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TOWARD MONOTONICITY ANALYSIS BETWEEN TWO VAGUE CONCEPTS BASED ON FUZZY ROUGH SETS

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ABSTRACT. A fuzzy rough set approach has been proposed without using any fuzzy logical connectives. By this approach, gradual decision rules are induced from a given decision table. Fuzzy-rough modus ponens and fuzzy-rough modus tollens have been formulated based on the gradual decision rules. The equivalence condition between fuzzy-rough modus ponens and fuzzy-rough modus tollens is discussed. The condition is related to the monotonicity between membership degrees of fuzzy sets. Monotonicity indices are proposed to measure the degree of monotonicity between fuzzy sets.

1. INTRODUCTION

Rough sets [13, 14] and fuzzy sets [19] have been proposed to treat different kinds of uncertainties. The former deals with uncertainty of information resulting from ambiguity caused by granular description of objects, while the latter treats the uncertainty of information as result of linguistic categories with vague boundaries. Because of this difference, many attempts in the combination of rough sets and fuzzy sets have been done [1, 3, 6, 7, 11, 12, 15, 16, 17, 18]. In many of them, some fuzzy logical connectives (t-norm, t-conorm, fuzzy implication) are employed.

Recently, the authors [4] proposed fuzzy rough sets without using any fuzzy logical connective. The fuzzy rough sets are associated with gradual decision rules. The gradual decision rules are characterized by modifier functions induced from a given decision table. Using modifier functions associated with the gradual decision rules, we formulated fuzzy-rough modus ponens and fuzzy-rough modus tollens and demonstrated their similarities [8]. The equivalence of fuzzy-rough modus ponens and fuzzy-rough modus tollens has been discussed [9]. The necessary and sufficient condition for the equivalence of fuzzy-rough

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modus ponens and fuzzy-rough modus tollens in a sub-domain of modifier functions has been shown. The condition is related to the monotonicity between membership degrees of two fuzzy sets.

In this paper, we extend the discussion about the equivalence of fuzzy-rough modus ponens and fuzzy-rough modus tollens to the monotonicity analysis between two vague concepts expressed by fuzzy sets. We show that the combination of equivalence conditions for two pairs of fuzzy-rough modus ponens and fuzzy-rough modus tollens implies the monotonicity relation between two membership degrees of two fuzzy sets. Based on this fact, we propose to use modifier functions associated with modus ponens and modus tollens, for measuring the degree of monotonicity between two vague concepts.

In the next section, we introduce fuzzy rough sets as well as fuzzy-rough modus ponens and fuzzy-rough modus tollens. We show the equivalence condition for fuzzy-rough modus ponens and fuzzy-rough modus tollens, and we discuss its relation with the monotonicity between membership functions of two fuzzy sets in Section 3. In Section 4, measures of monotonicity are defined.

2. Fuzzy-rough Modus Ponens and Fuzzy-rough Modus Tollens

Suppose that we want to approximate knowledge contained in fuzzy set Y using knowledge about fuzzy set X over a finite set U of all objects in a given decision table. Let us also adopt the hypothesis that X is positively related to Y. Then, we can define the lower approximation $\underline{App}^{+}(X,Y)$, and the upper approximation $\overline{App}^{+}(X,Y)$ of Y by the following membership functions [4, 8]:

$$\mu[\underline{App}^{+}(X,Y),x] = \min_{z\in U:\mu_X(z)\geq\mu_X(x)}\{\mu_Y(z)\},$$

$$\mu[\overline{App}^{+}(X,Y),x] = \max_{z\in U:\mu_X(z)\leq\mu_X(x)}\{\mu_Y(z)\},$$

where μ_X and μ_Y are membership functions of X and Y.

Similarly, if we adopt the hypothesis that X is negatively related to Y, then we can define the lower approximation $\underline{App}^{-}(X,Y)$, and the upper approximation $\overline{App}^{-}(X,Y)$ of Y by the following membership functions [4, 8]:

$$\mu[\underline{App}^{-}(X,Y),x] = \min_{z\in U:\mu_X(z)\leq\mu_X(x)}\{\mu_Y(z)\},$$

$$\mu[\overline{App}^{-}(X,Y),x] = \max_{z\in U:\mu_X(z)\geq\mu_X(x)}\{\mu_Y(z)\}.$$

The lower and upper approximations defined above can serve to induce lower and upper gradual decision rules over all possible objects $\hat{U}\supseteq U$ in the following way. Let us remark that inferring lower and upper gradual decision rules is equivalent to finding modifier functions $f^{+}_{Y|X}$, $f^{-}_{Y|X}$, $g^{+}_{Y|X}$ and $g^{-}_{Y|X}$ associated with lower and upper approximations. These can be defined as follows: for each $\alpha \in [0,1]$,

$$f^{+}_{Y|X}(\alpha) = \begin{cases} \max\limits_{x \in U : \mu_X(x) \le \alpha} \mu[\underline{App}^{+}(X,Y), x], & \text{if } \alpha > 0 \text{ and } \exists x \in U,\ \mu_X(x) \le \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

$$f^{-}_{Y|X}(\alpha) = \begin{cases} \max\limits_{x \in U : \mu_X(x) \ge \alpha} \mu[\underline{App}^{-}(X,Y), x], & \text{if } \alpha < 1 \text{ and } \exists x \in U,\ \mu_X(x) \ge \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

$$g^{+}_{Y|X}(\alpha) = \begin{cases} \min\limits_{x \in U : \mu_X(x) \ge \alpha} \mu[\overline{App}^{+}(X,Y), x], & \text{if } \alpha < 1 \text{ and } \exists x \in U,\ \mu_X(x) \ge \alpha, \\ 1, & \text{otherwise,} \end{cases}$$

$$g^{-}_{Y|X}(\alpha) = \begin{cases} \min\limits_{x \in U : \mu_X(x) \le \alpha} \mu[\overline{App}^{-}(X,Y), x], & \text{if } \alpha > 0 \text{ and } \exists x \in U,\ \mu_X(x) \le \alpha, \\ 1, & \text{otherwise,} \end{cases}$$

Note that $\mu[\underline{App}^{+}(X,Y), x] = f^{+}_{Y|X}(\mu_X(x))$ and $\mu[\overline{App}^{-}(X,Y), x] = g^{-}_{Y|X}(\mu_X(x))$ hold for $x \in U$ such that $\mu_X(x) > 0$, while $\mu[\underline{App}^{-}(X,Y), x] = f^{-}_{Y|X}(\mu_X(x))$ and $\mu[\overline{App}^{+}(X,Y), x] = g^{+}_{Y|X}(\mu_X(x))$ hold for $x \in U$ such that $\mu_X(x) < 1$. We assume that data in a given decision table is only a sample. Therefore, it seems reasonable to define $f^{+}_{Y|X}(0) = f^{-}_{Y|X}(1) = 0$ and $g^{+}_{Y|X}(1) = g^{-}_{Y|X}(0) = 1$ for possible existence of objects $y,\ z \in \hat{U}$ such that $\mu_X(y) = 0,\ \mu_X(z) = 1,\ \mu_Y(y) < \min\{\mu_Y(x) \mid \mu_X(x) = 0,\ x \in U\}$ and $\mu_Y(z) > \max\{\mu_Y(x) \mid \mu_X(x) = 1,\ x \in U\}$.

Using $f^{+}_{Y|X}$, $f^{-}_{Y|X}$, $g^{+}_{Y|X}$ and $g^{-}_{Y|X}$, we may have the following decision rules:

- LP-rules: "if $\mu_X(x) \ge \alpha$ then $\mu_Y(x) \ge f^{+}_{Y|X}(\alpha)$",
- LN-rules: "if $\mu_X(x) \le \alpha$ then $\mu_Y(x) \ge f^{-}_{Y|X}(\alpha)$",
- UP-rules: "if $\mu_X(x) \le \alpha$ then $\mu_Y(x) \le g^{+}_{Y|X}(\alpha)$",
- UN-rules: "if $\mu_X(x) \ge \alpha$ then $\mu_Y(x) \le g^{-}_{Y|X}(\alpha)$",

where the LP-rule can be regarded as a gradual decision rule [2]; it can be interpreted as: "the more object x is X, the more it is Y". In this case, the relationship between credibility of premise and conclusion is positive and certain. The LN-rule can be interpreted in turn as: "the less object x is X, the more it is Y", so the relationship is negative and certain. On the other hand, the UP-rule can be interpreted as: "the more object x is X, the more it could be Y", so the relationship is positive and possible. Finally, the UN-rule can be interpreted as: "the less object x is X, the more it could be Y", so the relationship is negative and possible.

Based on the decision rules, we formulated in [8] the following fuzzy-rough modus ponens (MP):

$$(LP\text{-}MP)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\ge\alpha,\text{ then } \mu_Y(x)\ge f^+_{Y|X}(\alpha)\\ \text{and } \mu_X(x)\ge\alpha'\end{array}}{\text{then } \mu_Y(x)\ge f^+_{Y|X}(\alpha')}$$

$$(LN\text{-}MP)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\le\alpha,\text{ then } \mu_Y(x)\ge f^-_{Y|X}(\alpha)\\ \text{and } \mu_X(x)\le\alpha'\end{array}}{\text{then } \mu_Y(x)\ge f^-_{Y|X}(\alpha')}$$

$$(UP\text{-}MP)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\ge\alpha,\text{ then } \mu_Y(x)\le g^+_{Y|X}(\alpha)\\ \text{and } \mu_X(x)\le\alpha'\end{array}}{\text{then } \mu_Y(x)\le g^+_{Y|X}(\alpha')}$$

$$(UN\text{-}MP)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\le\alpha,\text{ then } \mu_Y(x)\le g^-_{Y|X}(\alpha)\\ \text{and } \mu_X(x)\ge\alpha'\end{array}}{\text{then } \mu_Y(x)\le g^-_{Y|X}(\alpha')}$$

Moreover, the following fuzzy-rough modus tollens (MT) were obtained in [8]:

$$(LP\text{-}MT)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\ge\alpha,\text{ then } \mu_Y(x)\ge f^+_{Y|X}(\alpha)\\ \text{and } \mu_Y(x)<\beta\end{array}}{\text{then } \mu_X(x)<(f^+_{Y|X})^{-1}_{\epsilon}(\beta)}$$

$$(LN\text{-}MT)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\le\alpha,\text{ then } \mu_Y(x)\ge f^-_{Y|X}(\alpha)\\ \text{and } \mu_Y(x)<\beta\end{array}}{\text{then } \mu_X(x)>(f^-_{Y|X})^{-1}_{\epsilon}(\beta)}$$

$$(UP\text{-}MT)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\le\alpha,\text{ then } \mu_Y(x)\le g^+_{Y|X}(\alpha)\\ \text{and } \mu_Y(x)>\beta\end{array}}{\text{then } \mu_X(x)>(g^+_{Y|X})^{-1}_{\epsilon}(\beta)}$$

$$(UN\text{-}MT)\qquad \frac{\begin{array}{l}\text{if } \mu_X(x)\ge\alpha,\text{ then } \mu_Y(x)\le g^-_{Y|X}(\alpha)\\ \text{and } \mu_Y(x)>\beta\end{array}}{\text{then } \mu_X(x)<(g^-_{Y|X})^{-1}_{\epsilon}(\beta)}$$

where ϵ is an infinitely small positive number and the inverse functions are defined as:

$$(f^+_{Y|X})^{-1}_{\epsilon}(\beta)=\begin{cases}\inf\{\alpha\in[0,1]\mid f^+_{Y|X}(\alpha)\ge\beta\},\\ \qquad\qquad \text{if } \{\alpha\in[0,1]\mid f^+_{Y|X}(\alpha)\ge\beta\}\neq\emptyset,\\ 1+\epsilon,\quad \text{otherwise,}\end{cases}$$

$$(f^-_{Y|X})^{-1}_\epsilon(\beta) = \begin{cases} \sup\{\alpha \in [0,1] \mid f^-_{Y|X}(\alpha) \geq \beta\}, \\ \qquad\qquad \text{if } \{\alpha \in [0,1] \mid f^-_{Y|X}(\alpha) \geq \beta\} \neq \emptyset, \\ 0 - \epsilon, \quad \text{otherwise}, \end{cases}$$

$$(g^+_{Y|X})^{-1}_\epsilon(\beta) = \begin{cases} \sup\{\alpha \in [0,1] \mid g^+_{Y|X}(\alpha) \leq \beta\}, \\ \qquad\qquad \text{if } \{\alpha \in [0,1] \mid g^+_{Y|X}(\alpha) \leq \beta\} \neq \emptyset, \\ 0 - \epsilon, \quad \text{otherwise}, \end{cases}$$

$$(g^-_{Y|X})^{-1}_\epsilon(\beta) = \begin{cases} \inf\{\alpha \in [0,1] \mid g^-_{Y|X}(\alpha) \leq \beta\}, \\ \qquad\qquad \text{if } \{\alpha \in [0,1] \mid g^-_{Y|X}(\alpha) \leq \beta\} \neq \emptyset, \\ 1 + \epsilon, \quad \text{otherwise}. \end{cases}$$

Considering the fact that we induce gradual decision rules from a given decision table, we may find some similarities between the above fuzzy-rough MT and fuzzy-rough MP based on $f^+_{X|Y}$, $f^-_{X|Y}$, $g^+_{X|Y}$ and $g^-_{X|Y}$. In fact, the fuzzy-rough MP based on $f^+_{X|Y}$, $f^-_{X|Y}$, $g^+_{X|Y}$ and $g^-_{X|Y}$ are similar to inference patterns (*LP-MT*), (*LN-MT*), (*UP-MT*) and (*UN-MT*) in which X and Y are exchanged.

In what follows, we concentrate the discussion on fuzzy-rough MP and fuzzy-rough MT when two fuzzy sets X and Y are positively related. In the case of negative relation, a similar discussion can be drawn by the following relations:

$$f^-_{X|Y}(\alpha) = f^+_{X|Y^c}(1-\alpha), \qquad g^-_{X|Y}(\alpha) = 1 - f^+_{X^c|Y}(\alpha),$$
$$(f^-_{Y|X})^{-1}_\epsilon(\alpha) = (g^+_{Y^c|X})^{-1}_\epsilon(1-\alpha), \qquad (g^-_{Y|X})^{-1}_\epsilon(\alpha) = 1 - (g^+_{Y|X^c})^{-1}_\epsilon(\alpha).$$

3. The Equivalence Conditions and The Monotonicity

The necessary and sufficient condition for equivalence of (*UP-MP*) and (*LP-MT*) in a sub-domain of modifier functions has been discussed in [9]. To compare (*UP-MP*) and (*LP-MT*), we exchange X and Y in (*UP-MP*) so that both inference patterns conclude the upper bound of $\mu_X(x)$. Since (*UP-MP*) and (*LP-MT*) are characterized by $g^+_{X|Y}$ and $(f^+_{Y|X})^{-1}_\epsilon$, respectively, the condition is reduced to the necessary and sufficient condition for infinitely small difference between $g^+_{X|Y}$ and $(f^+_{Y|X})^{-1}_\epsilon$. The condition is shown in what follows based on the result from [9].

In order to describe the necessary and sufficient condition, we define the following items:

$$m^*(Y|X) = \min\{\mu_Y(z) \mid \mu_X(z) = \max\{\mu_X(y) \mid y \in U\},\ z \in U\},$$
$$V(X|Y) = \{x \in U \mid \nexists y \in U,\ \mu_X(y) > \mu_X(x) \text{ and } \mu_Y(y) = \mu_Y(x) \leq m^*(Y|X)\},$$
$$M_0(X|Y) = \begin{cases} \max\{\mu_X(x) \mid \mu_Y(x) = 0,\ x \in U\}, & \text{if } \exists x \in U, \mu_Y(x) \leq 0, \\ -\epsilon, & \text{otherwise}, \end{cases}$$

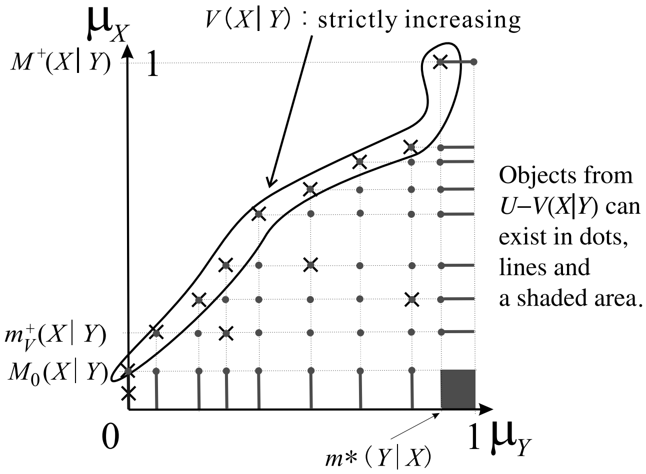

FIGURE 1. An area satisfying conditions in Proposition 1

$m_V^+(X|Y) = \min\{\mu_X(y) \mid \mu_Y(y) > 0,\ y \in V\}$,

$M^+(X|Y) = \max\{\mu_X(y) \mid \mu_Y(y) > 0,\ y \in U\}$.

Based on the result from [9], the equivalence condition is obtained as follows.

Proposition 1. *Assume that there exists $y \in U$ such that $\mu_Y(y) > 0$. Then, the necessary and sufficient condition for*

$$(f^+_{Y|X})^{-1}_\epsilon(\alpha) - g^+_{X|Y}(\alpha) \le \epsilon,\ \forall \alpha \in (0, 1] \tag{1}$$

is that the following conditions are all satisfied:

(a) *For any x, $y \in V(X|Y)$ such that $\mu_Y(x)$ and $\mu_Y(y) \in [0, m^*(Y|X)]$, $\mu_Y(x) > \mu_Y(y)$ implies $\mu_X(x) > \mu_X(y)$,*
(b) *for all $x \in U - V(X|Y)$ such that $\mu_Y(x) > 0$ and $\mu_X(x) \in [m_V^+(X|Y), M^+(X|Y)]$, there exists $y \in V$ such that $\mu_Y(y) \in (0, \mu_Y(x)]$ and $\mu_X(y) = \mu_X(x)$,*
(c) *$M_0(X|Y) \ge \max\{\mu_X(x) \mid \mu_X(x) < m_V^+(X|Y),\ x \in U\}$,*
(d) *if there exists $y \in U$ such that $\mu_Y(y) = 1$, then $M^+(X|Y) = 1$,*
(e) *'there exists $y \in U$ such that $\mu_X(y) = 1$' or 'for all $x \in U$ such that $\mu_X(x) = M^+(X|Y)$, there is no $y \in U$ such that $\mu_Y(y) > \mu_Y(x)$'.*

An area satisfying conditions in Proposition 1 is depicted in Figure 1.

Similarly, the necessary and sufficient condition that (*LP-MP*) and (*UP-MT*) are equivalent is obtained as the necessary and sufficient condition that the difference between $f^+_{X|Y}$ and $(g^+_{Y|X})^{-1}_\epsilon$ is infinitely small. Therefore, we define further items:

$$m_*(Y|X) = \max\{\mu_Y(z) \mid \mu_X(z) = \min\{\mu_X(y) \mid y \in U\},\ z \in U\},$$

$$\Lambda(X|Y) = \{x \in U \mid \nexists y \in U,\ \mu_X(y) < \mu_X(x) \text{ and } \mu_Y(y){=}\mu_Y(x){\geq}m_*(Y|X)\},$$

$$M_1(X|Y) = \begin{cases} \min\{\mu_X(x) \mid \mu_Y(x) = 1,\ x \in U\}, & \text{if } \exists x \in U, \mu_Y(x) \geq 1, \\ 1 + \epsilon, & \text{otherwise,} \end{cases}$$

$$m^-_\Lambda(X|Y) = \max\{\mu_X(y) \mid \mu_Y(y) < 1,\ y \in \Lambda\},$$

$$M^-(X|Y) = \min\{\mu_X(y) \mid \mu_Y(y) < 1,\ y \in U\},$$

Then, we obtain the necessary and sufficient condition for equivalence of (*LP-MP*) and (*UP-MT*) as follows:

Proposition 2. *Assume that there exists $y \in U$ such that $\mu_Y(y) < 1$. Then, the necessary and sufficient condition for*

$$f^+_{X|Y}(\alpha) - (g^+_{Y|X})^{-1}_\epsilon(\alpha) \leq \epsilon,\ \forall\alpha \in [0, 1) \tag{2}$$

is that the following conditions are all satisfied:

(f) *For any x, $y \in \Lambda(X|Y)$ such that $\mu_Y(x)$ and $\mu_Y(y) \in [m_*(Y|X), 1]$, $\mu_Y(x) < \mu_Y(y)$ implies $\mu_X(x) < \mu_X(y)$,*

(g) *for all $x \in U - \Lambda(X|Y)$ such that $\mu_Y(x) < 1$ and $\mu_X(x) \in [M^-(X|Y), m^-_\Lambda(X|Y)]$, there exists $y \in \Lambda$ such that $\mu_Y(y) \in [\mu_Y(x), 1)$ and $\mu_X(y) = \mu_X(x)$,*

(h) *$M_1(X|Y) \leq \min\{\mu_X(x) \mid \mu_X(x) > m^-_\Lambda(X|Y),\ x \in U\}$,*

(i) *if there exists $y \in U$ such that $\mu_Y(y) = 0$, then $M^-(X|Y) = 0$,*

(j) *'there exists $y \in U$ such that $\mu_X(y) = 0$' or 'for all $x \in U$ such that $\mu_X(x) = M^-(X|Y)$, there is no $y \in U$ such that $\mu_Y(y) < \mu_Y(x)$'.*

Replacing X and Y in Propositions 1 and 2, we obtain equivalence conditions for $g^+_{Y|X}$ and $(f^+_{X|Y})^{-1}_\epsilon$ and for $f^+_{Y|X}$ and $(g^+_{X|Y})^{-1}_\epsilon$. However, we have the following proposition.

Proposition 3. *$(f^+_{Y|X})^{-1}_\epsilon(\alpha) = g^+_{X|Y}(\alpha)$ in the range $(0, 1)$ if and only if $f^+_{Y|X}(\alpha) = (g^+_{X|Y})^{-1}_\epsilon(\alpha)$ in the range $(0, 1)$. Similarly, $(g^+_{Y|X})^{-1}_\epsilon(\alpha) = f^+_{X|Y}(\alpha)$ in the range $(0, 1)$ if and only if $(f^+_{X|Y})^{-1}_\epsilon(\alpha) = g^+_{Y|X}(\alpha)$ in the range $(0, 1)$.*

This proposition follows from definitions of inverse functions $(f^+_{X|Y})^{-1}_\epsilon$ and $(g^+_{X|Y})^{-1}_\epsilon$.

Equations (1) and (2) simultaneously hold if and only if conditions (a)∼(j) are satisfied. An area satisfying conditions (a)∼(j) is depicted in Figure 2.

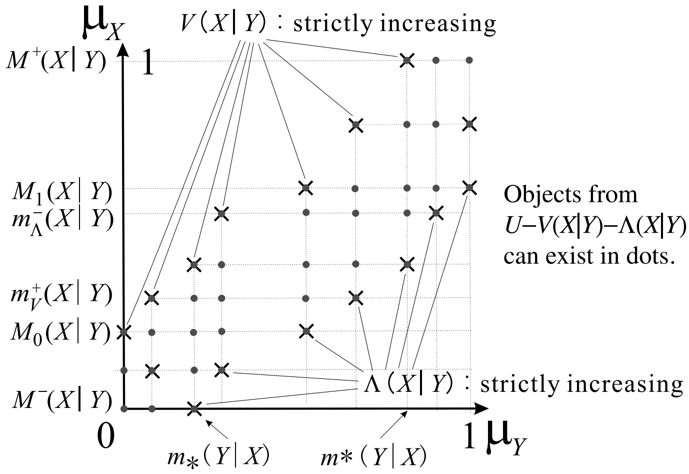

FIGURE 2. An area satisfying all conditions in Propositions 1 and 2

As shown in Figure 2, even if we combine conditions of Propositions 1 and 2, we cannot obtain the total monotonicity. However, the combined condition requests the monotonicity of lower and upper surfaces of the sample data. This implies that the conditions for the equivalence of fuzzy-rough MP and fuzzy-rough MT allow fluctuations in membership degree assessment to some extent. Moreover, when $V \cap \Lambda \neq \emptyset$, the combined condition approaches to the total monotonicity as shown in the following proposition.

Proposition 4. *When $V \cap \Lambda \neq \emptyset$, (a)∼(j) imply*

(k) *For any x, $y \in U$ such that $(\mu_Y(x), \mu_X(x))$, $(\mu_Y(y), \mu_X(y)) \in (0,1) \times (0,1)$, $\mu_Y(x) > \mu_Y(y)$ is equivalent to $\mu_X(x) > \mu_X(y)$.*

A similar condition of equivalence is stated in the following proposition.

Proposition 5. *When there are no x, y, z, $w \in U$ such that $\mu_Y(x) = 0$, $\mu_Y(y) = 0$, $\mu_X(z) = 0$ and $\mu_X(w) = 0$, (a)∼(j) imply*

(l) *For any x, $y \in U$ such that $\mu_Y(x)$, $\mu_Y(y) \in [0,1]$, $\mu_Y(x) > \mu_Y(y)$ is equivalent to $\mu_X(x) > \mu_X(y)$.*

The last proposition shows the closeness of conditions for (1) and (2) to the total monotonicity between membership functions of two fuzzy sets.

4. Measuring the lower and upper surface monotonicities

To evaluate the monotonicity, it will be convenient to measure the degree of monotonicity. Utilizing the modifier functions associated with fuzzy-rough *MP* and fuzzy-rough *MT*, we can define measures of lower and upper surface monotonicities. To this end, we define

$$N_Y = \{\mu_Y(z) \mid z \in U\}.$$

Now upper surface monotonicity ratio UR and lower surface monotonicity ratio LR can be defined as

$$UR = 1 - \frac{|\{\alpha \in N_Y \mid (f^+_{Y|X})^{-1}_\epsilon(\alpha) \neq g^+_{X|Y}(\alpha),\ \alpha \neq 0\}|}{|N_Y - \{0\}|},$$
$$LR = 1 - \frac{|\{\alpha \in N_Y \mid (g^+_{Y|X})^{-1}_\epsilon(\alpha) \neq f^+_{X|Y}(\alpha),\ \alpha \neq 1\}|}{|N_Y - \{1\}|}.$$

The larger these ratios, the higher the monotonicity degree is. They do not represent, however, the differences $|(f^+_{Y|X})^{-1}_\epsilon(\alpha) - g^+_{X|Y}(\alpha)|$ and $|(g^+_{Y|X})^{-1}_\epsilon(\alpha) - f^+_{X|Y}(\alpha)|$. To this end, we may define upper surface monotonicity grade UG and lower surface monotonicity grade LG by

$$UG = 1 - \frac{1}{|N_{\mu_Y} - \{0\}|} \sum_{\alpha \in N_Y, \alpha \neq 0} \frac{|(f^+_{Y|X})^{-1}_\epsilon(\alpha) - g^+_{X|Y}(\alpha)|}{W(\alpha)},$$
$$LG = 1 - \frac{1}{|N_{\mu_Y} - \{1\}|} \sum_{\alpha \in N_Y, \alpha \neq 1} \frac{|(g^+_{Y|X})^{-1}_\epsilon(\alpha) - f^+_{X|Y}(\alpha)|}{W(\alpha)},$$

where the width $W(\alpha)$ at level α is defined by

$$W(\alpha) = \min\left\{(f^+_{Y|X})^{-1}_\epsilon(\alpha), g^+_{X|Y}(\alpha)\right\} - \max\left\{(g^+_{Y|X})^{-1}_\epsilon(\alpha), f^+_{X|Y}(\alpha)\right\}.$$

The larger those grades, the higher the monotonicity degree.

5. Conclusions

In this paper, we investigated the equivalence condition of fuzzy-rough modus ponens and fuzzy rough modus tollens associated with a fuzzy rough set model. We showed that the conditions are related to a specific monotonicity between membership degrees of fuzzy sets. We proposed a few indices to measure this monotonicity. Future developments are related to a thorough analysis of this monotonicity and to the extension of the conditions for equivalence of fuzzy-rough modus ponens and fuzzy rough modus tollens with multiple premise fuzzy sets [5].

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