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595. A COMBINATORIAL IDENTITY OF L. N. ĐORĐEVIĆ*

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1. L. N. ĐORĐEVIĆ [1] has proved the following identity:

$$(1.1)\qquad \sum_{m_1=0}^{m}\sum_{m_2=0}^{m_1}\cdots\sum_{m_n=0}^{m_{n-1}}(-1)^{m_1}(m-m_1,\ m_1-m_2,\ \ldots,\ m_{n-1}-m_n,\ m_n)$$
$$\times\frac{(p_1)_{m_1-m_2}\cdots(p_{n-1})_{m_{n-1}-m_n}(p_n)_{m_n}}{(p_1+\cdots+p_n)_{m_1+1}}=\frac{m!}{(p_1+\cdots+p_n)_{m+1}},$$

where the p_i are real numbers satisfying $p_i>-1$ and

$$(1.2)\qquad (m-m_1,\ m_1-m_2,\ \ldots,\ m_{n-1}-m_n,\ m_n)=\frac{m!}{(m-m_1)!\,(m_1-m_2)!\cdots(m_{n-1}-m_n)!\,m_n!}.$$

The proof employs multiple integration over a certain infinite region in R_{n+1}.

In the present note we prove a slight generalization of (1.1) by a more elementary method, namely

$$(1.3)\qquad \sum_{m_1=0}^{m}\sum_{m_2=0}^{m_1}\cdots\sum_{m_n=0}^{m_{n-1}}(-1)^{m_1}(m-m_1,\ m_1-m_2,\ \ldots,\ m_{n-1}-m_n,\ m_n)$$
$$\times\frac{(p_1)_{m_1-m_2}\cdots(p_{n-1})_{m_{n-1}-m_n}(p_n)_{m_n}}{(\lambda)_{m_1+1}}=\frac{(\lambda-p_1-\cdots-p_n+1)_m}{(\lambda)_{m+1}},$$

where λ is not equal to zero or a negative integer but is otherwise arbitrary.

2. Put $m=k_0+m_1$ and

$$\begin{cases} m_1=k_1+k_2+\cdots+k_{n-1}+k_n\\ m_2=\qquad k_2+\cdots+k_{n-1}+k_n\\ \quad\vdots\\ m_{n-1}=\qquad\qquad k_{n-1}+k_n\\ m_n=\qquad\qquad\qquad k_n,\end{cases}$$

Then (1.3) becomes

$$(2.1)\qquad \sum_{k_0+k_1+\cdots+k_n=m}(-1)^{k_1+\cdots+k_n}(k_0,\ k_1,\ \ldots,\ k_{n-1},\ k_n)\cdot\frac{(p_1)_{k_1}(p_2)_{k_2}\cdots(p_n)_{k_n}}{(\lambda)_{k_1+\cdots+k_n+1}}$$
$$=\frac{(\lambda-p_1-\cdots-p_n+1)_m}{(\lambda)_{m+1}}.$$

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For brevity denote the left hand side of (2.1) by $A_n(m)$. Then

$$(2.2)\qquad \sum_{m=0}^{\infty} \frac{(\lambda)_{m+1}}{m!} A_n(m)\, x^m$$

$$= \sum_{m=0}^{\infty} x^m \sum_{k_0+k_1+\cdots+k_n=m} (-1)^{k_1+\cdots+k_n} \frac{(p_1)_{k_1}\cdots(p_n)_{k_n}}{k_1!\cdots k_n!} \cdot \frac{(\lambda+k_1+\cdots+k_n+1)_{k_0}}{k_0!}$$

$$= \sum_{k_0,k_1,\ldots,k_n=0}^{\infty} (-1)^{k_1+\cdots+k_n} \frac{(p_1)_{k_1}\cdots(p_n)_{k_n}}{k_1!\cdots k_n!} \cdot \frac{(\lambda+k_1+\cdots+k_n+1)_{k_0}}{k_0!} x^{k_0+k_1+\cdots+k_n}$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} (-1)^{k_1+\cdots+k_n} \frac{(p_1)_{k_1}\cdots(p_n)_{k_n}}{k_1!\cdots k_n!} x^{k_1+\cdots+k_n} \cdot (1-x)^{-\lambda-k_1-\cdots-k_n-1}$$

$$= \sum_{s=0}^{\infty} (-1)^s x^s (1-x)^{-\lambda-s-1} \sum_{k_1+\cdots+k_n=s} \frac{(p_1)_{k_1}\cdots(p_n)_{k_n}}{k_1!\cdots k_n!}.$$

By the multinomial theorem, the inner sum is equal to $\frac{(p_1+\cdots+p_n)_s}{s!}$, so that (2.2) becomes

$$\sum_{s=0}^{\infty} (-1)^s \frac{(p_1+\cdots+p_n)_s}{s!} x^s (1-x)^{-\lambda-s-1}$$

$$= (1-x)^{-\lambda-1}\left(1+\frac{x}{1-x}\right)^{-p_1-\cdots-p_n}$$

$$= (1-x)^{-(\lambda-p_1-\cdots-p_n+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(\lambda-p_1-\cdots-p_n+1)_m}{m!} x^m.$$

Thus we have proved that

$$\sum_{m=0}^{\infty} \frac{(\lambda)_{m+1}}{m!} A_n(m)\, x^m = \sum_{m=0}^{\infty} \frac{(\lambda-p_1-\cdots-p_n+1)_m}{m!} x^m$$

and therefore

$$(2.3)\qquad A_n(m) = \frac{(\lambda-p_1-\cdots-p_n+1)_m}{(\lambda)_{m+1}}.$$

This evidently proves (1.3).

3. Multiplying both sides of (2.1) by $(\lambda)_{m+1}$ we get the identity

$$(3.1)\qquad \sum_{k_0+k_1+\cdots+k_n=m} (-1)^{k_1+\cdots+k_n} (k_0, k_1, \ldots, k_n)\,(p_1)_{k_1}\cdots(p_n)_{k_n}$$

$$\times(\lambda+k_1+\cdots+k_n+1)_{k_0} = (\lambda-p_1-\cdots-p_n+1)_m,$$

which holds for all λ. In particular, for $\lambda=0$, (3.1) becomes

$$(3.2)\qquad \sum_{k_0+k_1+\cdots+k_n=m} (-1)^{k_1+\cdots+k_n}(k_0,\ k_1,\ \ldots,\ k_n)\frac{(p_1)_{k_1}\cdots(p_n)_{k_n}}{(k_1+\cdots+k_n)!}$$

$$=\frac{(-p_1-\cdots-p_n+1)_m}{m!}.$$

It is not difficult to give a direct proof of (3.2)

REFERENCE

1. L. N. Đorđević: *Proof of a combinatorial identity using multiple integration.* Same Publications, pp. 63—64.

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