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# Three-dimensional Ballistic Equations of Rocket Considering the Elastic Effects

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**Abstract.** Because the shell of modern rocket is more and more thin and the body length to diameter ratio increase constantly, influence to trajectory caused by elasticity effect cannot be ignored any more. Based on the model of Euler-Bernoulli beam, the dynamic equations of flexible rocket are derived by using of Hamilton principle. The influence of distributing mass and aerodynamic force and the coupling between longitudinal and transversal vibration are considered. Example shows ballistic range and altitude of rocket are influenced by elastic effects observably.

## Introduction

With the development of material science and material processing technology, structure of light alloys and composite materials are widely used in the rocket, the rocket shell become more and more thin. Because the increase of body length to diameter ratio and flying speed, effect of elasticity is more and more big when the modern rocket is on a launch flight, the deformation size of the rocket may be a lot more than its design value allowed, the flexible deformation cannot be ignored any more.

As a flexible deformation body, elastic deformation and ballistic couple to each other and interact with each other when the projectile is in flight, this may influence the flight performance of the projectile seriously. The consequences are increasing the trajectory scatter and reducing firing accuracy, even appearing the phenomenon of structural failure or losing the target due to projectile body's vibration. Therefore, the traditional rigid body model of projectile six degrees of freedom cannot adapt to the motion analysis and control of modern rocket and missile gradually [1,2].

In order to improve the rocket missile accuracy, it is urgent to establish a flexible rocket and missile model which consider influence of various elasticity factors. But there are not very much work in this domain recently [3-5].

## Flexible projectile motion analysis

Assume $oxyz$ is a launch coordinate system, $i$, $j$, $k$ is unit vector. Regardless of the rotation of the earth, then Launch coordinate system can be seen as the inertial coordinate system. Let axis $o'\xi$ always pass through the two ends of the projectile, $o'\xi\eta\zeta$ is conjoined coordinate system of the projectile body shown in Fig.1, e1,e2 and e3 are its unit vectors, then projectile body can be seen as a single beam supported on both ends in $o'\xi\eta\zeta$ coordinate system.

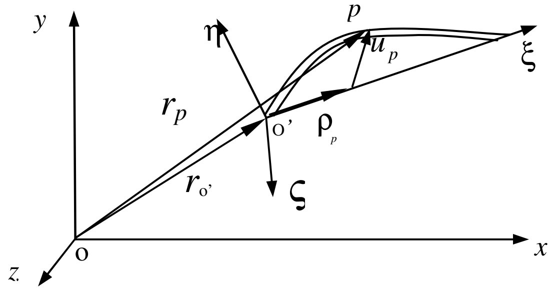



Fig.1. Model and coordinate system

In launch coordinate system, radius vector rp at some point p of the projectile body is described as:

$$\mathbf{r}_p = \mathbf{r}_{o'} + \mathbf{A}(\boldsymbol{\rho}_p + \mathbf{u}_p) \tag{1}$$

where:

$$\mathbf{r}_{o'} = [x_{o'} \quad y_{o'} \quad z_{o'}]^T \tag{2}$$

A is the direction cosine matrix which Conjoined coordinate system is related to the launch coordinate system. It is described as:

$$\mathbf{A} = \begin{bmatrix} \cos\varphi\cos\psi & \cos\varphi\sin\psi\sin\gamma - \sin\varphi\cos\gamma & \cos\varphi\sin\psi\cos\gamma + \sin\varphi\sin\gamma \\ \sin\varphi\cos\psi & \sin\varphi\sin\psi\sin\gamma + \cos\varphi\cos\gamma & \sin\varphi\sin\psi\cos\gamma - \cos\varphi\sin\gamma \\ -\sin\psi & \cos\psi\sin\gamma & \cos\psi\cos\gamma \end{bmatrix} \tag{3}$$

Here, φ, ψ and γ are pitching angle, yaw angle and roll angle respectively.

$\boldsymbol{\rho}_\mathbf{p} = [\xi \quad 0 \quad 0]^\mathrm{T}$ is the position of point p in the conjoined coordinate system before deformation. $\mathbf{u}_\mathbf{p} = [\mathrm{u}_1 \quad \mathrm{u}_2 \quad \mathrm{u}_3]^\mathrm{T}$ is the deformation displacement of point p. $u_1$,$u_2$ and $u_3$ are displacement components of $u_p$ along three axes of conjoined coordinate system.

then:

$$\mathbf{u}_p = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \tag{4}$$

Projectile can be considered as Bernoulli Euler beam, Assume the actual axial deformation and lateral deformation of the projectile are $w_1(x,t)$, $w_2(x,t)$ and $w_3(x,t)$ respectively. Considering the beam properties of geometric non-linear, the relationship between deformation displacement and the actual deformation can be established as follow:

$$\begin{cases} u_1 = w_1 - \int_0^x \frac{1}{2}(\frac{\partial w_2}{\partial x})^2 dx - \int_0^x \frac{1}{2}(\frac{\partial w_3}{\partial x})^2 dx \\ u_2 = w_2 \\ u_3 = w_3 \end{cases} \tag{5}$$

Application of modal expansion method , the deformation field can be expressed as:

$$\begin{cases} w_1(\xi,t) = \sum_{i=1}^{k} \varphi_{1i}(\xi) q_{1i}(t) \\ w_2(\xi,t) = \sum_{i=1}^{k} \varphi_{2i}(\xi) q_{2i}(t) \\ w_3(\xi,t) = \sum_{i=1}^{k} \varphi_{3i}(\xi) q_{3i}(t) \end{cases} \tag{6}$$

**Flexible dynamic equations of projectile**

From Eq. (1) , it can be obtained as follow:

$$\dot{\mathbf{r}}_p = \dot{\mathbf{r}}_{o'} + \dot{\mathbf{A}}(\boldsymbol{\rho}_p + \mathbf{u}_p) + \mathbf{A}\dot{\mathbf{u}}_p = [\mathbf{I} \quad \mathbf{B} \quad \mathbf{AN}]\dot{\mathbf{q}} \tag{7}$$

here, $\mathbf{B} = -\mathbf{A}\tilde{\mathbf{u}}'$, $\mathbf{u}' = \boldsymbol{\rho}_\mathbf{p} + \mathbf{u}_\mathbf{p}$, $\tilde{\mathbf{u}}'$ is the antisymmetric matrices of $\mathbf{u}'$, $\dot{\mathbf{q}} = [\dot{\mathbf{r}}_{\mathbf{o}'} \quad \boldsymbol{\omega}' \quad \dot{\mathbf{q}}_\mathbf{f}]^\mathrm{T}$ is generalized coordinate matrices, N is shape function matrices composed by $\varphi_{1i}$, $\varphi_{2i}$ and $\varphi_{3i}$, $\mathbf{q}_f$ is the modal coordinates composed by time-varying coefficient $q_{1i}$, $q_{2i}$ and $q_{3i}$.

$\boldsymbol{\omega}' = [\omega'_\xi \quad \omega'_\eta \quad \omega'_\zeta]^\mathrm{T}$, therein, $\omega'_\xi$, $\omega'_\eta$ and $\omega'_\zeta$ are components of the angular velocity vector on the three coordinate axes ξ , η and ζ in conjoined coordinate system.

Assuming linear density of projectile body mass along axes is ρ(ξ,t), The kinetic energy of the system is:

$$T = \frac{1}{2}\int_0^l \rho(\xi,t)\dot{\mathbf{r}}_p{}^T\dot{\mathbf{r}}_p d\xi = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} \tag{8}$$

where, mass matrix $\mathbf{M}=\int_0^l \rho(\xi,t)\begin{bmatrix} \mathbf{I} & \mathbf{B} & \mathbf{AN} \\ \mathbf{B}^T & \mathbf{B}^T\mathbf{B} & \mathbf{B}^T\mathbf{AN} \\ \mathbf{N}^T\mathbf{A}^T & \mathbf{N}^T\mathbf{A}^T\mathbf{B} & \mathbf{N}^T\mathbf{N} \end{bmatrix} d\xi$ (9)

After doing variation operation for Eq.8 and integration between [$t_0$, $t_1$], it can be obtained:

$$\int_{t_0}^{t_1}\delta T dt=\int_{t_0}^{t_1}[\frac{\partial}{\partial \mathbf{q}}(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}})\delta\mathbf{q}+\mathbf{M}\dot{\mathbf{q}}\delta\dot{\mathbf{q}}]dt=\int_{t_0}^{t_1}[\frac{\partial}{\partial \mathbf{q}}(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}})-\mathbf{M}\ddot{\mathbf{q}}]\delta\mathbf{q}dt \quad (10)$$

Potential energy of the system is:

$$V=\int_0^l[\frac{1}{2}EA(w_1')^2+\frac{1}{2}EI(w_2'')^2+\frac{1}{2}EI(w_3'')^2]d\xi-\boldsymbol{\rho}[\mathbf{r}_{o'}+\mathbf{A}(\boldsymbol{\rho}_p+\mathbf{u}_p)]=\frac{1}{2}\mathbf{q}_f{}^T\mathbf{K}_1\mathbf{q}_f-\boldsymbol{\rho}(\mathbf{r}_{o'}+\mathbf{B}) \quad (11)$$

where, $\mathrm{K}_1=\int_0^1\frac{\partial \mathbf{H}^T}{\partial\xi}\begin{bmatrix}\mathrm{EA} & & \\ & \mathrm{EI} & \\ & & \mathrm{EI}\end{bmatrix}\frac{\partial\mathbf{H}}{\partial\xi}d\xi$ (12)

$$\boldsymbol{\rho}=\begin{bmatrix}0 & \int_0^l\rho(\xi,t)d\xi & 0\end{bmatrix} \quad (13)$$

$$\mathbf{H}=\begin{bmatrix} \varphi_{11} \cdots \varphi_{1k} & 0 & 0 \\ 0 & \frac{\partial\varphi_{21}}{\partial\xi}\cdots\frac{\partial\varphi_{2k}}{\partial\xi} & 0 \\ 0 & 0 & \frac{\partial\varphi_{31}}{\partial\xi}\cdots\frac{\partial\varphi_{3k}}{\partial\xi} \end{bmatrix} \quad (14)$$

The engine thrust force $\mathrm{P}=[p_x\ p_y\ p_z]^T$, the aerodynamic force $\mathrm{R}=[X_1\ Y_1\ Z_1]^T$, the aerodynamic moment $\mathrm{M}_{st}=[M_x\ M_y\ M_z]^T$, damping moment $\mathrm{M}_d=[M_{dx}\ M_{dy}\ M_{dz}]^T$.

Elementary work made by nonconservative force is:

$$\delta W=\mathbf{P}\delta\mathbf{r}_{o'}-\mathbf{R}\delta\mathbf{r}_p+(\mathbf{M}_{st}-\mathbf{M}_d)\delta\boldsymbol{\pi} \quad (15)$$

According to Hamilton's principle , ,$\delta\int_{t_0}^{t_1}(\mathrm{T}-\mathrm{V})dt+\int_{t_0}^{t_1}\delta \mathrm{W}dt=0$, there is:

$$\int_{t_0}^{t_1}\{[\mathbf{M}\ddot{\mathbf{q}}-\frac{\partial}{\partial\mathbf{q}}(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}})]\delta\mathbf{q}+\mathbf{K}_1\mathbf{q}_f\delta\mathbf{q}_f-(\boldsymbol{\rho}-\mathbf{R})\mathbf{AN}\delta\mathbf{q}_f$$
$$-(\mathbf{P}+\boldsymbol{\rho}-\mathbf{R})\delta\mathbf{r}_{o'}-(\mathbf{M}_{st}-\mathbf{M}_d)\delta\boldsymbol{\pi}-(\boldsymbol{\rho}-\mathbf{R})\mathbf{A}\tilde{\mathbf{u}}'\boldsymbol{\omega}'\delta\boldsymbol{\pi}\}dt=0 \quad (16)$$

Finally, dynamic equation can be obtained:

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{F}+\mathbf{Q} \quad (17)$$

where, stiffness matrix $\mathbf{K}=\begin{bmatrix}\mathbf{0}_{6\times6} & \\ & \mathbf{K}_1\end{bmatrix}$, external active force $\mathbf{F}=\begin{bmatrix}\mathbf{P}+\boldsymbol{\rho}-\mathbf{R} \\ \mathbf{M}_{st}-\mathbf{M}_d+(\boldsymbol{\rho}-\mathbf{R})\mathbf{A}\tilde{\mathbf{u}}'\boldsymbol{\omega} \\ (\boldsymbol{\rho}-\mathbf{R})\mathbf{AN}\end{bmatrix}$, $\mathbf{Q}=\frac{\partial}{\partial\mathbf{q}}(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}})$ is generalized force related to quadratic term.

**Dynamics simulation**

Simulative ballistic conditions: assuming the takeoff weight of rocket is 5300.0 kg, post-boost mass is 1060.0 kg, engine working time is 50s, the fuel consumption during the engine working is uniform. theoretical length of the whole projectile shell is 9.1 m, the largest cross-sectional area of projectile body is 0.6082 $m^2$, the distance from pressure center of jet vane to theoretical of sharp point is 9.2 m.

Because of the displacement field w1, w2 and w3 is in the conjoined coordinate system, modal functions which meet the static boundary can be used to the discrete displacement field, and the boundary conditions can be processing as simply supported at both ends.

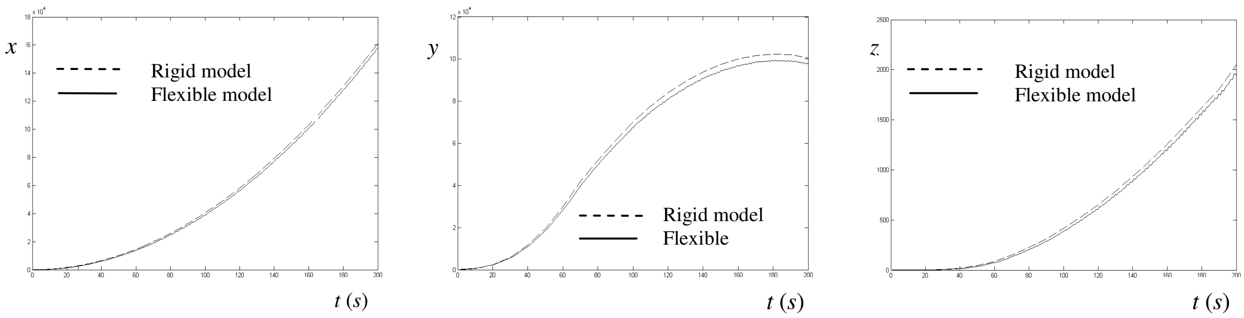
For easy calculating, first-order modal is selected as:

$$\varphi_1 = \cos\frac{\pi x}{l}, \quad \varphi_2 = \varphi_3 = \sin\frac{\pi x}{l}$$

Figure 2, 3 and 4 are simulation results of ballistic trajectory using traditional rigid model and flexible model which elastic effects are considered. The simulation time is 200s, it can be seen from the calculation results that the elastic effects have obvious effects on ballistic trajectory.

From the figure it can be known that in addition to the rigid body displacement, there is also elastic displacement coupled in the ballistic trajectory. Due to large range of movement and deformation coupling each other in the flexible model, so there are high frequencies wave overlapped in elastic deformation of the ballistic. As part of the energy is converted into elastic potential energy, therefore, compared with the rigid body model, range and height of the flexible model are smaller.

In rigid body model, the rocket reach the maximum height of 102.2Km at the 180th s, but in flexible model, the maximum height is only 100.02Km, the range in rigid body model is 166.4 Km at the 200 s, but it is only 156.2 Km in flexible model.




(a) (b) (c)

Fig.2. The relationship between flight time $t$ and 3 coordinates about centroid of projectile body (a) $t$ and coordinate of $x$ axis, (b) $t$ and coordinate of $y$ axis, (c) $t$ and coordinate of $z$ axis.

## Conclusion

Starting from continuum theory in this paper, coupling deformation is considered when calculating longitudinal deformation displacement, the vibration modal is used to describe the deformation field, Hamilton principle is used to derive the rocket flight dynamics equations considering the effect of elastic,

Numerical simulation results show that for a large range motion rockets, the space flight trajectory considering the elastic deformation coupling with rigid motion is different from the result of traditional rigid body model, the basic reasons for this difference is caused by the additional items related to the elastic deformation. Methods introduced in this paper have certain assistance to carry out high-precision trajectory analysis.

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