1-q)V_{\tau-}$, the fraction $(1-q)$ times the value of the same security directly before default. Unfortunately Duffie's model does not work for times of default that are predictable:
Assume the time of default is predictable, i.e. τ is $\mathcal{F}_{\tau-}$-measurable. We set $r \equiv 0$ to keep the example simple. Then the price of the bond just before τ is given by $V_{\tau-}$ and the price of the bond just after τ is $(1-q)V_{\tau-}$. Under the martingale measure we need $0 = \mathrm{E}[\,dV\,] = (1-q)V_{\tau-} - V_{\tau-} = qV_{\tau-}$. This implies that $V_{\tau-} = 0$, a zero recovery, if there is a positive loss $q > 1$ in default. A similar argument shows that Duffie's approach implies zero recovery whenever the time of default is not totally inaccessible (loosely speaking if $Q(\tau = t|\,\mathcal{F}_{t-}) > 0$ can arise). In many models (e.g. all models based on the firm's value approach) the time of default is predictable.

The model presented here can be extended to include the possibility of predictable times of default. (See the remarks at the end of section 5.) In addition to this we avoid the backward recursive stochastic integral equations that are necessary in the Duffie model.

Furthermore we include *magnitude risk* in our setup. The magnitude of the default is uncertain and the actual realisation of the loss q_i need not be predictable, it can be considered as a random draw at τ_i from the distribution $K(dq)$. This distribution may itself be stochastic. Except for the model of Madan and Unal [28] all models of default risk assume a constant or predictable recovery rate of the defaultable bond.

Assumption 5
In the presence of multiple defaults:
(i) The dynamics of the defaultable rates are given by assumption 1.1.
(ii) The dynamics of the defaultable bond prices are as in assumption 1.2. but with N' replacing N:

$$\frac{dC(t,T)}{C(t-,T)} = \mu(t,T)dt + \sum_{i=1}^{n} \eta_i(t,T)dW^i(t) - dN'.$$

(iii) The defaultable bank account is $c(t) = Q(t)\exp\{\int_0^t r^d(s)\,ds\}$.
(iv) The dynamics of the default risk free rates and bond prices are given by assumption 21. and 2.2.
(v) There is no total loss: $q_i < 1$ a.s.

Assumption 5 implies that all forward rates and the process $G(t,T)$ are continuous at times of default.

For the risk-free rates this is justifiable in most cases (unless a large souvereign debtor is concerned), but the risky rates should be modelled by explicitly allowing for dependence on the defaults μ. A default will usually discontinuously change the market's estimation of the future likelihood of defaults and thus the risky forward rates. This effect will be included in a later section, here we assume that the only direct effect of a default is the reduction of the face value of the defaultable debt. Nevertheless we allow the default to influence the diffusion parameters of the forward rates which distinguishes this setup from the literature on Cox processes (see Lando [25]).

6 Pricing with Recovery

The analysis of the pricing of defaultable zero coupon bonds goes along the lines of sections 3,4 and 5, where we increasingly have to introduce the theory of marked point processes. Standard references are Jacod and Shiryaev [22], and Bremaud [8] for the mathematical theory and Björk, Kabanov and Runggaldier [3] [4] for the application to interest rate theory[10].

6.1 Change of Measure

First, Girsanov's theorem (theorem 1) becomes [11]

[10] For other financial applications see also Merton [30] and Jarrow and Madan [23].
[11] See Jacod and Shiryaev [22] and Björk, Kabanov and Runggaldier [4].

THEOREM 4

Let λ be a n-dimensional predictable processes $\lambda_1(t), \dots, \lambda_n(t)$ and $\Phi(t,q)$ a strictly positive predictable function [12] *with*

$$\int_0^t \|\lambda(s)\|^2 ds < \infty, \qquad \int_0^t \int_E |\phi(s,q)|\, K(dq)\, h(s) ds < \infty$$

for finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \sum_{i=1}^{n} \lambda_i(t) dW^i(t) + \int_E (\Phi(t,q) - 1)(\mu(dt,dq) - \nu(dt,dq)).$$

Assume that $\mathrm{E}[\, L(t)\,] < \infty$ for finite t.
Then there is a probability measure Q equivalent to P with

$$dQ_t = L_t dP_t \tag{68}$$

such that

$$dW(t) - \lambda(t)dt = d\tilde{W}(t) \tag{69}$$

defines $\tilde{W}$ as Q-Brownian motion and

$$\nu_Q(dt,dq) = \Phi(t,q)\nu(dt,dq) \tag{70}$$

is the predictable compensator of μ under Q.
Every probability measure that is equivalent to P can be represented in the way given above.

Proof: BKR [4].
The only change to theorem 1 is the new predictable compensator $\Phi(t,q)\nu(dt,dq)$ of the marked point process. Instead of a single market price of risk for the jump risk we now have a market price of risk for each subset $e \in \mathcal{E}$ of the marker space. The market price of risk of a default with loss $q \in e$ is then $\int_e \Phi(q,t)K(dq)/\int_e K(dq)$ per unit of probability.

Note that now we have a much larger class of potential martingale measures, as for every (t,ω) a *function* $\Phi(t,q)$ has to be chosen and not just the value of the process $\phi(t)$. Typically we will have incomplete markets in this situation which poses entirely new problems for the hedging of contingent claims. See Björk, Kabanov and Runggaldier [4] [3] for a detailed analysis of trading strategies, hedging and completeness in bond markets with marked point processes.

As before, to save notation, we will assume that all dynamics are already specified with respect to the martingale measure.

[12] In functions of the marker q (like Φ here) *predictability* means measurable with respect to the σ-algebra $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$. Here $\mathcal{P}$ is the σ-algebra of the predictable processes.

6.2 Dynamics and Absence of Arbitrage

We start from the representation of the defaultable bond prices as (using the notation and results of the preceding sections)

$$C(t,T) = \exp\{-G(t,T)\}\, Q_t.$$

The dynamics of $C(t,T)$ are then

$$\begin{aligned}
\frac{dC(t,T)}{C(t-,T)} &= -dG(t,T) + d < G, G > -dN' \\
&= \left[-\hat{b}(t,T) + r^d(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\right] dt \\
&\quad + \sum_{i=1}^{n} a_i(t,T) dW^i(t) - dN'(t) \\
&= \left[-\hat{b}(t,T) + r^d(t) - q(t)h(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\right] dt \\
&\quad + \sum_{i=1}^{n} a_i(t,T) dW^i(t) - dM'(t),
\end{aligned}$$

where we used that G is continuous. Absence of arbitrage is here equivalent to

$$r(t) = -\hat{b}(t,T) + r^d(t) - q(t)h(t) + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T). \tag{71}$$

To show that the results of the preceding sections remain valid we only need to show that

$$h(t)q(t) = r^d(t) - r(t). \tag{72}$$

The argument goes exactly as before: The Doleans-Dade exponential of the martingale $M'(t) = N'(t) - \int_0^t q(s)h(s)\,ds$ is

$$\exp\{-\int_0^t q(s)h(s)ds\} \prod_{T_i \le t} (1 - q_i), \tag{73}$$

while the discounted value of the risky bank account is the Q-martingale

$$\bar{c}(t) := \frac{c(t)}{b(t)} = \exp\{\int_0^t r^d(s) - r(s)\,ds\} \prod_{T_i \le t} (1 - q_i). \tag{74}$$

This is the Doleans-Dade exponential of

$$\hat{M}(t) := -N'(t) + \int_0^t r^d(s) - r(s)\,ds. \tag{75}$$

Because $q_i < 1$ a.s. we have that $\bar{c}(t) > 0$ a.s. and therefore $\hat{M}$ is unique and well-defined as $\hat{M}(t) = \int_0^t \frac{d\bar{c}(s)}{\bar{c}(s-)}$. Again, we see that

$$M'(t) + \hat{M}(t) = \int_0^t r^d(s) - r(s) - q(s)h(s)\,ds \equiv 0 \tag{76}$$

(being a predictable martingale with initial value zero) must be constant and equal to zero. Therefore

$$h(t)q(t) = r^d(t) - r(t) > 0. \tag{77}$$

Equation (77) is the equivalent of equation (39), the key relationship which allowed for the derivation of conditions for the absence of arbitrage. These conditions are exactly the same as for zero recovery, the proof is the same as for theorem 2.

THEOREM 5
The following are equivalent:
1. The measure under which the dynamics are specified is a martingale measure.
2. (i) The short interest rate spread is the intensity of the default process multiplied with the locally expected loss quota. It is positive (for $q(t) > 0$).

$$q(t)h(t) = r^d(t) - r(t) > 0. \tag{78}$$

(ii) The drift coefficients of the defaultable forward rates satisfy for all $t \leq T$ equations (42) and (43).
(iii) The drift coefficients of the risk-free forward rates satisfy for all $t \leq T$ equation (44).
3. (i) $r^d(t) - r(t) = h(t)q(t) > 0$.
(ii) The dynamics of the risky bond prices are given by

$$\frac{dC(t,T)}{C(t-,T)} = r^d(t)dt + \sum_{i=1}^{n} a_i(t,T)dW^i(t) - dN'(t) \tag{79}$$

or, solving the s.d.e.

$$\begin{aligned} C(t,T) &= C(0,T)\,Q(t)\,\exp\left\{\int_0^t r^d(s)\,ds - \tfrac{1}{2}\sum_{i=1}^{n}\int_0^t a_i^2(s,T)\,ds \right. \\ &\quad \left. + \sum_{i=1}^{n}\int_0^t a_i(s,T)\,dW^i(s)\right\}. \end{aligned} \tag{80}$$

(iii) The dynamics of the risk-free bonds satisfy (47).

All no-arbitrage restrictions on the dynamics of the interest rates are exactly identical to the restrictions in theorem 2, although theorem 2 only concerned the situation with zero recovery. Thus theorem 5 allows us to directly transfer all results of sections 6 and 7 on the modelling of the spreads. The drift restrictions of the corollaries 1 and 2 and of theorem 3 are also valid in the present setup.

For the modelling of arbitrage-free dynamics of the risky interest rates $g(t,T)$ one need not be concerned with the specification of the recovery rates, it is sufficient to just model the interest rates subject to the positive spread restricition (41) or (78) and the drift restrictions (43) and (44). Again we see that the hard task of modelling an unobservable quantity (like the distribution of the loss quota q) can be replaced with a suitable model of the defaultable forward rates which are much more easily observed.

If one allows q to take on negative values, negative spot spreads $r^d - r$ are possible. A negative q means that the risky bond gains in value upon default. Of course such an event is very rare but in some cases there might be an early (and full) repayment of the debt which will result in $q < 0$, for instance if the proceeds of a liquidation are greater than the outstanding debt. The advantage of negative q is to allow a wider class of models to be used for h, e.g. the Gaussian models of Vasicek [33] and Ho and Lee [20].

6.3 Seniority

Bonds of different senoirity have different payoffs in default, the ones with higher seniority have a higher payoff than the ones with lower seniority. Strict seniority – junior debt has a positive payoff if and only if senior debt has full payoff – is rather rare in practice which is due to the various legal bankruptcy procedures, but in general senior debt has a higher payoff in default than junior debt.

With a loss quota of junior debt q^j that is higher than the loss of senior debt q^s, the risky instantaneous short rate of junior debt is greater than the short rate for senior debt: $r^{dj} > r^{ds}$.

For modelling junior and senior debt another stage is added to the usual risky debt modelling. First model the risk-free term-structure. Then model the spread to the senior bonds. Then (this is the new step) model the spread between junior and senior debt using the senior debt as 'risk-free' debt in the drift restrictions. The modelling restricitions we derived above still hold in this setup.

7 Instantaneous Short Rate Modelling

Going back to the representation (80) of the dynamics of the risky bond prices

$$\begin{aligned} C(t,T) &= C(0,T) \cdot \prod_{\tau_i \le t} (1-q_i) \exp\left\{ \int_0^t r^d(s)\, ds \right\} \\ &\cdot \exp\left\{ -\tfrac{1}{2} \sum_{i=1}^n \int_0^t a_i^2(s,T)\, ds + \sum_{i=1}^n \int_0^t a_i(s,T)\, dW^i(s) \right\}, \end{aligned}$$

one can evaluate this expression at $t = T$ and use the final condition $C(T,T) = \prod_{\tau_i \le T}(1-q_i)$ to reach

$$\begin{aligned} & \prod_{\tau_i \le T} (1-q_i) \exp\left\{ -\int_0^T r^d(s)\, ds \right\} \\ = \quad & C(0,T) \exp\left\{ -\tfrac{1}{2} \sum_{i=1}^n \int_0^T a_i^2(s,T)\, ds + \sum_{i=1}^n \int_0^T a_i(s,T)\, dW^i(s) \right\} \\ \cdot \quad & \prod_{\tau_i \le T} (1-q_i). \end{aligned} \tag{81}$$

If $q < 1$ a.s. we may divide both sides by $\prod_{\tau_i \le T}(1-q_i)$ and take expectations of both sides to reach

Corollary 3
If there is no total loss on the defaultable bond (i.e. $q_i < 1$), we have the following representation of the price of defaultable zero coupon bonds:

$$C(0,T) = \mathrm{E}\left[\exp\left\{ -\textstyle\int_0^T r^d(s)\, ds \right\} \mid \mathcal{F}_0 \right]. \tag{82}$$

(Here we used that the second exponential is a stochastic exponential of the martingale $\sum_i \int a_i(s,T) dW^i(s)$. Thus it is again a martingale with initial value 1.)

In the first sections with zero recovery we weren't able to derive this representation as r^d was not defined for times after the default. Obviosly the above representation of the prices of risky bonds is the exact analogue to the representation of the prices of risk-free bonds as discounted expected value of the final payoff 1. This representation is the starting point of all models of the term-structure of interest rates that are based on a model of the short rate[13] So far a result of this type has only been proved by Duffie, Schroder and Skiadas [14] (but in their valuation formula an additional jump term occurs), and by Lando [25] for the special case of a default that is triggered by the first jump of a Cox process. Here the Cox process assumption is not needed, the default process can have an intensity that conditions on previous defaults.

Alternatively to the modeling of defaultable interest rates in the HJM- framework of assumption 1 one can model the short rates directly. With any arbitrage-free short rate model for the risk-free short rate r and a positive short rate model for the spread h one can specify an arbitrage-free model framework. Because the model for the defaultable short rate will necessary be at least a two-factor model, the calibration of this model might become difficult and the HJM approach may be preferrable. On the other hand the short rate models need not worry about possibly negative forward spreads.

[13]Models of the short rate are by (among others): Vasicek [33], Cox, Ingersoll and Ross [10], Ho and Lee [20], Black, Derman, Toy [6], Hull and White [21] and Sandmann and Sondermann [32].

8 Jumps in the Defaultable Rates

In the presence of multiple defaults (with ensuing restructuration) it is more realistic to allow the risky rates to change discontinuously at times of default, the risky term structure must be allowed to change its *shape* at these events.

These jumps in the defaultable rates are not to be confused with the fractional loss at default. There are two distinct effects at a time of default which both cause a discrete change in the value process of the holders of risky bonds:
First there is the direct loss caused by the rescue plan and the reduction of the claims of the defaulted bondholders. This is modeled by the marker process q.
Secondly the market's valuation of the defaultable bonds may change due to the default. This is reflected in a discrete change in the yield curve but need not mean a irrecoverable loss. If there is no further default until maturity this jump in the value process is compensated. [14]

8.1 Dynamics

To reach the most general setup we use the marked point process $\mu(dq, dt)$. At every default there are jumps in the defaultable term structure $g(t,T)$ and the defaultable bond prices $C(t,T)$.

We replace assumption 1 with:

ASSUMPTION 6
(i) The dynamics of the defaultable forward rates are:

$$dg(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T)\, dW^i(t) + \int_E \delta(q;t,T)\, \mu(dq,dt) \tag{83}$$

(ii) The dynamics of $\hat{C}(t,T) := \exp\{-\int_t^T g(t,s)ds\}$ *are:*

$$\begin{aligned} \frac{d\hat{C}(t,T)}{\hat{C}(t-,T)} &= m(t,T)\, dt + \sum_{i=1}^{n} a_i(t,T)\, dW^i(t) \\ &\quad + \int_E \theta(q;t,T)\, \mu(dq,dt) \end{aligned} \tag{84}$$

(iii) The dynamics of the defaultable bond prices $C(t,T) = Q(t)\hat{C}(t,T)$ *are:*

$$\begin{aligned} \frac{dC(t,T)}{C(t-,T)} &= m(t,T)\, dt + \sum_{i=1}^{n} a_i(t,T)\, dW^i(t) \\ &\quad + \int_E (1-q)\theta(q;t,T)\, \mu(dq,dt) - \int_E q\, \mu(dq,dt) \end{aligned} \tag{85}$$

[14] The general methodology of modelling interest rates in the presence of marked point processes is taken from Björk, Kabanov and Runggaldier [4], who give an excellent account on risk-free interest rate modelling with marked point processes.

(iv) We assume sufficient regularity on the parameters to allow:
– differentiation (w.r.t. T) under the integral,
– interchange of order of integration,
– finite prices $C(t,T)$ almost surely.

We distinguish between a 'pseudo' bond price $\hat{C}(t,T)$ without fractional loss at default, and the 'real' bond price $C(t,T) = Q(t)\hat{C}(t,T)$. The dynamics of $C(t,T)$ in (85) follow directly from Itô's lemma.

These dynamics are interdependent due to the following result by BKR [4]:

PROPOSITION 3
Given the dynamics (83) of $g(t,T)$
(i) the dynamics of the defaultable short rate $r^d(t)$ are

$$\begin{aligned} dr^d(t) &= [\frac{\partial}{\partial T} g(t,t) + \alpha(t,t)]dt + \sum_{i=1}^{n} \sigma_i(t,t) dW^i(t) \quad (86) \\ &+ \int_E \delta(q;t,t)\mu(dq,dt). \quad (87) \end{aligned}$$

(ii) the dynamics of $\hat{C}(t,T)$ are

$$\begin{aligned} m(t,T) &= r^d(t) - \int_t^T \alpha(s,T)\,ds + \tfrac{1}{2}\sum_{i=1}^{n}(\int_t^T \sigma_i(s,T)\,ds)^2 \quad (88) \\ a_i(t,T) &= -\int_t^T \sigma_i(s,T)\,ds \quad (89) \\ \theta(q;t,T) &= \exp\left\{-\int_t^T \delta(x;s,T)\,ds\right\} - 1. \quad (90) \end{aligned}$$

(iii) The dynamics of the defaultable bond prices are given by assumption 6 (iii) with the specification of (ii) above.

Proof: see BKR [4] for (i) and (ii), point (iii) follows directly.

8.2 Absence of Arbitrage

The change of measure to the martingale measure is done according to theorem 4. The analysis leading to the key relation (78)

$$h(t)q(t) = r^d(t) - r(t) > 0$$

in section 6.2 is still valid, because the only defaultable security needed there is the defaultable bank account $c(t)$ which has no jump component in its development except the direct losses of q_i at default.

As usual we need for absence of arbitrage

$$\begin{aligned} r(t)\,dt &= \mathrm{E}\left[\tfrac{dC(t,T)}{C(t-,T)} \right] \\ &= r^d(t)dt - \hat{b}(t,T)\,dt + \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\,dt \\ &\quad + \int_E \theta(q;t,T)(1-q)\,K(dq)h(t)\,dt - q(t)h(t)dt. \end{aligned}$$

Substituting (78) and (90) yields:

Proposition 4
Under the martingale measure
(i) The short rate spread is given by

$$r^d(t) - r(t) = h(t)q(t). \tag{91}$$

(ii) The drift of the defaultable forward rates is restricted by

$$\hat{b}(t,T) = \tfrac{1}{2}\sum_{i=1}^{n} a_i^2(t,T) + \int_E \left(\exp\left\{ -\int_t^T \delta(q;t,v)\,dv \right\} - 1 \right) (1-q)\,K(dq)h(t), \tag{92}$$

or, differentiated,

$$\begin{aligned} \alpha(t,T) &= \sum_{i=1}^{n} \sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv \\ &- \int_E \delta(q;t,T) \exp\left\{ -\int_t^T \delta(q;t,v)\,dv \right\} (1-q)\,K(dq)h(t). \end{aligned} \tag{93}$$

(iii) The dynamics of the risky bond prices under the martingale measure are

$$\begin{aligned} \frac{dC(t,T)}{C(t-,T)} &= r^d(t,T)\,dt + \sum_{i=1}^{n} a_i(t,T)\,dW^i(t) \\ &\quad + \int_E (1-q)\theta(q;t,T)\,(\mu(dq,dt) - K(dq)\,h(t)\,dt) \\ &\quad - dN'(t) \end{aligned} \tag{94}$$

where $\theta(q;t,T)$ is defined as in (90).

Obviously the drift restriction (92) cannot be handled as easily as the other restrictions in theorems 2 and 5 before because of the integral over the jumps of the forward rates. As the defaults now have a jump-influence on the defaultable forward rates, the parameters of the default process do not disappear any more.

BKR [4] reach a quite similar restriction to (93) for the modelling of default-free interest rates in the presence of marked point processes. Their restriction is

$$
\begin{aligned}
\alpha'(t,T) &= \sum_{i=1}^{n} \sigma_i'(t,T) \int_t^T \sigma_i'(t,v)\,dv \\
&- \int_E \delta'(q;t,T) \exp\left\{ - \int_t^T \delta'(q;t,v)\,dv \right\} K(dq) h(t),
\end{aligned}
$$

and applies to the default-free interest rates. In this setup we assumed that the default-free interest rates do not jump (i.e. $\delta' = 0$) which reduces the restriction to the usual HJM-restriction (44).

The s.d.e. of the defaultable bond prices is of the usual type: There is a drift component of r^d and the default influence $-dN'(t)$. The other parts of the dynamics of the defaultable bond prices are local martingales.

9 Conclusion

In this paper we presented a new approach to the modelling of the price processes of defaultable bonds that was inspired by the Heath-Jarrow-Morton [19] model of the term structure of interest rates. This model avoids a precise specification of the mechanism that leads to default but rather gives necessary and sufficient conditions on the term structure of defaultable interest rates to ensure absence of arbitrage.

These restrictions show a striking similarity to the restrictions that are already well known from default-free term structure models. Specifically, the defaultable interest rates have to satisfy a drift restriction that is analogous to the HJM drift restriction, and a positive short spread restriction. Given these restrictions the model is arbitrage-free. In the implementation of the model one can therefore use the extensive machinery of risk-free interest rate modelling.

In the default mechanism the model presented here is the first to include multiple defaults and reduction of face value of the debt in default, it also recognises the magnitude risk in default.

As an alternative to the HJM-modelling approach it is shown that sufficient for the absence of arbitrage in a short rate model is a positive spread between the defaultable and the default-free short rate. Furthermore it is discussed how a defaultable bond model can be added to an existing model of risk-free bonds while keeping the combined model arbitrage-free.

There are several directions in which this approach can be extended. The inclusion of predictable times of default is an issue, as are econometric questions of the implementation. The question of negative forward spreads is still unresolved and a simple procedure for adding a spread model to an existing model of the default-free term structure is needed.

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