ON THE HYERS-ULAM STABILITY OF A DIFFERENTIABLE MAP

TAKESHI MIURA

Abstract. We consider a differentiable map f from an open interval I to a uniformly closed linear subspace A of $C(X)$, the Banach space of all complex-valued bounded continuous functions on a topological space X. Let ε be a non-negative real number, λ a complex number so that $\operatorname{Re}\lambda \neq 0$. Then we show that f can be approximated by the solution to A-valued differential equation $x'(t) = \lambda x(t)$, if $\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon$ holds for every $t \in I$.

1. Introduction

In this paper, I denotes an open interval of the real number field $\mathbb{R}$, unless the contrary is explicitly stated. That is $I = (a, b)$ for some $-\infty \leq a < b \leq +\infty$. The letters ε and λ denote a non-negative real number and a complex number, respectively. Let X be a topological space, $C(X)$ a Banach space of all complex-valued bounded continuous functions on X with respect to the pointwise operations and the supremum norm $\|\cdot\|_\infty$ on X. Throughout this paper, A denotes a uniformly closed linear subspace of $C(X)$.

Definition 1.1. *Let B be a Banach space, f a map from I into B. We say that f is differentiable, if for every $t \in I$ there exists an $f'(t) \in B$ so that*

$$\lim_{s \to 0} \left\| \frac{f(t+s) - f(t)}{s} - f'(t) \right\|_B = 0,$$

where $\|\cdot\|_B$ denotes the norm on B.

Let f be a differentiable function on I into $\mathbb{R}$. Alsina and Ger [1] gave all the solutions to the inequality $|f'(t) - f(t)| \leq \varepsilon$ for every $t \in I$. Then they showed that each solution to the inequality above was approximated by a solution to the differential equation $x'(t) = x(t)$. In accordance with [1], we define the Hyers-Ulam stability of Banach space valued differentiable map:

Definition 1.2. *Let B be a Banach space, f a differentiable map on I into B so that*

$$\|f'(t) - \lambda f(t)\|_B \leq \varepsilon, \quad (t \in I).$$

We say that the Hyers-Ulam stability holds for f, if there exist a $k \geq 0$ and a differentiable map x on I into B such that

$$x'(t) = \lambda x(t) \text{ and } \|f(t) - x(t)\|_B \leq k\varepsilon$$

holds for every $t \in I$.

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Let $C(X,\mathbb{R})$ be the Banach space of all real-valued bounded continuous functions on X and $C_0(X,\mathbb{R})$ the Banach space of all functions of $C(X,\mathbb{R})$ which vanish at infinity. Let r be a non-zero real number. In [2], we considered a differentiable map f on I into $C(X,\mathbb{R})$ (resp. $C_0(X,\mathbb{R})$) with the inequality $\|f^{'}(t) - rf(t)\|_\infty \leq \varepsilon$. Then we showed that the Hyers-Ulam stability held for f. That is, f can be approximated by a solution to $C(X,\mathbb{R})$ (resp. $C_0(X,\mathbb{R})$) valued differential equation $x^{'}(t) = rx(t)$.

In this paper, we consider a differentiable map f on I into A so that the inequality $\|f^{'}(t) - \lambda f(t)\|_\infty \leq \varepsilon$ holds for every $t \in I$. Unless $\operatorname{Re}\lambda = 0$, we show that the Hyers-Ulam stability holds for f. If $\operatorname{Re}\lambda = 0$, we give an example so that the Hyers-Ulam stability does not hold. Also we consider the Hyers-Ulam stability of an entire function.

2. Preliminaries

We give a characterization of the inequality $\|f^{'}(t) - \lambda f(t)\| \leq \varepsilon$.

Proposition 2.1. *Let B be a Banach space, f a differentiable map on I into B. Then the following conditions are equivalent.*

(i) $\|f^{'}(t) - \lambda f(t)\|_B \leq \varepsilon, \quad (t \in I)$.

(ii) *There exits a differentiable map g on I into B such that*

$$f(t) = g(t)e^{\lambda t} \text{ and } \|g^{'}(t)\|_B \leq \varepsilon e^{-(\operatorname{Re}\lambda)\, t},$$

for every $t \in I$.

Proof. (i) $\Rightarrow$ (ii) Put $g(t) = f(t)e^{-\lambda t}$ for every $t \in I$. Then we see that g is differentiable and

$$g^{'}(t) = \{f^{'}(t) - \lambda f(t)\}e^{-\lambda t}, \quad (t \in I).$$

By hypothesis, we have the inequality

$$\|g^{'}(t)\|_B \leq \varepsilon e^{-(\operatorname{Re}\lambda)\, t}$$

for every $t \in I$.

(ii) $\Rightarrow$ (i) If $f(t) = g(t)e^{\lambda t}$, we have

$$f^{'}(t) = \{g^{'}(t) + \lambda g(t)\}e^{\lambda t} = g^{'}(t)e^{\lambda t} + \lambda f(t)$$

for every $t \in I$. Since $\|g^{'}(t)\|_B \leq \varepsilon e^{-(\operatorname{Re}\lambda)\, t}$,

$$\|f^{'}(t) - \lambda f(t)\|_B \leq \varepsilon$$

holds for every $t \in I$. □

In particular, if we consider the case where $\varepsilon = 0$, then we have a solution of Banach space valued differential equation $f^{'}(t) = \lambda f(t)$. For the completeness we give a proof.

Proposition 2.2. *Let B be a Banach space, f a differentiable map on I into B. Then the following conditions are equivalent.*

(i) $f^{'}(t) = \lambda f(t), \quad (t \in I)$.

(ii) *There exists a $g \in B$ so that $f(t) = ge^{\lambda t}, \quad (t \in I)$.*

Proof. It is enough to show that the map $g(t)$ given in the condition (ii) of Proposition 2.1 is constant, if $g^{'}(t) = 0$ for every $t \in I$. Fix any $t_0 \in I$, then we define the function $\tilde{g}$ on I into $\mathbb{R}$ as

$$\tilde{g}(t) = \|g(t) - g(t_0)\|_B, \quad (t \in I).$$

We see that $\tilde{g}$ is differentiable and $\tilde{g}^{'}(t) = 0$ for every $t \in I$, since $g^{'}(t) = 0$. Therefore, $\tilde{g}$ is a constant function. Since $\tilde{g}(t_0) = 0$, we have $g(t) = g(t_0)$. Thus $g(t)$ is a constant function and this completes the proof. □

3. One point case

The results below are proved in case where $\operatorname{Re}\lambda > 0$, while corresponding ones hold in case where $\operatorname{Re}\lambda < 0$ and we omit them. In this section we consider the case where X is a singleton. In Lemma 3.1 and 3.2, g denotes a differentiable function on I into $\mathbb{C}$ so that

$$|g^{'}(t)| \le \varepsilon e^{-(\operatorname{Re}\lambda)\, t}$$

for every $t \in I$. Let u and v be the real part and the imaginary part of g, respectively. Unless $\operatorname{Re}\lambda = 0$, we define the functions $\tilde{u}$ and $\tilde{v}$ on I into $\mathbb{C}$ as

$$\begin{aligned} \tilde{u}(t) &= u(t) - \frac{\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\, t}, \\ \tilde{v}(t) &= v(t) - \frac{\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\, t}. \end{aligned}$$

Lemma 3.1. *Let $\operatorname{Re}\lambda \neq 0$ and $t_0 \in I$. Then we have the inequalities*

$$0 \le \tilde{u}(s) - \tilde{u}(t_0) \le \frac{2\varepsilon}{\operatorname{Re}\lambda} \left\{ e^{-(\operatorname{Re}\lambda)\, t_0} - e^{-(\operatorname{Re}\lambda)\, s} \right\},$$

$$0 \le \tilde{v}(s) - \tilde{v}(t_0) \le \frac{2\varepsilon}{\operatorname{Re}\lambda} \left\{ e^{-(\operatorname{Re}\lambda)\, t_0} - e^{-(\operatorname{Re}\lambda)\, s} \right\}$$

for every $s \in I$ with $t_0 \le s$.

Proof. Since $g^{'}(t) = u^{'}(t) + iv^{'}(t)$, we have

$$|u^{'}(t)|, |v^{'}(t)| \le |g^{'}(t)| \le \varepsilon e^{-(\operatorname{Re}\lambda)\, t}$$

for every $t \in I$. By definition,

$$\tilde{u}^{'}(t) = u^{'}(t) + \varepsilon e^{-(\operatorname{Re}\lambda)\, t}, \quad (t \in I).$$

Hence, we obtain the inequality

$$0 \le \tilde{u}^{'}(t) \le 2\varepsilon e^{-(\operatorname{Re}\lambda)\, t}$$

for every $t \in I$. We define the function U on I into $\mathbb{C}$ as

$$U(s) = -\tilde{u}(s) - \frac{2\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\, s} + \tilde{u}(t_0) + \frac{2\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\, t_0}, \quad (s \in I).$$

Then U is differentiable and

$$U^{'}(s) = -\tilde{u}^{'}(s) + 2\varepsilon e^{-(\operatorname{Re}\lambda)\, s} \ge 0$$

for every $s \in I$. Since $U(t_0) = 0$, we have $U(s) \ge 0$ if $s \ge t_0$. Since $\tilde{u}^{'}(s) \ge 0$, the inequality $\tilde{u}(t_0) \le \tilde{u}(s)$ holds if $t_0 \le s$. Therefore, we have

$$0 \le \tilde{u}(s) - \tilde{u}(t_0) \le \frac{2\varepsilon}{\operatorname{Re}\lambda} \left\{ e^{-(\operatorname{Re}\lambda)\, t_0} - e^{-(\operatorname{Re}\lambda)\, s} \right\},$$

if $t_0 \le s$. In a way similar to the above, we see that

$$0 \le \tilde{v}(s) - \tilde{v}(t_0) \le \frac{2\varepsilon}{\operatorname{Re}\lambda}\left\{e^{-(\operatorname{Re}\lambda)\,t} - e^{-(\operatorname{Re}\lambda)\,s}\right\}$$

holds, if $t_0 \le s$ and a proof is omitted. □

Lemma 3.2. *Let* $\operatorname{Re}\lambda > 0$*, then both* $\lim_{s\nearrow \sup I} \tilde{u}(s)$ *and* $\lim_{s\nearrow \sup I} \tilde{v}(s)$ *exist.*

Proof. As a first step, we show that $\sup_{t\in I} \tilde{u}(t)$ is finite. To this end fix any $t_0 \in I$, then by Lemma 3.1 we have the inequality

$$\begin{aligned}\tilde{u}(t) &\le \tilde{u}(t_0) + \frac{2\varepsilon}{\operatorname{Re}\lambda}\left\{e^{-(\operatorname{Re}\lambda)\,t_0} - e^{-(\operatorname{Re}\lambda)\,t}\right\}\\ &< \tilde{u}(t_0) + \frac{2\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\,t_0},\end{aligned}$$

if $t_0 \le t$. Since $\tilde{u}^{'}(t) \ge 0$ for every $t \in I$, we obtain $\tilde{u}(t) \le \tilde{u}(t_0)$ if $t < t_0$. Therefore,

$$\tilde{u}(t) \le \tilde{u}(t_0) + \frac{2\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\,t_0}$$

holds for every $t \in I$. Thus $\sup_{t\in I} \tilde{u}(t)$ is finite.

Next we show that $\lim_{s\nearrow \sup I} \tilde{u}(s) = \sup_{t\in I} \tilde{u}(t)$. In fact, for every $\eta > 0$ there exists an $s_0 \in I$ such that $\sup_{t\in I} \tilde{u}(t) - \eta < \tilde{u}(s_0)$. Since $\tilde{u}^{'}(t) \ge 0$ for every $t \in I$, we have

$$\sup_{t\in I} \tilde{u}(t) - \eta < \tilde{u}(s) < \sup_{t\in I} \tilde{u}(t) + \eta,$$

if $s_0 \le s$. Therefore,

$$\lim_{s\nearrow \sup I} \tilde{u}(s) = \sup_{t\in I} \tilde{u}(t)$$

holds. In a way similar to the above, we see that $\lim_{s\nearrow \sup I} \tilde{v}(s) = \sup_{t\in I} \tilde{v}(t)$ and a proof is omitted. □

Theorem 3.3. *Let* $\operatorname{Re}\lambda > 0$*,* f *a differentiable function on* I *into* $\mathbb{C}$ *so that*

$$|f^{'}(t) - \lambda f(t)| \le \varepsilon, \quad (t \in I).$$

Then there exists a $\theta \in \mathbb{C}$ *such that*

$$|f(t) - \theta e^{\lambda t}| \le \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}$$

holds for every $t \in I$.

Proof. By Proposition 2.1, there exists a differentiable function g on I into $\mathbb{C}$ such that

$$f(t) = g(t)e^{\lambda t} \text{ and } |g^{'}(t)| \le \varepsilon e^{-(\operatorname{Re}\lambda)t}, \quad (t \in I).$$

Let u and v be the real part and the imaginary part of g, respectively. We define the functions on I into $\mathbb{C}$ as

$$\begin{aligned}\tilde{u}(t) &= u(t) - \frac{\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\,t},\\ \tilde{v}(t) &= v(t) - \frac{\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\,t}.\end{aligned}$$

Then we see that both $\lim_{t\nearrow \sup I} \tilde{u}(t)$ and $\lim_{t\nearrow \sup I} \tilde{v}(t)$ exist, by Lemma 3.2. Note that for every $t \in I$ we have

$$0 \le \tilde{u}(s) - \tilde{u}(t) < \frac{2\varepsilon}{\operatorname{Re}\lambda}\, e^{-(\operatorname{Re}\lambda)\,t},$$

if $t \leq s$, by Lemma 3.1. Therefore, we obtain the inequality

$$\begin{aligned} \left| u(t) - \lim_{s \nearrow \sup I} \tilde{u}(s) \right| &= \lim_{s \nearrow \sup I} \left| \tilde{u}(t) + \frac{\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)t} - \tilde{u}(s) \right| \\ &\leq \frac{\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)t} \end{aligned}$$

for every $t \in I$. In a way similar to the above, we see that

$$\left| v(t) - \lim_{s \nearrow \sup I} \tilde{v}(s) \right| \leq \frac{\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)t}, \quad (t \in I).$$

Therefore, we have the inequality

$$\begin{aligned} &\left| f(t) - \lim_{t \nearrow \sup I} \{\tilde{u}(t) + i\tilde{v}(t)\} e^{\lambda t} \right| \\ = &\sqrt{\left\{ u(t) - \lim_{s \nearrow \sup I} \tilde{u}(s) \right\}^2 + \left\{ v(t) - \lim_{s \nearrow \sup I} \tilde{v}(s) \right\}^2} \, e^{(\operatorname{Re}\lambda)t} \\ \leq &\frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)t} e^{(\operatorname{Re}\lambda)t} = \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} \end{aligned}$$

for every $t \in I$. This completes the proof. □

4. General case

In this section we consider the case where X is any topological space.

Theorem 4.1. *Let* $\operatorname{Re}\lambda > 0$, f *a differentiable map on* I *into* A *so that*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon, \quad (t \in I).$$

If A *has constant functions, then there exists a* $\theta \in A$ *such that*

$$\|f(t) - \theta e^{\lambda t}\|_\infty \leq \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}$$

holds for every $t \in I$. *Unless* A *has constant functions, then there exists a* $\tilde{\theta} \in A$ *such that*

$$\|f(t) - \tilde{\theta} e^{\lambda t}\|_\infty \leq \frac{2\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}$$

for every $t \in I$.

Proof. For every $x \in X$ we define the induced function f_x on I into $\mathbb{C}$ as

$$f_x(t) = f(t)(x), \quad (t \in I).$$

Then f_x is a differentiable function, and for every $x \in X$

$$(f_x)'(t) = f'(t)(x), \quad (t \in I)$$

holds, by definition. Therefore, for every $x \in X$ we see that

$$|(f_x)'(t) - \lambda f_x(t)| \leq \|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon, \quad (t \in I).$$

By Proposition 2.1, for every $x \in X$ there corresponds a differentiable function g_x on I into $\mathbb{C}$ such that

$$f_x(t) = g_x(t) e^{\lambda t} \text{ and } |(g_x)'(t)| \leq \varepsilon e^{-(\operatorname{Re}\lambda)t}$$

for every $t \in I$. Let u_x and v_x be the real part and the imaginary part of g_x, respectively. We define the functions on I into $\mathbb{C}$ as

$$\begin{aligned} \tilde{u}_x(t) &= u_x(t) - \frac{\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)\,t}, \\ \tilde{v}_x(t) &= v_x(t) - \frac{\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)\,t}. \end{aligned}$$

By the proof of Theorem 3.3, for every $x \in X$ we have

$$\left| f_x(t) - \lim_{s \nearrow \sup I} \{\tilde{u}_x(s) + i\tilde{v}_x(s)\} e^{\lambda s} \right| \le \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}, \quad (t \in I).$$

We define the function θ on X into $\mathbb{C}$ as

$$\theta(x) = \lim_{s \nearrow \sup I} \{\tilde{u}_x(s) + i\tilde{v}_x(s)\}.$$

By definition, the inequality

$$\|f(t) - \theta e^{\lambda t}\|_\infty \le \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}$$

holds for every $t \in I$.

Let $\{t_n\}$ be a sequence of I so that $t_n \nearrow \sup I$. Then we define the function θ_n on X into $\mathbb{C}$ as

$$\theta_n(x) = \tilde{u}_x(t_n) + i\tilde{v}_x(t_n), \quad (x \in X).$$

Since $g_x(t_n) = f_x(t_n)e^{-\lambda t_n}$, we see that the function $x \mapsto g_x(t_n)$ belongs to A for every $n \in \mathbb{N}$.

We show that θ is an element of A, if A has constant functions. In fact, θ_n is an element of A for every $n \in \mathbb{N}$ by the definition of $\tilde{u}_x$ and $\tilde{v}_x$. Note that

$$|\tilde{u}_x(s) - \tilde{u}_x(t)|, |\tilde{v}_x(s) - \tilde{v}_x(t)| \le \frac{2\varepsilon}{\operatorname{Re}\lambda} |e^{-(\operatorname{Re}\lambda)s} - e^{-(\operatorname{Re}\lambda)t}|,$$

if $t \le s$, by Lemma 3.1. Therefore, we have

$$\begin{aligned} |\theta(x) - \theta_n(x)| &= \lim_{s \nearrow \sup I} \sqrt{|\tilde{u}_x(s) - \tilde{u}_x(t_n)|^2 + |\tilde{v}_x(s) - \tilde{v}_x(t_n)|^2} \\ &\le \frac{2\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} \left| \lim_{s \nearrow \sup I} e^{-(\operatorname{Re}\lambda)s} - e^{-(\operatorname{Re}\lambda)t_n} \right| \end{aligned}$$

for every $x \in X$ and every $n \in \mathbb{N}$. Hence θ is a uniform limit of $\{\theta_n\} \subset A$. Since A is uniformly closed, θ is an element of A.

Next we consider the case where A does not have constant functions. We define the functions $\tilde{\theta}$ and $\tilde{\theta}_n$ on X into $\mathbb{C}$ as

$$\tilde{\theta}(x) = \theta(x) + \frac{(1+i)\varepsilon}{\operatorname{Re}\lambda} \lim_{s \nearrow \sup I} e^{-(\operatorname{Re}\lambda)s},$$

$$\tilde{\theta}_n(x) = \theta_n(x) + \frac{(1+i)\varepsilon}{\operatorname{Re}\lambda} e^{-(\operatorname{Re}\lambda)t_n}.$$

Note that $\tilde{\theta}_n(x) = g_x(t_n)$ holds for every $x \in X$ and every $n \in \mathbb{N}$, hence $\{\tilde{\theta}_n\} \subset A$. Then we have

$$\begin{aligned} |\tilde{\theta}(x) - \tilde{\theta}_n(x)| &\le |\theta(x) - \theta_n(x)| + \frac{|1+i|\varepsilon}{\operatorname{Re}\lambda} \left| \lim_{s \nearrow \sup I} e^{-(\operatorname{Re}\lambda)s} - e^{-(\operatorname{Re}\lambda)t_n} \right| \\ &\le \frac{3\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} \left| \lim_{s \nearrow \sup I} e^{-(\operatorname{Re}\lambda)s} - e^{-(\operatorname{Re}\lambda)t_n} \right| \end{aligned}$$

for every $x \in X$ and every $n \in \mathbb{N}$. Since A is uniform closed, $\tilde{\theta}$ belongs to A. Moreover,

$$\begin{aligned}\|f(t) - \tilde{\theta} e^{\lambda t}\|_\infty &\leq \|f(t) - \theta e^{\lambda t}\|_\infty + \frac{|1+i|\varepsilon}{\operatorname{Re}\lambda}\left|\lim_{s \nearrow \sup I} e^{-(\operatorname{Re}\lambda)s} e^{\lambda t}\right| \\ &\leq \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} + \frac{\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda} = \frac{2\sqrt{2}\,\varepsilon}{\operatorname{Re}\lambda}\end{aligned}$$

holds for every $t \in I$. This completes the proof. □

Corollary 4.2. *Let* $\operatorname{Re}\lambda > 0$, f *a differentiable map on* $(a, +\infty)$, *for some* $-\infty \leq a < +\infty$, *into* A *so that*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon, \quad (t \in (a, +\infty)).$$

Then f *is uniquely approximated by a function of* A *in the sense of Theorem* 4.1.

Proof. By Theorem 4.1, it is enough to show that if $\theta_1, \theta_2 \in A$ so that

$$\|f(t) - \theta_j\, e^{\lambda t}\|_\infty \leq k_j\, \varepsilon, \quad (t \in (a, +\infty))$$

for some $k_j \geq 0$, $(j = 1, 2)$ then $\theta_1 = \theta_2$. In fact,

$$\begin{aligned}\|\theta_1 - \theta_2\|_\infty &\leq \|\theta_1 - f(t)e^{-\lambda t}\|_\infty + \|f(t)e^{-\lambda t} - \theta_2\|_\infty \\ &\leq (k_1 + k_2)\varepsilon e^{-(\operatorname{Re}\lambda)t} \to 0, \quad (t \to +\infty).\end{aligned}$$

Thus we have $\theta_1 = \theta_2$. This completes the proof. □

In general, the Hyers-Ulam stability does not hold if $\operatorname{Re}\lambda = 0$.

Example 4.1. *Let* $I = (0, +\infty)$, $\varepsilon > 0$ *and* f *be the function on* I *into* $\mathbb{C}$ *defined by*

$$f(t) = \varepsilon t e^{it}, \quad (t \in I).$$

Then the inequality $|f'(t) - if(t)| = \varepsilon$ *holds for every* $t \in I$. *On the other hand, the Hyers-Ulam stability does not hold. In fact, assume to the contrary that there exist a* $c \in \mathbb{C}$ *and* $k \geq 0$ *such that*

$$|f(t) - ce^{it}| \leq k\varepsilon, \quad (t \in I).$$

By the triangle inequality

$$|f(t)| \leq k\varepsilon + |c|$$

holds for every $t \in I$. *Though this is a contradiction, since* $|f(t)| = \varepsilon t$ *and since* $I = (0, +\infty)$.

If we consider the case where I is a finite interval, then the situation is different:

Theorem 4.3. *Let* $I = (a, b)$, *where* $-\infty < a < b < +\infty$, $\varepsilon \geq 0$ *and* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda = 0$. *If* f *is a differentiable map on* I *into* A *so that*

$$\|f'(t) - \lambda f(t)\|_\infty \leq \varepsilon, \quad (t \in I),$$

then there exists a $\theta \in A$ *such that*

$$\|f(t) - \theta e^{\lambda t}\|_\infty \leq \frac{(b-a)\varepsilon}{\sqrt{2}}$$

holds for every $t \in I$.

Proof. Let f_x, g_x, u_x and v_x be the differentiable function on I into $\mathbb{C}$, defined in the proof of Theorem 4.1. Then for every $x \in X$ we see that

$$f_x(t) = g_x(t)e^{\lambda t} \text{ and } |(g_x)'(t)| \le \varepsilon, \quad (t \in I),$$

by definition. Apply the mean value theorem to u_x and v_x respectively, then we have

$$\begin{aligned} \left| g_x(t) - g_x\left(\frac{a+b}{2}\right) \right| &= \left| (u_x)'(p)\left(t - \frac{a+b}{2}\right) + i(v_x)'(q)\left(t - \frac{a+b}{2}\right) \right| \\ &< \sqrt{2}\,\varepsilon \frac{b-a}{2} = \frac{(b-a)\varepsilon}{\sqrt{2}} \end{aligned}$$

for some $p, q \in I$. Since $\operatorname{Re}\lambda = 0$, the inequality

$$\left\| f(t) - g\left(\frac{a+b}{2}\right) e^{\lambda t} \right\|_\infty \le \frac{(b-a)\varepsilon}{\sqrt{2}}$$

holds for every $t \in I$. □

5. Hyers-Ulam stability of an Entire function

Recall that a function is entire if it is holomorphic in the whole plane $\mathbb{C}$. We may consider the Hyers-Ulam stability of an entire function.

Theorem 5.1. *Let f be an entire function so that*

$$|f'(z) - \lambda f(z)| \le \varepsilon, \quad (z \in \mathbb{C}).$$

Unless $\lambda = 0$, there exists a $\theta \in \mathbb{C}$ such that

$$|f(z) - \theta e^{\lambda z}| \le \frac{\varepsilon}{|\lambda|}$$

holds for every $z \in \mathbb{C}$. If we consider the case where $\lambda = 0$, then the Hyers-Ulam stability holds for f if and only if f is a constant function.

Proof. In a way similar to the proof of Proposition 2.1, we see that the inequality $|f'(z) - \lambda f(z)| \le \varepsilon$ holds for every $z \in \mathbb{C}$ if and only if there corresponds an entire function g so that

$$f(z) = g(z)e^{\lambda z} \text{ and } |g'(z)| \le \varepsilon |e^{-\lambda z}|, \quad (z \in \mathbb{C}).$$

Therefore $g'(z)e^{\lambda z}$ is a bounded entire function. Thus $g'(z)e^{\lambda z}$ is constant, by Liouville's theorem. Put $c_1 = g'(z)e^{\lambda z}$, then $|c_1| \le \varepsilon$.

Unless $\lambda = 0$, there exists a $c_2 \in \mathbb{C}$ such that

$$g(z) = c_2 - \frac{c_1}{\lambda} e^{-\lambda z}, \quad (z \in \mathbb{C}).$$

Therefore, we have the equality

$$f(z) = c_2 e^{\lambda z} - \frac{c_1}{\lambda}$$

for every $z \in \mathbb{C}$. Hence

$$|f(z) - c_2 e^{\lambda z}| \le \frac{\varepsilon}{|\lambda|}, \quad (z \in \mathbb{C}).$$

Next we consider the case where $\lambda = 0$. Then there exists a $c_3 \in \mathbb{C}$ so that

$$g(z) = c_1 z + c_3, \quad (z \in \mathbb{C}).$$

Therefore $f(z) = c_1 z + c_3$ for every $z \in \mathbb{C}$, since $\lambda = 0$. Then it is easy to see that the Hyers-Ulam stability holds for f, if and only if f is a constant function, and a proof is omitted. □

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Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan
E-mail address: miura@dip.yz.yamagata-u.ac.jp