A Separation Principle for Non-UCO Systems: The Jet Engine Stall and Surge Example

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Abstract

The problem of controlling surge and stall in jet engine compressors is of fundamental importance in preventing damage and lengthening the life of these components. In this theoretical study, we illustrate the application of a novel output feedback control technique to the Moore-Greitzer mathematical model for these two instabilities assuming that the plenum pressure rise is measurable. This problem is particularly challenging since the system is not *uniformly completely observable* and, hence, none of the output feedback control techniques found in the literature can be applied to recover the performance of a full state feedback controller.

1 Introduction and Problem Description

We consider the problem of controlling two instabilities which occur in jet engine compressors, namely rotating stall and surge. Rotating stall develops when there is a region of stagnant flow rotating around the circumference of the compressor causing undesired vibrations in the blades and reduced pressure rise of the compressor. Surge is an axisymmetric oscillation of the flow through the compressor that can cause undesired vibrations in other components of the compression system and damage to the engine. In [17] Moore and Greitzer developed a three-state finite dimensional Galerkin approximation of a nonlinear PDE model describing the compression system. Since its development, several researchers have used the Moore-Greitzer three state model (MG3) to design stabilizing controllers for stall and surge. The available control approaches may be divided into three main categories: 1) Linearization and linear perturbation models (e.g., [21, 18, 5] among others); 2) Bifurcation analysis (e.g., [11, 12, 6, 16, 1]); and 3) Lyapunov based methods (e.g., [8, 4, 20]). Most existing results focus on the development of state feedback controllers which may not be implementable. The work in [6] investigates the use of sensor arrays (2D sensing) to implement a state feedback control law depending on the squared amplitude of the first harmonic of asymmetric flow and the derivative of the air flow through the compressor. In [8] a partial state feedback controller simplifies practical implementation by only requiring measurements of the mass flow and plenum pressure rise (hence 2D sensing is not needed).

To the best of our knowledge, available solutions to the output feedback control problem using only plenum pressure rise (see [2] and Sections 12.6, 12.7 in [7]) do not rely on the estimation of the entire state of the system, and it seems that no attempt has been made to design a stabilizing output feedback controller (using only plenum pressure rise feedback) based on a full-state feedback control law. This is probably due to the fact that MG3 becomes unobservable when there is no mass flow through the compressor, i.e., the system is not uniformly completely observable (UCO), and none of the techniques found in the output feedback control literature (e.g., [19, 3]) can be employed for the solution of this problem. In this paper we introduce a new globally stabilizing full state feedback control law for MG3, and we employ the theory developed in [15, 14] for the output feedback control of non-UCO systems (i.e., system that are not globally observable) to regulate stall and surge by using only pressure measurements. Thus, this work explores the applicability of our method to an idealized version of the surge and stall control problem: the details of a practical design and implementation are not within its scope.

The MG3 model is described by (see [9, 8] for an analogous exposition)

$$\begin{aligned} \dot{\Phi} &= -\Psi + \Psi_C(\Phi) - 3\Phi R \\ \dot{\Psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T) \\ \dot{R} &= \sigma R(1 - \Phi^2 - R),\ \ R(0) \geq 0 \end{aligned} \tag{1}$$

where Φ represents the mass flow, Ψ is the plenum pressure rise, $R \geq 0$ is the normalized stall cell squared amplitude, Φ_T is the mass flow through the throttle, $\sigma = 7$, and $\beta = 1/\sqrt{2}$. The functions $\Psi_c(\Phi)$ and $\Phi_T(\Psi)$ are the compressor and throttle characteristics, respectively, and are defined as $\Psi_C(\Phi) = \Psi_{C_0} + 1 + 3/2\Phi - 1/2\Phi^3$, $\Psi = \frac{1}{\gamma^2}(1 + \Phi_T(\Psi))^2$, where Ψ_{C_0} is a constant and γ is the throttle opening, the control input. Our control objective is to stabilize system (1) around the critical equilibrium $R^e = 0, \Phi^e = 1, \Psi^e = \Psi_C(\Phi^e) = \Psi_{C_0} + 2$, which achieves the peak operation on the compressor characteristic.

We shift the origin to the desired equilibrium with the change of variables $\phi = \Phi - 1, \psi = \Psi - \Psi_{C_0} - 2$. System (1) then becomes

$$\begin{aligned}
\dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \\
\dot{\phi} &= -\psi - 3/2\phi^2 - 1/2\phi^3 - 3R\phi - 3R \\
\dot{\psi} &= \frac{1}{\beta^2}\left(\phi - \gamma\sqrt{\psi + \Psi_{C_0} + 2} + 2\right)
\end{aligned} \tag{2}$$

We assume the pressure rise (and hence ψ) to be the only measurable state variable. It is then readily seen that this system is input output feedback linearizable with relative degree one (the first derivative of ψ contains the input γ), and its zero-dynamics are nonminimum phase.

2 State Feedback Control Design

Next, notice that Assumption A2 in [15, 14] is satisfied since, for example, a stabilizing control law for (2) is given in [9] by means of backstepping design. However, the control law proposed in [9] turns out to be quite complex. In [8], it is shown that a partial state feedback control law of the type

$$\gamma = \frac{2 + d_1\psi - d_2\phi}{\sqrt{\psi + \Psi_{C_0} + 2}}$$

achieves either a unique asymptotically stable equilibrium point with domain of attraction $\{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$ or two equilibria on the axisymmetric and stall characteristic, with domains of attraction $\{(R, \phi, \psi) \in \mathbb{R}^3 | R = 0\}$ and $\{(R, \phi, \psi) \in \mathbb{R}^3 | R > 0\}$, respectively (see Theorem 3.1 in [8]). Here, by viewing system (2) as an interconnection of two subsystems, namely the R-subsystem and the (ϕ, ψ)-subsystem, we design a full-state feedback controller which makes the origin of (2) an asymptotically stable equilibrium point with domain of attraction $\{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$, as seen in the next theorem.

Theorem 1 *For system (2), with the choice of the control law*

$$\bar{\gamma} = \frac{2 + (1 - \beta^2 k_1 k_2)\phi + \beta^2 k_2 \psi + 3\beta^2 k_1 R\phi}{\sqrt{\psi + \Psi_{C_0} + 2}} \tag{3}$$

where k_1 and k_2 are positive scalars satisfying the inequalities,

$$k_1 > \frac{17}{8} + \frac{(2C\sigma + 3)^2}{2} \tag{4}$$

$$\left(C\sigma - \frac{105}{64}\right)k_1^2 + \frac{3}{4}\left(-\frac{1}{2}C\sigma + \frac{21}{4}\right)k_1 - (C\sigma + 3)^2 > 0 \tag{5}$$

$$k_2 > k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4k_1 - 9/2} + \frac{(k_1^2 - 1)^2}{4} \tag{6}$$

$$C > \frac{3}{2\sigma} \tag{7}$$

the origin is an asymptotically stable equilibrium point with domain of attraction $\mathcal{A} = \{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$.

Proof. Without loss of generality let

$$u = \frac{1}{\beta^2}\left(\phi - \gamma\sqrt{\psi + \Psi_{C_0} + 2} + 2\right),$$

so that the last equation in (2) becomes $\dot{\psi} = u$. Next, notice that system (2) can be viewed as the interconnection of two subsystems:

$$[S_1]\ \dot{R} = -\sigma R^2, \qquad [S_2]\begin{cases}\dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3\\ \dot{\psi} = -u\end{cases}$$

A Lyapunov function for $[S_1]$, defined on the domain $\{R \in \mathbb{R} \mid R \geq 0\}$, is $V_1 = R$, and its time derivative is readily found to be $\dot{V}_1 = -\sigma R^2$ thus showing that the origin of $[S_1]$ is an asymptotically stable equilibrium point of $[S_1]$, and its domain of attraction is $\{R \in \mathbb{R} \mid R \geq 0\}$. As for subsystem $[S_2]$ the analysis found in Section 2.4.3 in [9] suggests using $V_2 = \frac{1}{2}\phi^2 + \frac{k_1}{8}\phi^4 + \frac{1}{2}(\phi - k_1\psi)^2$, where k_1 is a positive design constant. Furthermore, in [9], a stabilizing control law for $[S_2]$ is found to be $u = -c_1\phi + c_2\psi$, where c_1 and c_2 are two appropriate positive constants. In the following we will show that, in order to stabilize the interconnection of systems $[S_1]$ and $[S_2]$, one needs to add to $u = -c_1\phi + c_2\psi$ a term which is proportional to the product $R\phi$. Based on these considerations, consider the following candidate Lyapunov function for system (2),

$$V = CV_1 + V_2 = CR + \frac{1}{2}\phi^2 + \frac{k_1}{8}\phi^4 + \frac{1}{2}(\psi - k_1\phi)^2 \tag{8}$$

where $C > 0$ is a scalar. After noticing that V is positive definite on the domain $\mathcal{A}$, and letting $\tilde{\psi} = \psi - k_1\phi$, we calculate the time derivative of V as follows,

$$\begin{aligned}\dot{V} = &- C\sigma R^2 - C\sigma R(2\phi + \phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right)\left(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R\right) + \\ &+ \tilde{\psi}\left(u + k_1\psi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1R\phi + 3k_1R\right)\end{aligned} \tag{9}$$

Here, as in [9], we use the identity $-\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 = -\frac{1}{2}\left(\phi + \frac{3}{2}\right)^2\phi + \frac{9}{8}\phi$ to eliminate the potentially destabilizing term $-\left(\phi + k_1/2\phi^3\right)3/2\phi^2$. Next, substituting (3) into (9) (after taking in account the definition of u), letting $\bar{k}_1 = k_1 - 9/8$, and using the definition of $\tilde{\psi}$, we get

$$\begin{aligned}\dot{V} = &- C\sigma R^2 - C\sigma R(2\phi + \phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right)\left(-\tilde{\psi} - \bar{k}_1\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\right)^2\phi - 3R\phi - 3R\right) + \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + k_1^2\phi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1R\right)\end{aligned} \tag{10}$$

Now notice that the expression $-\left(\phi + \frac{k_1}{2}\phi^3\right)\frac{1}{2}\left(\phi + \frac{3}{2}\right)^2$ can be discarded since it is negative definite, and that the term $\frac{k_1}{2}\phi^3\tilde{\psi}$ cancels out. After collecting the remaining terms, we get

$$\begin{aligned}\dot{V} \leq &- C\sigma R^2 - (2C\sigma + 3)R\phi - (C\sigma + 3)R\phi^2 - \bar{k}_1\phi^2 - \left(\frac{k_1\bar{k}_1}{2} + \frac{3k_1}{2}R\right)\phi^4 - \frac{3k_1}{2}R\phi^3 + \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + (k_1^2 - 1)\phi + \frac{3}{2}k_1\phi^2 + 3k_1R\right)\end{aligned} \tag{11}$$

By using Young's inequality[1] five times we obtain the following inequalities

$$
\begin{aligned}
&-(2C\sigma+3)R\phi \leq \frac{1}{2}R^2 + \frac{(2C\sigma+3)^2}{2}\phi^2, \qquad -\frac{3k_1}{2}R\phi^3 \leq \frac{3k_1}{2}\left(\frac{R\phi^2}{4} + R\phi^4\right), \\
&(k_1^2-1)\phi\tilde{\psi} \leq \phi^2 + \frac{(k_1^2-1)^2}{4}\tilde{\psi}^2, \quad 3k_1R\tilde{\psi} \leq R^2 + \frac{9}{4}k_1^2\tilde{\psi}^2, \quad \frac{3}{2}k_1\phi^2\tilde{\psi} \leq \frac{k_1\bar{k}_1}{4}\phi^4 + \frac{9k_1}{4\bar{k}_1}\tilde{\psi}^2.
\end{aligned}
\tag{12}
$$

Applying the inequalities above to (11) we get

$$
\begin{aligned}
\dot{V} \leq & -\left(C\sigma - \frac{3}{2}\right)R^2 - \left(\bar{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 - \left(k_2 - k_1 - \frac{9}{4}k_1^2 - \frac{9k_1}{4\bar{k}_1} - \frac{(k_1^2-1)^2}{4}\right)\tilde{\psi}^2 + \\
& - \left(C\sigma + 3 - \frac{3}{8}k_1\right)R\phi^2 - \frac{k_1\bar{k}_1}{4}\phi^4, \\
\leq & -\begin{bmatrix} R \\ \phi^2 \end{bmatrix}^\top \begin{bmatrix} C\sigma - \frac{3}{2} & \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) \\ \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) & \frac{1}{4}k_1\bar{k}_1 \end{bmatrix} \begin{bmatrix} R \\ \phi^2 \end{bmatrix} - \left(\bar{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 + \\
& - \left(k_2 - k_1 - \frac{9}{4}k_1^2 - \frac{9k_1}{4\bar{k}_1} - \frac{(k_1^2-1)^2}{4}\right)\tilde{\psi}^2
\end{aligned}
\tag{13}
$$

Hence, $\dot{V}$ is negative definite on the domain $\mathcal{A}$, provided that the quadratic form above is positive definite and that the coefficients multiplying ϕ^2 and $\tilde{\psi}^2$ be positive. By imposing the positive definiteness of the quadratic form we obtain $C\sigma - \frac{3}{2} > 0$, $\left(C\sigma - \frac{3}{2}\right)\frac{1}{4}k_1\bar{k}_1 - \frac{1}{4}\left(C\sigma + 3 - \frac{3}{8}k_1\right)^2 > 0$, while by imposing the positivity of the coefficients of the remaining two terms we get $\bar{k}_1 > \frac{(2C\sigma+3)^2}{2} + 1$, $k_2 > k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4\bar{k}_1} + \frac{(k_1^2-1)^2}{4}$. By using the definition of $\bar{k}_1$, inequalities (4), (5), (6), and (7) follow. In conclusion, if k_1, k_2, and C are chosen so that (4)-(7) hold, we have that $\dot{V}$ is negative definite on $\mathcal{A}$ which contains the origin. This and the fact that the boundary of $\mathcal{A}$, $\partial\mathcal{A} = \{(R,\phi,\psi) \mid R = 0\}$, is an invariant manifold prove that the origin of the closed-loop system in an asymptotically stable equilibrium point and the set $\{(R,\phi,\psi) \mid V \leq K\} \cap \mathcal{A}$ is its region of attraction for any positive real number K. This in turn shows that $\mathcal{A}$ is the domain of attraction of the origin of the closed-loop system.

■

Remark 1: Inequalities (4)-(7) in Theorem 1 represent conservative lower bounds on k_1 and k_2. In practice, k_1 and k_2 can be chosen significantly smaller than their theoretical lower bounds. Choosing $\beta = 7$ and $\sigma = 1/\sqrt{2}$, we found that the smallest values of k_1 and k_2 satisfying (4)-(7) are given by $k_1 = 20.43$, $k_2 = 4.43 \cdot 10^4$ ($C = 0.2179$). However, simulations of the closed-loop system for several different initial conditions (see Figure 1 for one such simulation) indicate that k_1 and k_2 can be chosen as low as 10. The simulation results in Figure 1 also suggest that from a practical viewpoint smaller values of k_1 and k_2 are desirable because they yield smaller overshoot.

Remark 2: Generally a full-state feedback controller may yield a better closed-loop performance than one using partial-state feedback because it uses more information about the state of the system. When comparing our full-state feedback controller to the partial-state feedback controller developed in [8], however, this claim cannot be made without a rigorous analysis which is beyond the scope of this

[1] For any real numbers a and b, and any positive real k, one has that $ab \leq \frac{a^2}{4k} + kb^2$.

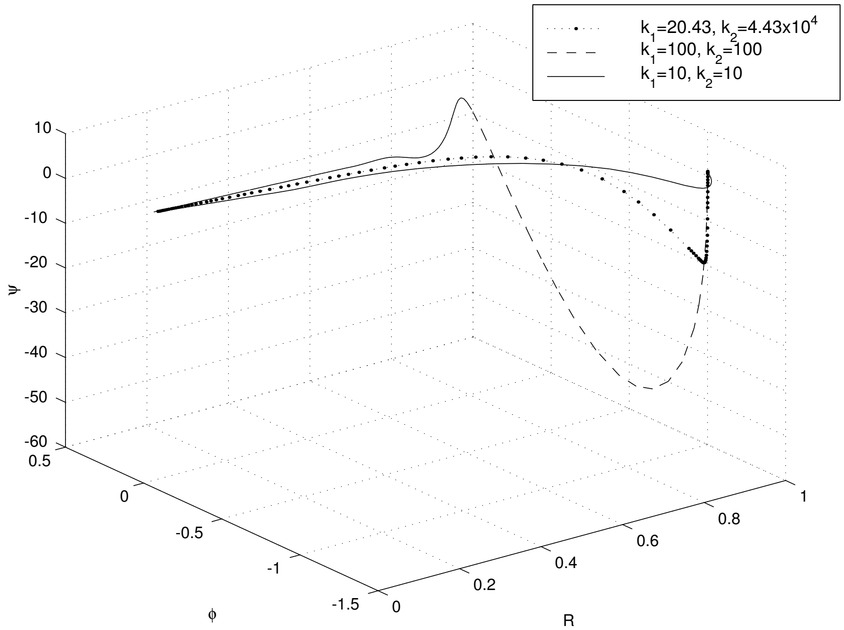

Figure 1: One trajectory of the closed-loop system using controller (3) for different choices of the parameters k_1 and k_2.

paper. Our main objective here is rather to show that one *can* recover the performance of a full-state feedback controller by using output feedback despite the fact that the system is not *uniformly completely observable* (this point will be made clear in the following). The methodology presented in this paper can be employed to recover the performance of any other full-state feedback controller.

3 Output Feedback Design

In this section we apply the methodology developed in [15, 14] to recover the performance of the state feedback controller (3) using output feedback. In what follows, we summarize the main result found in [14].

3.1 A Separation Principle for non-UCO Systems

Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{14}$$

where $x \in \mathbb{R}^n, u, y \in \mathbb{R}$, f and h are known smooth functions, and $f(0,0) = 0$. Our control objective is to construct a stabilizing controller for (14) without the availability of the system states x. In order to do so, we need an observability assumption. Define the observability mapping

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{H}\left(x, u, \ldots, u^{(n_u-1)}\right) \triangleq \begin{bmatrix} h(x,u) \\ \varphi_1(x,u,u^{(1)}) \\ \vdots \\ \varphi_{n-1}\left(x, u, \ldots, u^{(n_u-1)}\right) \end{bmatrix} \tag{15}$$

($y^{(n-1)}$ is the $n-1$-th derivative) where

$$\begin{aligned} \varphi_1(x,u,u^{(1)}) &= \frac{\partial h}{\partial x} f(x,u) + \frac{\partial h}{\partial u} u^{(1)} \\ &\vdots \\ \varphi_{n-1}\left(x, u, \ldots, u^{(n_u-1)}\right) &= \frac{\partial \varphi_{n-2}}{\partial x} f(x,u) + \frac{\partial \varphi_{n-2}}{\partial u} u^{(1)} + \ldots + \frac{\partial \varphi_{n-2}}{\partial u^{(n_u-2)}} u^{(n_u-1)} \end{aligned} \tag{16}$$

and $0 \leq n_u \leq n$ ($n_u = 0$ indicates that there is no dependence on u). Next, augment the system dynamics with n_u integrators at the input side, which corresponds to using a compensator[2] of order n_u. System (14) can be rewritten as follows,

$$\dot{x} = f(x, z_1),\ \dot{z}_1 = z_2,\ \ldots,\ \dot{z}_{n_u} = v \tag{17}$$

Define the extended state variable $X = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$, and the associated *extended system*

$$\begin{aligned} \dot{X} &= f_e(X) + g_e v \\ y &= h_e(X) \end{aligned} \tag{18}$$

where $f_e(X) = [f^\top(x, z_1),\ \ z_2, \ldots, z_{n_u}, 0]^\top$, $g_e = [0, \ldots, 1]^\top$, and $h_e(X) = h(x, z_1)$. Now, we are ready to state our first assumption.

Assumption A1(Observability): System (14) is observable over an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping $\mathcal{F} : \mathcal{O} \to \mathcal{Y}$ (where $\mathcal{Y} = \mathcal{F}(\mathcal{O})$) defined by

$$Y = \begin{bmatrix} y_e \\ \ldots \\ z \end{bmatrix} = \mathcal{F}(X) = \begin{bmatrix} \mathcal{H}(x,z) \\ z \end{bmatrix} \tag{19}$$

has a smooth inverse $\mathcal{F}^{-1} : \mathcal{Y} \to \mathcal{O}$,

$$\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(y_e, z) = \begin{bmatrix} \mathcal{H}^{-1}(y_e, z) \\ z \end{bmatrix}. \tag{20}$$

[2]The chain of integrators can be replaced by *any* regular (i.e., invertible) compensator in lower triangular form with relative degree equal to n_u

Following the terminology in [19], when $\mathcal{O} = \mathbb{R}^{n+n_u}$ we say that the system is *uniformly completely observable (UCO).*

Assumption A2(Stabilizability): The origin of (14) is locally stabilizable (or stabilizable) by a static function of x, i.e., there exists a smooth function $\bar{u}(x)$ such that the origin is an asymptotically stable (or globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \bar{u}(x))$.

Using A2, the knowledge of a Lyapunov function for (14) with $u = \bar{u}(x)$, and the integrator backstepping lemma (see, e.g., [10]), one may design a smooth control law

$$v = \phi(x, z) = \phi(X)$$

which makes the origin of (18) an asymptotically stable equilibrium point. In particular, from the application of the integrator backstepping lemma one also gets a Lyapunov function $\bar{V}(X)$. Given any scalar $c > 0$, let Ω_c denote the generic level set of $\bar{V}$, i.e.,

$$\Omega_c \triangleq \{X \in \mathbb{R}^{n+n_u} \,|\, \bar{V} \leq c\}.$$

Our last assumption concerns the topology of the "observability set" $\mathcal{O}$.

Assumption A3(Topology of $\mathcal{O}$): Assume that there exists a constant $c_2 > 0$ and a set $\mathcal{C}$ such that

$$\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C} \subset \mathcal{Y} \;\; (= \mathcal{F}(\mathcal{O})), \tag{21}$$

where $\mathcal{C}$ has the following properties

(i) The boundary of $\mathcal{C}$, $\partial\mathcal{C}$, is class C^1, i.e., there exists a C^1 function $g : \mathcal{C} \to \mathbb{R}$ such that $\partial\mathcal{C} = \{Y \in \mathcal{C} \,|\, g(Y) = 0\}$, and $(\partial g/\partial Y)^\top \neq 0$ on $\partial\mathcal{C}$.

(ii) Each slice $\mathcal{C}^{\bar{z}} = \{y_e \in \mathbb{R}^n \,|\, [{y_e}^\top, \bar{z}^\top]^\top \in \mathcal{C}\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

(iii) 0 is a regular value of $g(\cdot, \bar{z})$ for each fixed $\bar{z} \in \mathbb{R}^{n_u}$, i.e., $[\partial g/\partial y_e(y_e, \bar{z})]^\top$ does not vanish anywhere on the boundary of each slice $\mathcal{C}^{\bar{z}}$.

(iv) $\bigcup_{\bar{z}\in\mathbb{R}^{n_u}} \mathcal{C}^{\bar{z}}$ is compact.

We are now ready to introduce the output feedback controller for the extended system (18):

$$\dot{\hat{x}}^P = \begin{cases} \left[\dfrac{\partial \mathcal{H}}{\partial \hat{x}^P}\right]^{-1} \left\{ \dot{\hat{y}}_e|_{\hat{x}^P} - \Gamma \dfrac{N_{y_e}(\hat{Y}^P)\left(N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z}\right)}{N_{y_e}(\hat{Y}^P)^\top \Gamma N_{y_e}(\hat{Y}^P)} - \dfrac{\partial \mathcal{H}}{\partial z}\dot{z} \right\} \\ \qquad \text{if } N_{y_e}(\hat{Y}^P)^\top \dot{\hat{y}}_e|_{\hat{x}^P} + N_z(\hat{Y}^P)^\top \dot{z} \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \hat{f}(\hat{x}^P, z, y) = f(\hat{x}^P, z_1) + \left[\dfrac{\partial \mathcal{H}(\hat{x}^P, z)}{\partial \hat{x}^P}\right]^{-1} \mathcal{E}^{-1} L\left[y(t) - h(\hat{x}^P, z_1)\right] \qquad \text{otherwise} \end{cases} \tag{22}$$

$$v = \phi(\hat{x}^P, z), \tag{23}$$

where the various parameters are defined in the following table

$\hat{y}_e^P = \mathcal{H}(\hat{x}^P, z)$	$\hat{Y}^P = \mathcal{F}(\hat{x}^P, z) = [\hat{y}_e^{P\top}, z^\top]^\top$	$N_{y_e}(\hat{Y}^P) = \left[\frac{\partial g}{\partial \hat{y}_e^P}(\hat{y}_e^P, z)\right]^\top$
$N_z(\hat{Y}^P) = \left[\frac{\partial g}{\partial z}(\hat{y}_e^P, z)\right]^\top$	$\dot{\hat{y}}_e\|_{\hat{x}^P} = \frac{\partial \mathcal{H}}{\partial \hat{x}^P}\hat{f}(\hat{x}^P, z, y) + \frac{\partial \mathcal{H}}{\partial z}\dot{z}$	$\mathcal{E} = \mathrm{diag}\left[\rho, \ldots, \rho^n\right], \rho > 0$
L is any $n \times 1$ Hurwitz vector	$\Gamma = \rho^{2n}(S\mathcal{E})^{-1}\,(S\mathcal{E})^{-\top}$	$S = S^\top = P^{\frac{1}{2}}$

and P is the solution of the Lyapunov equation $P(A_c - LC_c) + (A_c - LC_c)^\top P = -I$, where (A_c, C_c) is the canonical observable pair.

Remark 3: The controller (23) has a certainty equivalence structure, whereby the non-measurable state x in the controller $\phi(x, z)$ is replaced by its estimate $\hat{x}^P$, generated by the observer (22). The observer incorporates a dynamic projection which constrains the estimate $\hat{x}^P$ inside the set $\mathcal{H}^{-1}(\mathcal{C}) \subset \mathcal{O}$ and thus guarantees its well-definiteness. This feature is particularly useful when $\mathcal{O}$ is not all of $\mathbb{R}^{n+n_u}$ (that is, when the system is not UCO) and other output feedback control approaches based on a separation principle such as [19, 3] cannot be employed. In the next section we will show that MG3 is not UCO and will use the methodology presented here to solve the output feedback stabilization problem.

The following result states that (22) and (23) guarantee closed-loop stability.

Theorem 2 ([14]) *For the closed-loop system (18), (22), (23), satisfying assumptions A1, A2, and A3, for any $0 < c_1 < c_2$ there exists a scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the set $\Omega_{c_1} \times \mathcal{H}^{-1}(\mathcal{C})$ is contained in the region of attraction of the origin $(x = 0, z = 0, \hat{x}^P = 0)$.*

3.2 Application of the Separation Principle to MG3

Following the procedure outlined in the previous section, we verify that assumptions A1- A3 are verified for MG3.

Observability: We start by forming the mapping $\mathcal{H}$ from the measurable output $y = \psi$

$$y_e = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \mathcal{H}\left([R, \phi, \psi]^\top, \gamma, \dot{\gamma}\right) = \begin{bmatrix} \psi \\ 1/\beta^2\,(\phi - \theta(\psi, \gamma)) \\ 1/\beta^2\left(-\psi - 3/2\phi^2 - 1/2\phi^3 - 3R\phi - 3R - \dot{\theta}\right) \end{bmatrix}, \tag{24}$$

where, for convenience, we denoted $\theta(\psi, \gamma) = \gamma\sqrt{\psi + \Psi_{C_0} + 2} - 2$ and $\dot{\theta} = (\partial\theta/\partial\psi)\,\dot{\psi} + (\partial\theta/\partial\gamma)\,\dot{\gamma}$. Recall that γ is the control input and note that both γ and $\dot{\gamma}$ appear in $\mathcal{H}$, thus $n_u = 2$. Next, we need to augment the system with $n_u = 2$ integrators at its input side. To simplify the integrator backstepping design, we employ a chain of two integrators with a *modified output*:

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = v, \qquad \gamma = \frac{z_1 + 2}{\sqrt{\psi + \Psi_{C_0} + 2}}, \tag{25}$$

so that θ and $\dot{\theta}$ in (24) are replaced by z_1 and z_2, respectively, and the augmented system becomes the following cascade interconnection of two subsystems $[P_1]$ and $[P_2]$

$$
\begin{aligned}
&[P_1]\begin{cases} \dot{R} = -\sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} = -\psi - 3/2\phi^2 - 1/2\phi^3 - 3R\phi - 3R \\ \dot{\psi} = \frac{1}{\beta^2}\left(\phi - z_1\right) \end{cases} \\
&[P_2]\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v. \end{cases}
\end{aligned}
\tag{26}
$$

Note that the dynamic extension (25) is well-defined in an output feedback context because the output of (25) is a function of the measurable variables z_1 and ψ. Next, the mapping $\mathcal{F}$ is given by

$$
Y = \mathcal{F}\left([R, \phi, \psi]^\top, [z_1, z_2]^\top\right) = \begin{bmatrix} \mathcal{H}\left([R, \phi, \psi]^\top, z_1, z_2\right) \\ z_1 \\ z_2 \end{bmatrix}.
$$

Now notice that the observability assumption A1 is satisfied on the set

$$
\mathcal{O} = \left\{ [R, \phi, \psi]^\top \in \mathbb{R}^3, z \in \mathbb{R}^2 \mid \phi > -1 \right\}.
$$

Since the system *is not UCO* other output feedback approaches based on a separation principle cannot be employed to recover the performance of a full-state feedback controller. It is easy to check that, when $\phi = -1$ and hence $\Phi = 0$, $\mathcal{F}$ does not depend on R and hence it is not invertible. Hence, when there is no mass flow through the compressor ($\Phi = 0$) the normalized stall cell squared amplitude R cannot be observed. Clearly, $\Phi = 0$ is a condition we would like to avoid during normal engine operation.

Stabilizability: Following the same notation used in the previous section we let $x = [R, \phi, \psi]^\top$. Rewrite $[P_1]$ in (26) as $\dot{x} = f_1(x) + g_1(x)z_1$ (also, let $f(x, z_1) = f_1(x) + g_1(x)z_1$) and notice that Theorem 1 guarantees that the stabilizability assumption A2 is satisfied by the controller $\bar{\gamma}(x)$. Next, recalling that $z_1 = \theta$, in order to design a stabilizing control law for the extended system (26) one can view $[P_1]$ as a subsystem with input θ and stabilizing controller $\bar{\theta} = \bar{\gamma}(x)\sqrt{\psi + \Psi_{C_0} + 2} - 2$ and apply integrator backstepping. Doing so, one obtains the stabilizing control law

$$
v = \dot{\alpha} - \tilde{z}_1 - k_4 \tilde{z}_2 \triangleq \phi(x, z), \tag{27}
$$

where $\tilde{z}_1 = z_1 - \bar{\theta}(x)$, $\alpha(x, z_1) = -k_3\tilde{z}_1 - \dfrac{\partial V}{\partial x} g(x) + \dfrac{\partial \bar{\theta}}{\partial x}[f(x) + g(x)\, z_1]$, $\tilde{z}_2 = z_2 - \alpha(x, z_1)$, and k_3, k_4 are arbitrary positive constants. This completes the design of a stabilizing state feedback for the extended system (26). The Lyapunov function of the closed-loop extended system is $\bar{V} = V + \frac{1}{2}\tilde{z}_1^2 + \frac{1}{2}\tilde{z}_2^2$, where V is defined in the proof of Theorem 1. Following the same reasoning as in the proof of Theorem 1, we conclude that the origin of the extended system is asymptotically stable with domain of attraction $\mathcal{D} = \mathcal{A} \times \mathbb{R}^2$.

Topology of the Observability Set: Noting that $\mathcal{Y} = \mathcal{F}(\mathcal{O}) = \{y_e \in \mathbb{R}^3, z \in \mathbb{R}^2 \mid y_{e,2} > \frac{1}{\beta^2}(-1 - z_1)\}$, it is readily seen that the set

$$\mathcal{C} = \Bigg\{ [{y_e}^\top, z^\top]^\top \in \mathbb{R}^5 \,|\, y_{e,1} \in [a_1, b_1], y_{e,2} \in \left[\frac{a_2 - z_1}{\beta^2}, \frac{b_2 - z_1}{\beta^2}\right],$$
$$y_{e,3} \in \left[\frac{1}{\beta^2}(-z_2 + a_3), \frac{1}{\beta^2}(-z_2 + b_3)\right], z_1 \in [a_4, b_4], z_2 \in [a_5, b_5] \Bigg\},$$

parameterized by the set of scalars $\{a_i, b_i \in \mathbb{R} \,|\, a_i < b_i, i = 1, \ldots, 5\}$, is contained in $\mathcal{Y}$ for all $a_2 > -1$. Furthermore, each slice $\mathcal{C}^{\bar{z}}$ obtained from $\mathcal{C}$ by holding z constant at $\bar{z}$ is convex (it is a parallelepiped in $\mathbb{R}^3$), thus satisfying requirement (ii) in A3. The union of all slices $\mathcal{C}^{\bar{z}}$ is the set

$$\bigcup_{\bar{z} \in \mathbb{R}^2} \mathcal{C}^{\bar{z}} = \left\{ y_e \in \mathbb{R}^3 \,|\, y_{e,1} \in [a_1, b_1], y_{e,2} \in \left[\frac{a_2 - b_4}{\beta^2}, \frac{b_2 - a_4}{\beta^2}\right], y_{e,3} \in \left[\frac{1}{\beta^2}(-b_5 + a_3), \frac{1}{\beta^2}(-a_5 + b_3)\right] \right\},$$

which is clearly compact, thus satisfying requirement (iv). Notice that the boundary of the set $\mathcal{C}$ defined above does not fully satisfy requirement (i) because it is continuous but not differentiable at some corners. This, in general, may generate some numerical problems in the projection, since at the points when the boundary is not differentiable the vectors N_{y_e} and N_z defined in the previous section are not uniquely defined. Should numerical problems arise, one may slightly modify the definition of $\mathcal{C}$ by smoothing out its corners. The vectors N_{y_e} and N_z are given by

$$N_{y_e}(\hat{y}_e, z) = \begin{cases} [1,0,0]^\top & \hat{y}_{e,1} = b_1 \\ [-1,0,0]^\top & \hat{y}_{e,1} = a_1 \\ [0,1,0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(b_2 - z_1^m) \\ [0,-1,0]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(a_2 - z_1^M) \\ [0,0,1]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(-z_2^m + b_3) \\ [0,0,-1]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(-z_2^M + a_3) \end{cases} \quad N_z(\hat{y}_e, z) = \begin{cases} [0,0]^\top & \hat{y}_{e,1} = b_1 \text{ or } \hat{y}_{e,1} = a_1 \\ \left[\frac{1}{\beta^2}, 0\right]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(b_2 - z_1) \\ \left[-\frac{1}{\beta^2}, 0\right]^\top & \hat{y}_{e,2} = \frac{1}{\beta^2}(a_2 - z_1) \\ \left[0, \frac{1}{\beta^2}\right]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(b_3 - z_2) \\ \left[0, -\frac{1}{\beta^2}\right]^\top & \hat{y}_{e,3} = \frac{1}{\beta^2}(a_3 - z_2) \\ [1,0]^\top & z_1 = b_4 \\ [-1,0]^\top & z_1 = a_4 \\ [0,1]^\top & z_2 = b_5 \\ [0,-1]^\top & z_2 = a_5. \end{cases} \tag{28}$$

where $z_1^m = \min\{z_1, a_4\}$, $z_1^M = \max\{z_1, b_4\}$, $z_2^m = \min\{z_2, a_5\}$, $z_2^M = \max\{z_2, b_5\}$.

Note that N_{y_e} never vanishes so in particular it does not vanish on any slice $\mathcal{C}^{\bar{z}}$, and thus requirement (iii) is fulfilled. In conclusion, in order for A3 to be satisfied, it remains to use the Lyapunov function $\bar{V}$ to find the largest value of c_2 such that $\Omega_{c_2} \subset \mathcal{O}$ (implying that $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{F}(\mathcal{O})$) and subsequently pick values for the scalars a_i, b_i, $i = 1, \ldots, 5$ such that $a_2 > -1$ and $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C}$. A more practical way to address the design of $\mathcal{C}$ entails running a number of simulations for the closed-loop system under state feedback corresponding to several initial conditions $(x(0), z(0))$ and calculating upper and lower bounds for $\psi(t)$, $\phi(t)$, $-\psi(t) - 3/2\phi^2(t) - 1/2\phi^3(t) - 3R(t)\phi(t) - 3R(t)$, $z_1(t)$, $z_2(t)$. These will provide the values of a_i, b_i, $i = 1, \ldots, 5$, respectively. By doing that, we found that whenever $[x(0)^\top, z(0)^\top]^\top \in \Omega_0 \overset{\triangle}{=} \{[x(0)^\top, z(0)^\top]^\top \in \mathbb{R}^5 \,:\, R \in [0, 0.1], \phi \in [-0.1, 0.1], \psi \in [-0.5, 0.5], z_1 \in [-0.1, 0.1], z_2 \in [-0.1, 0.1]\}$, we have that $a_1 = -1.15$, $b_1 = 0.5$, $a_2 = -0.3$, $b_2 = -0.1$, $a_3 = -0.75$, $b_3 = 0.4$, $a_4 = -2$, $b_4 = 7$, $a_5 = -70$, $b_5 = 250$. We must point out that our choice of Ω_0 is rather conservative and is made primarily for the sake of illustration.

Observer Design: Having verified that assumptions A1-A3 are verified and having selected the set $\mathcal{C}$, we are ready to design observer (22) for MG3. Denoting by $\hat{x}^P$ the vector $[\hat{R}^P, \hat{\phi}^P, \hat{\psi}^P]^\top$, the vector field $\hat{f}(\hat{x}^P, z, y)$ is given by

$$\hat{f} = \begin{bmatrix} -\sigma(\hat{R}^P)^2 - \sigma\hat{R}^P(2\hat{\phi}^P + (\hat{\phi}^P)^2) - \dfrac{l_1/\rho + \beta^2 l_2/\rho^2(3\hat{\phi}^P + 3\hat{R}^P + 3/2(\hat{\phi}^P)^2) + \beta^2 l_3/\rho^3}{3(1+\hat{\phi}^P)}(\psi - \hat{\psi}^P) \\ -\hat{\psi}^P - 3/2\,(\hat{\phi}^P)^2 - 1/2\,(\hat{\phi}^P)^3 - 3\hat{R}^P\hat{\phi}^P - 3\hat{R}^P + \beta^2 l_2/\rho^2(\psi - \hat{\psi}^P) \\ -\dfrac{z_1 - \hat{\phi}^P}{\beta^2} + l_1/\rho(\psi - \hat{\psi}^P) \end{bmatrix}$$

This, together with the expressions for N_{y_e} and N_z in (28) and the definition of $\mathcal{H}$ in (24), concludes the design of the observer and dynamic projection. In conclusion, the output feedback controller design is given by $\hat{v} = \phi(\hat{x}^P, z)$, where the function ϕ is defined in (27).

4 Simulation Results

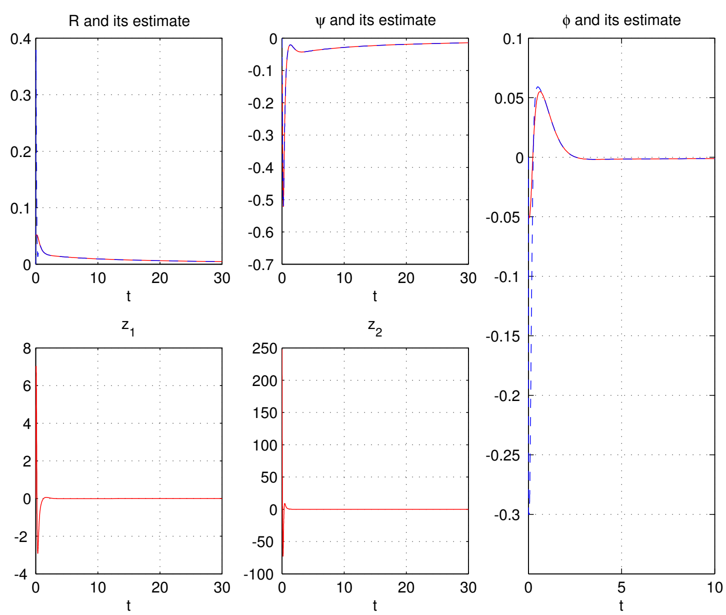

Figure 2: Closed-loop system trajectories when $\rho = 1/5$.

Here we present the simulation results when the output feedback controller developed in the previous section is applied to system (2). We choose $k_1 = 20.43$, $k_2 = 4.43 \cdot 10^4$ to fulfill inequalities (4)-(7) in Theorem 1, and $L = [6, 12, 8]^\top$ so that the associated polynomial $s^3 + l_1 s^2 + l_2 s + l_3 = 0$ is Hurwitz.

In Figure 2 system and controller states, together with the control input, are plotted for $\rho = 1/5$. The figure clearly shows the operation of the projection which prevents the observer from peaking and guarantees that $\hat{\phi} > -0.3$, and thus is bounded away from the singularity in -1. Figure 3 depicts the evolution of the observer estimation error for $\rho = 1/10$ and $\rho = 1/50$, confirming the theoretical predictions of Theorem 1 and Lemma 1 in [15] concerning the arbitrary fast rate of convergence of the observer with projection (22). Finally, in Figure 4 the orbits of (R, ϕ, ψ) are plotted for decreasing values

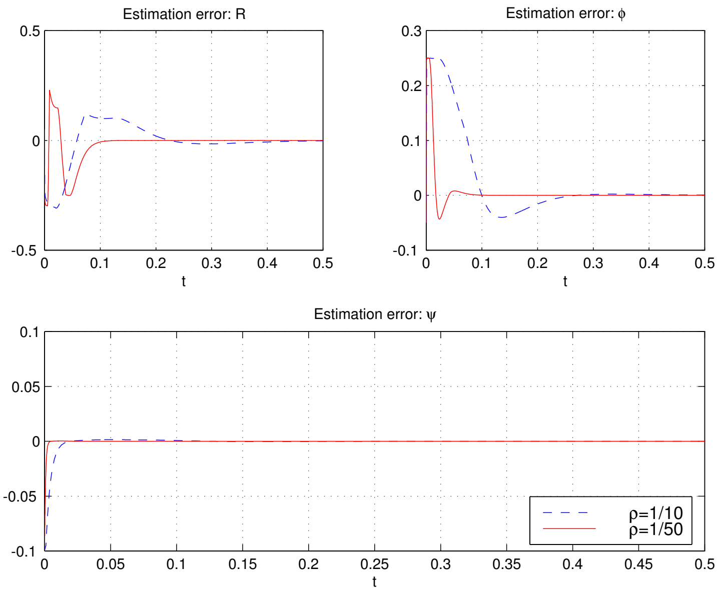

Figure 3: Observer estimation error for $\rho = 1/10$ and $\rho = 1/50$.

of ρ.

5 Concluding Remarks

While existing separation principle approaches, including [19] and [3], cannot be applied to recover the performance of full-state feedback controllers for MG3, they can be employed to recover the performance of *any* partial state feedback controller which does not use R, since the (ϕ, ψ) subsystem *is* UCO (whereas, as shown in Section 3.2, R *is not* observable when $\phi = -1$). Thus, in particular, the approach in [19] can be applied to the controller presented in [8] to obtain an output feedback controller for MG3. Additionally, without resorting to a separation principle, one can employ the technique developed in [7], Sections 12.6, 12.7 and obtain semiglobal stabilization of the origin of the closed-loop system system, or the one presented in [2], based on a globally convergent observer and a small-gain design.

The modularity of our approach and, specifically, the availability of an estimate for the *full state* of the system provides some design flexibility in that it allows using available state feedback control design techniques. On the other hand, the results presented here present some limitations that need to be addressed. First, the methodology in [14] (as well as the approaches developed in [19, 3]) requires adding two integrators at the input side of MG3. This unnecessarily complicates the state feedback design and yields a complex expression for the resulting control law $\phi(x, z)$. We have shown in [13] that, by sacrificinng asymptotic stability, one can avoid adding integrators by *estimating* the time derivatives of θ (recall that $z_1 = \theta$ and $z_2 = \dot{\theta}$). Additionally, our design relies on the perfect knowledge of the compressor characteristic[3], which is not a realistic assumption, and the absence of disturbances acting on the model. We are currently working on removing these obstacles. Finally, finding a set $\mathcal{C}$ satisfying A3 may be at times a difficult task. Developing numerical methods to automate the design of $\mathcal{C}$ and the

[3]Note that the works [8], [7], [2] share the same limitation.

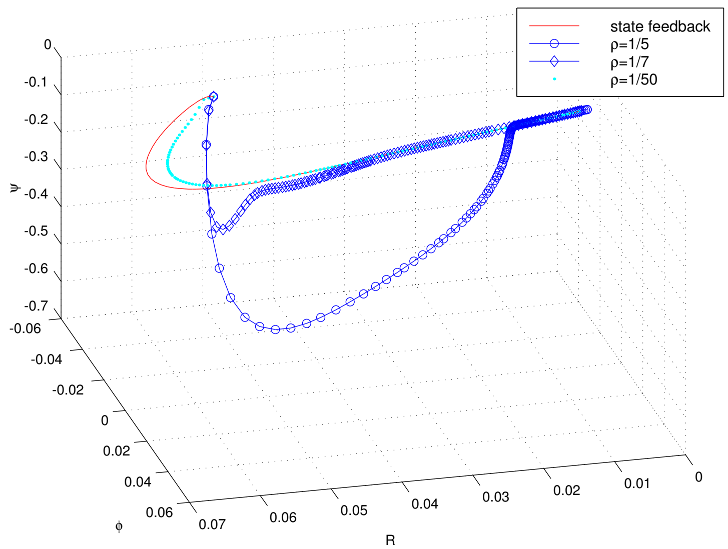

Figure 4: State feedback trajectories and output feedback trajectories for several choices of ρ.

associated dynamic projection is one of our research directions.

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