Regularity lemma basics (bibliography)

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This is a brief account of bibliography used for the three lectures on regularity lemma at CWI. Whenever possible, I tried to point out interesting papers for further reading. For an account of recent developments see surveys of applications [24, 23] and papers of Rödl et al. [31], Gowers [14], Tao [44], and Lovász and Szegedy [26, 27].

1. The introduction (motivation by van der Waerden Theorem and Erdős – Turán conjecture) was inspired by initial two chapters of 'Ramsey Theory' [15]. The Erdős – Turán conjecture ($r_k(n) = o(n)$, where $r_k(n)$ is the size of a largest subset of $\{1, \dots, n\}$ without an arithmetic progression of length k) appeared in [8]. Roth proved this conjecture for $k = 3$ in [36] and in a stronger form in [37]. Szemerédi's proof of the case $k = 4$ is in [39] and the full conjecture is proved in [40][1].

2. Basic properties of ε-regular pairs are discussed in many papers dealing with applications of the regularity lemma (see, e.g., [5, 24]). For some advanced ones, such as the existence of long paths in an ε-regular pair, see, e.g., [28]. The proof of the regularity lemma was adopted from Diestel's book [4] (The original proof [41] is equally good, some technicalities are treated differently.).

3. Applications: Ruzsa-Szemerédi's $(6, 3)$-Theorem[2] is from [38]. They also observed that it implied $r_3(n) = o(n)$. Erdős, Frankl, Rödl [6]

[1] Tao wrote a very nice, expository paper [42] about this proof. I would suggest to read it before attempting to read [40]. Anybody interested in this part of combinatorial number theory should also check out Tao's web-page and his expository and research papers.

[2] If G is an n-vertex 3-uniform hypergraph containing no 6 points with at least 3 triples, then $e(G) = o(n^2)$.

showed some extensions of this result, including the so-called 'removal lemma' for graphs[3].

The embedding lemma[4] appears, again, in various forms in many papers, but nice proofs are in [4, 23]. Examples of its applications include Erdős-Stone Theorem (original [7], proof with regularity lemma [4], proof without regularity lemma [25]) and linearity of Ramsey number for graphs with bounded degrees (original in [3], see also [4, 23]). In order to find large graphs as subgraphs, one needs more complicated embedding argument, such as Blow-up lemma of Komlos et al. [19] (see [20] for an algorithmic version and Rödl, Ruciński and Wagner [32, 33] for different proofs). A nice (but complicated) application of the Blow-up lemma is the proof of Pósa-Seymour conjecture [22, 21].

4. The first algorithmic version of the regularity lemma was proved by Duke et al. [5]. Check Alon et al. [1] for a slightly faster algorithm, Frieze and Kannan [11] for a simpler one, and Kohayakawa et al. [18] for optimal one. Do not forget to see [2, 10] for some nice applications and further references.

5. The original regularity lemma says nothing about sparse graphs (i.e., about n-vertex graphs with $o(n^2)$ edges). Kohayakawa [16] and Rödl (unpublished) found a regularity lemma for sparse graphs. Its main disadvantage is that it is hard to find a corresponding embedding lemma. For applications, further references, and state-of-the-art see [17, 12].

6. Frankl, Rödl [9] observed that a 'hypergraph version of the removal lemma' implies Erdős-Turán conjecture. In the same paper they proved both the regularity lemma and counting lemma for 3-uniform hypergraphs (see also [29]). This became a motivation to extend their result to k-uniform hypergraphs. This was done in [34, 30]) and the 'hypergraph removal lemma' was proved in [35]. There were other approaches, using different methods, with the same consequences: by

[3] If G is an n-vertex graph not containing a given fixed graph H, then it is possible to remove $o(n^2)$ edges from G and obtain a subgraph with no $K_{\chi(H)}$.

[4] The simples form is as follows: Suppose that G is an ℓ-partite graph with ℓ-partition $V_1 \cup \cdots \cup V_\ell$ such that all pairs (V_i, V_j) are ε-regular and with edge density at least d. If $\varepsilon \ll d$ and all V_i's are large, then G must contain K_ℓ as a subgraph. Modifications of this lemma ([4, 23]) allows to find subgraphs whose size is allowed to grow with the number of vertices of G.

Gowers [14, 13], and by Tao [43, 44] (see also papers by Lovász and Szegedy [26, 27]).

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