MetaHeuristics for Stochastic Shortest Path Problem

Deolinda M.L. Rasteiro[*] *António Batel Anjo*[†]

[*] Dep. Física/Matemática, ISEC
Coimbra
Email: dml@isec.pt

[†] Dep. Matemática
Universidade de Aveiro
Email: batel@ua.pt

1 Introduction

Different classes of stochastic optimal path problems are considered in the literature. The first work known in this area is due to Frank [1] where the shortest optimal path over a probabilistic graph is determined.

The most common criterion to determine the optimal path is the one that maximizes the expected value of an utility function. This criterion stems from the formulation presented by Von-Newman-Morgenstern where evaluations should be made under uncertainty [2].

Algorithms for the linear and quadratic utility functions have been presented by Mirchandini and Soroush [3], Murthy and Sarkar [4, 5, 6] and more recently by Deolinda Dias Rasteiro, António Batel Anjo anf Helena Ramalhinho Lourenço [7].

The present paper proposes a mathematical model where the expected value of a nonlinear utility function is maximized over an oriented network. The theoretical formulation presented is based on multicriteria models and computational results stem from the application of metaheuristics, as Entropy by driven Annealing (EdA) to obtain the solution.

2 Problem Definition

Given an oriented network $(\mathcal{N}, \mathcal{A})$ where to each arc $(i,j) \in \mathcal{A}$ is associated a real random variable X_{ij}, called random cost of arc $(i,j) \in \mathcal{A}$, we define the random cost of path $p \in \mathcal{P}$ as $X_p = \sum_{(i,j)\in p} X_{ij}$ ($\mathcal{P}$ is the set of all paths in the given network).

We will consider the case where the real random variables $X_{ij}, i = 1, \ldots, n; j = 1, \ldots, n$ have gaussian distributions with mean $\mu_{ij} \in \mathbb{R}$ and variance σ_{ij}^2 ($\sigma_{ij} \in \mathbb{R}^+$). Therefore the random cost of each path p is also a gaussian variable and we can state that

$$X_p \sim N\left(\sum_{(i,j)\in p} \mu_{ij}, \sum_{(i,j)\in p} \sigma_{ij}^2\right).$$

We will denote by μ_p and σ_p^2 respectively the mean and the variance of X_p.

Associated to each path p an **utility function** $\mathcal{U} : \mathcal{P} \longrightarrow \mathbb{R}$ is also considered. For each path p, $\mathcal{U}(X_p)$ depends on the random variables X_{ij} where $(i,j) \in p$.

We will assume that the utility of a path p is zero after some time-value d, called deadline, imposed by the decision maker.

The density function associated with p will be:

$$g_p(x) = \frac{1}{\sqrt{2\pi}\sigma_p} \quad \exp\left(-\frac{1}{2}\left(\frac{x-\mu_p}{\sigma_p}\right)^2\right), \forall x \in \mathbb{R}.$$

The expected value of the utility function is given by

$$\begin{aligned} \mathrm{E}\left(\mathcal{U}(X_p)\right) &= \int_{\mathbb{R}} \mathcal{U}(x) g_p(x)\, dx \\ &= \int_{-\infty}^{d} \mathcal{U}(x) \frac{1}{\sqrt{2\pi}\sigma_p} \quad \exp\left(-\frac{1}{2}\left(\frac{x-\mu_p}{\sigma_p}\right)^2\right) dx. \end{aligned}$$

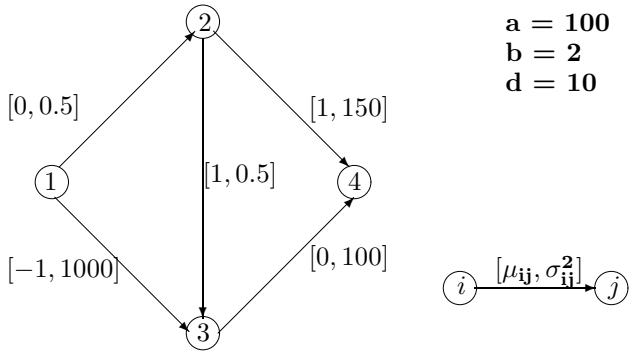
The Stochastic Shortest Path Problem, (SSOP), can be defined as

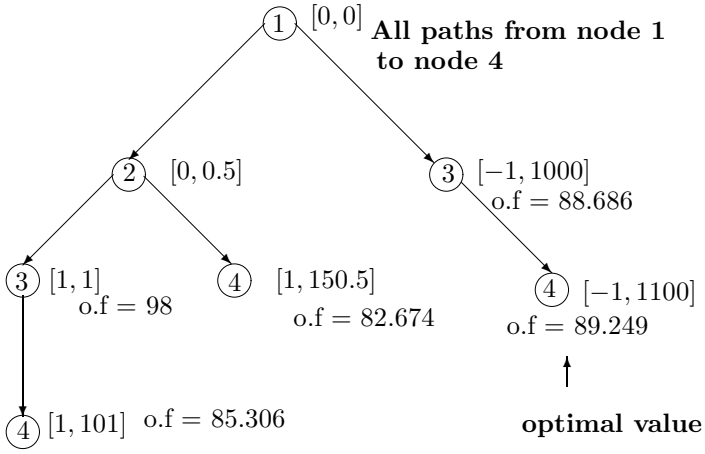
$$\max_{p \in \mathcal{P}} \mathrm{E}(\mathcal{U}(X_p)) \qquad \text{(SSOP)}$$

where $\mathrm{E}(\mathcal{U}(X_p))$ is given by the expression above.

3 Solving the Problem

Observe that (SSOP) does not verify the optimality principle as shown in the following example:





In this example we can observe that the path from 1 to 4 that maximizes the expected value of a linear utility function ($\mathcal{U}(X_p) = a - bX_p$) is $\langle 1, (1,3), 3, (3,4), 4\rangle$ with optimal value 89.249. Although it is not formed by optimal subpaths once the subpath $\langle 1, (1,3), 3\rangle$ has value 88.686 and exists another $\langle 1, (1,2), 2, (2,3), 3\rangle$ with value 98. Therefore labelling algorithms can not be straightforward applied to solve the problem.

Once the objective function of (SSOP) depends on the parameters μ_p and σ_p^2 we will have, at most, four biobjective problems to be solved in order to obtain its optimal solution. Those four problems have their origin on the possible signs of the first order partial derivatives of $\mathrm{E}(\mathcal{U}(X_p))$, with respect to μ_p and σ_p^2. As an example, if

$$\frac{\partial \mathrm{E}(\mathcal{U}(X_p))}{\mu_p} < 0 \qquad ; \qquad \frac{\partial \mathrm{E}(\mathcal{U}(X_p))}{\sigma_p^2} > 0$$

then the expected value of the utility function will be non-increasing, respectively non-decreasing, on the mean and on the variance. Thus we want to minimize the mean and maximize the variance over all paths $p \in \mathcal{P}$ and so it follows that the optimal solution for problem (SSOP) will be the non dominated path of the resulting four biobjective problems.

In order to solve those problems we will apply the referred metaheuristics and present the obtained computational results.

References

[1] H. Frank. Shortest paths in probabilistic graphs. *Oper. Res.*, 17:583–599, 1969.

[2] R. P. Loui. Optimal paths in graphs with stochastic or multidimensional weights. *Commun. ACM*, 26:670–676, 1983.

[3] P. B. Mirchandini and H. Soroush. Optimal paths in probabilistic networks: a case with temporary preferences. *Computers Oper. Res.*, 12:365–383, 1985.

[4] I. Murthy and S. Sarkar. A relaxation based prunning technique for a class of stochastic shortest path problems. *Transportation Sci.*, 30:220–236, 1996.

[5] I. Murthy and S. Sarkar. Exact algorithms for the stochastic shortest path problem with a decreasing deadline function. *European J. Oper. Res.*, 103:209–229, 1997.

[6] I. Murthy and S. Sarkar. Stochastic shortest path problems with piecewise linear concave utility functions. *Management Sci.*, 44/11:125–136, 1998.

[7] Deolinda Dias Rasteiro, António Batel Anjo and Helena Ramalhinho Lourenço. Stochastic shortest path. *under preparation*, 2001.