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Existence and uniqueness of global positive solutions to the stochastic functional Kolmogorov-type system

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In general, any population systems are often subject to various environmental noise. To examine whether the presence of such noise affects these systems significantly, we examine a stochastic functional Kolmogorov-type population system with the general stochastic perturbation. In this paper, we show that the environmental noise may suppress the potential population explosion and guarantee global existence of positive solutions when the noise intensity is strongly dependent on the population size. However, when the noise intensity is weakly dependent on the population size, the stochastic population system behaves similarly to the corresponding deterministic system. To illustrate our idea clearly, we also discuss some stochastic Lotka–Volterra systems as special cases.

Keywords: Kolmogorov-type system; Lotka–Volterra system; stochastic functional differential equations; global positive solutions.

1. Introduction

The classical Kolmogorov-type system for n interacting species is described by the n-dimensional differential equation

$$\dot{x}(t) = \operatorname{diag}(x_1(t), \ldots, x_n(t))\, f(x(t)), \tag{1.1}$$

where $x = (x_1, \ldots, x_n)^\top$ and $\operatorname{diag}(x_1, \ldots, x_n)$ represents the $n \times n$ matrix with all elements zero except those on the diagonal which are $x_1, \ldots, x_n$ and $f = (f_1, \ldots, f_n)^\top$. There is an extensive literature concerned with dynamics of this system and the Lotka–Volterra-type system (e.g. Gary, 2004; Han *et al.*, 2006; Hastings, 1978; Teng, 2000). Considering the maturation period of biological species, it is often more realistic to use delay differential equations to describe population systems. There are also many authors to consider delay population systems (e.g. Beretta & Takeuchi, 1988; Jin & Ma, 2002; Kuang & Smith, 1993; Meng *et al.*, 2008; Muroya, 2004). To generalize these delay system, Tang & Kuang (1996) examine the following functional population system:

$$\dot{x}(t) = \operatorname{diag}(x_1(t), \ldots, x_n(t))\, f(x_t), \tag{1.2}$$

where $x_t = x_t(\theta) \in C([-\tau, 0]; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$.

In (1.1) and (1.2), the vector x often denotes the population size of the n species. We are therefore not only interested in the positive solutions but also require the solutions not to explode at a finite time, namely, there exists a global positive solution. Under some conditions, He & Gopalsamy (1997) consider global positive solutions for a 2D Lotka–Volterra system. Chen *et al.* (2004) and Yang & Cao

(2004) examine positive periodic solutions to the Lotka–Volterra system. The global positive solution is a natural requirement from the biological point of view.

However, any population systems are often subject to environmental noise. Recently, the stochastic Lotka–Volterra system has received the increasing attention. Bahar & Mao (2004a) and Mao *et al.* (2002) reveal that the environmental noise may suppress the potential population explosion and guarantee the global positive solution to stochastic Lotka–Volterra system. Under another environmental noise perturbation, Bahar & Mao (2004b) reveals that the stochastic delay Lotka–Volterra system behaves similarly to the corresponding deterministic system. Mao (2005) reviews these two classes of models and indicates clearly that different structure of environmental noise may have different effects on the population dynamics. Rudnicki (2003) and Rudnicki & Pichor (2007) considers a stochastic prey–predator model and examines the existence of the stationary distribution.

Recall that $f_i(x_t)$ represents the inherent net birth rate of the ith species since (1.2) implies that $\dot{x}_i(t) = x_i(t) f_i(x_t)$. It is often affected by various unpredictable factors. According to the well-known central limited theorem, the sum of these factors follows a normal distribution. We can therefore replace the net birth rate $f_i(x_t)$ by

$$f_i(x_t) + g_i(x_t)\dot{w}(t),$$

where $\dot{w}(t)$ is a white noise (i.e. $w(t)$ is a scalar Brownian motion) and $g_i(x_t)$ represents this noise intensity. Hence, (1.2) becomes the following stochastic functional Kolmogorov-type system:

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[f(x_t)\mathrm{d}t + g(x_t)\mathrm{d}w(t)] \tag{1.3}$$

on $t \geqslant 0$, where

$$f = (f_1, \ldots, f_n)^{\top}: C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n, \quad g = (g_1, \ldots, g_n): C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n.$$

For more biological motivation on this type of modelling in population dynamics, we refer the reader to Allen (2007), Gard (1984, 1992) and Ladde & Sambandham (2003). Here, we hope to emphasize that both f and g will be redefined as $f, g : \mathbb{R}^n \to \mathbb{R}^n$ if $\tau = 0$. Clearly, (1.3) therefore includes the following three equations:

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[f(x_t)\mathrm{d}t + g(x(t))\mathrm{d}w(t)], \tag{1.4}$$

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[f(x(t))\mathrm{d}t + g(x_t)\mathrm{d}w(t)], \tag{1.5}$$

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}w(t)]. \tag{1.6}$$

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by $A^{\top}$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\top}A)}$. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, +\infty)$, and let $\tau > 0$. Denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|$, which forms a Banach space. For any $a \in \mathbb{R}$, let $a^+ = a \vee 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let $w(t)$ be a scalar Brownian motion defined on this probability space. If $x(s)$ is an $\mathbb{R}^n$-valued stochastic process on $s \in [-\tau, \infty)$, we let $x_t = \{x(t + \theta): -\tau \leqslant \theta \leqslant 0\}$ for $t \geqslant 0$. In addition, throughout this paper, const always represents a positive constant, whose precise value is not important. In this paper, we also impose the standard local Lipschitz condition on f and g.

ASSUMPTION 1.1 For every integer k, there is a positive constant c_k such that

$$|f(\varphi) - f(\phi)| \vee |g(\varphi) - g(\phi)| \leqslant c_k \|\varphi - \phi\|$$

for all $t \geqslant 0$ and these $\varphi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ with $\|\varphi\| \vee \|\phi\| \leqslant k$.

To show our idea clearly, Section 2 gives a result of existence and uniqueness of the global positive solution for general stochastic functional differential equations. Applying the result of Section 2, we give the conditions under which the stochastic functional Kolmogorov system almost surely admits a global positive solution in Section 3. As applications of Section 3, Section 4 discusses some stochastic Lotka–Volterra systems as special cases.

2. A general result

In the population dynamic system, the global positive solution implies that the population will not explode nor be extinct in any finite time. This property is therefore important. To show our idea about existence and uniqueness of global positive solutions clearly, we first consider the following general n-dimensional stochastic functional differential equation:

$$\mathrm{d}x(t) = F(t, x_t)\mathrm{d}t + G(t, x_t)\mathrm{d}B(t) \tag{2.1}$$

on $t \geqslant 0$, where $B(t)$ is an m-dimensional Brownian motion defined on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$F = (F_1, \ldots, F_n)^\top : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n, \quad G = [G_{ij}] : \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n \times m}.$$

Here, let $G_i = (G_{i1}, \ldots, G_{im})$. We also impose the following local Lipschitz condition on F and G.

ASSUMPTION 2.1 For every integer j, there is a positive constant c_j such that

$$|F(t, \varphi) - F(t, \phi)| \vee |G(t, \varphi) - G(t, \phi)| \leqslant c_j \|\varphi - \phi\|$$

for all $t \geqslant 0$ and these $\varphi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ with $\|\varphi\| \vee \|\phi\| \leqslant j$.

In order for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of this equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see Arnold, 1972; Mao, 1997) or a given non-Lipschitz condition and the linear growth condition (Fang & Zhang, 2005; Mao, 1994). These show that the linear growth condition plays an important role to suppress the potential explosion of solutions and guarantee the existence of global solutions. However, it is obvious that the coefficients of (1.3) do not satisfy the linear growth condition except that both f and g are bounded, so its solutions may explode at a finite time (cf. Khasminskii, 1981; Mao, 1994). It is therefore important to examine the existence and uniqueness of the global solution of (2.1) without the linear growth condition.

It is well known for stochastic differential equations that linear growth conditions for global solutions may be replaced by the use of the Lyapunov functions (see Khasminskii, 1981; Mao & Rassias, 2005; Shen *et al.*, 2006). However, the Lyapunov function method is sometimes not convenient in applications since the results are dependent on construction of this function. In this paper, we are concerned with the population dynamic system, so we are not only interested in the global existence of solutions but also require the solutions to be positive, namely, there exists a global positive solution. We therefore need to choose the special function to establish the existence of the global positive solution. To show this idea, we first need the concept of local solutions (see Mao, 1994).

DEFINITION 2.1 Set $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \leqslant t \leqslant 0$ and let $x(t)$, $-\tau \leqslant t \leqslant \tau_e$ be a continuous $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process. It is called a local strong solution of (2.1) with initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$ if $x(t) = \xi(t)$ on $-\tau \leqslant t \leqslant 0$ and

$$x(t \wedge \tau_k) = \xi(0) + \int_0^{t\wedge\tau_k} F(s, x_s)\mathrm{d}s + \int_0^{t\wedge\tau_k} G(s, x_s)\mathrm{d}B(s), \quad \forall\, t \geqslant 0$$

for each $k \geqslant 1$, where $\{\tau_k\}_{k\geqslant 1}$ is a non-decreasing sequence of finite stopping times such that $\tau_k \to \tau_e$ almost surely as $k \to \infty$. If moreover, $\limsup_{t\to\tau_e} |x(t)| = \infty$ is satisfied almost everywhere when $\tau_e < \infty$, it is called a maximal local strong solution and τ_e is called the explosion time. A maximal local strong solution $x(t)$, $-\tau \leqslant t < \tau_e$, is said to be unique if for any other maximal local strong solution $\bar{x}(t)$, $-\tau \leqslant t < \bar{\tau}_e$, we have $\tau_e = \bar{\tau}_e$ and $x(t) = \bar{x}(t)$ for $-\tau \leqslant t < \tau_e$ almost surely.

By Mao (1994, Theorem 3.2.2, P95), we have the following theorem.

THEOREM 2.1 Under Assumption 2.1, for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n)$, (2.1) almost surely admits a unique maximal local strong solution.

By Khasminskii's idea, Khasminskii (1981), Mao & Rassias (2005) and Shen *et al.* (2006) examine global existence of solutions of stochastic differential equations. In this paper, our aim is to examine global existence of the positive solutions of (2.1), so we need to introduce a special Lyapunov function. For any $p > 0$, define a C^2-function $U: \mathbb{R}^n_{++} \to \mathbb{R}_+$ by

$$U(x) = \sum_{i=1}^n c_i u(x_i) \quad \text{for } x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n_{++}, \tag{2.2}$$

where $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n_{++}$, $u(x_i) = x_i^p - p\log x_i$, $p > 0$ is a constant. Clearly, the function u holds the following nice properties:

$$u(t) > 0, \quad u(0^+) = u(\infty) = \infty. \tag{2.3}$$

THEOREM 2.2 Under Assumption 2.1, if there exist $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n_{++}$, $p > 0$ $K, K_j, \alpha_j \geqslant 0$, and probability measures μ_j (j = 1, 2, …, J) on $[-\tau, 0]$ for any integer $J \geqslant 1$ such that, for any $t \geqslant 0$ and $\varphi = (\varphi_1, \ldots, \varphi_n)^\top \in C([-\tau, 0]; \mathbb{R}^n_{++})$,

$$\begin{aligned}\mathcal{L}U(t, \varphi) :&= \sum_{i=1}^n [p\varphi_i^p(0) - \rho]c_i\varphi_i^{-1}(0)F_i(t, \varphi) + \frac{1}{2}\sum_{i=1}^n [p(p-1)\varphi_i^p(0) + \rho]c_i\varphi_i^{-2}(0)|G_i(t, \varphi)|^2 \\ &\leqslant K + \sum_{j=1}^J K_j \left[\int_{-\tau}^0 |\varphi(\theta)|^{\alpha_j}\, \mathrm{d}\mu_j(\theta) - |\varphi(0)|^{\alpha_j}\right],\end{aligned} \tag{2.4}$$

then for any given initial data $x_0 = \xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global solution $x(t)$ to (2.1) with $\tau_e = \infty$, and this solution will remain in $\mathbb{R}^n_{++}$ with probability one, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

Proof. Since both F and G satisfy the local Lipschitz condition, for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, by Theorem 2.1, there exists a unique maximal local strong solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is

the explosion time. To show that the solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let k_0 be a sufficiently large positive number such that

$$\frac{1}{k_0} < \min_{-\tau \leqslant \theta \leqslant 0} |\xi(\theta)| \leqslant \max_{-\tau \leqslant \theta \leqslant 0} |\xi(\theta)| < k_0.$$

For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\{t \in [-\tau, \tau_e): x_i(t) \notin (1/k, k) \quad \text{for some } i = 1, 2, \ldots, n\} \tag{2.5}$$

with the traditional setting $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \to \tau_\infty \leqslant \tau_e$ a.s. If we can show $\tau_\infty = \infty$, then $\tau_e = \infty$ a.s., which implies the desired result. This is also equivalent to proving that, for any $t > 0$, $\mathbb{P}(\tau_k \leqslant t) \to 0$ as $k \to \infty$. By condition (2.4), applying the Itô formula to the function $U(\cdot)$ yields

$$\begin{aligned}\mathbb{E}U(x(t \wedge \tau_k)) &= \mathbb{E}U(x(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} \mathcal{L}U(s, x_s)\mathrm{d}s \\ &\leqslant \mathbb{E}U(\xi(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} \left\{ K + \sum_{j=1}^{J} K_j \left[\int_{-\tau}^{0} |x(s+\theta)|^{\alpha_j}\, \mathrm{d}\mu_j(\theta) - |x(s)|^{\alpha_j} \right] \right\} \mathrm{d}s.\end{aligned}$$

By the Fubini theorem and a substitution technique,

$$\begin{aligned}&\int_0^{t\wedge\tau_k}\int_{-\tau}^{0} |x(s+\theta)|^{\alpha_j}\, \mathrm{d}\mu_j(\theta)\mathrm{d}s - \int_0^{t\wedge\tau_k} |x(s)|^{\alpha_j}\, \mathrm{d}s \\ &\quad\leqslant \int_{-\tau}^{0} \mathrm{d}\mu_j(\theta) \int_{-\tau}^{t\wedge\tau_k} |x(s)|^{\alpha_j}\, \mathrm{d}s - \int_0^{t\wedge\tau_k} |x(s)|^{\alpha_j}\, \mathrm{d}s \\ &\quad= \int_{-\tau}^{0} |\xi(\theta)|^{\alpha_j}\, \mathrm{d}\theta.\end{aligned}$$

We therefore have

$$\begin{aligned}\mathbb{E}U(x(t \wedge \tau_k)) &\leqslant \mathbb{E}U(\xi(0)) + Kt + \sum_{j=1}^{J} K_j \int_{-\tau}^{0} \mathbb{E}|\xi(\theta)|^{\alpha_j}\, \mathrm{d}\theta \\ &=: K_t,\end{aligned}$$

where K_t is independent of k. By the definition of τ_k, $x_i(\tau_k)$ is a $\{k, 1/k\}$-valued random variable for some $1 \leqslant i \leqslant n$. Hence, by conditional expectation, we have

$$\begin{aligned}\min_{1\leqslant i\leqslant n}\{c_i\}\mathbb{P}(\tau_k \leqslant t)(u(k) \wedge u(1/k)) &\leqslant \mathbb{P}(\tau_k \leqslant t)\mathbb{E}U(x(\tau_k)) \\ &\leqslant \mathbb{E}U(x(t \wedge \tau_k)) \\ &\leqslant K_t,\end{aligned}$$

which implies

$$\limsup_{k\to\infty} \mathbb{P}(\tau_k \leqslant t) \leqslant \lim_{k\to\infty} \frac{K_t}{\min_{1\leqslant i\leqslant n}\{c_i\}(u(k) \wedge u(1/k))} = 0.$$

We therefore obtain the desired result. □

3. Stochastic functional Kolmogorov-type systems

This section mainly applies the result of Section 2 to examine existence and uniqueness of the global positive solution to (1.3) as well as (1.4) and (1.5). To apply Theorem 2.2, we also impose the following assumption on both f and g.

ASSUMPTION 3.1 There exist constants $\alpha, \beta, \zeta, \kappa, \eta, \lambda, \lambda_1, \sigma, \sigma_1 \geqslant 0$ and probability measures μ, ν, ν_1 on $[-\tau, 0]$ such that, for any $\varphi \in C([-\tau, 0]; \mathbb{R}^n_{++})$,

$$|f(\varphi)| \leqslant \kappa \int_{-\tau}^{0} |\varphi(\theta)|^\alpha \, \mathrm{d}\mu(\theta) + \eta, \tag{3.1a}$$

$$|g(\varphi)| \leqslant \lambda \int_{-\tau}^{0} |\varphi(\theta)|^\beta \, \mathrm{d}\nu(\theta) + \sigma, \tag{3.1b}$$

$$g_i^2(\varphi) \geqslant \zeta \varphi_i^{2\beta}(0) - \lambda_1 \int_{-\tau}^{0} \varphi_i^{2\beta}(\theta) \mathrm{d}\nu_1(\theta) - \sigma_1, \quad 1 \leqslant i \leqslant n. \tag{3.1c}$$

In this assumption, conditions (3.1a) and (3.1b) show that both f and g are controlled by polynomial functions in which f and g satisfy the linear growth condition when $\alpha = \beta = 1$ (note that this does not imply that the coefficients of (1.3) satisfy the linear growth condition). Recall that g_i represents the noise intensity imposed on the ith population. Condition (3.1c) implies that the noise intensity is dependent on the population size with the order β at least. Then the following theorem follows.

THEOREM 3.1 Under Assumptions 1.1 and 3.1, if one of the following two conditions holds,

$$\alpha < 2\beta, \quad \zeta > \lambda_1, \tag{3.2a}$$

$$0 < \alpha = 2\beta < 2, \quad \zeta > \lambda_1 + 2\kappa\sqrt{n}, \tag{3.2b}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to (1.3) on $t \geqslant -\tau$ and the solution will remain in $\mathbb{R}^n_{++}$ with probability one, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

REMARK Condition (3.2a) shows that if the noise intensity is strongly dependent on the population size in the sense $\alpha < 2\beta$, then existence and uniqueness of the global positive solution is only determined by the noise perturbation with $\zeta > \lambda_1$ and independent of the parameters in the drift term. If the noise intensity is dependent on the population size in the sense $\beta = \alpha/2 \in (0, 1)$, then existence and uniqueness of the global positive solution is determined by both the drift term and the noise intensity with $\zeta > \lambda_1 + 2\kappa\sqrt{n}$.

Proof. Let $F(t, \varphi) = \mathrm{diag}(\varphi_1(0), \ldots, \varphi_n(0)) f(\varphi)$ and $G(t, \varphi) = \mathrm{diag}(\varphi_1(0), \ldots, \varphi_n(0))g(\varphi)$ (This requires $m = 1$ in (2.1)). Clearly, Assumptions 1.1 and 3.1 imply that F and G satisfy the local Lipschitz

condition. In Theorem 2.2, to test the condition (2.4), it is the key to estimate $\mathcal{L}U(t,\varphi)$. Choosing $p \in (0,1)$ and $c_i \equiv 1$ for all $1 \leqslant i \leqslant n$ in the function $U(x)$, then we have

$$\begin{aligned}\mathcal{L}U(t,\varphi) &= \sum_{i=1}^{n}[p\varphi_i^p(0) - p]\, f_i(\varphi) + \frac{1}{2}\sum_{i=1}^{n}[p(p-1)\varphi_i^p(0) + p]g_i^2(\varphi)\\ &= \sum_{i=1}^{n} p\varphi_i^p(0)\, f_i(\varphi) + \frac{p(p-1)}{2}\sum_{i=1}^{n}\varphi_i^p(0)g_i^2(\varphi) + \frac{p}{2}\sum_{i=1}^{n}[-2\, f_i(\varphi) + g_i^2(\varphi)]\\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.3}$$

Recall the following elementary inequalities: for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$ and $l > 0$,

$$|x|^l \leqslant \sum_{i=1}^{n} x_i^l \leqslant n^{\frac{2-l}{2}}|x|^l \qquad (0 < l \leqslant 2), \tag{3.4a}$$

$$n^{\frac{2-l}{2}}|x|^l \leqslant \sum_{i=1}^{n} x_i^l \leqslant |x|^l \quad (l \geqslant 2). \tag{3.4b}$$

Then by condition (3.1a) in Assumption 3.1 and the Young inequality,

$$\begin{aligned} I_1 &\leqslant p \max_{1\leqslant i\leqslant n}\{|\varphi_i(0)|^p\}\sum_{i=1}^{n}|\, f_i(\varphi)|\\ &\leqslant p\sqrt{n}|\, f(\varphi)|\left(\sum_{i=1}^{n}|\varphi_i(0)|^2\right)^{\frac{p}{2}}\\ &\leqslant p\sqrt{n}|\varphi(0)|^p|\, f(\varphi)|\\ &\leqslant p\sqrt{n}\left[\kappa\int_{-\tau}^{0}|\varphi(0)|^p|\varphi(\theta)|^\alpha\, \mathrm{d}\mu(\theta) + \eta|\varphi(0)|^p\right]\\ &\leqslant p\sqrt{n}\Big[\frac{\kappa p}{\alpha+p}|\varphi(0)|^{\alpha+p} + \frac{\alpha\kappa}{\alpha+p}\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+p}\, \mathrm{d}\mu(\theta) + \eta|\varphi(0)|^p\Big]. \end{aligned} \tag{3.5}$$

Noting that $p \in (0,1)$, by condition (3.1c) in Assumption 3.1 and the Young inequality,

$$\begin{aligned} I_2 &\leqslant \frac{p(p-1)}{2}\sum_{i=1}^{n}\varphi_i^p(0)\left[\zeta\varphi_i^{2\beta}(0) - \lambda_1\int_{-\tau}^{0}\varphi_i^{2\beta}(\theta)\mathrm{d}\nu_1(\theta) - \sigma_1\right]\\ &= \frac{p(p-1)}{2}\sum_{i=1}^{n}\left[\zeta\varphi_i^{2\beta+p}(0) - \lambda_1\int_{-\tau}^{0}\varphi_i^p(0)\varphi_i^{2\beta}(\theta)\mathrm{d}\nu_1(\theta) - \sigma_1\varphi_i^p(0)\right] \end{aligned}$$

$$\leqslant \frac{p(p-1)}{2}\sum_{i=1}^{n}\left[\zeta\varphi_i^{2\beta+p}(0)-\sigma_1\varphi_i^{p}(0)-\frac{\lambda_1}{2\beta+p}\int_{-\tau}^{0}(p\varphi_i^{2\beta+p}(0)+2\beta\varphi_i^{2\beta+p}(\theta))\mathrm{d}\nu_1(\theta)\right]$$

$$=\frac{p(p-1)}{2}\sum_{i=1}^{n}\left[\left(\zeta-\frac{p\lambda_1}{2\beta+p}\right)\varphi_i^{2\beta+p}(0)-\sigma_1\varphi_i^{p}(0)-\frac{2\beta\lambda_1}{2\beta+p}\int_{-\tau}^{0}\varphi_i^{2\beta+p}(\theta)\mathrm{d}\nu_1(\theta)\right]. \quad (3.6)$$

By condition (3.1b) in Assumption 3.1, applying the Hölder inequality gives

$$|g(\varphi)|^2 \leqslant 2\lambda^2\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta}\mathrm{d}\nu(\theta)+2\sigma^2. \quad (3.7)$$

It therefore follows that

$$\begin{aligned} I_3 &\leqslant \text{const}[|f(\varphi)|+|g(\varphi)|^2] \\ &\leqslant \text{const}\left[1+\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha}\,\mathrm{d}\mu(\theta)+\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta}\,\mathrm{d}\nu(\theta)\right]. \end{aligned} \quad (3.8)$$

Substituting (3.5), (3.6) and (3.8) into (3.3) yields

$$\begin{aligned} \mathcal{L}U(t,\varphi) \leqslant{}& \frac{\beta\lambda_1 p(1-p)}{2\beta+p}\sum_{i=1}^{n}\left[\int_{-\tau}^{0}\varphi_i^{2\beta+p}(\theta)\mathrm{d}\nu_1(\theta)-\varphi_i^{2\beta+p}(0)\right] \\ &+\frac{\alpha\kappa p\sqrt{n}}{\alpha+p}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+p}\,\mathrm{d}\mu(\theta)-|\varphi(0)|^{\alpha+p}\right]+\text{const}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha}\,\mathrm{d}\mu(\theta)-|\varphi(0)|^{\alpha}\right] \\ &+\text{const}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta}\,\mathrm{d}\nu(\theta)-|\varphi(0)|^{2\beta}\right]+I(\varphi(0)), \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} I(x) &= \frac{p(p-1)}{2}\left(\zeta-\frac{p\lambda_1}{2\beta+p}-\frac{2\beta\lambda_1}{2\beta+p}\right)\sum_{i=1}^{n}x_i^{2\beta+p}+\kappa p\sqrt{n}|x|^{\alpha+p} \\ &\quad+\text{const}[1+|x|^p+|x|^{\alpha}+|x|^{2\beta}] \\ &= -\frac{p(1-p)(\zeta-\lambda_1)}{2}\sum_{i=1}^{n}x_i^{2\beta+p}+\kappa p\sqrt{n}|x|^{\alpha+p}+\text{const}[1+|x|^p+|x|^{\alpha}+|x|^{2\beta}]. \end{aligned}$$

By (3.4a) and (3.4b), for any $x=(x_1,\ldots,x_n)^{\top}\in\mathbb{R}^n_{++}$ and $l>0$, there exist constants $\bar{k}_1,\bar{k}_2>0$ such that

$$\bar{k}_1|x|^l \leqslant \sum_{i=1}^{n}x_i^l \leqslant \bar{k}_2|x|^l. \quad (3.10)$$

If condition (3.2a) holds, we therefore have

$$I(x) \leqslant -\frac{p\bar{k}_1(1-p)(\zeta-\lambda_1)}{2}|x|^{2\beta+p}+\kappa p\sqrt{n}|x|^{\alpha+p}+\text{const}[1+|x|^p+|x|^{\alpha}+|x|^{2\beta}].$$

By the bounded property of polynomial functions, there exists a constant H_1 such that $I(x) \leqslant H_1$. If condition (3.2b) holds, which implies that $\alpha < 2$ and $\zeta > \lambda_1 + 2\kappa\sqrt{n}$, choosing p sufficient small such that $\alpha + p = 2\beta + p < 2$ and $(1-p)(\zeta - \lambda_1) - 2\kappa\sqrt{n} > 0$, by (3.4a), we have

$$I(x) \leqslant -\frac{p}{2}[(1-p)(\zeta-\lambda_1) - 2\kappa\sqrt{n}]|\varphi(0)|^{\alpha+p} + \text{const}[1 + |x|^p + |x|^\alpha + |x|^{2\beta}].$$

Similarly, there exists a constant H_2 such that $I(x) \leqslant H_2$. (3.9) therefore implies that the condition (2.4) holds, so applying Theorem 2.2 gives the desired result. □

Similarly, we may also examine the global positive solution of (1.4). But we need to replace Assumption 3.1 by the following assumption:

ASSUMPTION 3.2 There exist constants $\alpha, \beta, \zeta, \kappa, \eta, \sigma_1 \geqslant 0$ and the probability measure μ such that, for any $\varphi \in C([-\tau, 0]; \mathbb{R}^n_{++})$ and $x \in \mathbb{R}^n_{++}$, (3.1a) and

$$\limsup_{|x|\to\infty} |x|^\beta |g(x)| < \infty, \tag{3.11a}$$

$$g_i^2(x) \geqslant \zeta x_i^{2\beta} - \sigma_1, \quad 1 \leqslant i \leqslant n \tag{3.11b}$$

hold.

Then the following theorem follows:

THEOREM 3.2 Under Assumptions 1.1 and 3.2, if one of the two conditions

$$\alpha < 2\beta, \qquad \zeta > 0, \tag{3.12a}$$

$$0 < \alpha = 2\beta < 2, \quad \zeta > 2\kappa\sqrt{n} \tag{3.12b}$$

holds, then for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to (1.4) on $t \geqslant -\tau$, and the solution will remain in $\mathbb{R}^n_{++}$ with probability one, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

This proof is similar to Theorem 3.1, so we omit it. By the same method, we may also examine the existence and uniqueness of the global solution of (1.5) except that condition (3.1a) in Assumption 3.1 is replaced by

$$\limsup_{|x|\to\infty} |x|^{-\alpha} |f(x)| < \infty. \tag{3.13}$$

Here, we leave this result and its proof to the reader.

In Theorems 3.1 and 3.2, conditions (3.1c) and (3.11b) are very important. This shows that the noise plays an important role to guarantee the global positive solution, in the case where the noise intensity is strongly dependent on the population size in the sense $\alpha \leqslant 2\beta$. In the following, we examine existence and uniqueness of the global positive solution of (1.3) when the noise intensity is weakly dependent on the population size in the sense $\alpha > 2\beta$. To show this idea, we impose another condition on f. We describe them as the following assumption.

ASSUMPTION 3.3 There exist $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n_{++}$, $p > 0$, $\alpha, \beta, \zeta, \kappa, \kappa_1, \eta, \eta_1, \lambda, \sigma \geqslant 0$, and probability measures μ, μ_1, ν on $[-\tau, 0]$ such that for any $\varphi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, (3.1a), (3.1b) and the

following condition:

$$\sum_{i=1}^{n} c_i \varphi_i^p(0) f_i(\varphi) \leqslant -\zeta|\varphi(0)|^{\alpha+p} + \kappa_1 \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha+p} \,\mathrm{d}\mu_1(\theta) + \eta_1 \tag{3.14}$$

hold.

Then the following theorem follows.

THEOREM 3.3 Under Assumptions 1.1 and 3.3, if

$$\alpha > 2\beta, \quad \zeta > \kappa_1, \tag{3.15}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to (1.3) on $t \geqslant -\tau$, and this solution will remain in $\mathbb{R}^n_{++}$ with probability one, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

REMARK Condition (3.15) shows that if the noise intensity is weakly dependent on the population size in the sense $\alpha > 2\beta$, then existence and uniqueness of the global positive solution is only determined by the drift term with $\zeta > \kappa_1$ and independent of the parameters in the noise intensity.

Proof. Similar to the proof of Theorem 3.1, we also need to estimate $\mathcal{L}U(t, \varphi)$. It is easy to compute

$$\begin{aligned}\mathcal{L}U(t,\varphi) &= \sum_{i=1}^{n} c_i p \varphi_i^p(0) f_i(\varphi) + \frac{p(p-1)}{2} \sum_{i=1}^{n} c_i \varphi_i^p(0) g_i^2(\varphi) + \frac{p}{2} \sum_{i=1}^{n} c_i [-2 f_i(\varphi) + g_i^2(\varphi)] \\ &=: \bar{I}_1 + \bar{I}_2 + \bar{I}_3. \end{aligned} \tag{3.16}$$

By condition (3.14) in Assumption (3.3),

$$\bar{I}_1 \leqslant -\zeta p|\varphi(0)|^{\alpha+p} + \kappa_1 p \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha+p} \,\mathrm{d}\mu_1(\theta) + \eta_1 p. \tag{3.17}$$

By condition (3.1b), (3.7), the Young inequality and the elementary inequality (3.10),

$$\begin{aligned}\bar{I}_2 &\leqslant \frac{p(p-1)^+}{2} \sum_{i=1}^{n} c_i \varphi_i^p(0) |g(\varphi)|^2 \\ &\leqslant p(p-1)^+ \bar{k} |\varphi(0)|^p \left[\lambda^2 \int_{-\tau}^{0} |\varphi(\theta)|^{2\beta} \,\mathrm{d}\nu(\theta) + \sigma^2 \right] \\ &\leqslant \frac{\lambda^2 p^2 (p-1)^+ \bar{k}}{2\beta + p} |\varphi(0)|^{2\beta+p} + p\bar{k}\sigma^2 (p-1)^+ |\varphi(0)|^p \\ &\quad + \frac{2\beta\lambda^2 p(p-1)^+ \bar{k}}{2\beta + p} \int_{-\tau}^{0} |\varphi(\theta)|^{2\beta+p} \,\mathrm{d}\nu(\theta), \end{aligned} \tag{3.18}$$

where $\bar{k} := \max_{1 \leqslant i \leqslant n}\{c_i\}\bar{k}_2$. By the same calculation as for I_3 in Theorem 3.1,

$$\bar{I}_3 \leqslant \text{const} \left[1 + \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha} \,\mathrm{d}\mu(\theta) + \int_{-\tau}^{0} |\varphi(\theta)|^{2\beta} \,\mathrm{d}\nu(\theta) \right]. \tag{3.19}$$

Substituting (3.17), (3.18) and (3.19) into (3.16) gives

$$\begin{aligned}\mathcal{L}U(t,\varphi) \leqslant \kappa_1 p\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+p}\,\mathrm{d}\mu_1(\theta) - |\varphi(0)|^{\alpha+p}\right] \\ + \frac{2\beta\lambda^2 p(p-1)^+\bar{k}}{2\beta+p}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta+p}\mathrm{d}\nu(\theta) - |\varphi(0)|^{2\beta+p}\right] \\ + \text{const}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha}\mathrm{d}\mu(\theta) - |\varphi(0)|^{\alpha}\right] \\ + \text{const}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta}\mathrm{d}\nu(\theta) - |\varphi(0)|^{2\beta}\right] + \bar{I}(\varphi(0)),\end{aligned} \tag{3.20}$$

where

$$\bar{I}(x) := -p(\zeta-\kappa_1)|x|^{\alpha+p} + \text{const}[1+|x|^p+|x|^{\alpha}+|x|^{2\beta}+|x|^{2\beta+p}].$$

By condition (3.15), the bounded property of polynomial functions implies that there exists a constant $\bar{H}$ such that $\bar{I}(x) \leqslant \bar{H}$, so (3.20) implies condition (2.4). Theorem 2.2 therefore gives the desired assertion. □

By the similar method, we may also examine existence and uniqueness of the solution of (1.5). But we need to replace Assumption 3.3 by the following assumption.

ASSUMPTION 3.4 Assume that there exist $c = (c_1,\ldots,c_n)^{\top} \in \mathbb{R}^n_{++}$, $p > 0$, α, β, ζ, η_1, λ, $\sigma \geqslant 0$, and the probability measure ν on $[-\tau,0]$ such that for any $\varphi \in C([-\tau,0];\mathbb{R}^n_{++})$ and $x \in \mathbb{R}^n_{++}$, (3.13), (3.1b) and the following condition:

$$\sum_{i=1}^{n} c_i x_i^p f_i(x) \leqslant -\zeta|x|^{\alpha+p} + \eta_1 \tag{3.21}$$

hold.

The following theorem therefore follows.

THEOREM 3.4 Under Assumptions 1.1 and 3.4, if

$$\alpha > 2\beta, \quad \zeta > 0, \tag{3.22}$$

then for any initial data $\xi \in C([-\tau,0];\mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to (1.5) on $t \geqslant -\tau$, and this solution will remain in $\mathbb{R}^n_{++}$ with probability one, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

Applying the same method to (1.4) may produce the similar result except that condition (3.1b) is replaced by condition (3.11a) in Assumption 3.3. We also leave this result and its proof to the reader.

4. Some special equations

In this section, as applications of the previous results, we examine some stochastic Lotka–Volterra systems as the special cases. First, repeating the process of Theorems 3.1 and 3.3, we may obtain the global positive solution to (1.6).

THEOREM 4.1 Let f and g satisfy the local Lipschitz condition. Assume that there exist α, $\beta \geqslant 0$ such that conditions (3.11a) and (3.13) hold. Assume also that there exist $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n_{++}$, η, $\sigma \geqslant 0$ and $\zeta > 0$. For any $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n_{++}$, if one of the two following conditions holds,

$$\alpha < 2\beta, \quad g_i^2(x) \geqslant \zeta x_i^{2\beta} - \sigma, \quad \zeta > 0, \tag{4.1a}$$

$$\alpha > 2\beta, \quad \sum_{i=1}^n c_i x_i^p f_i(x) \leqslant -\zeta |x|^{\alpha+p} + \sigma, \quad \zeta > 0 \text{ for } p > 0, \tag{4.1b}$$

then for any initial value $x(0) \in \mathbb{R}^n_{++}$, there almost surely exists a unique global positive solution $x(t)$ to (1.6) on $t \geqslant -\tau$, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

Now, let us discuss some stochastic Lotka–Volterra systems as examples to show the applications of our results. Firstly, consider the n-dimensional stochastic functional Lotka–Volterra equation

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t)) \left[\left(b + \int_{-\tau}^0 Ax(t+\theta)\mathrm{d}\mu(\theta) \right) \mathrm{d}t + g(x(t))\mathrm{d}w(t) \right], \tag{4.2}$$

where $b \in \mathbb{R}^n$, $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, μ is the probability measure on $[-\tau, 0]$, $g = (g_1, \ldots, g_n)^\top$ and $g_i(x) = \sum_{j=1}^n q_{ij} x_j^\beta$ satisfying

$$q_{ij} \geqslant 0, \quad q_{ii} > 0, \quad 1 \leqslant i, j \leqslant n. \tag{4.3}$$

This equation may be regarded as the stochastically perturbed equation of the deterministic functional Lotka–Volterra system

$$\dot{x}(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t)) \left(b + \int_{-\tau}^0 Ax(t+\theta)\mathrm{d}\mu(\theta) \right), \tag{4.4}$$

which is discussed by Tang & Kuang (1996). Define $f(\varphi) = b + \int_{-\tau}^0 A\varphi(\theta)\mathrm{d}\mu(\theta)$. Then (4.2) may be rewritten as (1.4). Since

$$|f(\varphi)| \leqslant |b| + |A| \int_{-\tau}^0 |\varphi(\theta)|\mathrm{d}\mu(\theta),$$

the condition (3.1a) holds with $\alpha = 1$ and $\eta = |b|$. Clearly, by (3.10), for any $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n_{++}$,

$$\begin{aligned} \limsup_{|x|\to\infty} |x|^{-\beta}|g(x)| &= \limsup_{|x|\to\infty} |x|^{-\beta} \left(\sum_{i=1}^n g_i^2(x) \right)^{\frac{1}{2}} \\ &\leqslant \limsup_{|x|\to\infty} \max_{1\leqslant i,j\leqslant n} \{q_{ij}\} |x|^{-\beta} \left(n\bar{k}_2^2 |x|^{2\beta} \right)^{\frac{1}{2}} \\ &= \max_{1\leqslant i,j\leqslant n} \{q_{ij}\} \sqrt{n}\bar{k}_2, \end{aligned}$$

$$g_i^2(x) = \left(\sum_{j=1}^n q_{ij} x_j^\beta \right)^2 \geqslant q_{ii}^2 x_i^{2\beta},$$

so both (3.11a) and (3.11b) hold with $\zeta = q_{ii}^2$ and $\sigma = 0$. Applying Theorem 3.2 gives the following theorem.

THEOREM 4.2 Let condition (4.3) hold. If one of the two conditions

$$\beta > \frac{1}{2}, \tag{4.5a}$$

$$\beta = \frac{1}{2}, \quad q_{ii}^2 > 2|A|\sqrt{n}, \tag{4.5b}$$

for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution to (4.2) on $t \geqslant -\tau$, namely, $x(t) \in \mathbb{R}^n_{++}$ for all $t \geqslant -\tau$ almost surely.

Let μ be the Dirac measure on the point $-\tau$ and $\beta = 1$.Equation (4.2) may be rewritten as the stochastic delay Lotka–Volterra system (see Bahar & Mao, 2004a)

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t-\tau))\mathrm{d}t + Qx(t)\mathrm{d}\omega(t)], \tag{4.6}$$

where $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$. By Theorem 4.2, under condition (4.3), for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to this equation on all $t \geqslant -\tau$ (also see Theorem 2.1 in Bahar & Mao, 2004a).

Let μ be the Dirac measure on the zero and $\beta = 1$. Equation (4.2) becomes the following stochastic ordinary Lotka–Volterra system (also see Mao *et al.*, 2002):

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t))\mathrm{d}t + Qx(t)\mathrm{d}\omega(t)]. \tag{4.7}$$

Similarly, by Theorem 4.2, under condition (4.3), for any initial value $x(0) \in \mathbb{R}^n_{++}$, there almost surely exists a unique global positive solution $x(t)$ to this equation on all $t \geqslant 0$ (also see Theorem 2.1 in Mao *et al.*, 2002).

To show that the environmental noise may suppress the potential explosion of the population, we consider the following deterministic scalar logistic equation:

$$\dot{x}(t) = x(t)(b + ax(t)) \tag{4.8}$$

on $t \geqslant 0$ with initial value $x(0) = x_0 > 0$. Since here $x(t)$ represents population size, only positive solutions are of interest. It is obvious that if $a > 0$ and $b > 0$, (4.8) has only the local solution

$$x(t) = \frac{b}{-a + \mathrm{e}^{-bt}(b + ax_0)/x_0} \quad \text{for } 0 \leqslant t < \tau_\mathrm{e},$$

which explodes to infinity at the finite time

$$\tau_\mathrm{e} = -\frac{1}{b}\log\left(\frac{ax_0}{b + ax_0}\right).$$

However, if we stochastically perturb the system (4.8) into the following stochastic scalar logistic equation:

$$\dot{x}(t) = x(t)[(b + ax(t))\mathrm{d}t + \sigma x^\beta(t)\mathrm{d}\omega(t)] \tag{4.9}$$

by Theorem 4.2, for any initial value $x(0) \in \mathbb{R}^n_{++}$, there almost surely exists a unique global positive solution $x(t)$ to this system on all $t \geqslant 0$ when $\beta > 1/2$ and $\sigma > 0$. Clearly, this existence and uniqueness is independent of the parameters a and b.

We also consider the following deterministic delay logistic equation:

$$\dot{x}(t) = x(t)\left(b + \frac{a}{2}x(t) + \frac{a}{2}x(t-\tau)\right) \tag{4.10}$$

on $t \geqslant 0$ with initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$. When $a, b > 0$, by the comparison theorem and the solution of (4.8), the solution of this equation will also explode to infinity at a finite time. Now let μ_1 and μ_2 be the Dirac measures on the points zero and $-\tau$. Let $\mu = (\mu_1 + \mu_2)/2$. Then μ is also a probability measure and (4.10) may be rewritten as

$$\dot{x}(t) = x(t)\left(b + a\int_{-\tau}^{0} x(t+\theta)\mathrm{d}\mu(\theta)\right). \tag{4.11}$$

If we stochastically perturb this system into the following stochastic scalar delay logistic equation:

$$\dot{x}(t) = x(t)\left[\left(b + a\int_{-\tau}^{0} x(t+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}t + \sigma x^{\beta}(t)\mathrm{d}w(t)\right], \tag{4.12}$$

by Theorem 4.2, for any initial data $\xi \in C([-\tau, 0]; \mathbb{R}^n_{++})$, there almost surely exists a unique global positive solution $x(t)$ to this system on all $t \geqslant -\tau$ when $\beta > 1/2$ and $\sigma > 0$. These results all show that the environmental noise may suppress the explosion of the population size.

When $\beta < 1/2$, we consider the following n-dimensional Lotka–Volterra equation:

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t))\mathrm{d}t + g(x(t))\mathrm{d}w(t)], \tag{4.13}$$

where b, A and g have the same definitions in (4.2). Define $f(x) = b + Ax$. We may compute that

$$\begin{aligned}\sum_{i=1}^{n} c_i x_i f_i(x) &= \sum_{i=1}^{n} b_i c_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{ij} x_i x_j \\ &= \sum_{i=1}^{n} b_i c_i x_i + x^{\top}\left(\frac{\bar{C}A + A^{\top}\bar{C}}{2}\right)x \\ &\leqslant \sum_{i=1}^{n} b_i c_i x_i + \check{\lambda}|x|^2,\end{aligned}$$

where $\bar{C} = \mathrm{diag}(c_1, \ldots, c_n)$, $\check{\lambda} = \lambda^+_{\max}((\bar{C}A + A^{\top}\bar{C})/2)$. Here, we give the definition of $\lambda^+_{\max}(A)$ as follows:

$$\lambda^+_{\max}(A) = \sup_{x \in \mathbb{R}^n_+, |x|=1} x^{\top}Ax \quad \text{for a symmetric } n \times n \text{ matrix } A \in \mathbb{R}^{n\times n},$$

which was introduced by Bahar & Mao (2004b). Let us emphasize that this is different from the largest eigenvalue $\lambda_{\max}(A)$ of the matrix A. Recall the nice property of the largest eigenvalue:

$$\lambda_{\max}(A) = \sup_{x \in \mathbb{R}^n, |x|=1} x^{\top}Ax.$$

Clearly, $\lambda^+_{\max}(A) \leqslant \lambda_{\max}(A)$. If

$$\check{\lambda} = \lambda^+_{\max}\left(\frac{\bar{C}A + A^\top \bar{C}}{2}\right) < 0, \tag{4.14}$$

choosing $\varepsilon \in (0, -\check{\lambda})$ sufficiently small, then there exists a constant σ such that

$$\sum_{i=1}^{n} c_i x_i f_i(x) \leqslant \sum_{i=1}^{n} b_i c_i x_i - \varepsilon|x|^2 + (\check{\lambda} + \varepsilon)|x|^2$$

$$\leqslant (\check{\lambda} + \varepsilon)|x|^2 + \sigma.$$

Choose $\zeta = -(\check{\lambda} + \varepsilon)$, $\alpha = 1$. Clearly, condition (4.1b) holds with $p = 1$, so by Theorem 4.1 and noting an arbitrary property of ε, we may obtain the following result.

THEOREM 4.3 Under condition (4.14), if $\beta < 1/2$ and there is a positive vector $c \in \mathbb{R}^n_{++}$ such that

$$\lambda^+_{\max}(\bar{C}A + A^\top \bar{C}) < 0,$$

where $\bar{C} = \operatorname{diag}(c_1, \ldots, c_n)$, then for any given initial value $x(0) \in \mathbb{R}^n_{++}$, there almost surely exists a unique global positive solution $x(t)$ to (4.13) on $t \geqslant 0$, namely, $x(t) \in \mathbb{R}^n_{++}$ on all $t \geqslant 0$ almost surely.

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REFERENCES

ALLEN, E. (2007) *Modeling with Itô Stochastic Differential Equations*. Dordrecht, The Netherlands: Springer.

ARNOLD, L. (1972) *Stochastic Differential Equations: Theory and Applications*. New York: Wiley.

BAHAR, A. & MAO, X. (2004a) Stochastic delay Lotka–Volterra model. *J. Math. Anal. Appl.*, **292**, 364–380.

BAHAR, A. & MAO, X. (2004b) Stochastic delay population dynamics. *Int. J. Pure Appl. Math.*, **11**, 377–400.

BERETTA, E. & TAKEUCHI, Y. (1988) Global asymptotic stability of Lotka–Volterra diffusion models with continuous time delay. *SIAM J. Appl. Math.*, **48**, 627–651.

CHEN, S., WANG, T. & ZHANG, J. (2004) Positive periodic solution for non-autonomous competition Lotka–Volterra path system with time delay. *Nonlinear Anal. Real World Appl.*, **5**, 409–419.

FANG, S. & ZHANG, T. (2005) A study of a class of stochastic differential equations with non-Lipschitzian coefficients. *Prob. Theory Related Fields*, **132**, 356–390.

GARY, B. M. (2004) A functional equation characterizing monomial functions used in permanence theory for ecological differential equation. *Univ. Lagel. Acta Math.*, Fasciculus XLII, 69–76.

GARD, T. C. (1984) Persistence in stochastic food web models. *Bull. Math. Biol.*, **46**, 357–370.

GARD, T. C. (1992) Stochastic models for toxicant-stressed populations. *Bull. Math. Biol.*, **54**, 827–837.

HAN, X., TENG, Z. & XIAO, D. (2006) Persistence and average persistence of a nonautonomous Kolmogorov system. *Chaos Solitons Fractals*, **30**, 748–758.

HASTINGS, A. (1978) Global stability in Lotka–Volterra systems with diffusion. *J. Math. Biol.*, **6**, 163–168.

HE, X. & GOPALSAMY, K. (1997) Persistence, attractivity, and delay in facultative mutualism. *J. Math. Anal. Appl.*, **215**, 154–173.

JIN, Z. & MA, Z. (2002) Stability for a competitive Lotka–Volterra system with delays. *Nonlinear Anal. Theory Methods Appl.*, **51**, 1131–1142.

KHASMINSKII, R. Z. (1981) *Stochastic Stability of Differential Equations*. Alphen aan den Rijn, The Netherlands: Sijthoff and Noordhoff.

KUANG, Y. & SMITH, H. L. (1993) Global stability for infinite delay Lotka–Volterra type system. *J. Diff. Equ.*, **103**, 221–246.

LADDE, G. S. & SAMBANDHAM, M. (2003) *Stochastic Versus Deterministic Systems of Differential Equations*. New York: Marcel Dekker.

MAO, X. (1994) *Exponential Stability of Stochastic Differential Equations*. New York: Marcel Dekker.

MAO, X. (1997) *Stochatic Differential Equations and Applications*. Chichester: Horwood.

MAO, X. (2005) Delay population dynamics and environmental noise. *Stoch. Dyn.*, **5**, 149–162.

MAO, X., MARION, G. & RENSHAW, E. (2002) Environmental noise suppresses explosion in population dynamics. *Stoch. Process. Appl.*, **97**, 95–110.

MAO, X. & RASSIAS, M. J. (2005) Khasminskii-type theorems for stochastic differential delay equations. *Stoch. Anal. Appl.*, **23**, 1045–1069.

MENG, X., JIAO, J. & CHEN, L. (2008) Global dynamics behaviors for a nonautonomous Lotka–Volterra almost periodic dispersal system with delays. *Nonlinear Anal. Theory Methods Appl.*, **68**, 3633–3645.

MUROYA, Y. (2004) Persistence and global stability in Lotka–Volterra delay differential systems. *Appl. Math. Lett.*, **17**, 759–800.

SHEN, Y., LUO, Q. & MAO, X. (2006) The improved LaSalle-type theorems for stochastic functional differential equations. *J. Math. Anal. Appl.*, **318**, 134–154.

RUDNICKI, R. (2003) Long-time behaviour of a stochastic prey-predator model. *Stoch. Process. Appl.*, **108**, 93–107.

RUDNICKI, R. & PICHOR, K. (2007) Influence of stochastic perturbation on prey-predator systems. *Math. Biosci.*, **206**, 108–119.

TANG, B. & KUANG, Y. (1996) Permanence in Kolmogorov-type systems of nonautonomous functional differential equations. *J. Math. Anal. Appl.*, **197**, 427–447.

TENG, Z. (2000) The almost periodic Kolmogorov competitive Systems. *Nonlinear Anal. Theory Methods Appl.*, **42**, 1221–1230.

YANG, Z. & CAO, J. (2004) Positive periodic solutions of neutral Lotka–Volterra system with periodic delays. *Appl. Math. Comput.*, **149**, 661–687.