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# A short proof on the local detachment theorem

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**Abstract**

A simplified and shortened proof is presented for a theorem of Jordán and Szigeti [2] on detachments preserving local edge-connectivity.

## 1 Introduction

Let $G = (V + s, E)$ be a graph. A *degree specification* for $s$ is a sequence $f(s) = (d_1, ..., d_p)$ of positive integers with $\sum_{j=1}^{p} d_j = d_G(s)$. An $f(s)$-*detachment* of $G$ at $s$ is the graph $G'$ obtained from $G$ by replacing $s$ by a set $s_1, ..., s_p$ of independent vertices and distributing the edges incident to $s$ among them in such a way that $d_{G'}(s_i) = d_i$ $(1 \leq i \leq p)$. Note that all the other ends of the edges in $G$ remain the same. For a requirement function $r : V \times V \to Z_+$, we say that $G$ is $\mathbf{r}$-*edge-connected* if $\lambda_G(u, v) \geq r(u, v)$ $\forall u, v \in V$, where $\lambda_G(u, v)$ is the *local edge-connectivity* between $u$ and $v$ in $G$, that is the size of a minimum edge cut separating $u$ and $v$ in $G$. The following theorem characterizes graphs having an $\mathbf{r}$-edge-connected $f(s)$-detachment.

**Theorem 1.1 (Jordán, Szigeti [2]).** *Let $r$ be a requirement function for $G = (V + s, E)$ with $r(u, v) \geq 2$ $\forall u, v \in V$. Let $f(s) = (d_1, ..., d_p)$ be a degree specification for $s$ with $d_i \geq 2$ $\forall i$. Let $\varphi = \sum_1^p \lfloor \frac{d_i}{2} \rfloor$. Then there exists an $\mathbf{r}$-edge-connected $f(s)$-detachment of $G$ at $s$ if and only if*

$$G \quad is \quad \mathbf{r}\text{-}edge\text{-}connected, \tag{1}$$

$$G - s \quad is \quad (\mathbf{r}-\boldsymbol{\varphi})\text{-}edge\text{-}connected. \tag{2}$$

The aim of this paper is to provide a short proof for Theorem 1.1. We mention that Theorem 1.1 is a common generalization of Mader's theorem [4] on splitting off preserving local edge-connectivities between vertices in $V$ ($f(s) = (2, d_G(s)-2), r(u, v) = \lambda_G(u, v)$ $\forall u, v \in V$) and Fleiner's theorem [1] on $k$-edge-connected detachments ($r(u, v) = k$ $\forall u, v \in V$). This paper does not provide a new proof for Mader's theorem because it applies it. For a new proof on a generalization of Mader's threorem the reader is referred to [5].

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# 2 Definitions, preliminary results

Recall that $G = (V + s, E)$ is **r**-edge-connected if $\lambda_G(u, v) \geq r(u, v)$ $\forall u, v \in V$. Note that it is equivalent to $h^r_G(X) \geq 0$ $\forall X \subseteq V$, where $h^r_G(X) := d_G(X) - R(X)$ and $R(X) := \max\{r(u, v) : u \in X, v \in V - X\}$. The following basic property "skew-submodularity" of the function $h$ (see in [3]) will be usefull. $d_G(X, Y)$ denotes the number of edges between $X - Y$ and $Y - X$, $\overline{d}_G(X, Y) = d_G(X \cap Y, V + s - (X \cup Y))$.

For any two subsets $X, Y \subseteq V$ at least one of the following inequalities holds:

$$h^r_G(X) + h^r_G(Y) \geq h^r_G(X \cap Y) + h^r_G(X \cup Y) + 2d_G(X, Y), \quad (3)$$
$$h^r_G(X) + h^r_G(Y) \geq h^r_G(X - Y) + h^r_G(Y - X) + 2\overline{d}_G(X, Y). \quad (4)$$

If $X \cup Y = V$ then (4) always holds (with equality).

For $X \subset V$, the *cut* $\delta_G(X)$ is the set of edges leaving $X$. For $T \subset \delta_G(s)$, the *T-split* of $G$ is the $(|T|, d_G(s) - |T|)$-detachment $G'$ of $G$ at $s$ where $\delta_{G'}(s_1) = T$. Let $e(T, X) := |\delta_G(X) \cap T|$.

# 3 The proof

**Proof. Necessity:** Let $G' := (V + \{s_1, ..., s_p\}, E)$ be an **r**-edge-connected $f(s)$-detachment of $G$ at $s$. Since the identification of $\{s_1, ..., s_p\}$ does not destroy **r**-edge-connectivity in $V$, (1) is satisfied. Applying for every vertex $s_i$ $1 \leq i \leq p$ that the deletion of $s_i$ can decrease the local edge-connectivities in $V$ by at most $\lfloor \frac{d_i}{2} \rfloor$ it follows that (2) is satisfied. □

**Sufficiency:** Wlog. $p \geq 2$ and $\varphi \geq 2$. As we already mentioned, (1) and (2) can be reformulated as

$$h^r_G(X) \geq 0 \quad \forall X \subseteq V, \quad (5)$$
$$h^{r-\varphi}_{G-s}(X) \geq 0 \quad \forall X \subseteq V. \quad (6)$$

We shall use induction on $z(G) := |V| + d_G(s)$. Note that

$$h^{r-\varphi}_{G-s}(X) = h^r_G(X) - d_G(s, X) + \varphi \quad \forall X \subseteq V. \quad (7)$$

**Lemma 3.1.** *We may assume that*

$$\textit{every set } X \textit{ with } h^r_G(X) = 0 \textit{ is a singleton.} \quad (8)$$

**Proof.** Suppose there exists a set $Q$ with $h^r_G(Q) = 0$ and $|Q| > 1$. Then let $\hat{G} := (\hat{V}, \hat{E})$ be obtained from $G$ by contracting $Q$ into a vertex $q$ and let $\hat{r}(u, v) := r(u, v)$ if $u, v \in \hat{V} - q$, and $\max\{r(w, x) : w \in Q\}$ if $q \in \{u, v\}$ where $x = \{u, v\} - q$. It can be verified easily that $\hat{R}(\hat{X}) = R(X)$ $\forall \hat{X} \subseteq \hat{V}$, so (5) and (6) are satisfied for $\hat{G}$ and $\hat{r}$. Since $|Q| > 1$, $z(\hat{G}) < z(G)$ and hence, by induction, $\hat{G}$ has an $\hat{\boldsymbol{r}}$-edge-connected $f(s)$-detachment $\hat{G}'$. We show that the graph $G'$ obtained from $\hat{G}'$ by "blowing up" $Q$ is $\boldsymbol{r}$-edge-connected and we are done. Let $X' \subseteq V'$. Using that $h^r_{G'}(Q) = h^r_G(Q) = 0$,

the skew-submodularity of $h^r_{G'}$ and the fact that if $X'$ and $Q$ are not intersecting then $h^r_{G'}(X') \geq 0$ (because if $X' \subset Q$ then $h^r_{G'}(X') = h^r_G(X) \geq 0$ by (5) and if $Q \subseteq X'$ or $Q \cap X' = \emptyset$ then $h^r_{G'}(X') = h^{\hat{r}}_{\hat{G}'}(\hat{X}') \geq 0$ since $\hat{G}'$ is $\hat{\boldsymbol{r}}$-edge-connected) we get that $h^r_{G'}(X') \geq 0$ as we wanted. □

**Lemma 3.2.** *There exists $T \subset \delta_G(s)$ with $|T| = 3$ if $f(s) = (3, 3, ..., 3)$ and $|T| = 2$ otherwise such that the graph $G'$ obtained from $G$ by the $T$-split satisfy*

$$G' \quad is \quad \boldsymbol{r'}\text{-edge-connected in } V', \tag{9}$$

$$G' - s \quad is \quad (\boldsymbol{r'} - (\boldsymbol{\varphi} - \mathbf{1}))\text{-edge-connected in } V', \tag{10}$$

*where $\boldsymbol{r'}(u, v) := r(u, v)$ if $u, v \in V$ and 2 otherwise and $V' = V \cup s_1$.*

**Proof.**

**Claim 3.3.** *(9) and (10) are equivalent to*

$$h^r_G(X) \geq 2e(T, X) - |T| \quad \forall X \subset V, \tag{11}$$

$$e(T, C) \geq 1 \qquad \forall C \in \mathcal{C}, \tag{12}$$

*where $\mathcal{C}$ is defined as the minimal sets $X$ with $h^{r-\varphi}_{G-s}(X) = 0$.*

**Proof.** (9) is satisfied if and only if $0 \leq h^{r'}_{G'}(X')$ wlog. $\forall s_1 \in X'$ which is, by $h^{r'}_{G'}(X') = h^r_G(X) - e(T, X) + (|T| - e(T, X))$ with $X = X' - s_1$, equivalent to (11). (10) is satisfied if and only if $0 \leq h^{r'-\varphi'}_{G'-s}(X)$ wlog. $\forall s_1 \notin X'$ which is, by $h^{r'-\varphi'}_{G'-s}(X) = h^{r-\varphi}_{G-s}(X) + e(T, X) - 1$, equivalent to (12). □

**Claim 3.4.** *The following are true for $\mathcal{C}$ :*

$$\text{the sets in } \mathcal{C} \text{ are pairwise disjoint,} \tag{13}$$

$$d_G(s, C) \geq \varphi \text{ for each } C \in \mathcal{C}, \tag{14}$$

$$|\mathcal{C}| \in \{0, 2, 3\}, \tag{15}$$

$$\text{if } |\mathcal{C}| = 3 \text{ then } f(s) = (3, 3, ..., 3) \text{ and } h^r_G(C) = 0 \quad \forall C \in \mathcal{C}. \tag{16}$$

**Proof.** By the submodularity of $h^{r-\varphi}_{G-s}(X)$, the minimality of the sets in $\mathcal{C}$ and (6), (13) follows. (7) and (5) imply (14). By (14), (13) and $d_i \geq 2$, $|\mathcal{C}|\varphi \leq \sum_{C \in \mathcal{C}} d_G(s, C) \leq d_G(s) = \sum_{i=1}^p d_i \leq 3 \sum_{i=1}^p \lfloor \frac{d_i}{2} \rfloor = 3\varphi$ that is $|\mathcal{C}| \leq 3$. Moreover, if $X \in \mathcal{C}$, then there exists $Y \subseteq V - X$ with $Y \in \mathcal{C}$ implying (15). It also follows that if $|\mathcal{C}| = 3$ then each $d_i = 3$, that is $f(s) = (3, 3, ..., 3)$ and for every $C \in \mathcal{C}$, $d_G(s, C) = \varphi$, so by (7), $h^r_G(C) = 0$. □

By (15), either $|\mathcal{C}| = 3$ or $|\mathcal{C}| \in \{0, 2\}$. If $\boldsymbol{|\mathcal{C}| = 3}$**,** then, by (16), $f(s) = (3, 3, ..., 3)$. By (14), there exists $T \subset \delta_G(s)$ with $|T| = 3$ that satisfies (12). $T$ also satisfies (11). Indeed, by (16), (8) and (14), $d_G(s, X) \geq \varphi\, e(T, X)$. So, by (7), (6) and $\varphi \geq 2$, $h^r_G(X) \geq d_G(s, X) - \varphi \geq \varphi(e(T, X) - 1) \geq 2(e(T, X) - 1) \geq 2e(T, X) - |T|$. From now on $\boldsymbol{|\mathcal{C}| \in \{0, 2\}}$**.**

**Lemma 3.5.** *There exists $T = \{su, sv\}$ that satisfies (11) and (12).*

**Proof.** If $|\mathcal{C}| = 0$, then, by Mader theorem [4], there exists $T \subset \delta_G(s)$ with $|T| = 2$ that satisfies (11) and in this case (12) is automatically satisfied. If $\mathcal{C} = \{C_1, C_2\}$, then, by (14), there exists $T \subset \delta_G(s)$ with $|T| = 2$ that satisfies (12). We claim that *$T$ satisfies (11).* Suppose $X \subset V$ violates (11). Then $e(T, X) = 2$ and $h^r_G(X) \leq 1$. Wlog. $C_1 - X \neq \emptyset$, otherwise $C_1 \cup C_2 \subset X$ so, by (7) (6) and (13) (14), $1 \geq h^r_G(X) \geq d_G(s, X) - \varphi \geq d_G(s, C_1 \cup C_2) - \varphi \geq 2\varphi - \varphi \geq 2$, contradiction. Since $C_1 \in \mathcal{C}, h^{r-\varphi}_{G-s}(C_1 - X) \geq 1$. Then, by (7), $h^r_G(C_1 - X) = h^{r-\varphi}_{G-s}(C_1 - X) + d_G(s, C_1 - X) - \varphi \geq 1 + d_G(s, C_1) - d_G(s, C_1 \cap X) - \varphi = h^r_G(C_1) + 1 - d_G(s, C_1 \cap X)$. Suppose (4) applies for $C_1$ and $X$. Then, by (5), $1 + h^r_G(C_1) \geq h^r_G(X) + h^r_G(C_1) \geq h^r_G(X - C_1) + h^r_G(C_1 - X) + 2\overline{d}_G(X, C_1) \geq h^r_G(C_1 - X) + 2d_G(X \cap C_1, s) \geq h^r_G(C_1) + 1 + d_G(s, C_1 \cap X) \geq h^r_G(C_1) + 2$, contradiction. So (3) applies for $C_1$ and $X$ and $C_1 \cup X \neq V$. Since $\mathcal{C} = \{C_1, C_2\}, h^{r-\varphi}_{G-s}(C_1 \cup X) = h^{r-\varphi}_{G-s}(V - (C_1 \cup X)) \geq 1$. Then, by (7), $h^r_G(C_1 \cup X) = h^{r-\varphi}_{G-s}(C_1 \cup X) + d_G(s, C_1 \cup X) - \varphi \geq 1 + d_G(s, C_1) + d_G(s, C_2 \cap X) - \varphi \geq h^r_G(C_1) + 2$. Then, by (5), $1 + h^r_G(C_1) \geq h^r_G(X) + h^r_G(C_1) \geq h^r_G(X \cap C_1) + h^r_G(C_1 \cup X) \geq h^r_G(C_1 \cup X) \geq h^r_G(C_1) + 2$, contradiction. □

If $f(s) \neq (3, 3, ..., 3)$, then we are done. From now on $\boldsymbol{f(s) = (3, 3, ..., 3).}$ Then $d_G(s) = 3\varphi$.

**Lemma 3.6.** *$T$ can be extended to $T' \subset \delta_G(s)$ with $|T'| = 3$ such that $T'$ satisfies (11).*

**Proof.** First suppose that $\boldsymbol{\Gamma(s) = \{u, v\}.}$ Since $d_G(s) = 3\varphi$ and $\varphi \geq 2$, wlog. $d_G(s, u) \geq \varphi + 1$ and hence there exists another copy $e'$ of $su$. Then $T' := T \cup e'$ satisfies (11). Hence $\boldsymbol{\Gamma(s) \neq \{u, v\}.}$ Suppose indirect that there exists a minimal set $\mathcal{M}$ of subsets of $V$ such that for every $z_i \in \Gamma(s) - \{u, v\}$ there exists a set $M_i \in \mathcal{M}$ violating (11) for $T' := T \cup sz_i$. Then, by the fact that $T$ satisfies (11) and by (8), $e(T', M_i) = 3$ so $\{u, v, z_i\} \subseteq M_i$ and $h^r_G(M_i) \leq 2$. Since $\Gamma(s) \neq \{u, v\}, |\mathcal{M}| \geq 1$. By (7), (6), $h^r_G(M_i) \leq 2$ and $\varphi \geq 2$, $|\mathcal{M}| \geq 2$.

**Claim 3.7.** *If $M_i, M_j \in \mathcal{M}$, then*

$$h^r_G(M_i - M_j) = 0, \text{ (so, by (8), } M_i - M_j = z_i, ) \quad (17)$$

$$\overline{d}_G(M_i, M_j) = 2, \quad (18)$$

$$d_G(z_i, M_i - z_i) \geq 1. \quad (19)$$

**Proof.** $2 \geq h^r_G(M_i), 2 \geq h^r_G(M_j), h^r_G(M_i \cap M_j) \geq 2e(T, M_i \cap M_j) - |T| \geq 2 \times 2 - 2 = 2$ (by (11) and $\{u, v\} \subset M_i \cap M_j$), $h^r_G(M_i \cup M_j) \geq 3$ (by the minimality of $\mathcal{M}$), so (3) cannot be satisfied for $M_i$ and $M_j$. Then $M_i$ and $M_j$ satisfy (4) implying (17) and (18). Moreover, $2 \leq h^r_G(z_i) + h^r_G(M_i \cap M_j) = h^r_G(z_i) + h^r_G(M_i - z_i) \leq h^r_G(M_i) - 2 + 2d_G(z_i, M_i - z_i) \leq 2d_G(z_i, M_i - z_i)$. □

**Claim 3.8.** $|\mathcal{M}| \geq 3$.

**Proof.** Suppose $\mathcal{M} = \{M_1, M_2\}$. Then, by (7),(17),(6),(18), $3\varphi = d_G(s) = d_G(s, z_1) + d_G(s, z_2) + d_G(s, M_1 \cap M_2) = h^r_G(z_1) - h^{r-\varphi}_{G-s}(z_1) + \varphi + h^r_G(z_2) - h^{r-\varphi}_{G-s}(z_2) + \varphi + d_G(s, M_1 \cap$

$M_2) \leq 2\varphi + 2 \leq 3\varphi$. It follows that $h^{r-\varphi}_{G-s}(z_1) = 0$, so $z_1 \in \mathcal{C}$, that is (12) is violated for $T$, contradiction. □

Let $M_1, M_2, M_3 \in \mathcal{M}$. Then, by (19), (17), (18), $1 \leq d_G(M_3 - z_3, z_3) = d_G(M_1 \cap M_2, z_3) \leq \overline{d}_G(M_1, M_2) - d_G(M_1 \cap M_2, s) \leq 2 - 2 = 0$, contradiction. This completes the proof of Lemma 3.6. □

Since $T$ satisfies (12), so does $T'$ and the proof of Lemma 3.2 is complete. □

Let $G'$ be obtained from $G$ by the T-split from Lemma 3.2. Let us denote the new vertex of $G'$ of degree $|T|$ by $t$. Wlog. $d_1 \geq d_2 \geq ... \geq d_p$. If $d_p = |T|$ then let $f'(s) := (d_1, ..., d_{p-1})$ otherwise ($|T| = 2, d_1 \geq 4$) let $f'(s) := (d_1 - 2, d_2, ..., d_p)$. Then $(G', f'(s))$ satisfies (9) and (10) and $z(G') < z(G)$, so by induction, $G'$ has an $\mathbf{r}$-edge-connected $f'(s)$-detachment $G''$. Then, in the former case $G''$, in the latter case the graph obtained from $G''$ by identifying $s_1$ and $t$, is an $\mathbf{r}$-edge-connected $f(s)$-detachment of $G$. □

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