EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN SL/2001-002 AP
CERN-NUFACT-Note-067

Focusing in Recirculating Muon Accelerators

Eberhard Keil

Abstract

This report discusses the focusing in the FODO cells along the linear accelerators in a recirculating muon accelerator that is part of the ongoing study of a neutrino factory in Europe. The quadrupoles for the focusing in the FODO cells alternate with RF cavities that accelerate the muon beam. The report derives the parameters of the FODO cells that accommodate the muon beam with the assumed emittance. Two different styles of focusing are considered, constant betatron wavelength and constant quadrupole gradient.

Geneva, Switzerland

January 25, 2001

1 INTRODUCTION

This report discusses the focusing in FODO cells along the linear accelerators in a recirculating muon accelerator that is part of the ongoing study of a neutrino factory in Europe [1]. The quadrupoles for the focusing in the FODO cells alternate with RF cavities that accelerate the muon beam. In our design of the recirculating muon accelerators [2], we give the parameters of the FODO cells that accommodate the assumed muon beam emittance, but we did not much discuss the relations between the parameters of the quadrupoles, the RF cavities and the muon beam. They are discussed in Section 2.

2 FODO CELL PARAMETERS

Let us first consider the first FODO cell at the entrance of the linear accelerator, and ignore the acceleration. The muon beam enters the cell with a normalised emittance ε_n that we assume to be equal in the horizontal and vertical directions, and relativistic parameters β and γ. Hence, the RMS beam radius σ at a place where the transverse amplitude function has value $\beta_\perp$ becomes:

$$\sigma = \sqrt{\frac{\varepsilon_n \beta_\perp}{\beta\gamma}} \tag{1}$$

Note that (1) implies my definition of $\varepsilon_n = \sigma^2 \beta\gamma/\beta_\perp$.

2.1 FODO Cell Length and Aperture

The aperture radius b of the FODO cells must be larger than σ by a factor N_σ. We typically take $N_\sigma = 3$. In thin-lens approximation, the period length L, the phase advance μ, assumed to be equal in the horizontal and vertical plane, and the transverse amplitude function $\beta_\perp$ in the focusing quadrupole are related by:

$$\beta_\perp = \frac{L(1+\sin\frac{\mu}{2})}{\sin\mu} \tag{2}$$

We choose μ such that the ratio $\beta_\perp/L$ becomes a minimum. This happens at $\mu/2\pi \approx 0.212071$ when $\sin\frac{\mu}{2} = (\sqrt{5}-1)/2$. Here, the right hand side is the Golden ratio [3]. Numerically, we have $\beta_\perp/L \approx 1.6651$. Putting the pieces together and solving for L, we find in full generality:

$$L = \left(\frac{b}{N_\sigma}\right)^2 \frac{\beta\gamma\sin\mu}{\varepsilon_n(1+\sin\frac{\mu}{2})} \tag{3}$$

With $N_\sigma = 3$, $\mu/2\pi = 0.212071$, and $\beta_\perp/L = 1.6651$, this becomes numerically for muons with momentum p:

$$L/\mathrm{m} \leq 631.558 (b/\mathrm{m})^2 (p/\mathrm{GeV/c})/(\varepsilon_n/\mathrm{mm}) \tag{4}$$

We observe that the period length L is proportional to the square of the beam port radius b, and hence to the square of the RF wavelength λ_{RF}, and proportional to the ratio p/ε_n. We use the $\leq$ sign in (4) to make it clear that it gives the upper limit for the period length L. We may verify (4) by inserting the parameters of the recirculating linear accelerator cells [2]. With $p = 2$ GeV/c, $\varepsilon_n = 1.83$ mm, and $b = 0.09925$ m we retrieve the published value $L = 6.809$ m for RLA1, while with $p = 8$ GeV and $\varepsilon_n = 1.67$ mm we again find the published value $L = 29.793$ m for RLA2. At the period lengths needed for the emittance, there is just enough space for one four-cell LEP cavity between the quadrupoles of RLA1, and for a module of four such cavities in RLA2.

The radius b of the beam ports of the RF cavities is related to the frequencies of the RF systems in the whole neutrino factory. The currently favoured set of harmonically related frequencies is 44, 88, 176 and 352 MHz, where the highest frequency is that of the LEP RF systems [4, 5]. The value of b above is the radius of the beam ports in the super-conducting LEP cavities, once the conical end pieces are removed.

The recirculating muon accelerators for a $\mu^+\mu^-$collider must accelerate the μ^+ and the μ^- in opposite directions at the same time. In this case, the spacing of the RF cavities must be a multiple of $\lambda_{\mathrm{RF}}/2$ as in LEP. It probably is a good idea to impose the same condition on the recirculating muon accelerator for a neutrino factory, although it is not strictly necessary, since the μ^+ and μ^- are accelerated at different times, and possibly both in the same direction. In the current design, we have $L/2 = 4\lambda_{\mathrm{RF}}$ in RLA1 and $L/2 = 17.5\lambda_{\mathrm{RF}}$ in RLA2. Fig. 1 shows the optical functions in the first cells of RLA1 and RLA2.

The super-conducting LEP cavities are only a firm basis for the design study of the recirculating muon accelerators, since much is known about them. The total RF voltage in the recirculating muon accelerators is about 12 GV, much more than that in LEP. Hence, the construction of a neutrino factory will use RF cavities with frequencies and beam port radii that are optimised for the purpose of accelerating muon beams with a large emittance. If the shape of the cavities is independent of their frequency, then the ratio between b and the RF wavelength λ_{RF} is constant. In the LEP cavities, b/λ_{RF} is about 0.117, in the 200 MHz single-cell cavity under study [6] it is about 0.133.

So far, we have discussed only the focusing in the first linear accelerator, paying much attention to the RMS beam radii and the muon momentum at its entrance. At the entrance of the second linear accelerator, the muon momentum is a factor 3/2 higher [2]. According to (3) and (4), the period length L there could also be a factor 3/2 larger than in the first linear accelerator, if this is convenient.

2.2 Quadrupole Length

The focal length f of the quadrupoles in a FODO lattice with equal phase advances μ in the horizontal and vertical planes is in thin-lens approximation:

$$f = \frac{L}{4 \sin \frac{\mu}{2}} \tag{5}$$

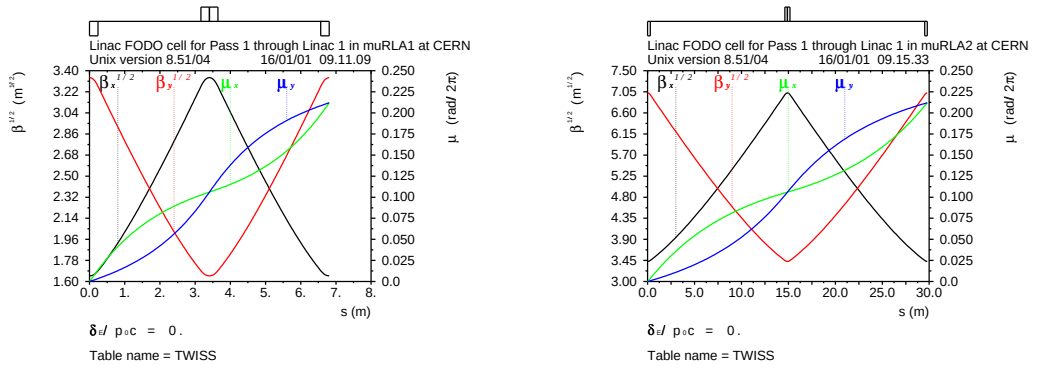

Figure 1: Amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ and phases $\mu_x/2\pi$ and $\mu_y/2\pi$ in the first cell of the first linear accelerators of RLA1 (left) and RLA2 (right)

We may use it to calculate the length of the quadrupoles ℓ_Q in the first FODO cell at the entrance of the linear accelerator, assuming that it has a field B_Q at the aperture radius b, i.e. a gradient B_Q/b, and find:

$$\ell_Q = \frac{2m_\mu N_\sigma^2 \varepsilon_n}{becB_Q} \frac{1+\sin\frac{\mu}{2}}{\cos\frac{\mu}{2}} \tag{6}$$

Here $m_\mu/e \approx 0.105658389$ GV is the rest voltage of the muon, and c is the speed of light.Using the radius of the beam ports of the RF cavities also for the quadrupole aperture b is slightly pessimistic. Note that (6) is independent of the muon momentum. With $\mu/2\pi = 0.212071$ and $N_\sigma = 3$, we find numerically:

$$\ell_Q/\mathrm{m} = \frac{0.013057(\varepsilon_n/\mathrm{mm})}{(B_Q/\mathrm{T})(b/\mathrm{m})} \tag{7}$$

We may verify (7) by inserting the parameters of the recirculating linear accelerator cells [2]. With $B_Q = 0.5$ T, $b = 0.09925$ m, and $\varepsilon_n = 1.83$ mm we retrieve the published value $\ell_Q = 0.481$ m for RLA1, and with $\varepsilon_n = 2.09$ mm $\ell_Q = 0.549$ m for RLA2. The lengths of the quadrupoles along the linear accelerators depend on the assumption about the variation of the focusing there, which will be discussed in Section 3.

3 FOCUSING ALONG THE LINEAR ACCELERATORS

There are two extreme possibilities for the focusing along the linear accelerator on the first pass:

Constant betatron wavelength: The quadrupole strengths $B_Q\ell_Q$ are varied along the linear accelerator in proportion to the increasing muon momentum, such that the betatron wavelength is constant.

Constant strength: The quadrupole strengths $B_Q \ell_Q$ are constant along the linear accelerator. The betatron wavelength increases roughly in proportion to the increasing muon momentum.

Hybrid schemes with constant strengths in sections of a linear accelerator can also be envisaged.

3.1 Constant Wavelength Lattice

We have chosen a linear accelerator lattice with constant betatron wavelength on the first pass [2]. In the first linear accelerator, the muon momentum increases from 2 to 3 GeV/c. Hence, the product $\ell_Q B_Q$ must also increase by a factor 3/2 between the first and the last quadrupoles there. Similarly, in the second linear accelerator, the muon momentum increases from 3 to 4 GeV/c, and $\ell_Q B_Q$ increases by a factor 4/3. Fig. 2 shows the optical functions through the first linear accelerator of RLA1 and RLA2 on the first pass. Because of the constant betatron wavelength, the graphs are rather dull.

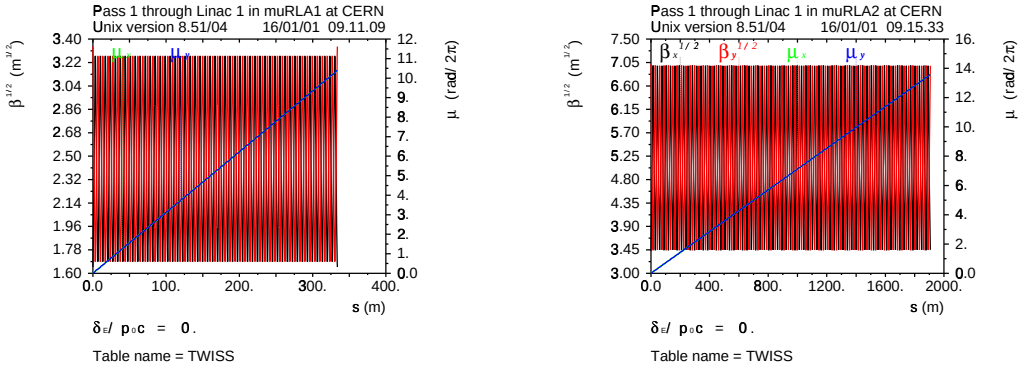

Figure 2: Amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ and phases $\mu_x/2\pi$ and $\mu_y/2\pi$ through the first linear accelerators of RLA1 (left) and RLA2 (right) on the first pass in the case of constant betatron wavelength

On the second, third and fourth pass through the linear accelerators, the muon beam is less focused, because the muon momenta are higher than on the first pass. Hence, the betatron functions β_x and β_y at the entrance to the linear accelerators increase from pass to pass. Furthermore, the relative increase of the muon momentum along the linear accelerators is less than the relative increase of the product $\ell_Q B_Q$. Hence, the betatron functions β_x and β_y decrease along the linear accelerators. Fig. 3 shows the optical functions through the first linear accelerator of RLA1 and RLA2 on the fourth pass, and demonstrates these effects.

The RMS beam radius, and hence the aperture needed, do not increase between passes although the focusing becomes less. The reason is that the maximum value of the β-function increases less rapidly than the betatron wavelength, because the ratio between maximum and average β-function in a FODO cell becomes smaller when

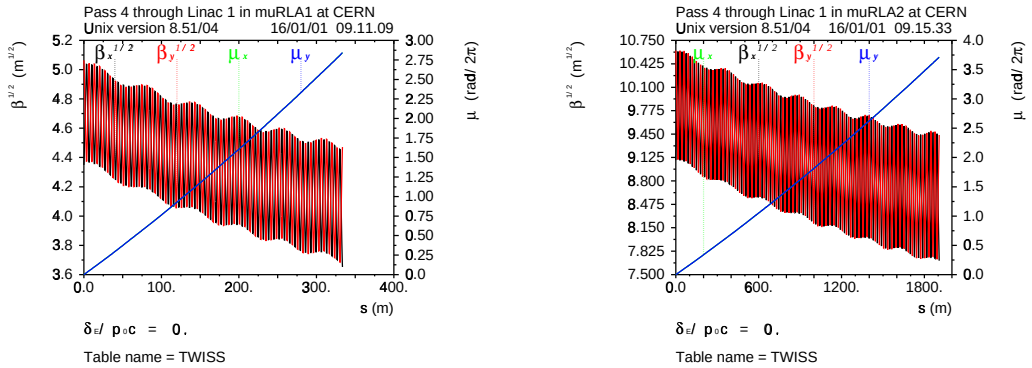

Figure 3: Amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ and phases $\mu_x/2\pi$ and $\mu_y/2\pi$ through the first linear accelerators of RLA1 (left) and RLA2 (right) on the fourth pass in the case of constant betatron wavelength

the phase advance decreases, and because the unnormalized emittance is proportional to the reciprocal of the muon momentum. This justifies a posteriori calculating the aperture needed from the first cell at the entrance of the linear accelerator on the first pass.

3.2 Constant Gradient Lattice

The engineering of the second possibility, constant strength, is easier, because all quadrupoles in a linear accelerator are identical. Fig. 4 shows the optical functions through the first linear accelerator of RLA1 and RLA2 on the first pass. Because of the constant quadrupole gradients, the betatron wavelength and the β-functions increase along the linear accelerators. However, the RMS beam radii are still largest at the entrance of the linear accelerators, because the increase in the β-functions is smaller than the increase in the muon momentum.

As was the case with constant betatron wavelength along the linear accelerators, the muon beam is less focused on the second, third and fourth pass through the linear accelerators, because the muon momenta are higher than on the first pass. Hence, the betatron functions β_x and β_y at the entrance to the linear accelerators increase from pass to pass. Because of the relative increase of the muon momentum along the linear accelerators, the betatron functions β_x and β_y also increase along the linear accelerators. Fig. 5 shows the optical functions through the first linear accelerator of RLA1 and RLA2 on the fourth pass, and demonstrates these effects. The behaviour of the optical functions through the second and third pass is intermediate between that on the first and fourth pass, as one would expect.

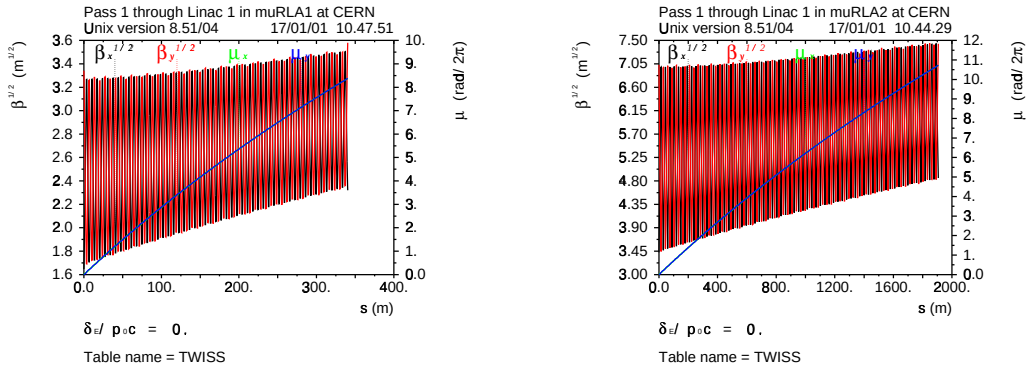

Figure 4: Amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ and phases $\mu_x/2\pi$ and $\mu_y/2\pi$ through the first linear accelerators of RLA1 (left) and RLA2 (right) on the first pass in the case of constant quadrupole gradient

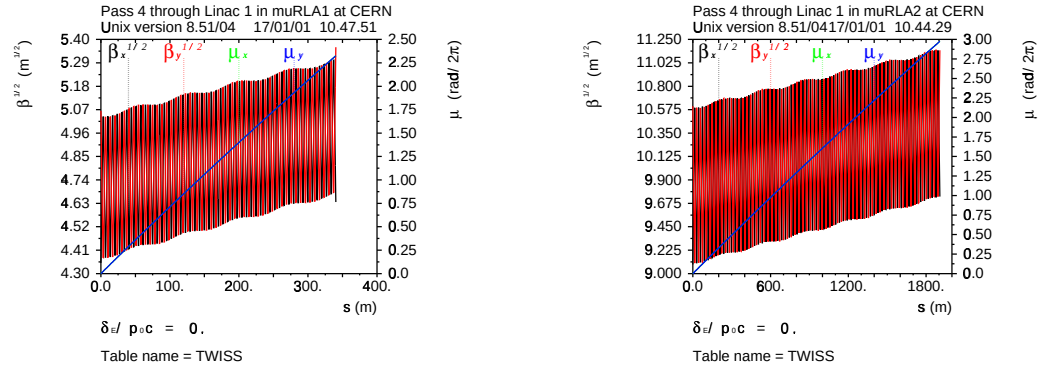

Figure 5: Amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ and phases $\mu_x/2\pi$ and $\mu_y/2\pi$ through the first linear accelerators of RLA1 (left) and RLA2 (right) on the fourth pass in the case of constant quadrupole gradients

4 PARAMETER LIST

Tab. 1 shows the parameters of the linear accelerators in RLA1 and RLA2. The injection and ejection momenta are nice round numbers. Of course, the ejection momentum from RLA2 has to be equal to the muon momentum in the storage ring. However, the injection momenta into both RLA1 and RLA2 ought to be the result of a calculation, rather than being fixed at the beginning. The number of passes is four in both RLA1 and RLA2. It might be better to adopt a smaller number for RLA1 and a larger one for RLA2. In RLA1 and RLA2, an RF module consists of one and four four-cell LEP cavities, respectively. The lengths of the FODO cells are 8 and 35 RF wavelength, respectively. The total length of the linear accelerators is the sum of the length of the two linear accelerators in the two opposite straight sections of the racetrack-shaped RLA1

Table 1: Parameters of the linear accelerators. Input parameters are marked by an asterisk ($*$).

	RLA1	RLA2	
$*$Injection momentum	2	10	GeV/c
$*$Number of passes	4	4	
$*$Ejection momentum	10	50	GeV/c
$*$Frequency of RF system	352.209	352.209	MHz
$*$Radius of beam ports	0.09925	0.09925	m
$*$Number of RF modules	200	256	
$*$Length of FODO cell	6.809	29.791	m
Total length of linacs	680.942	3813.28	m
$*$Phase advance in first pass/2π	0.212071	0.212071	
Focal length of quadrupoles	2.754	12.0508	m
$*$Quadrupole field	0.5	0.5	T
Quadrupole length	0.48076	0.549445	m
Maximum β-function in linac	11.338	49.605	m
Normalized acceptance	1.8298	2.08839	mm

and RLA2. The length of each straight section is half that total length. The lengths of the FODO cells are smaller than the values calculated in section 2.1 above. Hence the normalized acceptances are larger than the normalized emittance $\varepsilon_n = 1.667$ mm. In the case of constant quadrupole gradient, the figure for quadrupole field B_Q and length ℓ_Q apply to all quadrupoles in the first linear accelerator. In the case of constant betatron wavelength, they only apply to the very first quadrupole next to the entrance of the first linear accelerator. The product $B_Q \ell_Q$ increases along the linear accelerator. At its exit, it is a factor 3/2 higher than at the entrance. All the other figures in Tab. 1 apply to both cases, constant gradient and constant wavelength.

5 SUMMARY OF CELL DESIGN PROCEDURE

For given frequency of the RF system and cavity design, one has a fair idea about likely values for the beam port radius b. The normalized emittance ε_n of the muon beam, and to a lesser extent also its momentum at injection are also given. One can then use (3) or (4), and (6) or (7) to find an upper limit for the length of the FODO cells and a lower limit for the length of the quadrupoles ℓ_Q. From these figures, one gets an idea about the space available for the RF cavities, and other equipment needed. One can then decide how to choose the number of cells in an RF cavity, and how to package them in cryostats. Note that the length of RF cavities is quantized in steps of half an RF wavelength. Fine adjustments are better done by small changes to the

muon momentum at injection. It is always best to pack the equipment into the FODO cells as closely as possible, since this increases the accepted normalized emittance, and minimizes the muon decay losses. We have done this in our designs of RLA1 and RLA2 [2], where we assume four cells in a cavity as in LEP, packaged into cryostats that contain one and four such cavities, respectively, and achieve emittances larger than those required.

For the focusing along the linear accelerators, two schemes are presented: constant betatron wavelength and constant quadrupole gradient. The engineering of the latter scheme looks simpler, since the layout of the FODO cells and the RF cavities does not change along the whole linear accelerator.

References

[1] H. Haseroth, *Status of Studies for a Neutrino Factory at CERN*, Proc. EPAC'2000 (2000) 253.

[2] R. Palmer, C. Johnson and E. Keil, *A Cost-Effective Design for a Neutrino Factory*, Nucl.Instr.Meth. **A451** (2000) 265, also CERN-SL/99-070-AP (1999).

[3] L. Michelotti, *50 TeV High Field Lattice: Observations from a Golden Cell*, Fermilab-FN-658 (Sep 1997).

[4] *LEP Design Report, Vol.II The LEP Main Ring*, CERN-LEP/84-01 (1984).

[5] LEP 2 Team, *LEP Design Report, Vol.III LEP 2*, CERN-AC/96-01(LEP 2) (1996).

[6] E. Chiaveri, private communication (Jan 2001).