Network Localization with Biased Range Measurements

Anthony J. Weiss[*], and Joseph S. Picard

***Abstract* - We discuss the challenge of node localization in ad-hoc wireless networks using range measurements and a few anchors. The measurements are assumed to be noisy with unknown *bias*. We propose an iterative Maximum Likelihood algorithm that is statistically efficient under these conditions. The algorithm is related to phase retrieval and can be viewed as an adaptation of the Gerchberg-Saxton Iterations alternating projection procedure. As a byproduct the algorithm estimates the bias of the range measurements. Closed form expressions for the Fisher Information Matrix are derived. Finally, simulations are used to corroborate the analysis.**

***Index Terms* - Wireless Sensor Network, Location Estimation, Ad Hoc Networks, Maximum Likelihood Estimation, Phase Retrieval, Alternating Projections, Cramér-Rao Bound**

[*] The authors are with Tel Aviv University, School of Electrical Engineering, Tel Aviv, 69978, Israel. Phone: +97236407460 Fax: +97236405027, email: ajw@eng.tau.ac.il.

I. Introduction

We consider ad-hoc wireless networks. In particular, we are interested in estimating the network nodes (stations) location. In order to determine the location, each node collects range measurement between itself and other nodes in its vicinity. Since each particular node is connected to a subset of the network nodes, not all possible range measurements are available. The challenge is to find the best estimates of the nodes position given the range measurements. The problem can be solved using a few nodes with known location (anchors) or it can be treated with no anchors and then only relative positions are found. A closely related problem exists in the context of wireless sensor networks. Sensor networks applications include environmental sensing (temperature, barometric pressure, humidity, etc.), water quality monitoring, inventory management, traffic monitoring, herd control, wild life behavior studies and many more. In this work we consider the effect of bias in the range measurements. Range measurements may be based on received signal strength (RSS) [3] or on time difference of arrival between a radio frequency signal and an ultrasonic signal [23]. However, most often range measurements are obtained by measuring the round trip propagation time from the sensor to its neighbor and back. Here we focus on the last method. In Rayleigh channels due to multipath and the absence of line-of-sight (LOS), the propagation time tends to be longer than line-of-sight propagation time and therefore the range measurements are usually biased. The situation is better in Ricean channels due to the existence of the LOS component but multipath components can still cause bias if the multipath is close to the LOS and strong enough. This inaccuracy source, which is often overlooked, is discussed here.

Ad-hoc wireless networks positioning attracts much interest in the research community. A discussion of all previous publications is beyond the scope of this contribution. Some representative algorithms can be found in [1]- [12] and a comprehensive review in [21].

Maximum Likelihood Estimation and the Cramér-Rao Bound have been previously discussed in [3] for unbiased range measurements.

Following [20], in this paper the problem is coached using complex numbers for planar geometry, *i.e*. all nodes are contained within a single plane. Obviously, all the results can be extended to three dimensions using real numbers. We assume that a rough initial estimate of the nodes position is available. For obtaining the initial estimate one can employ the technique described in [20] or any other technique such as the one described in [2]. For obtaining the Maximum likelihood geolocation of each node we modify the Gerchberg-Saxton algorithm [13] to handle biased measurements. We show that the algorithm is guaranteed to converge. Under mild regularity conditions, Maximum Likelihood estimates are statistically efficient and asymptotically approach the Cramér Rao Lower Bound (CRLB) [18]. We show that this is the case for the problem discussed here. Finally, we obtain compact expressions for the CRLB.

II. Problem Formulation

Consider a network consisting of n stations with unknown positions and m stations with known positions, termed "anchors". If there are no stations with known positions, it is possible to select three inter-connected stations and use them as anchors, see [2] for details. Assume that all stations are confined to a plane. The location of each station can then be represented by a complex number where the real and the imaginary parts represent the x,y coordinates of a Cartesian system. Thus, the location of all stations can be described by the $(n+m)\times 1$ complex column vector, $\mathbf{x}=[x_1, x_2, \cdots x_{n+m}]^T \in \mathbb{C}^{n+m\times 1}$, and the location of the anchors can be described by the $m\times 1$ column vector, $\mathbf{z}=[x_{n+1}, x_{n+2}, \cdots x_{n+m}]^T \in \mathbb{C}^{m\times 1}$. The location difference between any two stations can be described by the $(n+m)\times 1$ column vector, $\mathbf{e}_{ij}$, whose i-th entry is 1 and whose j-th entry is -1, while all other entries are zero. Thus,

$$\mathbf{e}_{ij}^T\mathbf{x} = x_i - x_j \tag{1}$$

Assume that a set of L distance measurements between pairs of stations is available. It is emphasized that not all possible distance measurements are available. The distance measurement between the i-th station and the j-th station is given by d_{ij}. Thus, we have the relation $d_{ij} = |x_i - x_j|$, if the measurements are precise.

Collecting all the differences associated with available distance measurements we get from (1),

$$\mathbf{Ex} = \mathbf{y}, \tag{2}$$

where the rows of the real matrix $\mathbf{E}$ are the vectors $\mathbf{e}_{ij}^T$ and $\mathbf{y}$ is a complex vector. The absolute value of each entry in $\mathbf{y}$ is given by the appropriate distance, d_{ij}, and the angle of each entry is unknown. Denote by $\mathbf{d}$ the vector consisting of the absolute values of the entries of $\mathbf{y}$.

The problem we are addressing can be concisely described by:

Estimate the n unknown complex elements of $\mathbf{x} \in \mathbb{C}^{n+m\times 1}$ given $\mathbf{E} \in \mathbb{R}^{L\times n+m}$, $\mathbf{z} \in \mathbb{C}^{m\times 1}$ and noisy measurements of $\mathbf{d} \in \mathbb{R}^{L\times 1}$.

As a first step towards a solution we partition the vector $\mathbf{x}$ into two sub-vectors, one sub-vector includes the unknown node positions, denoted by $\mathbf{u}$, and one sub-vector includes the anchor positions, denoted by $\mathbf{z}$. Thus, $\mathbf{x} = [\mathbf{u}^T, \mathbf{z}^T]^T$. We also partition the matrix $\mathbf{E}$ into two sub-matrices, the first sub-matrix with n columns and the second sub-matrix with m columns, $\mathbf{E} = [\mathbf{E}_1, \mathbf{E}_2]$, thus (2) becomes,

$$\mathbf{E}_1\mathbf{u} + \mathbf{E}_2\mathbf{z} = \mathbf{y} \tag{3}$$

Since $\mathbf{E}_2\mathbf{z}$ is a known vector it is convenient to replace it by $\mathbf{v} \triangleq \mathbf{E}_2\mathbf{z}$, Thus (3) becomes,

$$\mathbf{E}_1\mathbf{u} + \mathbf{v} = \mathbf{y}; \qquad |\mathbf{y}| = \mathbf{d}, \tag{4}$$

where $|\mathbf{y}|$ denotes the vector whose entries are the absolute values of the entries in $\mathbf{y}$.

Range measurements based on propagation time are commonly modeled as Gaussian variables. This Gaussian model is justified by invoking the central limit theorem and backed up by experimental results [3], [22]. The algorithms we present here are realization of the Maximum

Likelihood Estimator only if the range measurements have normal distribution. If this is not the case the algorithm can be viewed as the least squares estimator.

Assume that the range measurements errors are independent, identically distributed, Gaussian random variables. Thus, the probability density function of the range measurements is given by

$$f\left(\hat{\mathbf{d}}\middle|\mathbf{u},b\right)=\prod_{j=1}^{L}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(\left|y_j\right|-\hat{d}_j+b)^2\right\} \tag{5}$$

where b is the errors mean, σ^2 is the errors variance and $\hat{\mathbf{d}}$ is the range measurement vector.

The negative log-likelihood function is given, up to additive and multiplicative constants by,

$$e(\mathbf{u},b)=\sum_{j=1}^{L}\left|\left|y_j\right|-\hat{d}_j+b\right|^2=\left\|\,\left|\mathbf{y}\right|-\hat{\mathbf{d}}+\mathbf{1}b\right\|_F^2 \tag{6}$$

where $\left\|\bullet\right\|_F$ stands for the Frobenius norm and $\mathbf{1}$ is a vector of ones.

In the sequel we assume that the bias satisfies $0\le b\le \min_j \hat{d}_j$

III. Maximum Likelihood Estimation

The Maximum Likelihood estimator is given by,

$$\hat{\mathbf{u}},\hat{b}=\arg\min_{\mathbf{u},b}\ e(\mathbf{u},b) \tag{7}$$

In other words, we are interested in minimizing the cost function $e(\mathbf{u},b)$.

We assume that the precise anchor locations are given. Moreover, we assume that an initial estimate of the nodes position was obtained by the method in [20] or the method in [2].

In order to eliminate the effect of the unknown bias in the range measurements we modify the Gerchberg-Saxton iterations (Alternating Projections) proposed in [20]. The minimization procedure of the cost function in (7) yields the Maximum Likelihood estimate.

Consider now the following minimization procedure for the cost function, where *k* indicates the iteration number,

$$
\begin{array}{ll}
1) & \mathbf{y}^{(k)} = \mathbf{E}_1\mathbf{u}^{(k)} + \mathbf{v} \\
2) & b^{(k)} = \underset{0\le b\le \min \hat{d}_j}{\arg\min} \left\| \left|\mathbf{y}^{(k)}\right| - \hat{\mathbf{d}} + \mathbf{1}b \right\|_F^2 \\
3) & \overline{y}_j^{(k)} = \dfrac{\left|\hat{d}_j - b^{(k)}\right| y_j^{(k)}}{\left|y_j^{(k)}\right|} \\
4) & \mathbf{u}^{(k+1)} = \left(\mathbf{E}_1^T\mathbf{E}_1\right)^{-1}\mathbf{E}_1^T\left(\overline{\mathbf{y}}^{(k)} - \mathbf{v}\right)
\end{array}
\tag{8}
$$

In words, given a node location vector $\mathbf{u}^{(k)}$ compute the vector $\mathbf{y}^{(k)}$ using the first line of (8). Find an estimate of the bias using the second line of (8) (a simple one dimensional search). Adjust the modulus of $\mathbf{y}^{(k)}$ to be equal to the measured ranges, $\hat{\mathbf{d}}$, minus the estimated bias, as shown in the third line. Use the fourth line in (8) to obtain a new estimate of the nodes locations.

Note that the minimization with respect to the bias, implied by the second step in (8), can usually be expressed analytically,

$$
b^{(k)} = \frac{1}{L}\mathbf{1}^T\left(\hat{\mathbf{d}} - \left|\mathbf{y}^{(k)}\right|\right) \quad \text{if} \quad 0 \le b^{(k)} \le \min \hat{d}_j \tag{9}
$$

However, if the result is negative bias estimate or yields a negative range estimate one must resort to line search.

Proof of Convergence

The error function defined in (6) can be written using the definitions of the procedure as

$$
\begin{aligned}
e(\mathbf{u}^{(k)}, b^{(k)}) &= \sum_{j=1}^{L}\left|\left|y_j^{(k)}\right| - \left|\hat{d}_j - b^{(k)}\right|\right|^2 = \sum_{j=1}^{L}\left|\left|y_j^{(k)}\right| - \left|\overline{y}_j^{(k)}\right|\right|^2 \\
&= \sum_{j=1}^{L}\left|y_j^{(k)} - \overline{y}_j^{(k)}\right|^2 = \left\|\mathbf{y}^{(k)} - \overline{\mathbf{y}}^{(k)}\right\|_F^2 \\
&= \left\|(\mathbf{y}^{(k)} - \mathbf{v}) - (\overline{\mathbf{y}}^{(k)} - \mathbf{v})\right\|_F^2
\end{aligned}
\tag{10}
$$

Also, according to the procedure

$$
\mathbf{y}^{(k+1)} - \mathbf{v} = \mathbf{E}_1\mathbf{u}^{(k+1)} = \mathbf{E}_1\left(\mathbf{E}_1^T\mathbf{E}_1\right)^{-1}\mathbf{E}_1^T\left(\overline{\mathbf{y}}^{(k)} - \mathbf{v}\right) \tag{11}
$$

Note that $\mathbf{y}^{(k+1)} - \mathbf{v}$ is the projection of $\overline{\mathbf{y}}^{(k)} - \mathbf{v}$ on the column space of $\mathbf{E}_1$.

Thus, $\mathbf{y}^{(k+1)} - \mathbf{v}$ is the closest vector in the $\mathbf{E}_1$ column space to $\overline{\mathbf{y}}^{(k)} - \mathbf{v}$. Similarly $\mathbf{y}^{(k)} - \mathbf{v}$ is also within the column space of $\mathbf{E}_1$ and therefore,

$$\overline{e}^{(k)} \triangleq \left\|(\mathbf{y}^{(k+1)} - \mathbf{v}) - (\overline{\mathbf{y}}^{(k)} - \mathbf{v})\right\|_F^2 \le \left\|(\mathbf{y}^{(k)} - \mathbf{v}) - (\overline{\mathbf{y}}^{(k)} - \mathbf{v})\right\|_F^2 = e(\mathbf{u}^{(k)}, b^{(k)}) \tag{12}$$

Note that

$$\begin{aligned} \overline{e}^{(k)} &\triangleq \left\|\mathbf{y}^{(k+1)} - \overline{\mathbf{y}}^{(k)}\right\|_F^2 \\ &\ge \left\| \left|\mathbf{y}^{(k+1)}\right| - \left|\overline{\mathbf{y}}^{(k)}\right| \right\|_F^2 \\ &= \left\| \left|\mathbf{y}^{(k+1)}\right| - \hat{\mathbf{d}} + \mathbf{1}b^{(k)} \right\|_F^2 \\ &\ge \left\| \left|\mathbf{y}^{(k+1)}\right| - \hat{\mathbf{d}} + \mathbf{1}b^{(k+1)} \right\|_F^2 \\ &= \left\| \left|\mathbf{y}^{(k+1)}\right| - \left|\overline{\mathbf{y}}^{(k+1)}\right| \right\|_F^2 \\ &= \left\|\mathbf{y}^{(k+1)} - \overline{\mathbf{y}}^{(k+1)}\right\|_F^2 \triangleq e(\mathbf{u}^{(k+1)}, b^{(k+1)}) \end{aligned} \tag{13}$$

where the first inequality is due to the triangle inequality, and the second inequality is due to the choice of $b^{(k+1)}$, which minimizes the squared Frobenius norm.

Combining (12) and (13), we get,

$$e(\mathbf{u}^{(k+1)}, b^{(k+1)}) \le \overline{e}^{(k)} \le e(\mathbf{u}^{(k)}, b^{(k)}) \tag{14}$$

Thus, in each iteration, the error reduces or stays the same and convergence is guaranteed. ■

Although we provided proof of convergence, the convergence point is not guaranteed to be the global minimum of $e(\mathbf{u}, b)$, or equivalently, the global maximum of the likelihood function. However, the value of $e(\mathbf{u}, b)$ at the convergence point tends to be small if the convergence point is the global minimum and tends to be large if the convergence point is a local minimum. Thus, the algorithm can be restarted with different initial values if the cost function value at the convergence point is not small compared to a predefined threshold.

IV. Computational load

In this section we evaluate the computational load of the proposed algorithm in terms of the number of multiplications. Since the complexity of inverting a $n \times n$ matrix is $O\left(n^3\right)$, the complexity of evaluating the term $\left(\mathbf{E}_1^T \mathbf{E}_1\right)^{-1} \mathbf{E}_1^T$ is also $O\left(n^3\right)$. Since $\mathbf{z}$ is complex and $\mathbf{E}_2$ may have up to two non-zero entries per line, the complexity induced by $\mathbf{v} = \mathbf{E}_2 \mathbf{z}$ is $4L$. At any iteration, the first line of the procedure have a complexity of $4L$. Denote by R the complexity of the calculation of $\left|y_j^{(k)}\right|$. Then, the computational load for the second step is $\left(R+3\right)L$. The third and last steps have $3L$ and nL multiplications, respectively. Observing that $L \geq 3n$, the complexity of I iterations of the full procedure is upper-bounded by,

$$C\left(n\right) \sim O\left(n^3\right) + 4L + (7+R)IL \tag{15}$$

The computational load of the proposed algorithm is relatively low since the matrix inversion is performed only once.

V. Compact Cramér-Rao Lower Bound

In order to obtain the Cramér-Rao Lower Bound on the error covariance of the nodes location estimation we define the real vector of parameters $[\mathbf{u}_r^T, \mathbf{u}_i^T]^T$ where $\mathbf{u}_r$ is the real part of $\mathbf{u}$ and $\mathbf{u}_i$ is the imaginary part. We have shown in [20] that the Fisher Information matrix (FIM) for unbiased measurements is given by,

$$\mathbf{F}_u = \frac{1}{\sigma^2} \begin{bmatrix} \mathbf{E}_1^T \mathbf{D}_r^2 \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_r \mathbf{D}_i \mathbf{E}_1 \\ \mathbf{E}_1^T \mathbf{D}_i \mathbf{D}_r \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_i^2 \mathbf{E}_1 \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} \mathbf{D}_r &= \mathrm{Re}\left\{\mathrm{diag}\left\{y_1/\left|y_1\right|, \cdots, y_L/\left|y_L\right|\right\}\right\} \\ \mathbf{D}_i &= \mathrm{Im}\left\{\mathrm{diag}\left\{y_1/\left|y_1\right|, \cdots, y_L/\left|y_L\right|\right\}\right\} \end{aligned} \tag{17}$$

When an unknown bias degrades the range measurements, we show in the Appendix that the FIM is given by:

$$\mathbf{F}_b = \frac{1}{\sigma^2}\begin{bmatrix} \sigma^2 \mathbf{F}_u & \mathbf{h} \\ \mathbf{h}^T & L \end{bmatrix}$$
$$\mathbf{h}^T \triangleq \begin{bmatrix} \mathbf{1}_L^T \mathbf{D}_r \mathbf{E}_1 & \mathbf{1}_L^T \mathbf{D}_i \mathbf{E}_1 \end{bmatrix} \tag{18}$$

Note that the upper-left matrix in $\mathbf{F}_b$ is equal to $\mathbf{F}_u$. Denote by $\mathbf{CRB}_u$ the inverse of $\mathbf{F}_u$, the Cramér-Rao lower Bound of $\mathbf{u}$ for unbiased measurements. Denote by $\mathbf{CRB}_b$ the upper-left matrix of $\mathbf{F}_b^{-1}$, *i.e.* is the Cramér-Rao lower bound on $\mathbf{u}$ for biased measurements. Using block matrix inversion properties, we obtain:

$$\mathbf{CRB}_b = \mathbf{F}_b^{-1} = \left(\mathbf{F}_u - \frac{1}{\sigma^2 L} \mathbf{h}\mathbf{h}^T \right)^{-1} = \left(\mathbf{CRB}_u^{-1} - \frac{1}{\sigma^2 L} \mathbf{h}\mathbf{h}^T \right)^{-1} \tag{19}$$

Invoking Woodbury's identity, equation (19) becomes,

$$\mathbf{CRB}_b = \mathbf{CRB}_u + \frac{1}{\sigma^2 L - \mathbf{h}^T\ \mathbf{CRB}_u\ \mathbf{h}} \mathbf{CRB}_u\ \mathbf{h}\mathbf{h}^T\ \mathbf{CRB}_u \tag{20}$$

Note that since $\mathbf{F}_b$ is assumed to be positive definite, by the matrix inversion lemma we have $\sigma^2 L - \mathbf{h}^T\ \mathbf{CRB}_u\ \mathbf{h} > 0$. Thus, as expected, the diagonal elements of $\mathbf{CRB}_b$ are not smaller than the diagonal elements of $\mathbf{CRB}_u$.

Inserting in equation (19) the expression of $\mathbf{F}_u$ given in (16) yields,

$$\mathbf{CRB}_b = \sigma^2 \begin{bmatrix} \mathbf{E}_1^T \mathbf{D}_r \mathbf{P}_1 \mathbf{D}_r \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_r \mathbf{P}_1 \mathbf{D}_i \mathbf{E}_1 \\ \mathbf{E}_1^T \mathbf{D}_i \mathbf{P}_1 \mathbf{D}_r \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_i \mathbf{P}_1 \mathbf{D}_i \mathbf{E}_1 \end{bmatrix}^{-1}$$
$$\mathbf{P}_1 \triangleq \mathbf{I} - \mathbf{1}_L \mathbf{1}_L^T / L \tag{21}$$

Note that $\mathbf{P}_1$ is a projection matrix.

Also note that the Cramér-Rao bound on the bias estimation is given by

$$\text{CRB(bias)} = \frac{\sigma^2}{L - (1/\sigma^2)\mathbf{h}^T \mathbf{CRB}_u \mathbf{h}} \tag{22}$$

It is evident that the CRB on the bias decreases with increasing number of measurements (L) and decreasing range variance, σ^2. Also, as L increases the effect of the bias on node accuracy estimation reduces.

In order to obtain further insight we will discuss a simple special case.

Analysis of a simple special case

Consider a single node, s_0, connected to m other nodes $\{s_1, s_2, \cdots s_m\}$. Obviously, the accuracy with which s_0 can be located depends on the accuracy with which $\{s_1, s_2, \cdots s_m\}$ are located. A lower bound on the accuracy with which s_0 can be located is obtained by assuming that the locations of $\{s_1, s_2, \cdots s_m\}$ is known with high precision. Nodes whose location is known precisely are anchors for all practical purposes. Therefore, we will analyze the case of s_0 connected to m anchors.

Recalling the definitions of $\mathbf{E}_1$ and $\mathbf{E}_2$ in (3) we have, $\mathbf{E}_1 = \mathbf{1}_m; \quad \mathbf{E}_2 = -\mathbf{I}_m$.

According to (16), the FIM for unbiased measurements is now given by,

$$\mathbf{F}_u = \frac{1}{\sigma^2}\begin{bmatrix} \mathbf{1}^T \mathbf{D}_r^2 \mathbf{1} & \mathbf{1}^T \mathbf{D}_r \mathbf{D}_i \mathbf{1} \\ \mathbf{1}^T \mathbf{D}_r \mathbf{D}_i \mathbf{1} & \mathbf{1}^T \mathbf{D}_i^2 \mathbf{1} \end{bmatrix} = \frac{1}{\sigma^2}\begin{bmatrix} \text{tr}\left\{\mathbf{D}_r^2\right\} & \text{tr}\left\{\mathbf{D}_r \mathbf{D}_i\right\} \\ \text{tr}\left\{\mathbf{D}_r \mathbf{D}_i\right\} & \text{tr}\left\{\mathbf{D}_i^2\right\} \end{bmatrix} \tag{23}$$

and the corresponding CRB is,

$$\mathbf{CRB}_u = \frac{\sigma^2}{\text{tr}\left\{\mathbf{D}_r^2\right\}\text{tr}\left\{\mathbf{D}_i^2\right\} - \text{tr}^2\left\{\mathbf{D}_r \mathbf{D}_i\right\}}\begin{bmatrix} \text{tr}\left\{\mathbf{D}_i^2\right\} & \text{-tr}\left\{\mathbf{D}_r \mathbf{D}_i\right\} \\ \text{-tr}\left\{\mathbf{D}_r \mathbf{D}_i\right\} & \text{tr}\left\{\mathbf{D}_r^2\right\} \end{bmatrix} \tag{24}$$

Note that $y_j / \left|y_j\right|$ are points on the unit circle and therefore can be expressed as $y_j / \left|y_j\right| = e^{j\theta_j}$, which yields,

$$\begin{aligned}
\operatorname{tr}\left\{\mathbf{D}_r^2\right\} &= \sum_{j=1}^{m}\cos^2\theta_j = \frac{1}{2}\sum_{j=1}^{m}1+\cos 2\theta_j = \frac{m(1+c)}{2} \\
\operatorname{tr}\left\{\mathbf{D}_i^2\right\} &= \sum_{j=1}^{m}\sin^2\theta_j = \frac{1}{2}\sum_{j=1}^{m}1-\cos 2\theta_j = \frac{m(1-c)}{2} \\
\operatorname{tr}\left\{\mathbf{D}_r\mathbf{D}_i\right\} &= \sum_{j=1}^{m}\cos\theta_j\sin\theta_j = \frac{1}{2}\sum_{j=1}^{m}\sin 2\theta_j = \frac{ms}{2}
\end{aligned} \tag{25}$$

where

$$c \triangleq \frac{1}{m}\sum_{j=1}^{m}\cos 2\theta_j; \quad s \triangleq \frac{1}{m}\sum_{j=1}^{m}\sin 2\theta_j \tag{26}$$

Note that $|c| \le 1;\ |s| \le 1$. Equations (24) and (25) lead to

$$\mathbf{CRB}_u = \frac{2\sigma^2}{m(1-c^2-s^2)}\begin{bmatrix} 1-c & -s \\ -s & 1+c \end{bmatrix} \tag{27}$$

If the anchors are placed evenly around the node we get $c \simeq s \cong 0$ and

$$\mathbf{CRB}_u \cong \frac{2\sigma^2}{m}\mathbf{I}_2 \tag{28}$$

This simple expression provides a fundamental bound on the location accuracy of any node in the network. We observe that the accuracy depends not only on the range measurements accuracy but also on the number of connections each node has. This result could be expected and agrees very well with intuition.

The FIM in the case of non-zero bias is given in equation (18). Repeating the same steps we get,

$$\mathbf{F}_b = \frac{m}{2\sigma^2}\begin{bmatrix} (1+c) & s & c_1 \\ s & (1-c) & s_1 \\ c_1 & s_1 & 2 \end{bmatrix} \tag{29}$$

where

$$c_1 = \frac{1}{m}\sum_{j=1}^{m}\cos\theta_j; \quad s_1 = \frac{1}{m}\sum_{j=1}^{m}\sin\theta_j \tag{30}$$

Note that $|c_1| \le 1;\ |s_1| \le 1$. One immediately notes that if $c_1 = s_1 = 0$ the bias does not affect the node position accuracy. Thus, if the anchors are distributed uniformly around the node the bias has a negligible effect on the node location accuracy. In this case the bias can be estimated with a variance of σ^2/m. Note again that the higher the connectivity (m) the better the bias estimation accuracy.

VI. Numerical Examples

In this section we evaluate the performance of the proposed algorithm and the effect of the bias on the quality of the localization.

Consider a random network consisting of n = 22 nodes and m = 3 anchors. The nodes are uniformly distributed within a square area of 5×5 distance units (DU) and the anchors are positioned at $(3.5,4.5)$[DU], $(4.5,3.5)$[DU] and $(4.5,4.5)$[DU]. The minimum distance between nodes is 0.2 DU. Each node is connected to all neighbors within a distance of 2.8 DU. See Fig. 1 for a single realization of the network topology. The algorithm receives as input range measurements with biased random errors. The bias was set to 0.09 DU, which is 45% of the minimal range measurement. The standard deviation was increased from 0 to 0.18 DU. For each value of the standard deviation we performed N=100 experiments. In each experiment we selected random values of range errors. We then calculated Root Mean Square Errors (RMSE) and Cramer-Rao Lower Bounds (CRLB),

$$\begin{aligned} \mathrm{RMSE} &= \sqrt{\frac{1}{N}\frac{1}{n}\sum_{j=1}^{N}\sum_{k=1}^{n}\varepsilon_{j,k}^2} \qquad (n \text{ nodes},\ N \text{ experiments}) \\ \mathrm{CRLB} &= \sqrt{\frac{1}{n}\operatorname{trace}\{\mathbf{C}\}} \end{aligned} \tag{31}$$

where $\varepsilon_{j,k}$ is the positioning error of the k-th node in the j-th experiment. We use $\mathbf{C}$ to denote the CRB matrix for the topology of Figure 1 which has been used in all experiments.

Fig. 2 shows the RMSE and the CRLB against the ranging standard deviation. Two algorithms are compared: the Maximum Likelihood Estimator (MLE) obtained in [20], ignoring the ranging bias (circles); and the MLE obtained in Section III which estimates the bias (squares). Both algorithms were initialized by phase I of the algorithm in [20]. In order to overcome a potential convergence to a local minimum 4 other initializations were performed. The other initializations were obtained by small random perturbations of the first initialization. Out of the 5 position estimates the one that achieved the smallest cost function value was selected.

Fig. 3 shows the RMSE of both algorithms, for two different bias values. As can be seen in Fig. 2 and Fig. 3, the algorithm of Section III that includes bias estimation outperforms the one proposed in [20] that ignores potential bias. As expected, the MLE performance reaches the CRLB for small ranging errors. Note that when the bias is small, in comparison with the ranging standard deviation, the inclusion of bias estimation slightly degrades the performance of the MLE procedure. This should be expected due to the over parameterization of the problem. For standard deviation much larger than the bias the effect of the bias is negligible and therefore the two algorithms achieve similar performance. Observe that the Cramer Rao bound does not depend on the bias value. Similarly, the accuracy of the proposed MLE algorithm is almost the same for any bias value.

It is of interest to check the algorithm on many different topologies. Hence, instead of performing simulations on one particular topology, we formed $N = 200$ randomly generated networks, each having 50 stations. Three of the stations are anchors, located at $(8.5, 4.5)$ [DU], $(9.5, 3.5)$ [DU] and $(9.5, 4.5)$ [DU], the other $n = 47$ stations are uniformly distributed in a rectangle of 10×5 [DU]. We calculated Root Mean Square Errors (RMSE) as in (31) and averaged the Cramer-Rao Lower Bounds (CRLB),

$$\mathrm{CRLB} = \sqrt{\frac{1}{n}\sum_{j=1}^{N} \mathrm{trace}\left\{\mathbf{C}_j\right\}} \qquad (n \text{ nodes, } N \text{ experiments}) \tag{32}$$

where $\mathbf{C}_j$ denotes the CRB matrix for the j-th network. We kept only networks for which all nodes are locatable. In Fig. 4, the RMSE and CRLB for both algorithms are plotted. As expected, the proposed algorithm has a superior performance.

In order to examine the number of iterations required for convergence we performed 2000 trials with random topologies. The ranging standard deviation was set to 0.09 DU and the bias set to 0.09 DU. Figure 5 shows the probability of convergence versus the number of iterations for a few stopping criteria. The stopping criteria, or stopping thresholds, can be selected according to the desired final accuracy of position estimation.

Finally, we address the dependence of the RMSE and the CRLB on the number of connections in the networks. The bias was set to 0.04 DU and the standard deviation set to 0.06 DU. Fig. 6 shows the decrease of the RMSE and the CRLB for biased ranging as the connectivity increases.

In conclusion the proposed procedure for joint nodes location and bias estimation outperforms the corresponding basic procedure which ignores the bias, as long as the bias value is significant enough in comparison with the ranging standard deviation.

VII. Conclusions

We have considered the localization of nodes in an ad-hoc network using biased range measurements. We have shown that the bias can be estimated together with the nodes location. We have shown that iterative Maximum Likelihood estimator can be implemented using a modification of the Gerchberg-Saxton algorithm for phase retrieval. Further, we derived a closed form compact expression for the Cramer Rao bound.

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Appendix: Cramér-Rao Bound

In [20] we obtained a compact expression for the Cramér-Rao bound for unbiased measurements and exact anchors location. In this work, the estimated parameters are not only $\mathbf{u}$, but also b. We define a real version of the parameter vector $\left[\mathbf{u}_r{}^T, \mathbf{u}_i{}^T, b\right]^T$, where $\mathbf{u}_r, \mathbf{u}_i$ are the real and imaginary parts of $\mathbf{u}$ and b is the bias of the range measurements. The log-likelihood function for these parameters is $\ell(\mathbf{u}, b) = -\frac{1}{2} e(\mathbf{u}, b)$

Denote by $\mathbf{q}_j^T$ the j-th row of $\mathbf{E}_1$ and $\mathbf{q}_j'^T$ the j-th row of $\mathbf{E}_2$. Using this definition we can write,

$$\left|y_j\right| = \left|\mathbf{q}_j^T\mathbf{u} + \mathbf{q}_j'^T\mathbf{z}\right| = \left[\mathbf{q}_j^T\mathbf{u}\mathbf{u}^H\mathbf{q}_j + \mathbf{q}_j^T\mathbf{u}\mathbf{z}^H\mathbf{q}_j' + \mathbf{q}_j'^T\mathbf{z}\mathbf{u}^H\mathbf{q}_j + \mathbf{q}_j'^T\mathbf{z}\mathbf{z}^H\mathbf{q}_j'\right]^{1/2} \tag{33}$$

The derivative of $\left|y_j\right|$ w.r.t. $\mathbf{u}_r$ is given by,

$$\begin{aligned} \frac{\partial\left|y_j\right|}{\partial\mathbf{u}_r} &= \frac{1}{2\left|\mathbf{q}_j^T\mathbf{u} + \mathbf{q}_j'^T\mathbf{z}\right|}\left[\mathbf{q}_j^T\mathbf{u} + \mathbf{q}_j^T\mathbf{u}^* + \mathbf{q}_j'^T\mathbf{z}^* + \mathbf{q}_j'^T\mathbf{z}\right]\mathbf{q}_j \\ &= \frac{1}{\left|y_j\right|}\left[\mathbf{q}_j^T\,\mathrm{Re}\{\mathbf{u}\} + \mathbf{q}_j'^T\,\mathrm{Re}\{\mathbf{z}\}\right]\mathbf{q}_j = \mathrm{Re}\left\{y_j\right\}\mathbf{q}_j / \left|y_j\right| \end{aligned} \tag{34}$$

Similarly,

$$\frac{\partial\left|y_j\right|}{\partial\mathbf{u}_i} = \mathrm{Im}\left\{y_j\right\}\mathbf{q}_j / \left|y_j\right| \tag{35}$$

The derivative of $\ell(\mathbf{u}, b)$ w.r.t. $\mathbf{u}_r$ is given by

$$\frac{\partial \ell}{\partial \mathbf{u}_r} = -\frac{1}{\sigma^2}\sum_{j=1}^{L}\left(\left|y_j\right| + b - d_j\right)\frac{\partial \left|y_j\right|}{\partial \mathbf{u}_r} \\ = -\frac{1}{\sigma^2}\sum_{j=1}^{L}\left(\left|y_j\right| + b - d_j\right)\mathrm{Re}\left\{y_j\right\}\mathbf{q}_j / \left|y_j\right| \tag{36}$$

Using the definitions (17) in (36) we get

$$\frac{\partial \ell}{\partial \mathbf{u}_r} = -\frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_r \left(\left|\mathbf{y}\right| + \mathbf{1}b - \mathbf{d}\right) \tag{37}$$

Similarly, the derivative of $\ell(\mathbf{u}, b)$ w.r.t. $\mathbf{u}_i$ is,

$$\frac{\partial \ell}{\partial \mathbf{u}_i} = -\frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_i \left(\left|\mathbf{y}\right| + \mathbf{1}b - \mathbf{d}\right) \tag{38}$$

The derivative of $\ell(\mathbf{u}, b)$ w.r.t. b is,

$$\frac{\partial \ell}{\partial b} = -\frac{1}{\sigma^2}\sum_{j=1}^{L}\left(\left|y_j\right| + b - d_j\right) \\ = -\frac{1}{\sigma^2}\mathbf{1}^T \left(\left|\mathbf{y}\right| + \mathbf{1}b - \mathbf{d}\right) \tag{39}$$

Note that,

$$E\left\{\left(\left|\mathbf{y}\right| + \mathbf{1}b - \mathbf{d}\right)\left(\left|\mathbf{y}\right| + \mathbf{1}b - \mathbf{d}\right)^T\right\} = \sigma^2 \mathbf{I} \tag{40}$$

The blocks of the FIM corresponding to $\mathbf{u}$ only are given by,

$$E\left\{\left(\frac{\partial \ell}{\partial \mathbf{u}_r}\right)\left(\frac{\partial \ell}{\partial \mathbf{u}_r}\right)^T\right\} = \frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_r^2 \mathbf{E}_1 \\ E\left\{\left(\frac{\partial \ell}{\partial \mathbf{u}_r}\right)\left(\frac{\partial \ell}{\partial \mathbf{u}_i}\right)^T\right\} = \frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_r \mathbf{D}_i \mathbf{E}_1 \\ E\left\{\left(\frac{\partial \ell}{\partial \mathbf{u}_i}\right)\left(\frac{\partial \ell}{\partial \mathbf{u}_i}\right)^T\right\} = \frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_i^2 \mathbf{E}_1 \tag{41}$$

The block of the FIM corresponding to b only is given by,

$$E\left\{\left(\frac{\partial \ell}{\partial b}\right)^2\right\} = \frac{L}{\sigma^2} \tag{42}$$

The blocks of the FIM corresponding to $\mathbf{u}$ and b are given by,

$$\begin{aligned} E\left\{\left(\frac{\partial \ell}{\partial \mathbf{u}_r}\right)\left(\frac{\partial \ell}{\partial b}\right)\right\} &= \frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_r \mathbf{1} \\ E\left\{\left(\frac{\partial \ell}{\partial \mathbf{u}_i}\right)\left(\frac{\partial \ell}{\partial b}\right)\right\} &= \frac{1}{\sigma^2}\mathbf{E}_1^T \mathbf{D}_i \mathbf{1} \end{aligned} \tag{43}$$

Finally, the FIM is given by sorting these blocks according to $\left[\mathbf{u}_r{}^T, \mathbf{u}_i{}^T, b\right]^T$:

$$\mathbf{F}_b = \frac{1}{\sigma^2}\begin{bmatrix} \mathbf{E}_1^T \mathbf{D}_r^2 \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_r \mathbf{D}_i \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_r \mathbf{1}_L \\ \mathbf{E}_1^T \mathbf{D}_r \mathbf{D}_i \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_i^2 \mathbf{E}_1 & \mathbf{E}_1^T \mathbf{D}_i \mathbf{1}_L \\ \mathbf{1}_L^T \mathbf{D}_r \mathbf{E}_1 & \mathbf{1}_L^T \mathbf{D}_i \mathbf{E}_1 & L \end{bmatrix} \tag{44}$$

The Cramér-Rao bound for biased measurement is given by the inverse of the FIM.

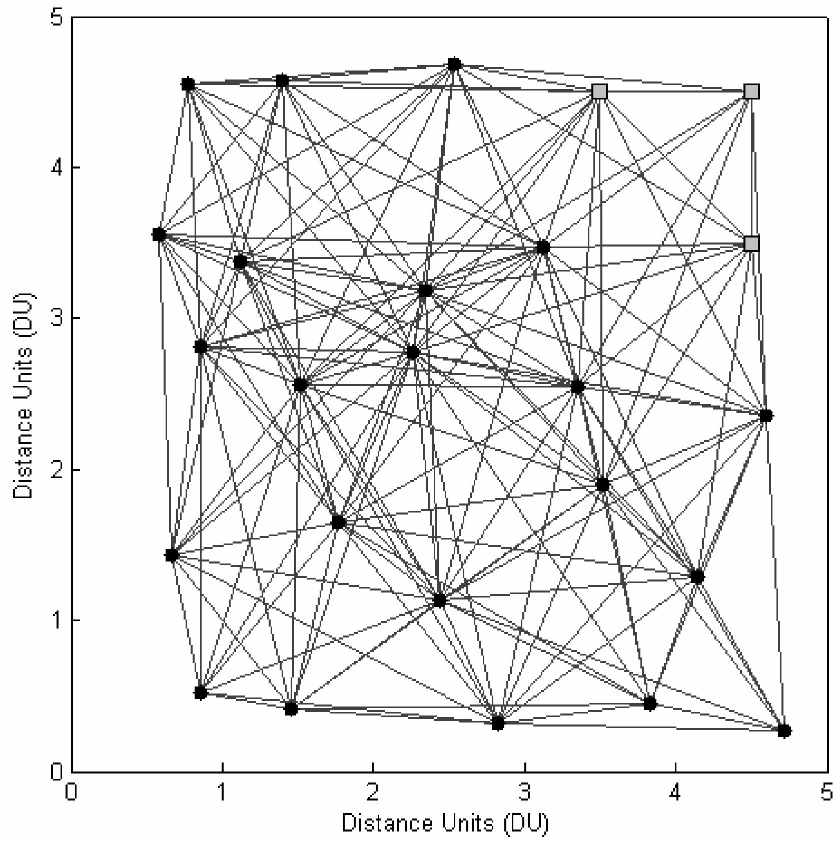

Figure 1: Random Network Realization, 22 nodes (circles), 3 anchors (squares)

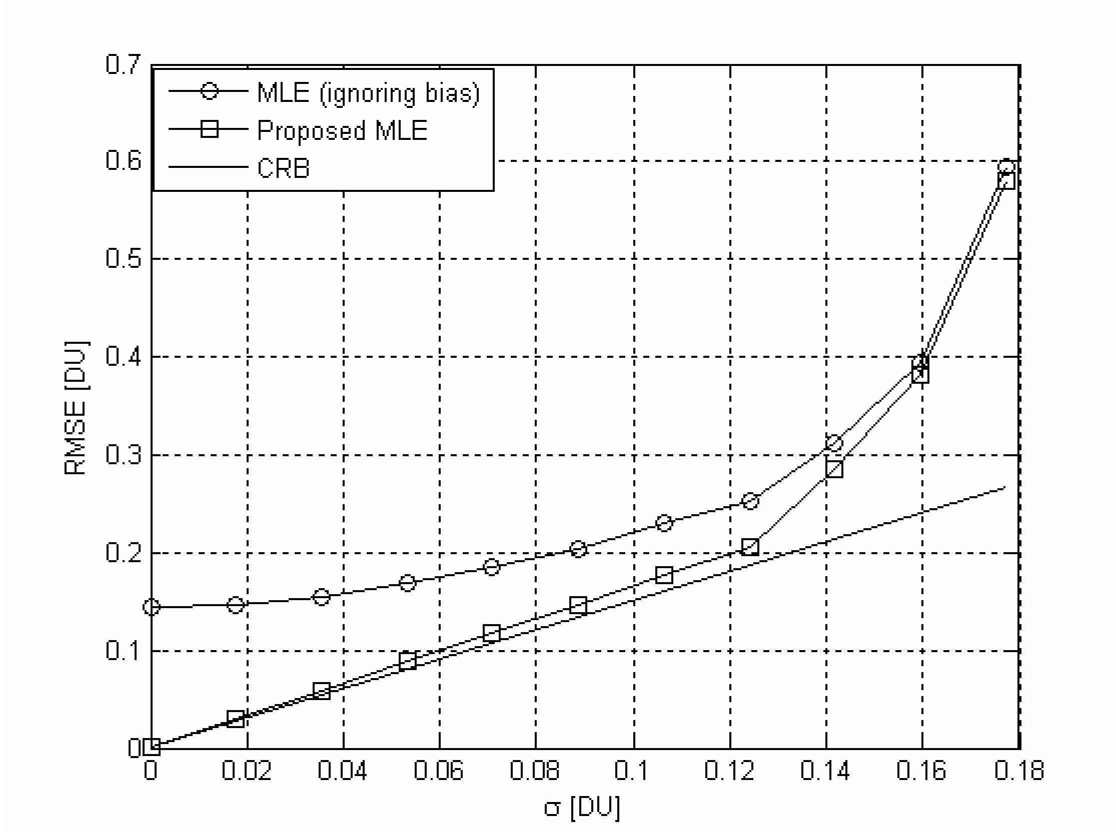

Figure 2: RMSE vs. Ranging STD, bias = 0.09 [DU], single topology

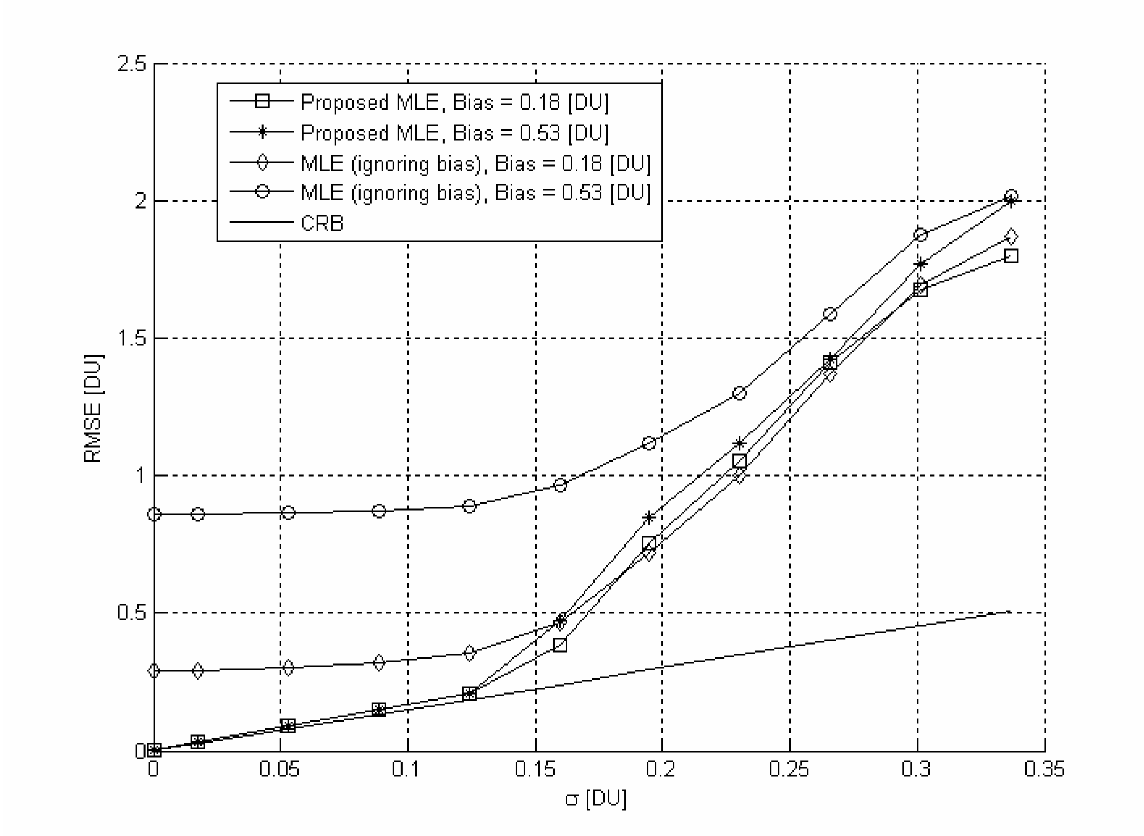

Figure 3: RMSE vs. Ranging STD for two different bias values, single topology

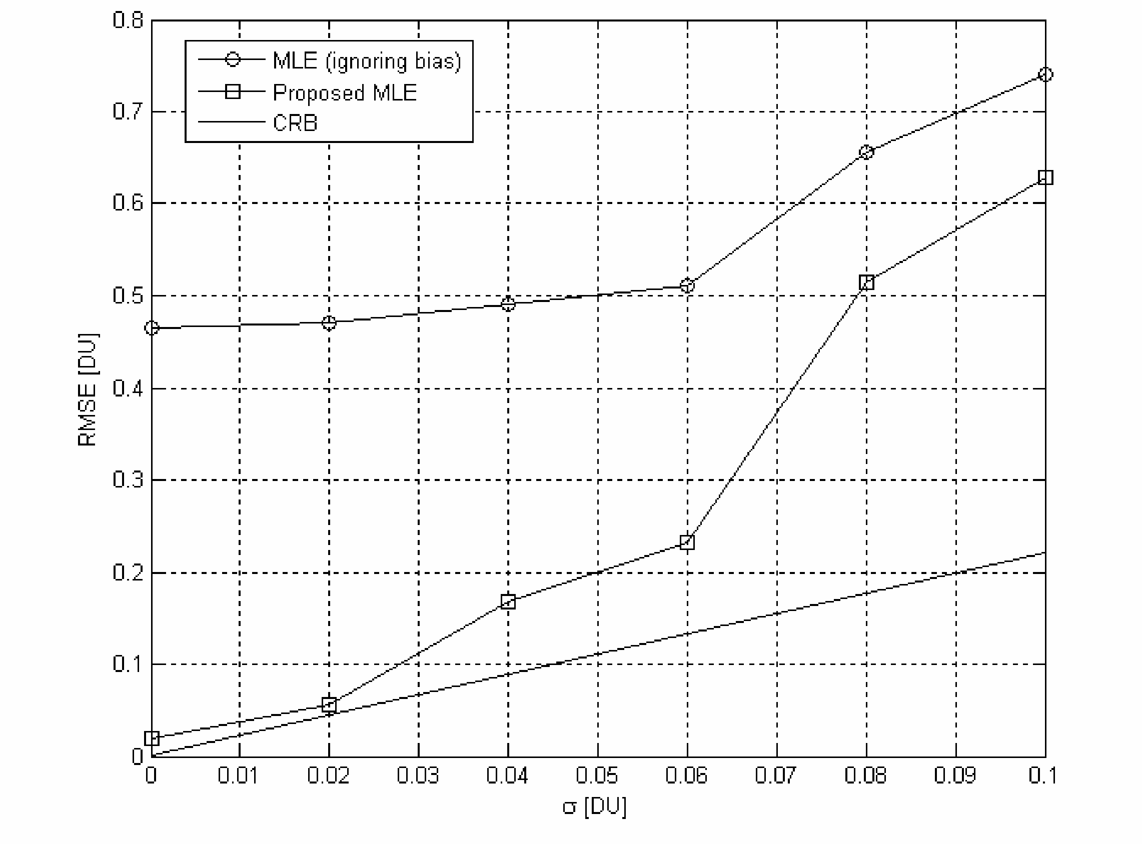

Figure 4: RMSE vs. Ranging STD, bias = 0.2 DU, averaged over 200 random topologies.

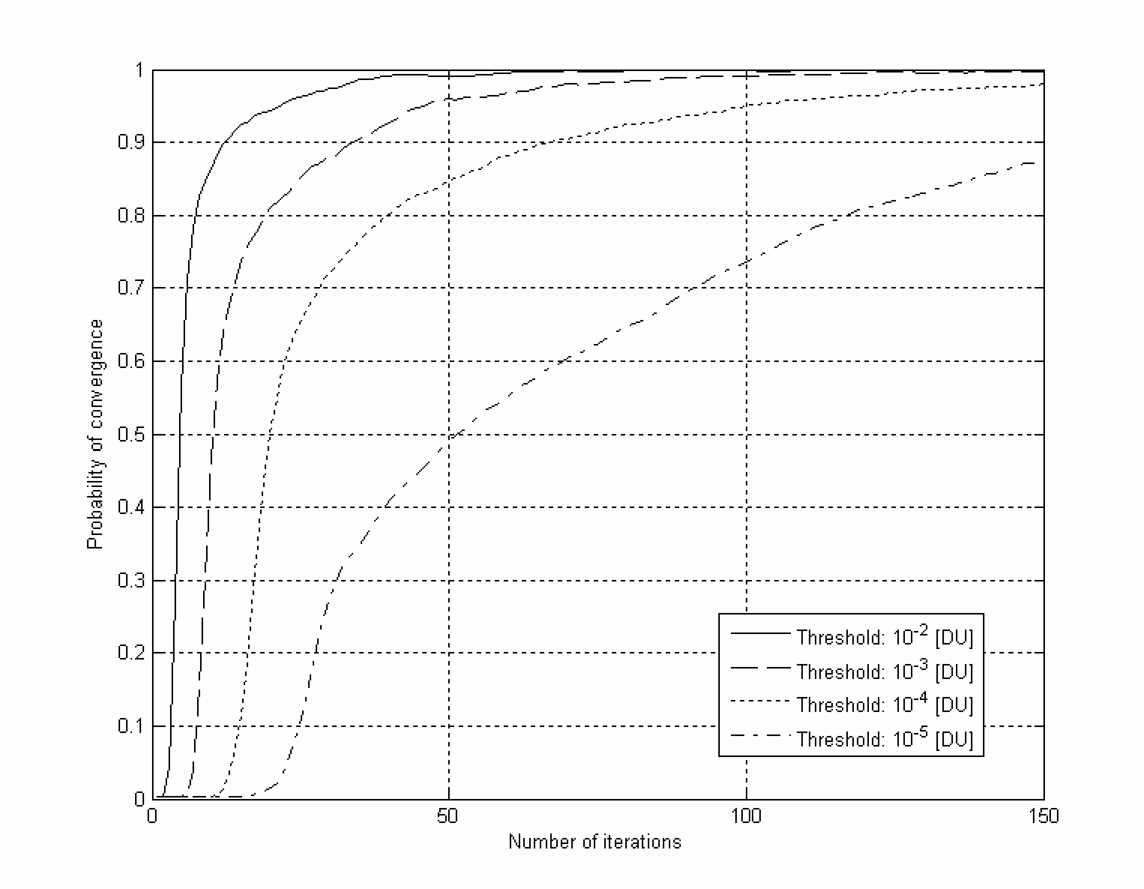

Figure 5: Probability of convergence vs. number of iterations for different stopping criteria (stopping threshold)

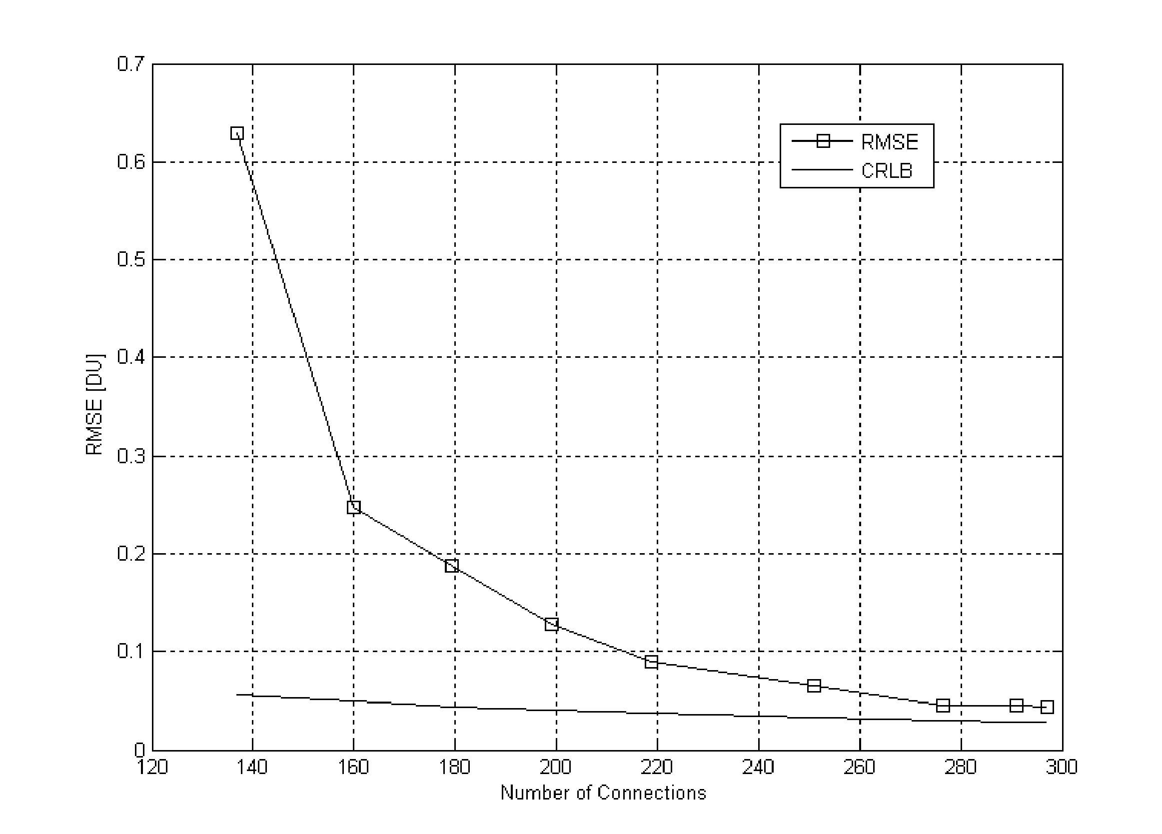

Figure 6: RMSE vs. Number of connections.