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ABSTRACT

A quasi-systematic strategy of devising rule sets for problem solving is applied to ruler and compass geometrical constructions. "Lower order" rules consisting of basic skills and "higher order" rules which govern the selection and combination of lower order rules are identified by an analysis of problem types; three types of construction problems are used to generate three specific rule sets. A second level of "higher order" rules, determining how various aspects of the individual rule sets can be combined, results in generalized rule set which describes solutions to a wide-range of instruction problems. This model seems intuitively to reflect the kinds of relevant knowledge that might be applied by successful problem-solvers. The results are suggestive of how the construction of at least certain artificial intelligence systems might be systematized. The results also identify the knowledge underlying reasonably complex kinds of problem solving which could be used as a basis for explicit instruction. (JP)

STRUCTURAL LEARNING SERIES

UNIVERSITY of PENNSYLVANIA
Philadelphia

HIGHER ORDER CHARACTERIZATION OF HEURISTICS FOR COMPASS AND STRAIGHT EDGE CONSTRUCTIONS IN GEOMETRY

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INTRODUCTION

According to the eminent mathematician George Polya (1962), perhaps the greatest value to be gained from the study of mathematics is the ability to solve problems. In spite of its importance, however, very little is known about how to teach people to solve problems. Indeed, one of the greatest mysteries of our time is why some people cannot solve problems. Specifically, why is it that some problem solvers succeed on problems for which they have all of the necessary component skills whereas others fail.

The answer, clearly, is that the former either know something which the others do not, or that they are in some way inherently superior individuals. If the latter is true, of course, there is little that educators can expect to do to overcome the problem, short of genetic regeneration. If one can identify missing knowledge (competence) as the source of trouble, however, there may be a great deal that can be done.

Although it is likely that both factors enter into problem solving ability to some extent, it is the implicit belief of many that problem solving is subject to training. Polya, for one, has believed in this possibility sufficiently to devote at least five books and numerous articles to the subject (cf. Polya, 1962). Indeed, his discussions of the role of heuristics in problem solving have had a great influence on many mathematics educators and computer scientists in the area of artificial intelligence.

By a "heuristic " Polya means a rule, technique, or method of attack which, while not guaranteed to work, leads to success sufficiently often as to warrant its use. "Working backward from the unknown," "the pattern of two loci," and "the Cartesian pattern" are three heuristics about which Polya has written.

In spite of the broad acclaim for Polya's work generally, and the intrinsic support for his notion of heuristics specifically, it has been difficult to capitalize on these ideas as fully as might be desired. Although

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sometimes useful, heuristics are little more than general hints, and leave much to be desired insofar as pinpointing what a subject must know in order to solve specific kinds of problems. Furthermore, in order to lend themselves to technological treatment, whether in computer assisted instruction or in artificial intelligence, heuristics must be transformed or encorporated into strictly mechanical procedures. Since there are any number of ways of doing this for any given heuristic, heuristics themselves provide only a general point of view or way of looking at problems, and are not prescriptive.

The highly diverse literature on artificial intelligence illustrates one role heuristics may play in problem solving. In this case, the aim is to come up with a program or set of programs which enable the computer to solve a given class of presumably complex problems. Heuristics such as means-ends analysis (Simon & Newell, in Scandura, 1973) and resolution (Nilsson, 1971) are built directly into the programs. Specifically, a rule (program, or set of programs) must be detailed, and where more than one program is involved, mechanisms must also be built into the machine which determine how the rules are to interact.[1] In general, neither the rules nor the mechanisms need reflect human behavior. Even in computer simulation of human behavior, there is no guarantee, just because computer outputs correspond roughly to human behavior, that the underlying procedures are the same.

Recent attempts to make artificial intelligence systems less mechanistic have centered on semantics (cf. Winston, 1972); that is, the construction of syntactic procedures constrained to reflect semantic reality. This is an important step forward, as the rules people use almost certainly reflect semantics. But, this does not address itself to the equally basic question of how the rules are to interact. Further, different people may and frequently do deal with the same problems in quite different ways. Machine intelligence is far from being able to deal with individual differences.

About all one can say with confidence about current artificial intelligence systems, then, is that they are highly precise. It would be dangerous to make inferences concerning human behavior about either the specific programs and heuristics used, or the mechanisms which determine how these programs interact. In logical reasoning, for example, there is little reason to suspect that

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1. Some computer scientists make no conceptual distinction among programs, mechanisms, and procedures.

human beings make use of resolution methods even though they are involved in many artificial intelligence systems (cf. Nilsson, 1971).

A THEORY OF PROBLEM SOLVING

Scandura's (1971, 1973) theory of structural learning provides a potential basis for increasing the closeness of fit between human behavior and sets of rules (rule sets). The theory spells out a mechanism which governs the way in which rules interact in certain kinds of human problem solving. This mechanism has been tested empirically under what have been called memory-free conditions. A basic tenet of this idealized (memory-free) theory is that problem solving ability can be traced to the presence or absence of higher order capabilities (higher order rules) which make it possible to combine the constituent parts (component rules) of problem solutions into coherent wholes which are adequate for solving the problems. Specifically, according to the theory, successful problem solving frequently requires higher order rules to combine the individual component rules. Further, these higher order rules are assumed to operate not just on individual (component) rules, but on classes of such rules.

Consider the following simple experiment which Scandura and Ackler (cf. Scandura, 1973, Chapter 7) recently conducted with elementary school children. At the beginning of the experiment, each child was taught a number of specific rules for trading objects. One such rule involved trading n + 2 candy bars for n balloons. Once having learned such a rule, a child was able to give the appropriate number of candy bars in return for any given number of balloons. (The numbers were small enough, n + 1, 2, ..., 5, so that the children had no difficulty with addition.) Certain pairs of these rules were such that the outputs of one could serve as inputs for the other, although the child was not told this explicitly.

The crucial test came after the child was taught one such pair (e.g., trading n + 3 pencils for n candy bars and trading n + 2 candy bars for n balloons). Could the child solve a problem (e.g., trading pencils for balloons) which required for its solution the corresponding composite rule (i.e., first performing the latter rule and then the former)? Six of the 31 subjects tested were able to solve this composite problem without any explicit instruction. (One subject failed to learn to interpret certain of the simpler rules and was therefore not considered in the analysis.)

According to the theory, these subjects entered the situation with an appropriate higher order capability. The problem was to identify that capability. Analysis of the task led to the hypothesis that given suitable pairs of component rules, the ability to form corresponding composite rules would provide a sufficient basis for solving such problems.

With this in mind, the remaining 24 subjects were randomly split into two groups. One of the groups was trained on the higher order rule. That is, they were shown which kinds of rules could be composed and how to compose them. Then, all of the subjects were presented with two completely new rules which could be composed. Finally, the subjects were tested on the corresponding composite task. It is important to note that the composite task was completely new to the subjects. As had been predicted, all but one of the subjects[2] who had been trained on the higher order rule were able to solve the transfer problem, whereas none of those who were not given this training succeeded.

According to Scandura's (1971, 1973) theory, these results can be explained in a simple way. The theory rests on the fundamental assumption that in problem solving people are attempting to achieve some goal. The basic hypotheses that govern the way in which available rules are put to use are as follows: (A) If a subject has a rule available which satisfies a given goal, then he will apply it. (As trivial as it sounds, this hypothesis is an assumption. It does not logically follow that just because a subject has a rule available for achieving a given goal that he will use it.) (B) If a subject does not have a rule available for achieving a given goal, then control automatically shifts to the higher order goal of deriving a procedure which will satisfy the original goal. (C) If a higher order goal has been satisfied, control reverts back to the previous goal. (When we say that a higher order goal has been satisfied, we mean that some new rule has been derived which contains the stimulus situation in its domain and whose outputs satisfy the original goal criterion.) The third hypothesis allows control to revert back to lower order goals once a higher order goal has been satisfied.[3]

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2. There was reason to believe that the discrepent subject had not actually learned the higher order rule. He had required an inordinate amount of help from the experimenter in order to reach criterion. He was therefore run through the experiment again a week later and this time he performed as expected.

3. For a more general and rigorously formulated set of hypotheses, see Scandura (1973, Chapter 7).

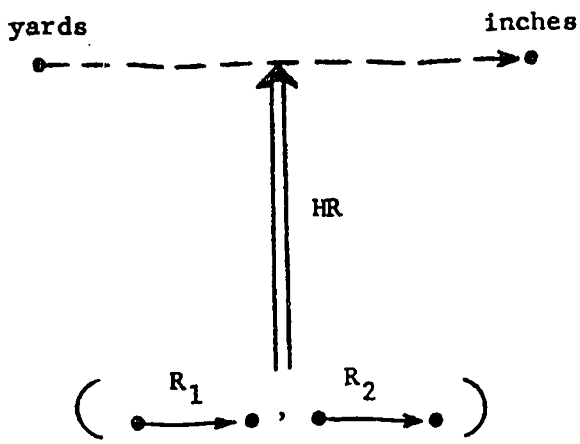
Putting all this together, we see that if an appropriate higher order rule is available when control shifts to a higher order goal, then the higher order rule will be applied and control will automatically revert to the original goal. The subject will then apply the newly derived rule and solve the problem. If the subject does not have a higher order rule available for deriving a procedure that works, then control is presumed to move to still higher levels (e.g., deriving a rule for deriving a rule that works). Although this process is assumed to go on indefinitely in the idealized theory, memory places strict limits on actual behavior.

These assumptions provide an adequate basis for generating predictions in a wide variety of problem solving situations. Consider the problem of converting a given number of yards into inches. There are two possible ways in which a subject might solve the problem. The first is to simply know, and have available, a rule for converting yards directly into inches: "Multiply the number of yards by 36." In this case, the subject need only apply the rule according to hypothesis (A). The other way is more interesting, and involves all of the mechanisms described above. Here, we assume that the subject has mastered one rule for converting yards into feet, and another for converting feet into inches. The subject is also assumed to have mastered the higher order composition rule above.

In the second situation the subject does not have an applicable rule which is immediately available, and, hence, according to hypothesis (B), he automatically adopts the higher order goal of deriving such a procedure. Then, according to the simple performance hypothesis (A), the subject applies the higher order composition rule to the rules for converting yards into feet and feet into inches. This yields a new composite rule for converting yards into inches. Next, control reverts to the original goal by hypothesis (C) and, finally, the subject applies the newly derived composite rule by hypothesis (A) to generate the desired response. This sequence of events is depicted in Figure 1.

INSERT FIGURE 1 ABOUT HERE

Figure 1



HYPOTHESIS

- (B)
- (A) (applied to R_1 and R_2)
- (C)
- (A) (applied to composite of R_1 and R_2)

It should be emphasized that the results are in no way peculiar to higher order composition rules. The basic mechanism has also been tested using higher order generalization rules which operate on rules with restricted domains and generate new rules with more encompassing domains (Scandura, 1972, 1973). A restricted rule, for example, might generate the sums of number series of the form $1 + 3 + 5 + \ldots + 2N - 1$, whereas a generalization of that rule might generate sums for all arithmetic series.

Although the theory places very definite constraints on the way in which rules may interact in problem solving, and in particular helps to insure the behavioral relevance of any rule set (with respect to a given class of problems), the theory provides no panacea. It provides a schema for identifying rule sets which account for given classes of problems, but not the rules themselves. In devising such rules, it seems clear that semantic considerations will play an important role. The rules people use almost certainly reflect their familiarity with the world they have confronted throughout their lives. Such considerations place important constraints on the kinds of rules allowed. In effect, the theorist is obliged to make intellectual guesses concerning the particular rules that a particular subject or group of subjects is likely to use. What makes this possible, presumably, is the common culture shared by the subjects and theorist. Any ultimate test of the adequacy of a particular rule set, of course, must deal with actual behavior.[4]

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4. The theoretical foundations for such tests have been worked out and tested empirically (Scandura, 1971, 1973; Scandura & Durnin, 1971; Durnin & Scandura, 1972). The basic idea is to determine each subject's behavior potential with respect to each rule in an identified rule set, and then to use the theory as a basis for making predictions concerning performance on problems which require interactions among the rules. The closeness of fit between the predictions and observed behavior would provide a direct test of the adequacy of the rule set. A study reported in Scandura (1973) on rule generality was of this type.

PURPOSE

The present study deals with the notion of competence quite apart from human behavior, albeit competence which to the extent the theory is adequate has direct behavioral relevance. Our specific goals were: (1) to devise a quasi-systematic strategy for devising rule sets, and (2) to illustrate this strategy in the analysis of geometry construction problems involving compass and straight edge.

We first consider those problems identified by Polya (1962, Chapter 1) as being soluble via the "pattern of two loci." Then we extend the analysis to encompass constructions involving similar and auxiliary figures problems.

In this analysis, a rule set is simply a set of rules which may include lower order rules, higher order rules, or both. A rule set is said to account for a class of problems if, for each problem in the class,(1) there is a solution rule in the rule set which has the problem in its domain and whose range contains the solution to the problem, or (2) there is a higher order rule in the rule set which applies to rules in the set and generates a solution rule. (For a generalization of this idea see Scandura (1973).)

METHOD OF ANALYSIS

It seems unlikely that algorithmic methods can be found for devising non-trivial competence theories. Indeed, Chomsky has argued persuasively that no such method exists for dealing with observables as complex as language. He suggests instead the more modest goal of determining criteria for evaluating alternative rule sets. Recent work in automatic programming, on the other hand, while it is quite far at present from a satisfactory solution, is proceeding on the assumption that significant progress in this direction can be made.

In the present case, the problem of devising rule sets is made simpler in several ways. First, and most important, the type of competence theory proposed imposes important constraints on the nature of the rule sets. Second, restricting the level of analysis to that of flow diagrams, rather than computer programs, makes it natural to represent the constituent operations and decision making capabilities at whatever level seems to most adequately reflect human knowledge rather than at a level predetermined by some programming language.[5]

Although the ultimate test of the behavioral adequacy of a given rule or rule set depends on human behavior, intuitive judgments can often serve to a first approximation. This is possible in many situations because of the common culture the competence theorist usually shares with his subject. Recent work in artificial intelligence (cf., Winston, 1972), for example, shows that attention to semantics can pay handsome dividends. Programs constructed without due concern for the way people sort the environment tend to lead to programs that are overly complex, mechanistic in nature, and have an ad hoc character.

With this in mind, our method of analysis went something as follows. First, we attempted to set some reasonably explicit bounds on the class of geometry construction problems to be considered. In particular, we considered only those problems in Chapter 1 of Polya (1962).

Our next step was to classify these problems on intuitive grounds. Our aim was to place similar problems in the same categories, at least to a first approximation. We were one step up in this regard, since Polya had already done a preliminary categorization for us. All of his problems can be solved by some variant or combination of the three general methods he describes: (1) the pattern of two loci, (2) the pattern of similar figures, and (3) the pattern of auxiliary figures.

After the various tasks had been classified, we made sure that the domains and ranges of each task were fairly explicit. Then we identified explicit procedures for solving each type of task. Care was taken to insure that these procedures reflected our intuitions as to how intelligent high school students might go about solving the problems. In some cases it was possible at this point to subclassify some of the tasks.

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5. Each of our flow diagrams has a unique start but we allow any finite number of stops. Operations are represented as rectangular boxes containing a description of the operation. Decision making capabilities, that is, capabilities for deciding which of a given number of predicates is satisfied by an element, are represented as diamond shapes (or elongated hexagons) containing statements of the predicates. In our analyses, it was convenient to consider only binary decisions. In some cases for simplicity, flow diagrams are represented as directed graphs in which the nodes correspond to decision making capabilities, and the arrows to operations. For more details, the reader is referred to Scandura (1973, Chapter 2).

The most critical step was to identify parallels among the procedures in each of the various classifications, and even more important to devise higher order rules which realized these parallels as formal procedures. The higher order rules so identified, together with the component lower order rules on which they act, constituted the characterizing rule sets. As we shall see, the lower order rules were primarily components of the various solution procedures, many of which were involved in a wide variety of different tasks.

The final step was to determine the adequacy of the resulting rule set by determining whether it provided an adequate account of all of the problems in the class. Where a rule set failed to meet this criterion, appropriate modifications were made. In view of our previous comments, it is not sufficient that a rule set just "work;" it also must be compatible with human knowledge. The only really adequate way of determining whether this is the case, is to effect a behavioral test; that is, to see whether the individual rules provide an adequate basis for assessing the behavior potential of individual subjects, and thereby make it possible to predict the behavior of individual subjects on new instances (of the rules). Since this was impractical in the present study, we adopted the weaker and less rigorous criterion of requiring that the rule sets be compatible with our intuition (cf. Chomsky, 1957).

PATTERN OF TWO LOCI

Our first step was to select a broad sampling of two-loci problems and to devise procedures for solving each type. For example, consider the problem: "Given a line and a point not on the line, and a radius R, construct a circle of radius R which is tangent to the given line and which passes through the given point." This problem can be solved according to the following procedure: "Construct the locus of points at distance R from the given point; construct the locus of points at distance R from the given line; construct a circle using the intersection point of the two loci as center, and the distance R as radius." (Table 1 in appendix A lists 11 two-loci problems taken from Polya (1962), and their solution procedures.)

This solution rule clearly involves the pattern of *two* loci. In this case, as with all of the problems in Polya's first category, two loci are determined one after the other; the point of intersection of these loci in turn makes it possible to construct the goal figure. In each case the goal figure is either a circle or a triangle.

Further analysis of the class of two-loci problems, however, revealed certain differences in the ways problems are solved that could have behavioral implications. In most solution rules, like the example above, the two loci can be found independently, in either order. Furthermore, at no point is it necessary to measure a distance during the course of applying the solution rule. Some form of distance measurement, however, is required in other tasks (8 through 11 in Table 1). Some of these tasks (8, 9, and 10) require measurement in order to construct the goal figure; the solution rule for another problem (Rule 11 in Table 1) involves measurement before the second locus can be found. In still another task (10 in Table 1), one of the loci is actually given, or equivalently, can be thought of as obtained by applying an identity rule. The goal figure in still another task (7 in Table 1) is simply the point of intersection of the two loci, so we can also think of the goal figure as being constructed by an identity rule. Finally, we mention that the rule for finding the locus of vertices of an angle of a given measure subtending a given line segment (See Tasks 5-7, Table 1) is particularly complex and is probably not immediately available to most beginning geometry students.

The Basic Rule Set

As a first step in constructing a characterizing rule set, we systematically went through the various solution rules for the pattern of two-loci tasks (see Table 1, Appendix A) and identified all of the different component rules that are used either (1) in constructing one of the loci or (2) in constructing a goal figure. (The eleven lower order rules we identified as involved in the eleven two-loci tasks are shown in Table 2 of Appendix A.)

All of the lower order component rules were used in at least one solution rule. Some were used to construct a needed locus, others were involved in constructing goal figures, and some served both functions.

The higher order rule in Figure 2 below shows schematically how the various solution rules may be constructed from the component rules.

INSERT FIGURE 2 HERE

The higher order rule in Figure 2 applies to the problem (i.e., the stimulus situation, S_o) and to the goal (G) itself, as well as to the lower order component rules.[6]

First, an arbitrary representation $\langle S_1, R_1 \rangle$ analogous to the solved problem is constructed. In our illustrative task, a sketch like Figure 3 would serve this purpose.

INSERT FIGURE 3 HERE

Note that constructing such a representation is <u>not</u> the same either as solving the problem, or as constructing a solution rule for the problem. The sketch in Figure 3, for example, can easily be generated by <u>first</u> drawing an arbitrary

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6. Strictly speaking, human subjects are presented with statements of problems as stimuli. Throughout this and our subsequent analyses we assume that the subject's initial subgoal is to interpret the goal statement (i.e., determine its meaning). The second subgoal is to solve the problem. In effect, the initial goal is divided into a pair of subgoals to be achieved in order. Our analysis is limited to the second part of this task. We assume that the given problem statements can be uniformly and correctly interpreted.

Although we do not pursue the question here, we have reason to believe that forming subgoals is closely related to the question of (problem) representation (cf. Amarel, 1968).

Figure 2

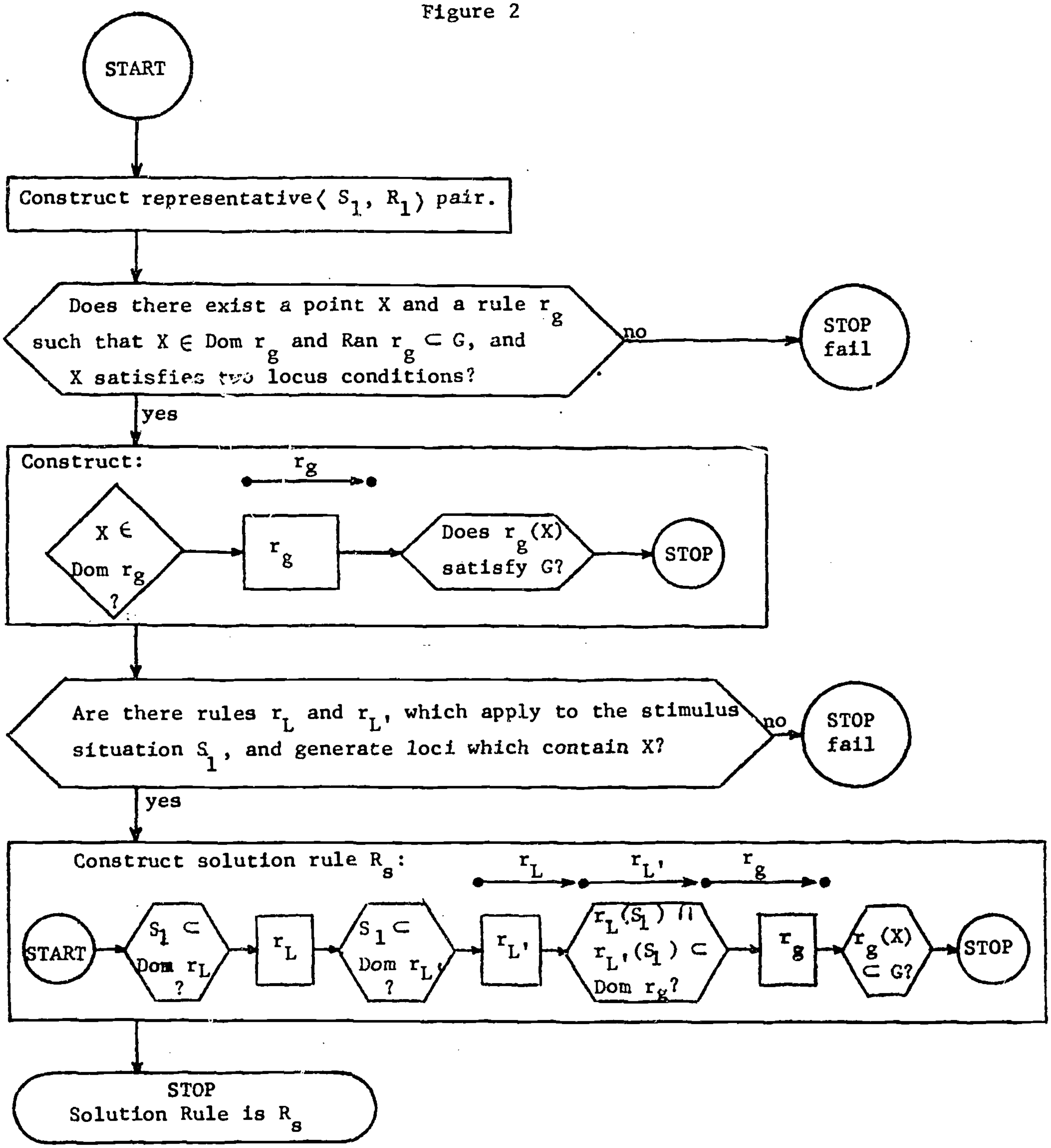
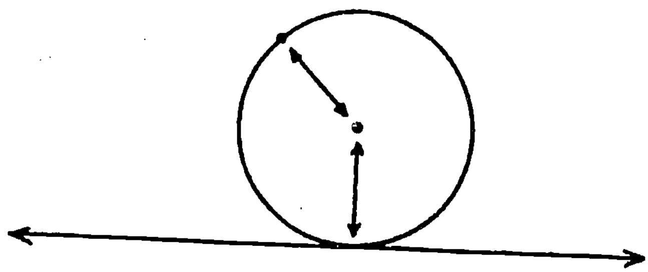

START
Construct representative ⟨ S1, R1 ⟩ pair.
Does there exist a point X and a rule rg such that X ∈ Dom rg and Ran rg ⊂ G, and X satisfies two locus conditions?
no
STOP fail
yes
Construct:
rg
X ∈ Dom rg ?
rg
Does rg(X) satisfy G?
STOP
Are there rules rL and rL' which apply to the stimulus situation S1, and generate loci which contain X?
no
STOP fail
yes
Construct solution rule Rs:
rL
rL'
rg
START
S1 ⊂ Dom rL ?
rL
S1 ⊂ Dom rL' ?
rL'
rL(S1) ∩ rL'(S1) ⊂ Dom rg?
rg
rg(X) ⊂ G?
STOP
STOP
Solution Rule is Rs

Figure 3



circle, then drawing an arbitrary line tangent to it, and placing an arbitrary point on it. More generally, an arbitrary representation (R_1) of the goal figure (R_o) is constructed first. Only then is a representation (S_1) of the information given in the stimulus situation (S_o) constructed in relation to the representation of the goal figure. In effect, the first operation in the higher order rule amounts to representing geometrically the meanings of goal situations (i.e., goals plus stimulus situations) by a "sketch," or some equivalent representation.

The second step is the question: "Is there a point in $\langle S_1, R_1 \rangle$ which satisfies two locus conditions - and, if so, is there a goal constructing rule (r_g) such that point X is contained in the domain of r_g (Dom r_g) and such that the range of r_g (Ran r_g) is contained in the goal, G?"

As shown in Scandura (1973), decision making capabilities can be characterized as partitions on a class of input situations; in the present case, each representation $\langle S_1, R_1 \rangle$ either contains a point X which satisfies two locus conditions or it does not. If it does satisfy two such conditions, then the next operation involves forming the rule consisting of (1) a decision which asks whether there is a point X in the domain of r_g which satisfies two locus conditions, (2) the rule r_g, and (3) a stopping decision which tests to determine whether the output of rule r_g (when applied to point X plus perhaps other entities) satisfies the goal G.

Next, the available rules in the lower order rule set are tested to see whether there are two of them which apply to the represented stimulus (S_1) and whether they generate loci which contain the point X. Given that such locus rules exist, the next operation constructs the solution rule R_s in which first one locus rule r_L is applied (after testing to see whether the stimulus situation is in its domain), then the other $r_{L'}$, and finally the goal construction rule r_g. (In actual applications of the higher order rule, it is assumed that the problem solver automatically tests the solution rule R_s to see if it satisfies the higher order goal condition. That is, is $S_o \in$ Dom R_s and Ran $R_s \subset G$.? If the higher goal is satisfied, control is assumed to revert to the original goal so that R_s will be applied.)

A More Rigorous Analysis

This level of description is sufficient to give one an intuitive feeling for how the higher order rule operates. But, unfortunately, the rule is ambiguous; the decision making capabilities are not well defined. As they stand, we cannot be sure, given a goal situation, that the higher order rule will generate a unique output.[7] In the first decision making capability, for example, it is not clear just what constitutes a locus condition. Similarly, in the second decision making capability the notion of a rule applying to a stimulus situation is something less than precise.

Closer perusal of the individual tasks (Table 1, Appendix A) make it possible to overcome these ambiguities. In many cases, the desired point X is a given distance from one or two given points and/or lines. In the example above (Task 1) the point X is a distance R from the given point and from the given line. As a second example, consider the task, "Given side a, the median M_a to side a, and the altitude H_a to side a, construct the triangle." In this case, the point X is a given distance (H_a) from a given line (side a) and another given distance (M_a) from a fixed point (the mid-point of side a). This suggested the following more rigorous characterization of the first decision making capability:

(1) Does there exist a point X in $\langle S_1, R_1 \rangle$ and a rule r_g such that (X, E) is contained in the domain of r_g where E is a given distance, and the range of r_g is contained in the goal (Ran $r_g \subset G$) such that X is a given distance from one or two given points and/or lines.

Similarly analysis suggested reformulating the second decision making capability as:

(2) Is there a rule r_L such that a pair consisting of given points, lines, and/or distances in S_1 is in the domain of r_L (Dom r_L) and such that X is a member of L (i.e., a point on L) where L is contained in the range of r_L ($X \in L \in$ Ran r_L)?

A similar characterization is required for $r_{L'}$.

A higher order rule incorporating these refinements can be used to generate solution rules for many two-loci problems (e.g., Tasks 1-4 in Table 1). For example, consider the problem mentioned earlier, "Given a line, a point

7. Many two-loci problems, of course, do not have a unique solution. Correspondingly, higher order rules could be required to generate more than one solution rule. There is no motivation for doing so here, however, so we do not consider the possibility further.

not on the line, and a radius R, find a circle having the given radius R, which is tangent to the line, and which passes through the point." There is certainly a point X in the representation $\langle S_1, R_1 \rangle$ which is at the given distance R from a given point and from a given line in S_1.

INSERT FIGURE 4 ABOUT HERE

It is also true that there is an r_g rule in the rule set (r_C, Table 2, Appendix A) which applies to the pair consisting of the point X and the given distance, and whose range consists of circles and is thereby contained in the goal.

Unfortunately, as it stands, the modified higher order rule does not provide an adequate means for characterizing solution rules for other two-loci tasks. In certain tasks (e.g., 8 and 9 in Table 1, Appendix A), for example, no distance is given. The important requirement in such cases is often that the point X be equidistant from a given pair of elements, points and/or lines, in two different instances (i.e., for two given pairs of elements). Thus, in the task, "Inscribe a circle in a given triangle," the desired point X is equidistant simultaneously from two different pairs of sides of the triangle, or

INSERT FIGURE 5 ABOUT HERE

equivalently, the point X is equidistant from the three sides. In the task, "Circumscribe a circle about a given triangle," the point is equidistant in regard to two pairs of points.

INSERT FIGURE 6 ABOUT HERE

To take these possibilities into account, the first part of decision making capability (1) must be modified to allow pairs consisting of the point X and another element E, where E may be either a point or a distance. For example, in the "circumscribed circle" task, any one of the vertices may serve as point E. The goal construction rule, r_g, involves determining the distance between the two given points (X and E) and applying the circle construction rule with the point X as center and the obtained distance as the radius. Sinc

Figure 4

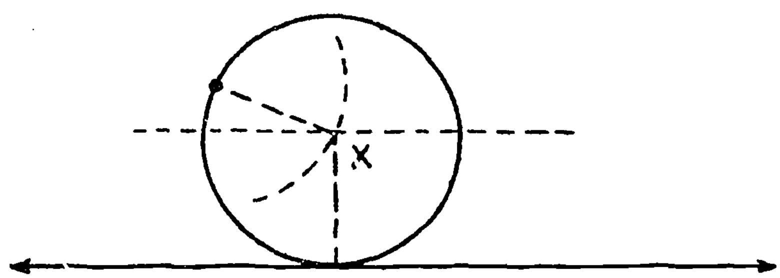


Figure 5

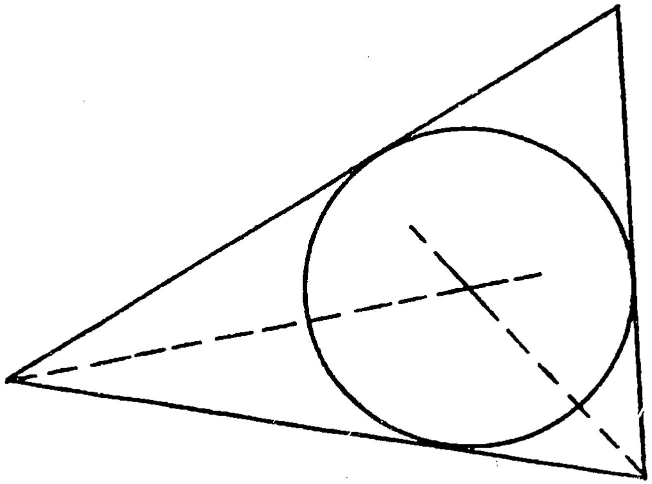
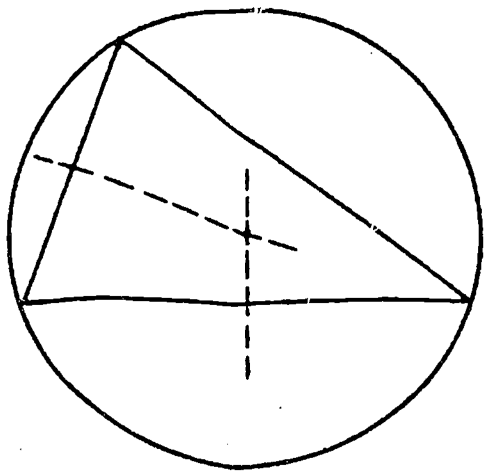


Figure 6



the point E in the "inscribed circle" task must be determined as part of the goal construction rule, we see that E may not be given initially (in S_o) but may have to be specified as part of r_g.

This modified higher order rule is generally more adequate, but still breaks down in certain situations. In the problem, "Given three intersecting lines, not all intersecting at a common point, construct a circle which is tangent to two of the lines and whose center is on the third," we have a situation where one of the loci, the line containing the point X, is already given.

INSERT FIGURE 7 ABOUT HERE

Here, the initial decision making capability (1) must be generalized slightly so that the desired point X may be a given distance from one point or line, or equidistant from a pair of points or lines, and another given distance from another point or line, or equidistant from another pair. This leads to the following reformulation of decision making capability (1): (1') Does there exist a point X in $\langle S_1, R_1\rangle$ and a rule r_g such that $(X, E) \in \text{Dom } r_g$ where E is a point or distance, and $\text{Ran } r_g \subset G$, and X satisfies two specific conditions (or the same condition twice, applied to different elements of $\langle S_1, R_1\rangle$ of types:

X is a given distance from a given point or line, and/or

X is equidistant from a given pair of points or lines?

A higher order rule incorporating these modifications appears in Figure 8.

INSERT FIGURE 8 ABOUT HERE

Unfortunately, even this reformulation is not adequate with regard to still other tasks, specifically tasks that involve the (lower order) rule for constructing the locus of vertices of an angle of given measure subtending a given line segment (cf. rule r_{AV}, Table 2, Appendix A). The task, "Given side a of a triangle, the median M_a, and the measure of angle A opposite side a, construct the triangle," is of this type. One locus, in this case, is an arc but the points on it are not at a fixed distance from any point on the given segment. Nor are the points of the locus equidistant from any two particular points on the line segment. Hence, in our final reformulation of the two-locus higher

Figure 7

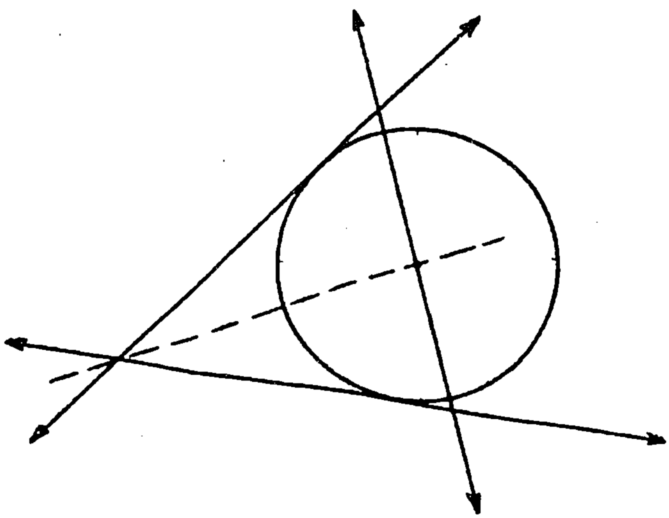
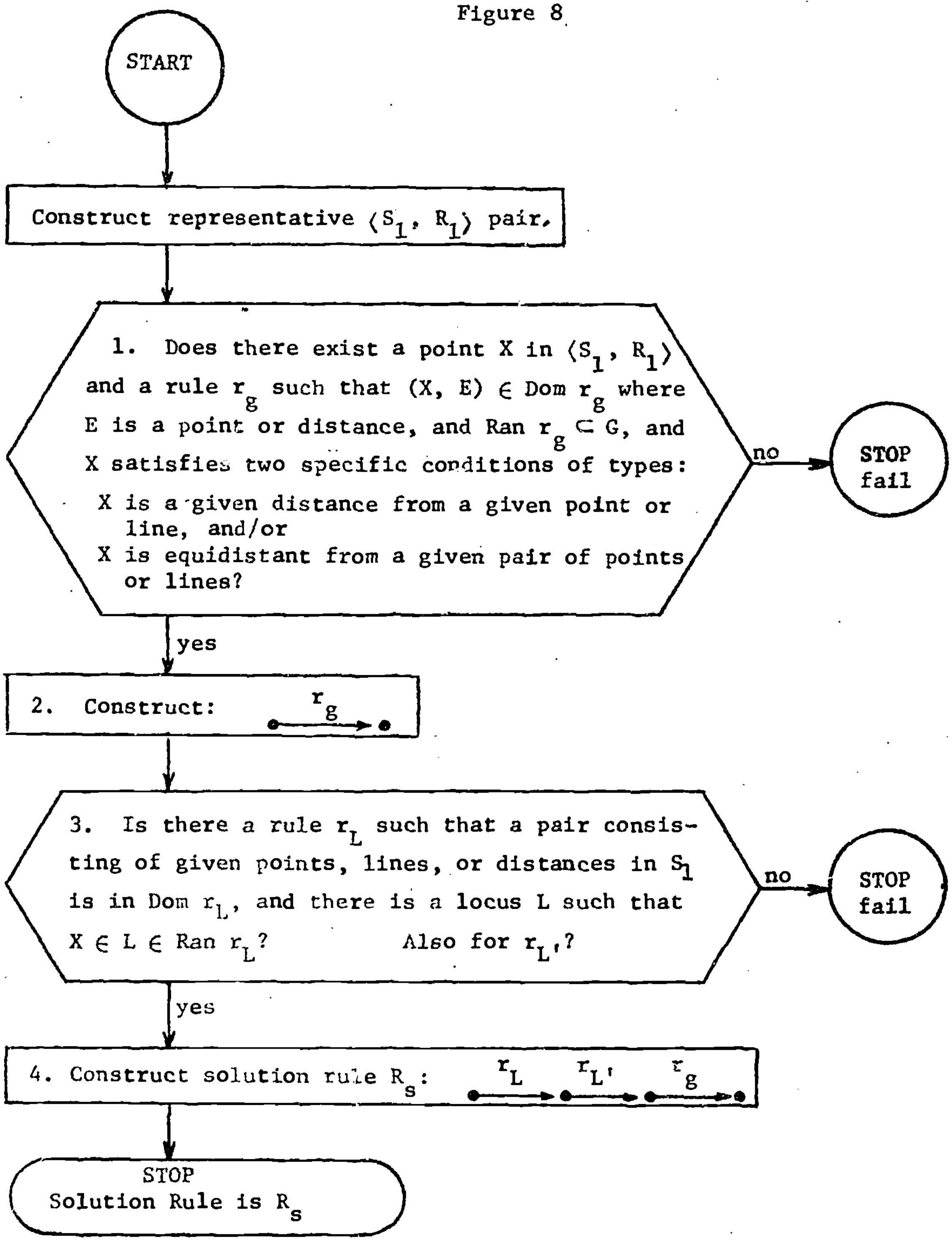


Figure 8


START
Construct representative ⟨S1, R1⟩ pair.
1. Does there exist a point X in ⟨S1, R1⟩ and a rule rg such that (X, E) ∈ Dom rg where E is a point or distance, and Ran rg ⊂ G, and X satisfies two specific conditions of types:
X is a given distance from a given point or line, and/or
X is equidistant from a given pair of points or lines?
no
STOP fail
yes
2. Construct:
rg
3. Is there a rule rL such that a pair consisting of given points, lines, or distances in S1 is in Dom rL, and there is a locus L such that X ∈ L ∈ Ran rL? Also for rL'?
no
STOP fail
yes
4. Construct solution rule Rs:
rL
rL'
rg
STOP
Solution Rule is Rs

order rule, the decision making capability is generalized so that the point X may serve as a vertex of an angle of given measure whose sides subtend (i.e., pass through the end points of) a given segment. Decision making capability (3) was also enriched so that pairs consisting of angle measures and segments could be in the domain of a locus rule.

With this modification, the rule set handled almost all of the pattern of two loci tasks we had sampled (Table 1, Appendix A). We ran into difficulty, however, with another task: "Given two parallel lines and a point between them, construct a circle which is tangent to the two lines and passes through the point." In this case, there is certainly a point X which is the same distance

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INSERT FIGURE 9 ABOUT HERE

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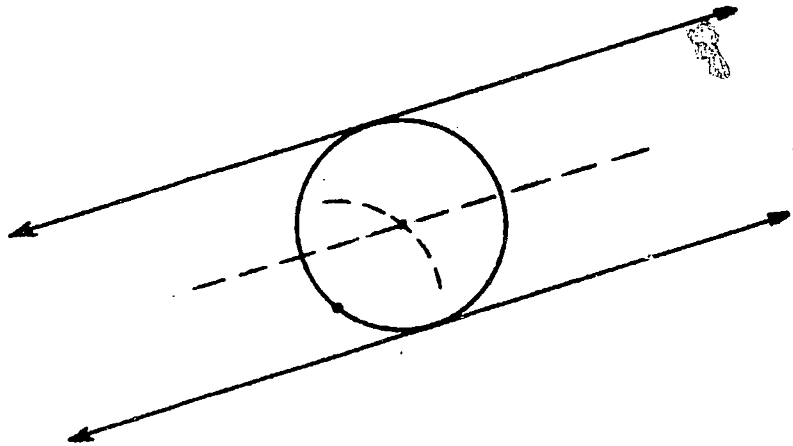
from a given point and from a given line so there is no problem there. Similarly, there is a point E such that (X, E) is in the domain of some r_g. The difficulty comes when we get to the second decision making capability (3). There is a pair of lines in the domain of one of the locus rules - the one which constructs the locus of points equidistant from the two given parallel lines. The second locus rule, however, requires that we first measure a distance between two parallel lines, one of which is not present in the stimulus S_o until after the first locus rule is applied. That is, we need to determine the distance between one of the parallel lines and the locus of points equidistant from the two given parallel lines. This distance serves as the desired radius.

Application of the higher order rule in this case (Task 11, Appendix A) results in failure at decision making capability (3). Fortunately, it is easy to modify the higher order rule to take this possibility into account. Furthermore, as we shall see, this modification serves an important purpose in dealing with the larger class of construction problems solvable either by the pattern of two loci or by the pattern of similar figures.

Instead of stopping when the second decision fails, we simply add another group of tests (A-C). (A) and (B) duplicate (1) and (2) except that X must satisfy only one specific condition. (C) asks: "Is there one rule in the rule set such that a pair of given points and/or lines is in the domain of that rule and is there a locus L such that the point X is part of L and L is contained in the range of r_L?" If the answer to this is no, we stop, but if the answer is

Figure 9



yes, we can ask whether there is another locus rule $r_{L'}$ such that the represented stimulus situation S_1, together with the preceding locus $r_L(S_1)$, contains a pair of given points and/or lines that are in the domain of $r_{L'}$.

A revised higher order rule which incorporates this addition is shown in Figure 10.

INSERT FIGURE 10 ABOUT HERE

The Importance of Lower Order Rules

The rule set consisting of this higher order rule and the lower order rules, provides an adequate account of all of the pattern of two loci problems sampled, as well as others. For example, consider Task A: "Given sides a, b, and c of a triangle, construct the triangle." In this case, application of the

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INSERT FIGURE 11 ABOUT HERE

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higher order rule generates the solution rule. This solution rule involves: (1) application of rule r_C, "Construct the locus of points at a given distance from a given point," to the end point of one line segment using another side as distance, followed by (2) another application of rule r_C to the other end point using the remaining side as radius. Then, of course, we apply rule r_T, "From a point not on a given segment, draw segments to the endpoints of the given segment" to the intersection of these two loci to obtain the desired goal figure.

Task B, "Given two intersecting lines and a point of tangency on one of the lines, construct a circle which is tangent to the two lines and which passes through the given point of tangency," also involves the pattern of two loci.

INSERT FIGURE 12 ABOUT HERE

Application of one lower order rule (r_{AB}, Table 2, Appendix A) constructs one locus, the angle bisector. The second locus is obtained by constructing a perpendicular bisector to one line at the given point of tangency. This is

Figure 10

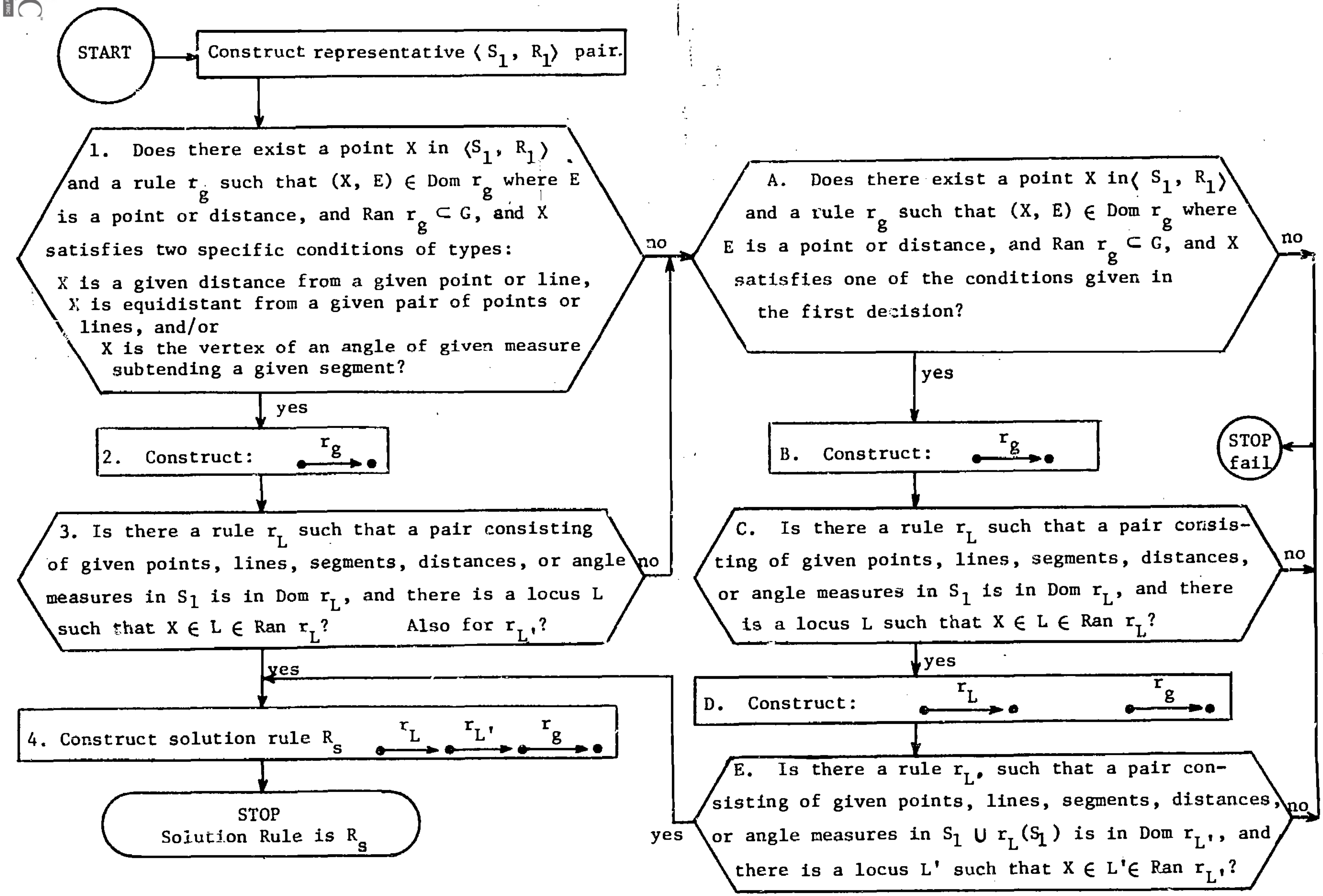

START
Construct representative ⟨S1, R1⟩ pair.
1. Does there exist a point X in ⟨S1, R1⟩ and a rule rg such that (X, E) ∈ Dom rg where E is a point or distance, and Ran rg ⊂ G, and X satisfies two specific conditions of types:
X is a given distance from a given point or line,
X is equidistant from a given pair of points or lines, and/or
X is the vertex of an angle of given measure subtending a given segment?
yes
no
2. Construct: rg
3. Is there a rule rL such that a pair consisting of given points, lines, segments, distances, or angle measures in S1 is in Dom rL, and there is a locus L such that X ∈ L ∈ Ran rL? Also for rL'?
yes
no
4. Construct solution rule Rs rL rL' rg
STOP
Solution Rule is Rs
A. Does there exist a point X in ⟨S1, R1⟩ and a rule rg such that (X, E) ∈ Dom rg where E is a point or distance, and Ran rg ⊂ G, and X satisfies one of the conditions given in the first decision?
yes
no
B. Construct: rg
STOP
fail
C. Is there a rule rL such that a pair consisting of given points, lines, segments, distances, or angle measures in S1 is in Dom rL, and there is a locus L such that X ∈ L ∈ Ran rL?
yes
no
D. Construct: rL rg
E. Is there a rule rL', such that a pair consisting of given points, lines, segments, distances, or angle measures in S1 ∪ rL(S1) is in Dom rL', and there is a locus L' such that X ∈ L' ∈ Ran rL'?
yes
no

Figure 11

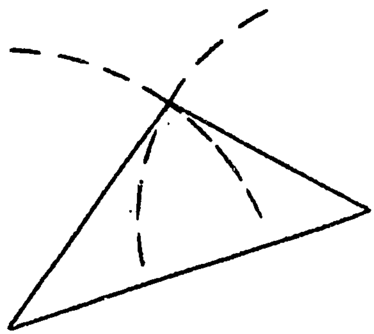
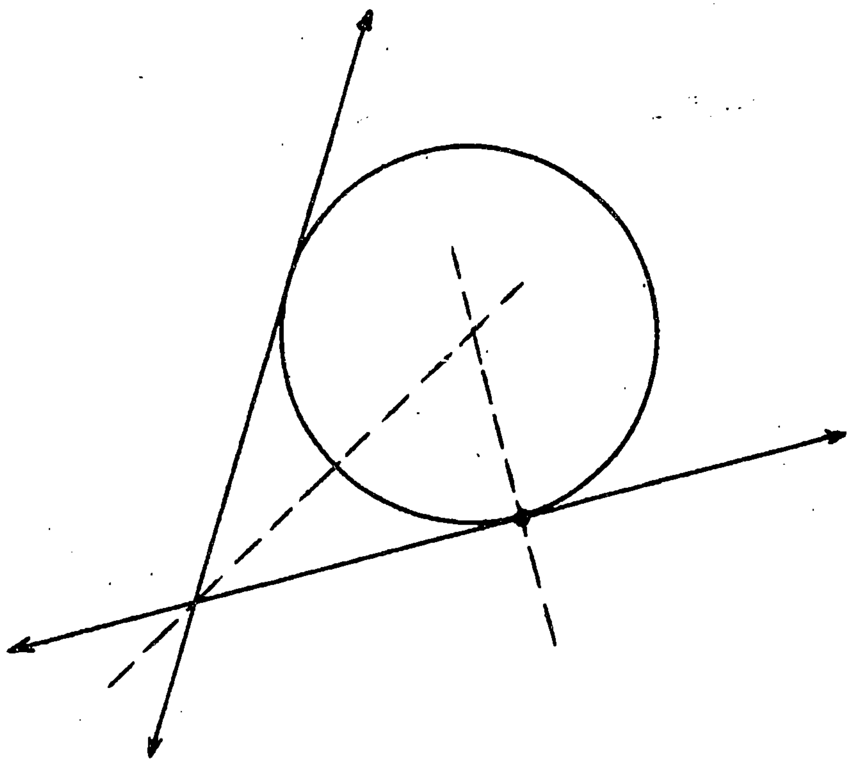


Figure 12



followed by application of a circle rule (r_C) to the intersection of the two loci to construct the goal circle.

Originally, we had not explicitly included in our rule set a rule for constructing perpendiculars to lines through points on the given lines, since none of the problems originally sampled required such a rule. One could argue, of course, that the needed rule is very similar to that involved in constructing the perpendicular bisector of a given segment (r_{PB}, Table 2, Appendix A). But, that would defeat the purpose of a rigorous analysis. To keep things complete, we would either be obliged to add a new lower order rule or to add a (higher order) rule which transforms rules of the latter type into the former. More generally, this illustrates that solving new two-loci problems may require the addition of new lower order rules.

Discussion

Aside from the possibility that new two-loci problems may require additional lower order rules, the rule set appears adequate. In particular, the higher order rule not only generates solution rules for each of the sampled two-loci problems, but also seems compatible with human knowledge.

As the form of the higher order rule suggests, the component decision making capabilities play a crucial role in deriving solution procedures for the problems.[9] These decision making capabilities are designed to reflect the underlying semantics of the problem situations by referring directly to figural representations of semantic information implicit in the problem descriptions. Those parts of a figural representation $\langle S_1, R_1 \rangle$ indicated in solid lines represent the meaning of a task statement and reflect the relation between the given stimulus (S_o) and the goal figure (R_o). Notice that while the relation between S_1 and R_1 will be the same as between S_o and R_o, S_1 and R_1 will not in general be the same as S_o and R_o, respectively.

For purposes of our analysis, the decision making capabilities were viewed as atomic although they can also be analyzed into more basic components. The first decision making capability in the second two loci higher order rule, for example, involves both a conjunction and disjunction of a number of simpler conditions. This decision making capability could be subdivided, for instance, into the following two decisions: (A) Is there a point X that is a given distance from a given point and/or line? (B) Is there a point X equidistant from a pair of given points or lines?[10] Instead of having one decision making capability involving conditions A and B, then, we could have one decision making capability involving A, and a subsequent one, B.[11, 12]

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9. Initially, we had failed to appreciate the critical importance of decision making capabilities in reflecting human knowledge. In our first attempts at higher order rule construction, the various rules were tried pretty much at random. This would be fine if all people did was to randomly try out various rule possibilities. But both our intuition and experience suggest otherwise. Effective problem solvers frequently have rather sophisticated bases for making the rule selections that they do. We think that the above higher order rule takes many of these capabilities into account.

10. We do not consider points equidistant from a point and a line. The loci in this case are parabolas which are not constructible by straightedge and compass.

11. For a discussion of how new decision making capabilities are learned from simpler ones, see Scandura (1973).

12. Such refinement may be useful in the assessment of behavior potential (Durnin & Scandura, 1972), specifically in increasing the precision of diagnostic testing.

Granting the adequacy of the higher order rule for purposes of our analysis, there are still some limitations in regard to the compatibility of the lower order rules with human knowledge. These limitations are all variants on a common theme: The lower order rules we have identified can be constructed from more basic components. This fact is reflected in at least three ways.

First, many of the simple rules have components in common. Several rules (e.g., r_C, r_{ML}, r_{PLC}, r_{PPC}, r_{LLC}, Table 2, Appendix A), for example, all involve constructing a locus of points (circle) at some distance from some point. The differences lie in whether or not the distance and/or center points are given directly or must be determined first. The construction rules needed to determine these distances and/or center points are quite basic and are apt to be useful in a wide variety of construction situations. Any reasonable account, designed to deal with a wider variety of problem situations, would undoubtedly include these construction rules directly in the rule set.

Second, certain of the identified lower order rules, particularly the rule for constructing the locus of vertices of an angle of a given measure subtending a given line segment (r_{AV}, Table 2, Appendix A), are complex in themselves and cannot automatically be assumed to be available to many problem solvers.

A third limitation is closely related to the first and was mentioned earlier: The lower order rules are to some degree specific to the tasks we have identified. To some extent this may be unavoidable because there are always certain problems which require "trick" solutions. It would be desirable, of course, to keep this to a minimum. In this regard, it should be emphasized that the simpler the lower order rules the greater the problem solving flexibility.

One way to modify our rule set to handle these limitations would be to "reduce" the lower order rules into their components and, correspondingly, to "enrich" the higher order rule by adding sub-routines for constructing the needed locus, r_L, and goal, r_g, rules.[13] Such a rule set would correspond to the type

13. In evaluating alternative rule set accounts for a given class of tasks, decisions must always be made concerning exactly how the computational load should be apportioned to the higher and lower order rules. Any number of alternatives exist; at one extreme, the lower order rules may do all the computation, in which case a separate rule would be needed for each type of problem, and, at the other extreme, the component lower order rules may be of minimal complexity with the higher order rule assuming most of the computational burden. The requirement of compatibility with human knowledge, of course, substantially reduces the number of plausible characterizations.

of knowledge that a person just having been taught the basic construction rules would need to have in order to generate solution rules directly.

For example, consider the rule: "Determine the distance between a given point and a given line and then construct the locus of points at the obtained distance from the given point." This rule can be divided into two subrules: (1) "Determine the distance between a point and a line," and (2) "Construct the locus of points at a given distance from a given point." (Also see r_C, r_{ML}, r_{PPE}, r_{LLC} - Table 2, Appendix A.) To compensate for the reduction in the latter case, the higher order rule could be "enriched" so that more complex r_L and r_g rules can be generated where needed. Specifically, instead of selecting a composite rule directly when it meets certain prescribed conditions, as we have done so far, we include in the higher order rule a simple sub-routine for combining component lower order rules. Consider, for example, application of the existing two-loci higher order rule to the task of inscribing a circle in a triangle. In this case, the above r_g rule is selected at the first decision making capability because its domain contains a point X which is some distance (not given) from a given line and its range (circles) is contained in the goal. A corresponding sub-routine would select (sub) rules until one is found such that X is in its domain (e.g., the distance measuring rule (1)), and another (e.g., the circle rule (2)) such that its range is contained in G. To make the search more efficient, it is natural to add the requirement that the range of the former be contained in the domain of the latter. After the component rules have been identified, the sub-routine would form the composite of these rules and, finally, would test the composite against the condition in the initial higher order rule.

As attractive as this possibility might appear at first, a little thought suggests its implausibility as a way of modeling human knowledge. This can be seen by noting that all geometric constructions with straightedge and compass are generated by just three basic operations: (a) using a straightedge (e.g., to draw a line, ray, or segment through two given points, or through one point, or intersecting a line, etc.), (b) drawing an arc given a compass set at some fixed radius, and (c) given two points, setting a compass to the distance between those points.

As we have seen, many of the lower order rules (e.g., the angle vertices rule r_{AV}) are really quite complex. Requiring a higher order rule, designed to reflect human knowledge, to generate such rules from elemental components is unrealistic. It is unlikely that a subject who is only able to perform the three indicated operations above would also have at his command a rather complex and sophisticated higher order rule. The acquisition of such higher order capabilities by naive subjects would almost certainly have to come about gradually through learning, presumably by interacting with problems in the environment.[14]

PATTERN OF SIMILAR FIGURES

Three Classes of Similar Figures Problems

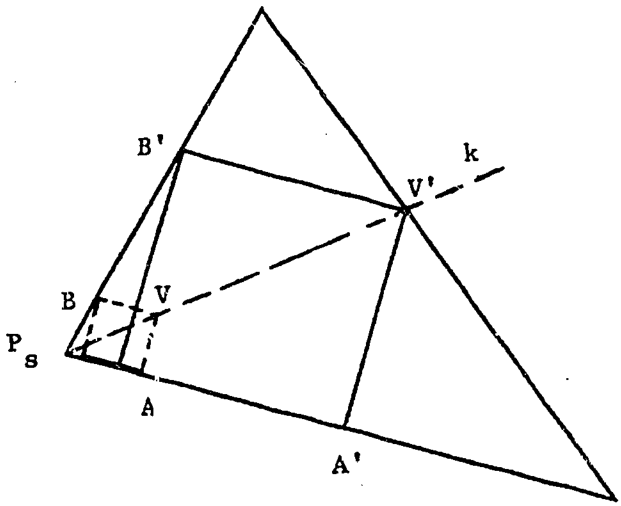
The pattern of similar figures problems were analyzed in similar fashion. Again, we began with a broad sampling of problems from Polya (1962). (The nine tasks shown in Table 3 of Appendix B . formed the basis for our analysis.) One of the problems identified was, "Given a triangle, inscribe a square in it such that one side of the square is contained in one side of the triangle and the two other opposite vertices of the square lie on the other two sides of the triangle." [illegible] The second step was to identify a solution rule for each of the problems. For the problem above the solution rule was, "Construct a square of arbitrary size such that one side is contained in the side of the triangle which is to contain the side of the goal square, and such that one vertex is on another side of the triangle. Draw a line through the point of intersection of those two sides of the triangle and through the fourth vertex of the arbitrary square. From the intersection of this line and the third side of the triangle (which is the fourth vertex of the goal square) construct a segment perpendicular to the side of the triangle which is to contain a side of the goal square. Complete the goal square using the length of the perpendicular segment as the length of the sides."

INSERT FIGURE 13 ABOUT HERE

Analysis of the similar figures problems revealed three relatively distinct classes of solution rules. In the sample problem above, and in other problems in the same class (problems 1-3 in Table 3, Appendix B), the solution

14. See the discussion following the treatment of the pattern of auxiliary figures.

Figure 13



rules all involve first constructing a square of arbitrary size which is in the same orientation as the desired goal square, and which meets as many of the task conditions as possible. (Rules of this type for constructing similar figures are denoted by r_{gs}.) The second step in each solution rule uses two pairs of corresponding points (e.g., A, A' and B, B') in the goal and similar figures (i.e., in $\langle S_1, R_1 \rangle$ superimposed with the similar figure) to determine the point of similarity (P_s), and then, constructs a line (e.g., k) through the point of similarity and a point on the similar figure (e.g., V) which corresponds to a needed point (e.g., V') of the goal figure. (In Table 4, Appendix B, point of similarity rules are denoted r_{PS}.) Finally, the obtained point on the goal figure (e.g., V') is used as a basis for constructing the goal square. (In Table 4, Appendix B, rules for constructing goal squares from obtained points or lines are denoted r_{GSQ}.)

The second class (problems 4-6 in Appendix B) is well represented by the problem, "Given angles B and C of a triangle, and the median M_a to side a, construct the triangle." The corresponding solution rules begin similarly by applying a similar figures rule (r_{gs}) to two given angles to construct an arbitrary sized triangle similar to the goal triangle, with medians, altitudes, etc., as required. Then a modified point of similarity rule (r_{PS}) is used to

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INSERT FIGURE 14 ABOUT HERE

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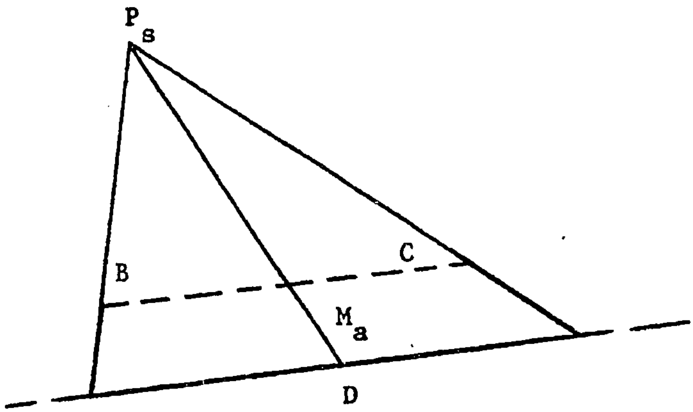
determine the point of similarity (P_s,the vertex of the non-given angle), and to construct the given segment (e.g., M_a), such that one endpoint of the segment is the point of similarity, and such that the segment coincides with the corresponding segment in the similar triangle. Finally, a line is constructed, through the other endpoint of the constructed segment (e.g., D), parallel to that side of the similar triangle (e.g., broken line) that is opposite to the point of similarity. The remaining sides of the goal triangle are obtained by extending two sides of the similar triangle to intersect the constructed parallel line.

The solution rules for the third class of problems (tasks 7-9 in Appendix B) differ in that the first step in each is to use an r_L rule to construct a locus of points which contains a critical point, specifically the center of the goal circle. In the problem, "Given a line and two points (A and B) on the same

Figure 14



side of the line, construct a circle tangent to the line which passes through the two given points," for example, the locus of points (L) equidistant from the two given points contains the center of the goal circle. Also, the point of similarity is the intersection of the locus and the given line. The second step

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INSERT FIGURE 15 ABOUT HERE

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is to construct a similar figure (circle, C_1), which satisfies part of the goal condition. In our example, a circle is constructed, with center on the constructed locus and tangent to the given line. Next, another version of the point of similarity rule is applied; this time the point of similarity (P_s) and a given point on the goal figure (e.g., B) are used to determine a corresponding point (B') on the <u>similar</u> circle. Then, parallel lines (e.g., k and k'), involving corresponding points (e.g., B'O and BO', respectively), are constructed to determine the center of the goal circle. Finally, the goal circle is actually constructed. (The component rules in these solution procedures are detailed in Table 4, Appendix B.)

The Characterizing Rule Set

The higher order rule shown in Figure 16, together with a set of ten

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INSERT FIGURE 16 ABOUT HERE

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additional lower order rules (cf. Appendix B), provides a sufficient basis for solving all of the sampled pattern of similar figures problems. Furthermore, in so doing, the higher order rule appears to reflect the underlying semantics. For example, let us see how a solution rule for the first illustrative problem above (inscribing a square in a triangle) can be generated by application of the higher order rule. The first decision making capability (A) asks essentially whether a point X is needed to serve as the center for a goal circle. As the goal figure is a square, the answer is obviously no. Decision making capability J then asks if there is a goal similar figure rule (r_{gs}) which applies to representing stimulus S_1 and generates squares that satisfy part (G_s) of the goal condition (i.e., the range of r_{gs} is contained in G_s which in turn contains G -

Figure 15

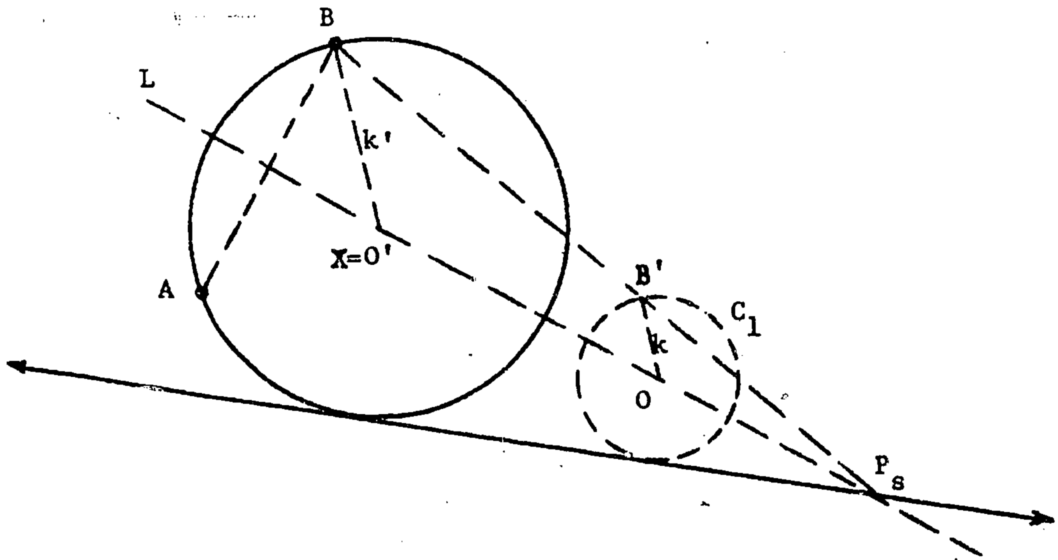
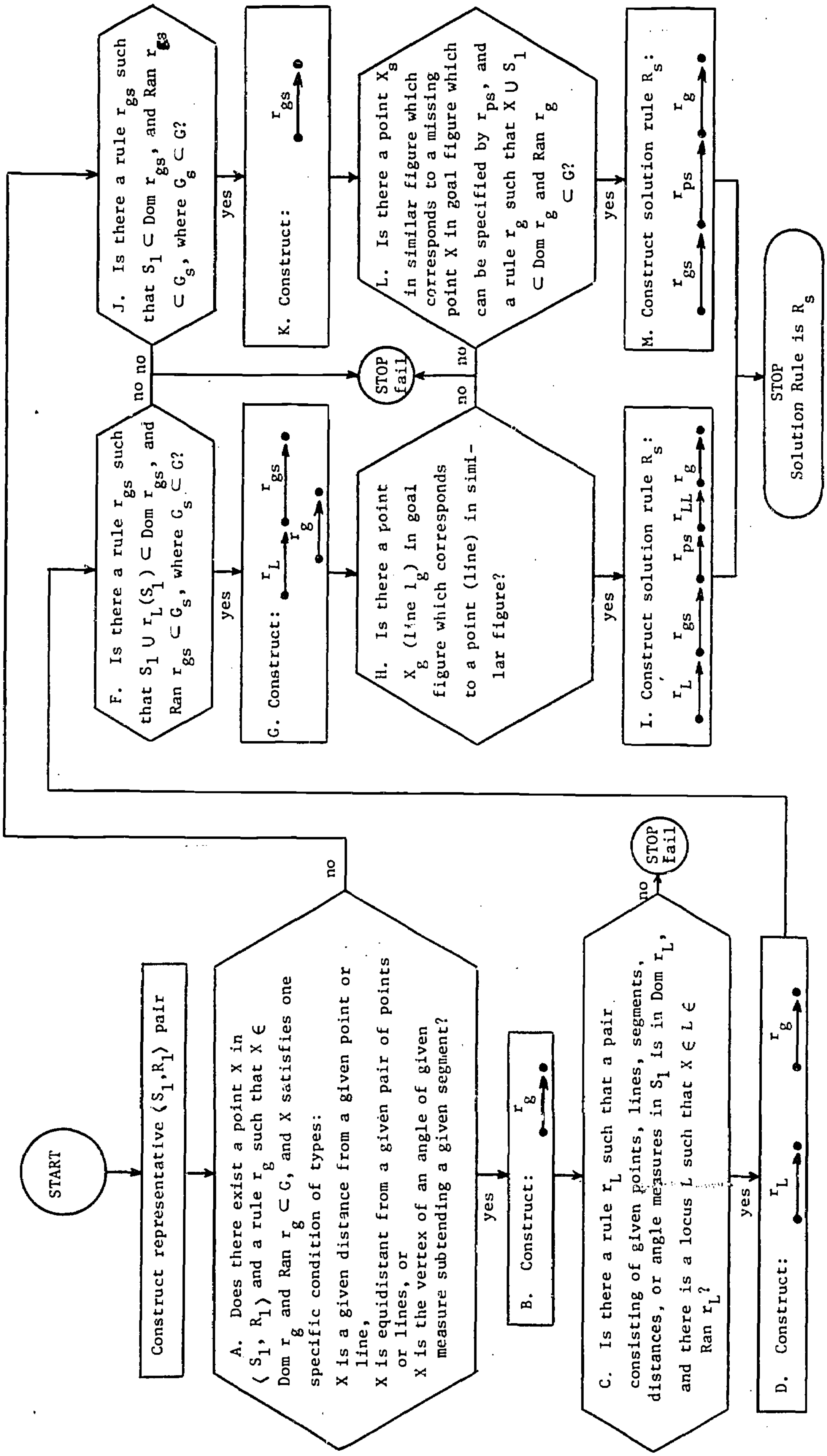


Figure 16


START
Construct representative ⟨S1,R1⟩ pair
A. Does there exist a point X in ⟨S1, R1⟩ and a rule rg such that X ∈ Dom rg and Ran rg ⊂ G, and X satisfies one specific condition of types:
X is a given distance from a given point or line,
X is equidistant from a given pair of points or lines, or
X is the vertex of an angle of given measure subtending a given segment?
yes
no
B. Construct:
rg
C. Is there a rule rL such that a pair consisting of given points, lines, segments, distances, or angle measures in S1 is in Dom rL, and there is a locus L such that X ∈ L ∈ Ran rL?
no
STOP fail
yes
D. Construct:
rL
rg
F. Is there a rule rgs such that S1 ∪ rL(S1) ⊂ Dom rgs, and Ran rgs ⊂ Gs, where Gs ⊂ G?
yes
no
G. Construct:
rL
rg
rgs
H. Is there a point Xg (line lg) in goal figure which corresponds to a point (line) in similar figure?
yes
no
I. Construct solution rule Rs:
rL
rgs
rps
rLL
rg
J. Is there a rule rgs such that S1 ⊂ Dom rgs, and Ran rgs ⊂ Gs, where Gs ⊂ G?
yes
no
K. Construct:
rgs
L. Is there a point Xs in similar figure which corresponds to a missing point X in goal figure which can be specified by rps, and a rule rg such that X ∪ S1 ⊂ Dom rg and Ran rg ⊂ G?
yes
no
STOP fail
M. Construct solution rule Rs:
rgs
rps
rg
STOP
Solution Rule is Rs

equivalently, anything which satisfies G, satisfies G_s, but not necessarily conversely. The lower order rule, "Construct a square in a triangle with one side coincident with one side of the triangle and one vertex on another side of the triangle" (rule r_{SS}, Table 4, Appendix B), satisfies these conditions so the rule is retained as indicated in operation K.

Decision making capability L asks two things: (1) Is there a point X_s which corresponds to a missing point X in the goal square? (2) Is there a rule r_g such that the stimulus S_1, supplemented with the point X ($X \cup S_1$), is in the domain of r_g, and r_g generates a goal-like figure (Ran $r_g \subset G$)? In short, is there a point X_s in the similar square which corresponds to a point X from which the goal square may be constructed? Clearly, there is such a point X_s and the rule, "Determine the distance from a point to a given line segment and construct a square with sides of that length" (r_{GSQ}, Table 4, Appendix B), satisfies the necessary conditions. Operation M forms the solution rule consisting of the two rules above (r_{SS} and r_{GSQ}), with the point of similarity rule (r_{PS}, Table 4, Appendix B) between them.

To see how the higher order rule works with the second class of problems, consider the second illustrative problem above (constructing a triangle, given two angles and a median). In this case, the answers to decision making capabilities A and J are again "no" and "yes", respectively. Here, r_{gs} is, "Construct a triangle of arbitrary size using two given angles and add parts corresponding to given segments," (rule r_{ST}, Table 4, Appendix B). The answer to decision making capability L is "yes". There is a point X in the goal figure, the end point of median M_a, which can be specified by r_{PS}. Operation M again forms the solution rule ($r_{ST} \circ r_{PS} \circ r_{GT}$).

Notice that the first two classes of problems involve the same path in the higher order rule. Each solution rule requires a goal similar figure rule (r_{gs}), the point of similarity rule (r_{PS}), and a goal constructing rule (r_g). The only difference is whether the goal and similar figures are squares or triangles, with all that implies for the particular r_{gs} and r_g rules required. In short, this example illustrates how what may appear initially to be basically different kinds of problems may turn out to have a common genesis.

The third sample problem (constructing a circle tangent to a given line and passing through two given points) illustrates the other path through the higher order rule. In this case, if we knew the center (X) of the desired circle we could solve the task. Furthermore, this missing point X is on a locus, namely the locus of points equidistant from the two given points. Hence, we answer "yes" to decision making capabilities A and C and retain the circle constructing rule (r_g) and the perpendicular bisector rule (r_{PB}, Table 2, Appendix A). Decision making capability F asks if there is a rule (r_{gs}) which applies to the stimulus S_1 as modified by the output of the locus rule (i.e., $S_1 \cup r_{PB}(S_1)$). Condition F is satisfied by a rule (r_{SC}, Table 4, Appendix B) that generates circles with centers on a given line (the locus) and tangent to another given line. The answer to the decision making capability H is also yes. The two given points on the goal figure obviously correspond to two points on the similar circle. By operation I, the solution rule follows directly: "Construct the locus of points equidistant from the two given points ($r_L = r_{PB}$); construct a circle with center on that locus tangent to the given line ($r_{gs} = r_{SC}$); apply the point of similarity rule, and then the parallel line rule (r_{PS}, r_{LL}) to determine the center of the goal circle; construct the goal circle using this center and the distance between it and a given point as radius ($r_g = r_C$)."

It should be noted that in one of the sampled tasks (Task 9, Table 3, Appendix B) the "locus" is given. The easiest way to handle this special case is to simply add an identity locus constructing rule. It would also be a simple matter to modify the higher order rule to take this possibility into account by asking, prior to or at decision making capability C, whether there is a line in S_1 which contains X. Clearly, similar additional modifications might have been called for had we not limited our analysis to the sampled problems.

Combined Rule Sets for Two-Loci and Similar Figures Problems

It would appear from our analysis that a rule set consisting of the lower order rules associated with the pattern of two loci and similar figures problems, together with the two higher order rules, would provide an adequate basis for solving the sampled problems and others like them. Indeed, there are two possible modes of solution in the case of one similar figures task (Task 1, Table 3, Appendix B): "Inscribe a square in a right triangle so that two sides of the square lie on legs of the triangle, and one vertex of the square lies on the hypotenuse." Instead of using the pattern of similar figures, as illustrated in our first example, the pattern of two loci rule can be used to construct the bisector of the right angle. The intersection of this locus with the hypotenuse (the other locus) is the "missing point" X and provides a sufficient basis for constructing the goal square (by r_{GSQ} in Table 4, Appendix B).

Although it is not always critical to distinguish between different modes of problem solving, any complete account must specify why one mode of solution is to be preferred over another (cf. Scandura, 1973, Chapter 8). In the present case, there are two possible ways of handling this. First, we can add a higher order selection rule to the rule set which says simply, if both higher order rules apply,[15] select the pattern of two loci. The rationale is that the pattern of two loci rule will generally yield a simpler method of solution.

A second way to handle the problem is to devise a single higher order rule which combines the advantages of both higher order rules. The higher order rules in Figures 10 and 16 can be combined to yield the higher order rule depicted in Figure 17. The path in this higher order rule designated ⟨1,2,3,4⟩

INSERT FIGURE 17 ABOUT HERE

corresponds to that path of the two loci higher order rule which deals with those cases where the two loci may be found in either order. The path ⟨1,2,3,A,B,C,D, E,4⟩ deals with those two-loci problems where one locus must be found before the other. The other two paths correspond to the similar figures higher order rule.

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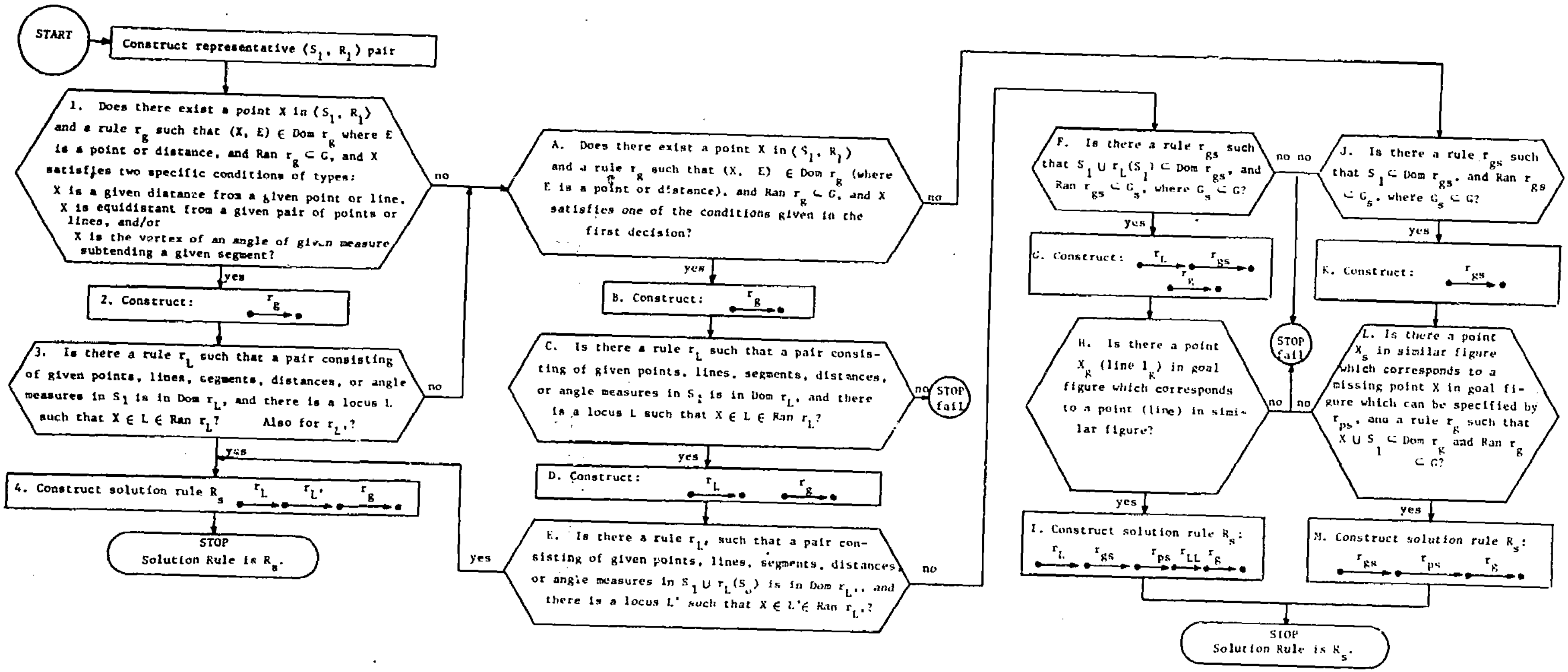
15. Such a selection rule presumably would center on conditions for distinguishing between problems according to which higher order rule(s) can be used to generate a solution rule. Although we shall not attempt to specify such conditions precisely, it would appear that the availability of two locus rules containing the missing point X (cf. decision making capability 1) would play a central role.

Figure 17


START
Construct representative (S_1, R_1) pair
1. Does there exist a point X in (S_1, R_1) and a rule r_g such that (X, E) ∈ Dom r_g where E is a point or distance, and Ran $r_g \subset G$, and X satisfies two specific conditions of types:
X is a given distance from a given point or line,
X is equidistant from a given pair of points or lines, and/or
X is the vertex of an angle of given measure subtending a given segment?
yes
no
2. Construct: r_g
3. Is there a rule r_L such that a pair consisting of given points, lines, segments, distances, or angle measures in S_1 is in Dom r_L, and there is a locus L such that X ∈ L ∈ Ran r_L? Also for $r_{L'}$?
4. Construct solution rule R_s r_L $r_{L'}$ r_g
STOP
Solution Rule is R_s.
A. Does there exist a point X in (S_1, R_1) and a rule r_g such that (X, E) ∈ Dom r_g (where E is a point or distance), and Ran $r_g \subset G$, and X satisfies one of the conditions given in the first decision?
B. Construct: r_g
C. Is there a rule r_L such that a pair consisting of given points, lines, segments, distances, or angle measures in S_1 is in Dom r_L, and there is a locus L such that X ∈ L ∈ Ran r_L?
STOP fail
D. Construct: r_L r_g
E. Is there a rule $r_{L'}$ such that a pair consisting of given points, lines, segments, distances, or angle measures in $S_1 \cup r_L(S_1)$ is in Dom $r_{L'}$, and there is a locus L' such that X ∈ L' ∈ Ran $r_{L'}$?
F. Is there a rule r_{gs} such that $S_1 \cup r_L(S_1) \subset$ Dom r_{gs}, and Ran $r_{gs} \subset G_s$, where $G_s \subset G$?
G. Construct: r_L r_{gs} r_g
H. Is there a point X_g (line l_g) in goal figure which corresponds to a point (line) in similar figure?
I. Construct solution rule R_s: r_L r_{gs} r_{ps} r_{LL} r_g
J. Is there a rule r_{gs} such that $S_1 \subset$ Dom r_{gs}, and Ran $r_{gs} \subset G_s$, where $G_s \subset G$?
K. Construct: r_{gs}
L. Is there a point X_s in similar figure which corresponds to a missing point X in goal figure which can be specified by r_{ps}, and a rule r_g such that $X \cup S_1 \subset$ Dom r_g and Ran $r_g \subset G$?
M. Construct solution rule R_s: r_{gs} r_{ps} r_g
STOP
Solution Rule is R_s.

PATTERN OF AUXILIARY FIGURES

Not all compass and straightedge problems can be solved via the pattern of two loci or the pattern of similar figures. In this section, we describe a higher order rule for dealing with the third class of problems identified by Polya (1962), the pattern of auxiliary figures. We also show how the combined rule set, developed in the previous section, may be extended to account for essentially all of the construction problems identified by Polya (1962).

Auxiliary Figures Higher Order Rule

Our initial analysis was based on a sample of five auxiliary figures problems (see Table 5, Appendix C). One of the problems used is, "Given the three medians of a triangle, construct the triangle." (The additional lower order rules required for the auxiliary figures problems are given in Table 6, Appendix C.)

The analysis proceeded as before. First, we identified a procedure for solving each problem. Then, we looked for similarities among the solution rules and identified the component rules involved. In general, the required goal figures were not constructable via either the two loci or similar figures higher order rules. However, in each case the goal figure could be obtained from an (auxiliary) figure that was constructable from the given information. In the problem above, for example, a triangle can be constructed from segments one-third the lengths of the given medians. The goal figure is obtained by extending two of the sides of this auxiliary triangle to the respective median lengths and drawing lines through the resulting endpoints.

The analysis resulted in the auxiliary figures higher order rule shown

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INSERT FIGURE 18 ABOUT HERE

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in figure 18. This higher order rule generates a solution rule for the illustrative task above as follows. First, an arbitrary representation for the solved problem (S_1, R_1) is constructed. In this case, an arbitrary triangle is "sketched," and its medians are represented on it, as shown in figure 19.

INSERT FIGURE 19 ABOUT HERE

Figure 18

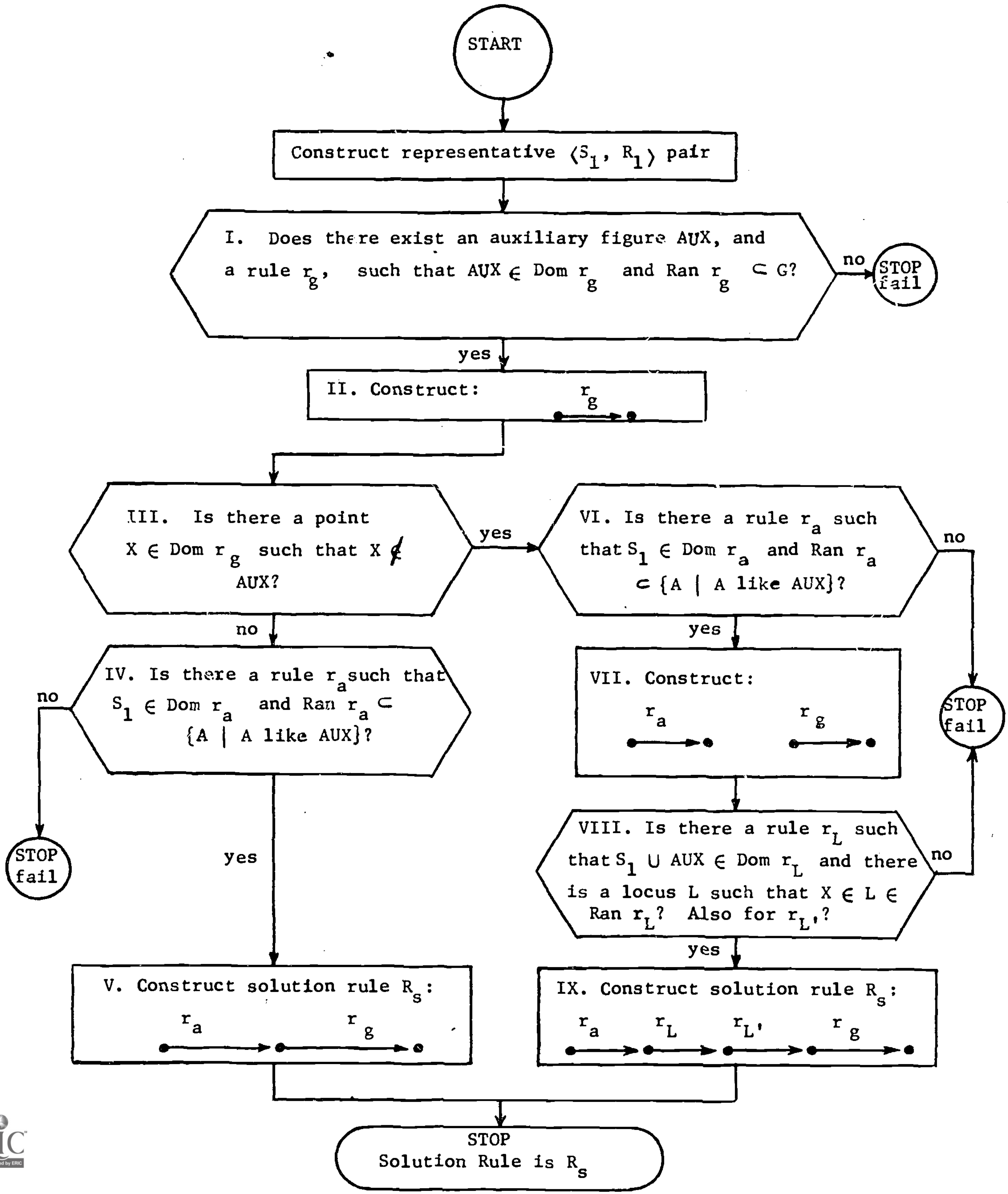
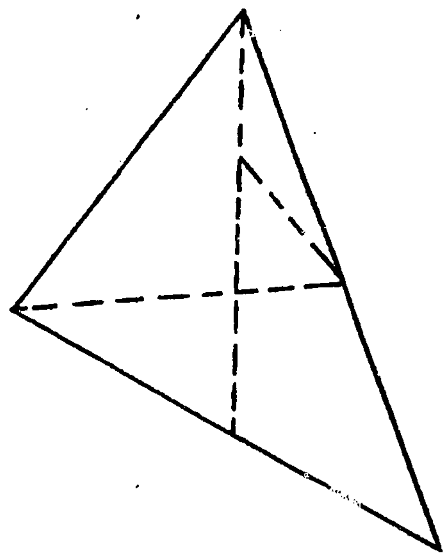

START
Construct representative ⟨S1, R1⟩ pair
I. Does there exist an auxiliary figure AUX, and a rule rg, such that AUX ∈ Dom rg and Ran rg ⊂ G?
no
STOP fail
yes
II. Construct:
rg
III. Is there a point X ∈ Dom rg such that X ∉ AUX?
yes
VI. Is there a rule ra such that S1 ∈ Dom ra and Ran ra ⊂ {A | A like AUX}?
no
no
IV. Is there a rule ra such that S1 ∈ Dom ra and Ran ra ⊂ {A | A like AUX}?
no
STOP fail
yes
yes
VII. Construct:
ra
rg
STOP fail
VIII. Is there a rule rL such that S1 ∪ AUX ∈ Dom rL and there is a locus L such that X ∈ L ∈ Ran rL? Also for rL'?
no
yes
V. Construct solution rule Rs:
ra
rg
IX. Construct solution rule Rs:
ra
rL
rL'
rg
STOP
Solution Rule is Rs

Figure 19



The first decision asks whether there is (1) an auxiliary figure, and (2) a rule r_g which operates on the auxiliary figure and generates the goal figure. In this task, there is such an auxiliary figure (indicated in Figure 19 by the broken line), a triangle having sides one-third the lengths of the given medians.[16] In addition, the rule, "Extend the constructed segments to their given lengths and draw lines through their endpoints " (r_{EG}, Table 6, Appendix C), satisfies condition (2). The next decision (III) asks whether or not a point is needed, in addition to the auxiliary figure, to construct the goal. Here, the answer is "no"; no other point is needed. Finally, decision IV asks if there is an auxiliary figure construction rule (r_a) available whose domain contains S_1 ($S_1 \in \text{Dom } r_a$) and whose range contains the auxiliary figure (i.e., Ran $r_a \subset$ $\{A | A$ is like AUX$\}$. In this case, the rule, "Construct a triangle from segments one-third the lengths of three given medians." (r_{MT}, Table 6, Appendix C), satisfies these conditions and operation V constructs the solution rule, "Construct a triangle having sides one-third the length of the given medians (r_{MT}); extend two segments of the constructed triangle to the respective median lengths, and draw lines through the endpoints of the medians to construct the goal triangle (r_{EG}).

The other path through the higher order rule may be illustrated using the task, "Given the four sides a,b,c,d of a trapezoid ($a < c$), construct the trapezoid." Again, the answer to decision I is "yes". (Where the answer is "no",

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INSERT FUGURE 20 ABOUT HERE

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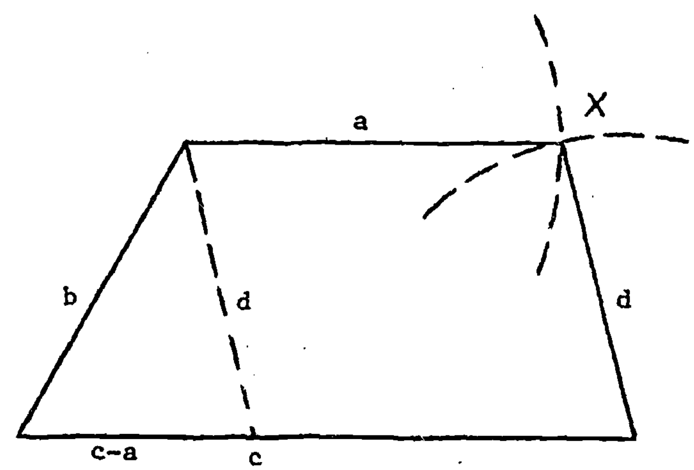
the higher order rule fails.) The triangle with c-a, b, d as sides serves as the auxiliary goal figure and the goal rule, "Through corner points of an auxiliary figure and through another point not in the auxiliary figure, draw segments to complete the goal " (r_{AXP}, Table 6, Appendix C), is selected. Unlike the first path, however, the answer to decision III is "yes" since the goal rule (r_g) acts on pairs ($X \cup$ AUX) consisting of an auxiliary figure and a critical point X.

16. We do not attempt to spell out the procedures necessary for finding auxiliary figures. However, in all of the sampled auxiliary figures problems, it was necessary to construct a line parallel to some "distinguished" line through some "distinguished" point not on that line. Such procedures also frequently require special knowledge - for example, that medians intersect at a common point that is 2/3 of the distance from the respective vertices to the midpoint of opposite sides. Such knowledge is frequently logically deducible, but for our purposes, may be represented in terms of simple "associations" - for example, between triangles with their medians and the common intersection property.

Figure 20



The next decision (IV) asks if there is a rule r_a that constructs the auxiliary figure from given information. This condition is satisfied by the r_a rule which constructs the auxiliary triangle from the sides of a trapezoid (Rule r_{TT}, Table 6, Appendix C). Decision VIII asks whether there are two locus rules (r_L and $r_{L'}$) which apply to the auxiliary figure and/or other given information (S_1) and whose ranges contain X. The circle rule (r_C), applied to different portions of $S_1 \cup$ AUX, plays the role of both locus rules. The solution rule (Operation IX) $r_{TT} \cdot r_C \cdot r_C \cdot r_{AXP}$ is a concatenation of the component rules.

Combined Two-Loci, Similar and Auxiliary Figures Higher Order Rule

Taken collectively, the three rule sets described above (including the higher order rules) account for a wide range of geometry construction problems. Furthermore, the rule sets appear compatible both with human behavior and with the heuristics originally identified by Polya (1962).

This is not meant to imply, however, that the three higher order rules are unrelated to one another. Both the needed point X in the pattern of two loci, and the similar figure in the pattern of similar figures can be regarded as special auxiliary figures. Indeed, one could modify the auxiliary figure higher order rule so that it, together with the relevant lower order rules, would account for all three classes of problems. In addition, the similar and auxiliary figures higher order rules may be viewed as progressive generalizations of the two-loci higher order rule. It is not difficult to conceive of third level higher order generalization rules which have the two loci higher order rule and a similar or auxiliary figure as inputs , and a more general higher order rule, in which a similar or auxiliary figure is substituted for the missing point X, as the corresponding output.

Alternatively, the combined two-loci, similar figures higher order rule (Figure 17) can be extended to include auxiliary figures. In fact, the extended

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INSERT FIGURE 21 ABOUT HERE

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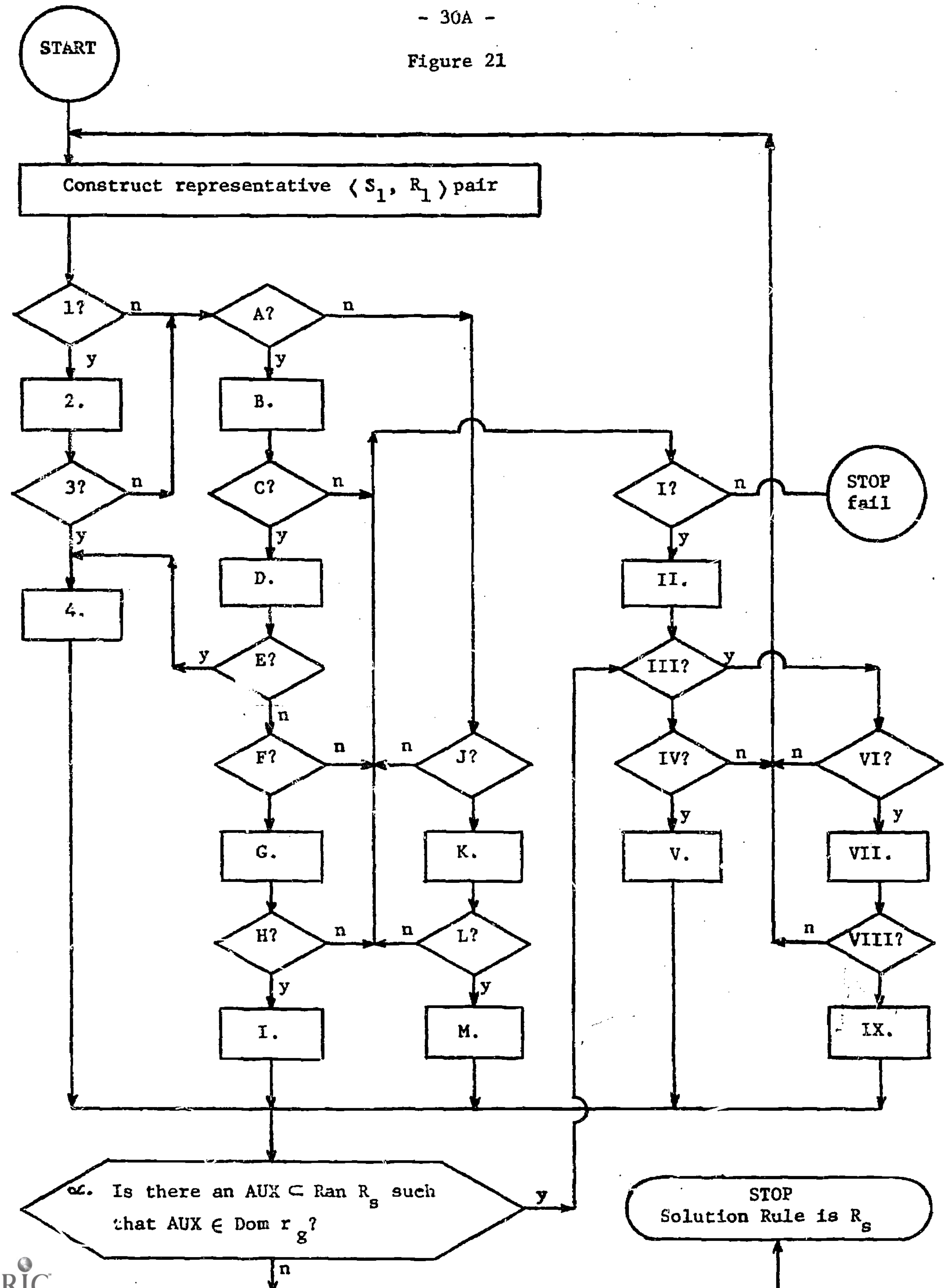
higher order rule depicted in Figure 21 allows recursion on the higher order rules. To see this, notice that the higher order rule shown in Figure 18 can terminate at several points without finding a solution rule. In some problems this is unavoidable; there may not be an auxiliary figure from which the goal figure can be constructed. Sometimes, however, there is an auxiliary

Figure 21


START
Construct representative ⟨ S1, R1 ⟩ pair
1?
2.
3?
4.
A?
B.
C?
D.
E?
F?
G.
H?
I.
J?
K.
L?
M.
I?
II.
III?
IV?
V.
VI?
VII.
VIII?
IX.
STOP fail
α. Is there an AUX ⊂ Ran Rs such that AUX ∈ Dom rg?
STOP Solution Rule is Rs
y
n

figure, but one which is not directly constructable from the given information. Such auxiliary figures can often be constructed via the pattern of two-loci, the pattern of similar figures, or the pattern of auxiliary figures itself. In those cases where such an auxiliary figure exists, we allow for this possibility by returning control to the start of the combined higher order rule in order to derive an r_a rule for constructing the auxiliary figure. Once an auxiliary figure (r_a) rule has been derived, the original procedure resumes.

To see how this higher order rule works, consider the following task, "Construct a trapezoid given the shorter base a, the base angles A and D, and the altitude H_t."

INSERT FIGURE 22 ABOUT HERE

As in the trapezoid example given earlier, the needed auxiliary figure is the triangle having sides c-a, b, and d. But, this triangle is not directly constructible from the given information. None of the assumed lower order rules is adequate, so the higher order rule breaks down at step VI. The flow of control therefore returns to step 1 with the aim of constructing the auxiliary figure.[17] Beginning here, the problem of constructing this auxiliary figure is a straightforward similar figures task, one in fact which we had sampled (Task 5, Table 3, Appendix B).

The higher order rule of figure 21 also generates solution rules for even more complex problems, provided we assume the necessary component rules. For example, consider the problem, "Given three noncollinear points A, B, and C, construct a line XY which intersects segment $\overline{AC}$ in the point X and segment $\overline{BC}$ in the point Y, such that segments $\overline{AX}$, $\overline{XY}$, and $\overline{YB}$ are all of the same length."

INSERT FIGURE 23 ABOUT HERE

It is instructive to consider the derivation of the solution rule for this rather difficult problem in some detail. Three recursions are required.

(I) The answers to decisions 1, A, and J of the combined higher order rule are all 'no'. Hence, control goes to decision I. At decision I, the segments $\overline{XZ}$ and $\overline{ZB}$ in Figure 23 can be identified as an auxiliary figure. (This is equivalent to, but less convenient than, using rhombus XZBY.) Operation II

17. This involves memory and is not indicated in the flow diagram.

Figure 22

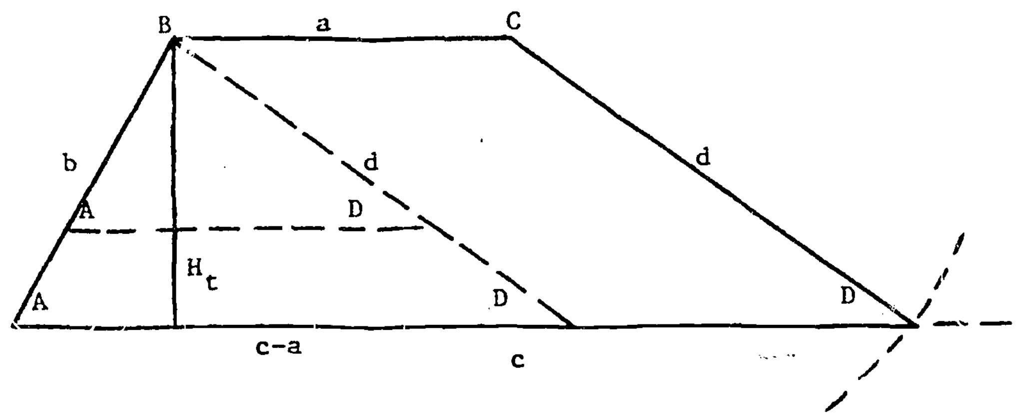
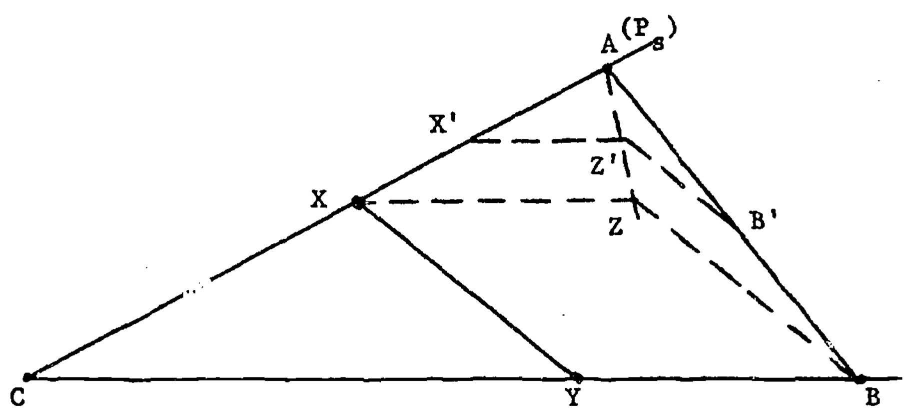


Figure 23



denotes a lower order r_g rule which can be used to construct the rest of rhombus XZBY (i.e., construct the goal figure, $\overline{AX} = \overline{XY} = \overline{YB}$) from S_1 supplemented with $\overline{XZ}$ and $\overline{ZB}$ ($S_1 \cup \overline{XZ}$ & $\overline{ZB}$). The following lower order rule would be both adequate for this purpose, and useful in other situations: "Given two fixed sides of a parallelogram (or two sides and the angle between them), construct the parallelogram."

The answers to decisions III and IV, however, are "no". Decision IV, in particular, asks if there is an r_A rule which operates on S_1 and generates an appropriate pair of segments ($\overline{XZ}$ and $\overline{ZB}$). There is no such lower order rule among those we identified: we could add one, but any such rule would be extremely complex and rather unnatural (i.e., highly specific to the problem).

(II) Instead, the auxiliary figure plays the role of goal and control returns to the start of the procedure. That is, on this loop the aim is to derive an r_a rule for constructing the required auxiliary figure (pair of segments). As before, the answers to decisions 1 and A are "no", but the answer to decision J depends on what lower order rules are available. If segments $\overline{X'Z'}$ and $\overline{Z'B'}$ can be constructed directly, we go through decision L where for our purposes we assume that the answer is "yes". Control then goes to question 3, "Is there an auxiliary figure in the range of the solution rule (AUX $\subset$ Ran R_s) such that the auxiliary figure (segments $\overline{XZ}$ and $\overline{ZB}$) is in the domain of an r_g rule (AUX $\in$ Dom r_g)." Since the rhombus constructing r_g rule above satisfies this condition control flows to decision III which refers to segments $\overline{XZ}$ and $\overline{ZB}$ and where the answer is "no". Decision IV refers again to the availability of a rule for constructing $\overline{XZ}$ and $\overline{ZB}$ from $\overline{X'Z'}$ and $\overline{Z'B'}$. This was assumed at decision L so the answer is yes. After getting a no at decision 3, the solution rule is formed by combining the derivations obtained on each loop.

(III) If, on the other hand, at decision J the similar figure (i.e., segments $\overline{X'Z'}$ and $\overline{Z'B'}$) cannot be constructed directly, a third loop may be required. For example, control goes to decision I where the answer is "yes". The point Z' is a degenerate auxiliary figure from which it is reasonable to assume that the similar figure $\overline{X'Z'}$ and $\overline{Z'B'}$ may be constructed. Suppose, however, that the answer to decision IV is "no". In this case, we start over again with what amounts to a two-loci problem. The answers to decisions 1 and 3 this time are "yes". A circle rule (applied to X' as center) and a rule for constructing parallel lines (e.g., $\overline{X'Z'}$) serve as locus rules r_L and $r_{L'}$. The circle rule also serves in the r_g role to complete the goal (similar) figure, only this ...

it is applied to Z' as center. From operation 4, of course, we go to decision [illegible] and complete the derivation as before. This time, however, we go back through the loop from decision III to decision [illegible] twice before forming the final solution rule.

DISCUSSION

Summary

In summary, a quasi-systematic method for devising rule-based accounts of problem solving was proposed and illustrated with compass and straightedge constructions in geometry. Higher order rules, together with corresponding sets of lower order rules, were constructed for the two-loci, similar figures and auxiliary figures problems identified by Polya (1962). First, rule sets were constructed to account for a broad sampling of two-loci problems. We saw how decision making capabilities (decisions),and particularly, how the conditions used to define decisions play a central role in higher order rules. Among other things, these conditions insure that derived solution rules satisfy the respective higher order goals - equivalently, that the higher order rules are deterministic relative to the higher order goals.[18]

Separate rule sets were similarly constructed for the similar figures and auxiliary figures problems. We also showed how the two-loci and similar figures higher order rules could be combined to form one higher order rule, which together with the lower order rules provides a basis for solving both kinds of problems. Finally, a combined two-loci, similar and/or auxiliary figures higher order rule was constructed. This higher order rule allows recursive returns to the components of the higher order rule, corresponding to the individual higher order rules, and was considerably more powerful than the others. Its use on some complex problems was illustrated.

Overall, the analyses demonstrated the viability of the analytic method and provide further evidence in support of the competence theory on which it is based. In particular, the form of competence theory proposed by Scandura (1971, 1973), in which rules operate on rules, places constraints on the kinds of rule sets possible and, correspondingly, on the methods which may be used to construct them. The higher order rules identified were precise, compatible with the heuristics identified by Polya, and intuitively seemed to reflect the kinds of relevant knowledge that successful problem solvers might have.

The central role played by semantics in the analysis should be emphasized. The meaning of each task was represented by a goal figure $\langle S_1, R_1 \rangle$ representing both the goal object (R_o) and the given information (S_o). The relations among,

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18. This feature has the important advantage of avoiding false starts requiring backup capabilities within given (higher order) rules (cf. Minsky & Papert, 1972).

and properties of, the elements of these figures, together with the domains and ranges of individual rules, were reflected directly in the higher order rules. Although little attention was given to the formal representation of semantic features, the goal figures clearly placed powerful constraints on the rules selected at each stage in applying the higher order rules. Representation in terms of some arbitrary (e.g., random) syntax, unconstrained by goal figures, would have necessitated backup capabilities[18] and, in principle, could easily increase the number of possible construction rules at each stage beyond any reasonable computational capability. That is, without the constraints imposed by the goal figures, the number of possible points, arcs, and lines that might be constructed could be almost unlimited. The effect of using goal figures is very much the same as that referred to by Winston (1972) in a recent paper on vision. He argued that although the number of combinatorially possible arrangements of vertex types (Guzman, 1968) is very large, the number of types that yield real figures is much smaller.

Limitations.

In spite of these advances, the present study has certain limitations which, in principle, and on the basis of the existing theory of structural learning (Scandura, 1973), could be overcome. First, all of the higher order operations were limited to compositions of rules. In future research, more attention should be given to other kinds of operations. Generalization, restriction, and selection rules (Scandura, 1973), for example, might well be expected to play an important role in problem solving.

There are a variety of ways in which such rules might enter. (a) In discussing the two-loci higher order rule, we have already seen how the scope of a decision (making capability) may be generalized to generate solution rules for a broader range of problems. In particular, we saw how the first decision, which was initially restricted to situations where the desired point X was a given distance from two given points, could be generalized, for example, to allow the point to be the same distance from two given points. It is not hard to envisage a generalization rule by which this shift might be made (see Scandura, 1973). The relationships observed previously between the missing points X and the similar and auxiliary figures, suggest another kind of generalization involving the identified higher order rules.

18. See footnote 18, page 34.

(b) There are a wide variety of construction problems which might require the independent derivation of more than one missing point X, similar figure, or auxiliary figure. As a simple example, consider the task of constructing two circles, one of which is to be inscribed in a given triangle and the other, to pass through its vertices (i.e., to circumscribe the triangle). In this case, the problem can be solved by applying the two-loci higher order rule twice. The higher order derivation rule here can be thought of as a generalization of the two-loci rule in which two or more applications (i.e., recursions) may be allowed. One can easily conceive of a simple higher order generalization rule which operates on rules and generates corresponding rules which are recursive. The combined two-loci, similar and auxiliary figures higher order rule is one possible consequence of applying some such higher order rule.

(c) If we had allowed unsolvable variants of the problems considered, truly viable solution rules would have to be appropriately restricted. The solution rule for "constructing a triangle with sides of predetermined length," for example, works only when the sum of each pair of sides of the triangle is greater than the third. A completely adequate solution rule would have to test this possibility. It is possible to conceive of higher order rules, which operate on rules of various kinds together with special restrictions (e.g., the triangle inequality) to generate correspondingly restricted rules.

(d) It is also possible to conceive of three dimensional analogues of compass and straightedge constructions. In this case, the higher order rules would operate on the usual two dimensional construction rules and would generate their three dimensional analogues. For example, a rule for constructing the locus of points equidistant from a given line (i.e., a pair of lines) corresponds to a three dimensional rule which constructs a cylinder about the line.[19]

A second major limitation derives from the underlying theory assumed. As we saw in the introduction, this theory has been tested empirically and provides an accurate account of behavior to the extent that memory is not a factor (i.e., to the extent that all of the necessary rules are readily available and can be checked for applicability (e.g., pattern matched) within some reasonable

19. Implicit in the above examples is another limitation to which we have indirectly referred previously. Our original analyses were limited almost exclusively to single higher order rules. In no case did we attempt to identify rules which may operate on higher order rules, although our examples make it clear that we could have done so. The problems involved in accomplishing this would be practical rather than theoretical.

predetermined period of time.) Hence, all of the lower order rules needed in each derivation had to be explicitly included in our analysis. This necessitated inclusion of a number of rather complex lower order rules (e.g., the locus of vertices of a given angle subtending a given segment).

Earlier, it was shown that this problem could be overcome by building subroutines into the higher order rules to construct complex lower order rules as needed. We rejected this alternative, however, as it led to rather complex and, to some extent, unnatural higher order rules.

Fortunately, an as yet untested but reasonable theoretical mechanism does exist which effectively overcomes this limitation (see Scandura, 1973, Chapter 10). Essentially, this mechanism allows control to shift automatically among goals in a somewhat more general fashion than is assumed in the memory free theory. In particular: (a) If a subject does not have a rule available for achieving a desired goal, control automatically shifts to the higher order goal of selecting a rule for deriving such a procedure. (b) If a rule satisfies a higher order goal but depends for its application on some rule in its domain, which is not available, then control automatically shifts to the domain goal of selecting a procedure which generates the needed lower order (domain) rule. (c) Once a domain goal has been satisfied, control reverts to the goal from which control was diverted. (d) Everything else is as before.

Instead of building subroutines directly into higher order rules, then, these assumptions allow for the derivation, by independent higher order rules, of needed lower order rules. For example, suppose in applying the two loci higher order rule that a subject has identified a missing point X, but that his lower order rule set does not contain one of the needed locus rules (e.g., the locus of vertices mentioned above). According to the extended hypothesis, control shifts to the domain goal of deriving the needed locus rule. Once this goal has been satisfied, control reverts to the original (higher order) goal of deriving a solution rule.

It is not necessary to incorporate either goal shift into the higher order rule itself. In the theory, this is assumed to take place automatically. Obviously, such a mechanism, if reflected in human behavior, would greatly increase the power and scope of applicability of the rule sets we have identified. In effect, fewer constraints would be imposed on the manner in which the individual rules may interact in solving problems. As we shall see shortly, however these constraints are far from idiosyncratic as is the case in many present artificial intelligence systems; they are presumed to be fixed once and for all.[20]

A third major limitation of this research is that the cumulative effects of learning were not considered. Although it is clear that newly derived rules are to be thought of as automatically added to existing rule sets, each problem in our analysis was considered as de novo (relative to the given rule set). As discussed above, this necessitated the inclusion of a number of fairly complex rules in our basic sets. Furthermore, and in many ways more important, such characterizations tend to lack flexibility. The atomic elements are so large, relatively speaking, that there are many intermediate level problems that cannot readily be solved using such rule sets exclusively. Also important from the standpoint of behavioral analysis, it is doubtful that such lower order rules would adequately reflect the knowledge had by most subjects assumed to know the identified higher order rules. Such subjects would almost certainly also know a wide variety of simpler construction rules, even though we have not explicitly included them in our rule sets.

An alternative which should be pursued in future work is to begin initially with rule sets composed of simpler rules, and to allow these rule sets to grow gradually by interacting with a problem environment.[21] In the present case, only three atomic lower order rules would be needed: (a) setting a compass to a given radius, (b) drawing a straight line (segment), and (c) using a set compass to make a circle. It is not immediately clear what the higher order rules should be but, presumably, any reasonably satisfactory rule set would include simple composition, conjunction, and generalization higher order rules, together, possibly, with variants of the two loci and other higher order rules identified above. It should be emphasized in this regard that the initial selection of rules would not in itself be sufficient; the choice and sequencing of problems may also be expected to have important effects on both the rate and type of knowledge acquisition. As this paper goes to press, plans are being made to develop this approach with compass and straightedge constructions, including computer implementation.

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20. An experimental test of this mechanism is being planned as this paper goes to press.

21. Such rule sets have been called innate bases (Scandura, 1973, Chapter 5). In general, innate bases lack the immediate, direct computing power of comparable rule sets composed of more complex rules but, theoretically at least, can grow to become more powerful.

IMPLICATIONS

In addition to strictly epistemological considerations, this research has important implications for work both in simulation and artificial intelligence, and in education.

Artificial Intelligence.

There are at least three ways in which this research, and particularly the underlying theory, might influence artificial intelligence research.

First, the results are suggestive of how the construction of at least certain artificial intelligence systems might be partially systematized. In this regard, the topic of compass and straightedge constructions is not nearly as important as is the fact that it serves as a prototype for the proposed method of analysis. At the present time this method is being used to analyze the the proofs contained in an experimental algebra I high school text based on axiomatics.

Second, the fact that the laws which govern the interactions among individual rules are assumed to be fixed once and for all has important implications for computer implementation. In particular, the fixed mode of interaction would make it possible to modify and/or to extend an artificial intelligence system rule by rule, without having to worry about the effects of these changes on other parts.

One of the major difficulties in current artificial intelligence research is that even minor changes in one part of a system may have unpredictable effects which may require compensating changes elsewhere. The switch to heterarchical systems (e.g., Minsky and Papert, 1972) in which control may shift among individual programs in some predetermined manner, does not alleviate this problem. In contrast to the above mechanism, the mode of control in heterarchical systems may vary from system to system, and worse, from the standpoint of debugging, may interact with the individual programs themselves. In short, the important point for artificial intelligence research is the potential advantage for implementation of a fixed mode of interaction.

Whether or not the mode of interaction is restricted to that proposed here is not the most crucial point.[22] On the assumption that artificial intelligence research might benefit by taking account of such mechanisms, psychological research aimed at discovering what these interactions are, under various behavioral boundary conditions, would appear to be a first order of business. Research related to the extended mechanism outlined above, or to an even richer theoretical mechanism which fully incorporates memory (see Scandura, 1973, Chapter 10), would seem particularly timely.

The implications of the proposed mechanism are in no way limited to pragmatic considerations of system development. A third major implication is that the proposed mechanism provides an explicit basis for learning. As new problems are solved, new (solution) rules are added to the knowledge base. While keeping in mind the possible limitations of the proposed learning mechanism indicated above, this mode of representation (i.e., rule sets plus mechanism) has a number of basic advantages not shared by the more familiar state-space representations of problem solving that have been so widely used in artificial intelligence research (Nilsson, 1971). In particular, attempts to deal with learning using state-space representations have been uniformly unsatisfactory.

Although having nothing directly to do with the learning mechanism, the successful use of flow diagramming as a mode of representation of individual rules suggests that perhaps such representation might play a somewhat larger role in the exposition of future artificial intelligence research. The routine use of a large number of different and highly technical programming languages is often enough to turn away outsiders who might otherwise be interested. The limitations of flow diagrams with regard to memory considerations are a small price to pay for a more neutral and familiar form of representation. Furthermore, flow diagrams have a flexibility as to level of representation which is not shared by particular programming languages. This makes it possible to more readily represent basic components at a level of atomicity tailored to immediate needs, and to psychological reality (cf. Scandura, 1973), rather than to basic components determined by some programming language. These comments, of course, apply only to psychological and expository considerations and say nothing of the more strictly technical problems of representation which must be dealt with in computer implementations.

22. To be sure, earlier research has shown that such a mechanism, as relatively simple as it is, provides an accurate account of actual human problem solving behavior under memory free conditions. In addition, the above analyses show that this mechanism suffices for analyzing some rather complex classes of problems.

Education

The results of this study have both long range and immediate implications for education. The promising nature of the results attests to the practicability of the proposed approach as a means of identifying the knowledge underlying reasonably complex kinds of problem solving. In addition to serving as a prototype, the identified rules themselves could be helpful in teaching high school students how to solve compass and straightedge construction problems.

By identifying precisely what it is that students must know (i.e., one possible knowledge base), these rules provide an explicit basis for both diagnosis and instruction. In particular, the methods of analysis formalized by Scandura (1973) and developed empirically by Scandura and Durnin (1971) and Durnin and Scandura (1972) can be applied directly to assess the behavior potential of individual subjects on the individual rules, including the higher order ones. Operationalizing the knowledge of individual subjects in this way, and comparing this knowledge with the initial competence theory (i.e., set of rules), provides an explicit basis for remedial instruction (Durnin & Scandura, 1972). In effect, each subject can be taught precisely those portions of each competence rule which testing indicates he has not mastered.

Although no special claim is made for the identified lower order rules, care was taken to help insure that the higher order rules reflect the kinds of ability individual subjects might have, or use. To the extent that the identified higher order rules are unknown to high school students, instruction in these rules ought to facilitate problem solving performance. A field test of the efficacy of these higher order rules is now underway and will be reported in due time by Ehrenpreis and Scandura.

The above discussion of how knowledge is acquired through interaction of the learner with a problem environment also has educational relevance. Specifically, by assigning values to various objectives and costs to particular kinds of instruction (or rules), it should be possible to study the problem of instructional sequencing and optimization in a way which is both precise and relevant to meaningful education. We view this as a critically important problem for future research.

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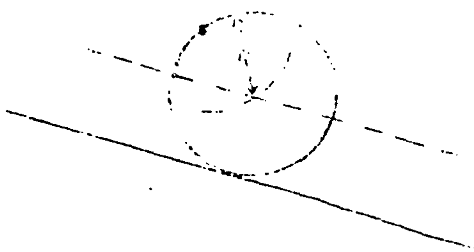
APPENDIX A

Table 1

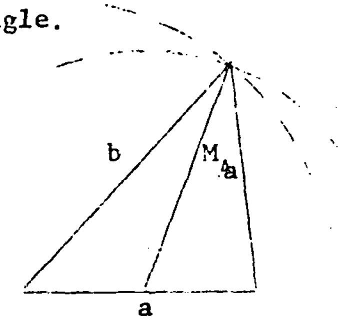
Two Loci Problems and Solution Rules

Task 1. Given a line and a point not on the line, and a radius R, find a circle having the given radius R, which is tangent to the line and passes through the point.



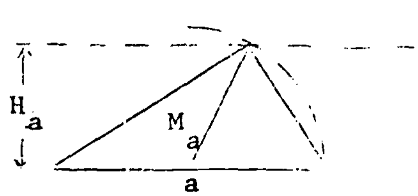
Rule 1. Construct the locus of points at distance R from the given point (r_C)[1]; construct the locus of points at distance R from the given line (r_{PL}); finally apply r_C, using the intersection of the two loci as center and the distance R as the radius.

Task 2. Given sides a and b of a triangle and the median M_a to side a, construct the triangle.



Rule 2. Construct the locus of points at distance b from one end point of the line segment a (r_C); find the midpoint of side a and apply r_C using the mid-point as the center and M_a as the radius; draw segments from the point of intersection to the end points of side a (r_T).

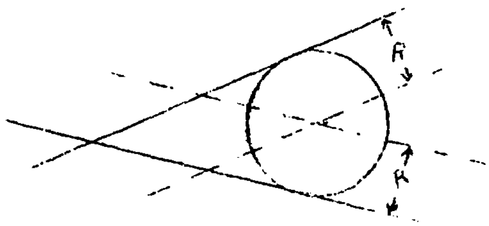
Task 3. Given side a of a triangle and the median M_a to side a, and the height H_a to side a, contruct the triangle.



Rule 3. Apply r_{PL} using segment a and the distance H_a; find the mid-point of segment a and apply r_C using the mid-point of a as center and M_a as radius, and apply r_T using the intersection point and side a.

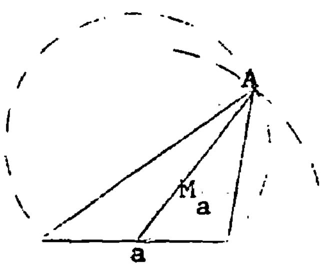
1. The subscript letters refer to Table 2, a list of component rules.

<u>Task 4</u>. Given two intersecting lines and a radius R, construct a circle with radius R tangent to the two given lines.



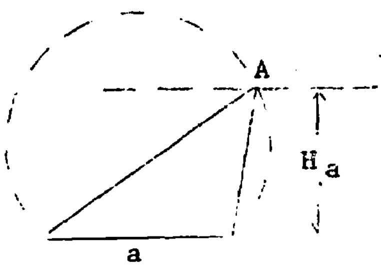
<u>Rule 4</u>. Apply r_{PL} using one line and the distance R; apply r_{PL} using the other line and distance R; apply r_C using the intersection as center and R as radius.

<u>Task 5</u>. Given a side a of a triangle and a median M_a to side a and the measure of angle A opposite side a, construct the triangle.



<u>Rule 5</u>. Apply r_C using the mid-point of a as center, and M_a as radius; construct the locus of possible vertices of angle A subtending segment a (r_{AV}) using a as the segment; and apply r_T using the intersection point and side a.

<u>Task 6</u>. Given side a of a triangle, the height H_a to side a, and the measure of angle A opposite side a, construct the triangle.

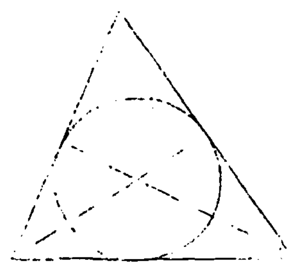


<u>Rule 6</u>. Apply r_{PL} to segment a using the distance H_a; apply r_{AV} to segment a and apply r_T using the intersection point and side a.

<u>Task 7</u>. Given a triangle, find the point inside the triangle such that each of the sides is subtended by an arc of 120° from the point.

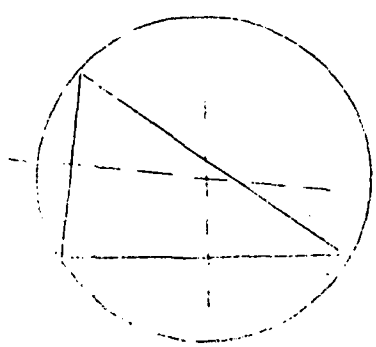
<u>Rule 7</u>. Apply r_{AV} using one side of the triangle; apply r_{AV} using another side of the triangle; the point of intersection is the desired point.

Task 8. Given a triangle, find the circle which is tangent to the three sides. (Inscribe a circle.)



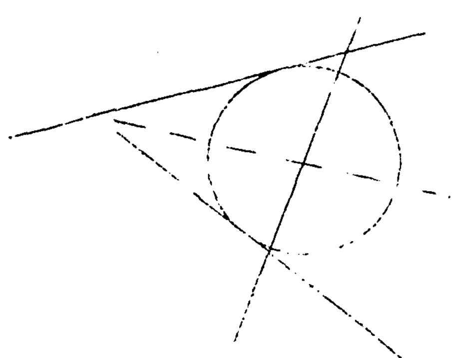
Rule 8. Construct the locus of points equidistant from two given lines (r_{AB}) using two sides of the triangle; apply r_{AB} to another pair of sides; measure the distance from the point of intersection of the two loci to a side of the triangle; and apply r_C using the point of intersection as center and the obtained distance as radius.

Task 9. Given a triangle, find a circle which passes through three vertices (circumscribe a circle).



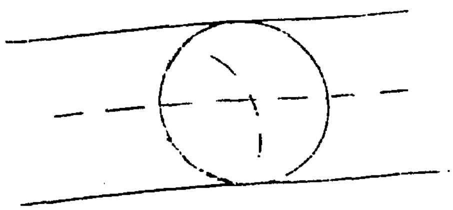
Rule 9. Construct the locus of points equidistant from two given points (r_{PB}) using two vertices of the triangle; apply r_{PB} using two other vertices of the triangle; determine the distance from the point of intersection to one of the vertices and apply r_C using the point of intersection as center and the obtained distance as radius.

Task 10. Given three intersecting lines, not all intersecting at a common point, construct a circle which is tangent to two of the lines and whose center is on the third.



Rule 10. Apply r_{AB} to two of the lines; apply the identity rule to the other line (i.e., use the other line as the locus); determine the distance from the point of intersection of the two loci to one of the other lines and apply r_C using the point of intersection as center and the obtained distance as radius.

Task 11. Given two parallel lines and a point between them, find a circle which is tangent to the two lines and passes through the point.



Rule 11. Construct the locus of points equidistant from two parallel lines (r_{PE}); determine the distance between the obtained locus and one of the lines, then apply r_C using the given point and the obtained distance; apply r_C using the intersection of the two loci as the center and the obtained distance as radius.

Table 2

Component Rules for Two Loci Tasks

r_C	Circle rule	Construct the locus of points at a given distance from a given point. Domain: Set of pairs consisting of one point and one distance. Range: Set of circles (arcs)
r_{ML}	Median locus Circle rule	Construct the midpoint of a given line segment and then construct the locus of points at a given distance from the midpoint. Domain: Set of pairs consisting of one line segment, one distance. Range: Set of circles (arcs).
r_{PLC}	Point-line circle rule	Determine the distance between a given point and a given line and then construct the locus of points at the obtained distance from the given point. Domain: Set of pairs consisting of one point and one line. Range: Set of circles (arcs)
r_{PPC}	Point-point circle rule	Determine the distance between two given points and then construct the locus of points at the obtained distance from one given point. Domain: Set of pairs consisting of two points. Range: Set of circles (arcs).
r_{LLC}	Line-line circle rule	Determine the distance between two given parallel lines and then construct the locus of points at the obtained distance from a given point. Domain: Set of triples consisting of two parallel lines and a point. Range: Set of circles (arcs).
r_{PL}	Parallel line rule	Construct the locus of points at a given distance from a given line. Domain: Set of pairs consisting of one line, one distance. Range: Set of straight lines.
r_{PB}	Perpendicular bisector rule	Construct the locus of points equidistant from two given points. Domain: Set of pairs consisting of two points. Range: Set of straight lines (perpendicular bisectors).
r_{AB}	Angle bisector rule	Construct the locus of points equidistant from two given intersecting lines. Domain: Set of pairs of nonparallel lines. Range: Set of straight lines (angle bisectors).

r_{AV} <u>Angle vertices rule</u>

Construct the locus of vertices of an angle of given measure subtending a given line segment.

<u>Domain</u>: Set of pairs consisting of a line segment and an angle of given measure.

<u>Range</u>: Set of arcs.

r_{PE} <u>Parallel equidistance rule</u>

Construct the locus of points equidistant from two given parallel lines.

<u>Domain</u>: Set of pairs of parallel lines.

<u>Range</u>: Set of straight lines.

r_T <u>Triangle rule</u>

From a point not on a given line segment, draw segments to the endpoints of the given segment (i.e., construct a triangle given a side and an opposite vertex).

<u>Domain</u>: Set of pairs consisting of a line segment and a point not on the segment.

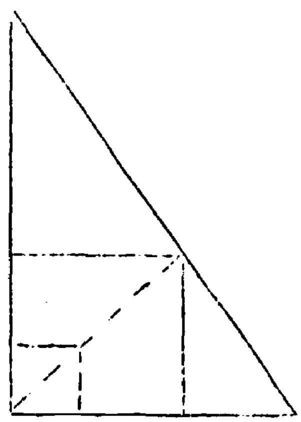
<u>Range</u>: Set of triangles.

APPENDIX B

Table 3

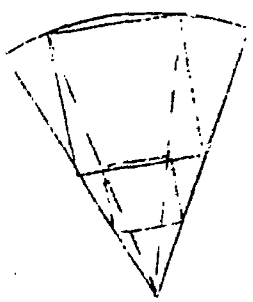
Similar Figures Tasks and Solution Rules

Task 1. Given a right triangle inscribe a square in which two sides coincide with the legs of the triangle.



Rule 1. Construct an arbitrary square such that two sides coincide with the legs of the triangle (r_{SS}).[1] Apply the similarity rule (r_{PS}) where r_{PS} is a general rule in which corresponding points of the goal square and the arbitrary square are used to determine a point of similarity. The missing point in the goal square is determined by drawing a line through the point of similarity and the corresponding point on the arbitrary square; determine the distance between this point of intersection and one of the legs of the right triangle, then construct a square with that distance as side (r_{GSQ}).

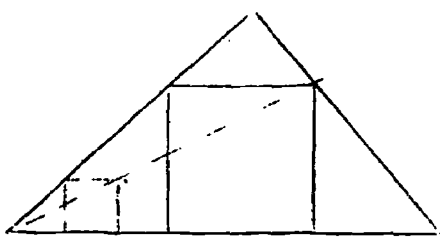
Task 2. Given a sector of a circle, inscribe a square in it.



Rule 2. Construct an arbitrary square such that two vertices are equidistant from the center of the sector and are on the sides of the sector ($r_{SS'}$); apply r_{PS} to determine the points of intersection of the two vertices of the goal square on the arc of the sector; determine the distance between the two vertices and use that distance as the side of the goal square ($r_{GSQ'}$).

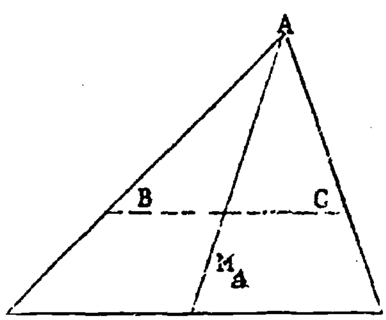
1. Rule subscripts refer to Table 4, Appendix B.

Task 3. Given a triangle, inscribe a square in it such that one side of the square coincides with one side of the triangle and the two opposite vertices of the square lie on the other two sides of the triangle.



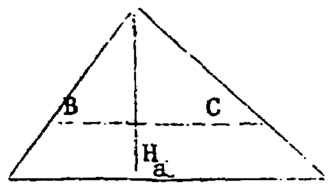
Rule 3. Construct an arbitrary square such that one side coincides with one side of the triangle and a third vertex of the square is on another side of the triangle (r_{SS}); apply r_{PS} to determine the vertex of the goal square corresponding to the fourth vertex of the arbitrary square; determine the distance between that intersection and the side which contains the side of the square; apply r_{GSQ} using this distance as side.

Task 4. Given angles B and C of a triangle and the length of the median M_a to side a, construct the triangle.



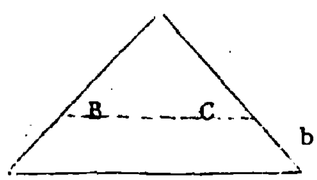
Rule 4. Construct an arbitrary triangle using the two given angles and construct the median M_a of the arbitrary triangle (r_{ST}); apply r_{PS} to the arbitrary triangle to construct the median of the goal triangle (i.e. construct the endpoint of the goal median opposite point A); construct a line parallel to side a of the arbitrary triangle through the endpoint of the median; extend the other sides of the arbitrary triangle to intersect the constructed parallel line (r_{GT}).

Task 5. Given angles B and C of a triangle and the altitude H_a to side a, construct the triangle.



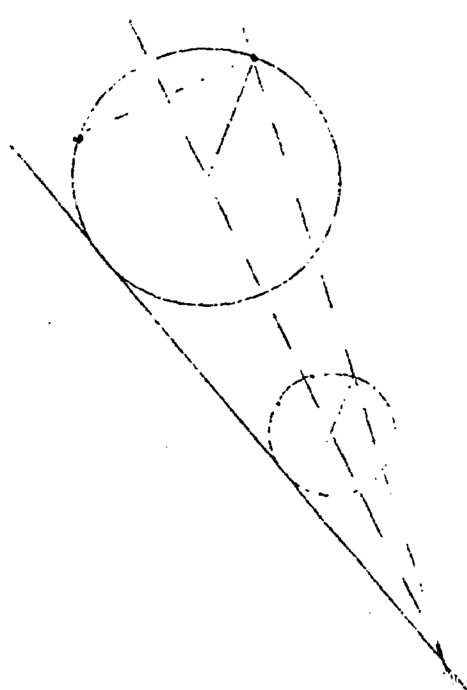
Rule 5. Apply r_{ST} to the given information to construct an arbitrary triangle with an altitude; apply r_{PS} to construct the altitude of the goal triangle; apply r_{GT} to the endpoint of the altitude to complete the triangle.

Task 6. Given angles B and C of a triangle and side b opposite angle B, construct the triangle.



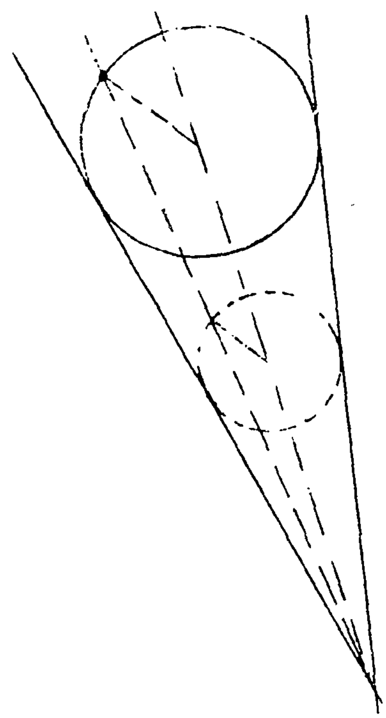
Rule 6. Apply r_{ST} to the given information to construct an arbitrary triangle, apply r_{PS} to construct side b of the goal triangle; apply r_{GT} to the endpoint of side b to complete the goal triangle.

Task 7. Given a line and two points on the same side of the line, construct a circle tangent to the line which passes through the two points.



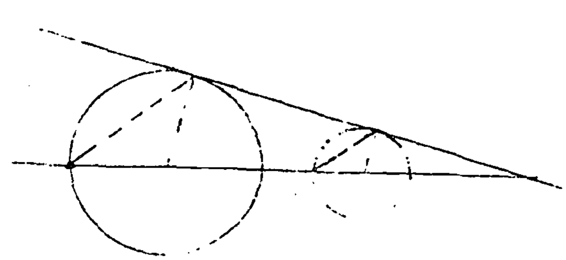
Rule 7. Construct the locus of points equidistant from the two given points (r_{PB}); choose an arbitrary point on the locus and determine the distance between that point and the given line; apply r_C using the point as center and the determined distance as radius; apply r_{PS} using the intersection of the locus with the given line as point of similarity; where the line from the point of similarity to one of the given points intersects the arbitrary circle, construct a line from that point to the center of the arbitrary circle and construct a line parallel to this line through the corresponding given point; apply r_C using the point of intersection of the parallel line with the locus given by r_{PB} as center and the distance between the center and a given point as radius.

Task 8. Given two intersecting lines and a point not on either line, construct a circle tangent to the two lines which passes through the point.



Rule 8. Construct the locus of points equidistant from two given lines (r_{AB}); choose an arbitrary point of this locus as center and determine the distance between the point and the given line, then apply r_C using the point as center and the obtained distance as radius to construct an arbitrary circle; apply r_{PS} to determine the point corresponding to the given point on the arbitrary circle, draw a line through the center of the arbitrary circle and the obtained point on the arbitrary circle, then construct a line parallel to this through the given point; apply r_C using the point of intersection of the parallel line and the locus given by r_{AB} as the center of the goal circle; determine the distance from this center to the given point and use it as radius.

Task 9. Given two intersecting lines and a point on one of the lines, construct a circle which has its center on the line containing the point and which passes through that point and which is tangent to the other line.



Rule 9. Apply an identity rule to the line containing the given point (i.e., take the line as the locus); choose an arbitrary point on the line, determine the distance between the point and the other given line, apply r_C using the arbitrary point as center and the obtained distance as radius; construct a chord from the point of tangency to the point on the arbitrary circle corresponding to the given point. From the given point construct a line parallel to this chord. From the point of intersection of this parallel line with the other given line construct a perpendicular segment to the given line [illegible]; apply r_C using the obtained point as center and the length of the perpendicular segment as radius.

Table 4

Component Rules for Similar Figures Tasks

r_{SS}	Similar square rule	Construct a square in a triangle having sides shorter than any side of the triangle and with two sides coincident with legs of a right triangle or with a side coincident with one side of the triangle and a vertex on another side of the triangle. Domain: Set of triangles. Range: Set of squares with two sides coincident with legs of right triangle or with one side coincident and a vertex on another side of the triangle.
r_{PS}	Point of similarity rule	Select a point of intersection of two lines through corresponding points of goal and similar figures as point of similarity, then construct a line through, or a given line segment from, the point of similarity through point (S) on the similar figure corresponding to a point X on the goal figure, from which the goal figure may be constructed. Domain: Set of pairs of lines to corresponding points of goal and similar figures with point S on similar figure corresponding to a point X on the goal figure. Range: Set of lines through point of similarity and given point.
r_{GSQ}	Goal square rule	Determine the distance from a point not on a line to a given line segment, then construct a square having that length as side with two sides coincident with perpendicular line segments of a right triangle or with one side coincident with a side of the triangle and two vertices on the other two sides of the triangle. Domain: Set of pairs of triangles and points on one side of the triangles "equidistant" from the other two sides. Range: Set of squares with two sides coincident with pair of perpendicular line segments of a right triangle or with one side coincident with a side of the triangle and two vertices on the other two sides of the triangle.
$r_{SS'}$	(2nd) similar square rule	Determine two points on the sides of a sector equidistant from the intersection of the two sides. Then determine the distance between the two points. Using that distance construct a square contained in the sector and with the two points as adjacent vertices. Domain: Sectors of circles. Range: Set of squares with adjacent vertices on respective lines and at same distance from point of intersection.

$r_{GSQ'}$ — (2nd) goal square rule

Construct a line parallel to a given line through point X on the arc of a sector. Then determine the distance between point X and the other point of intersection X_L of the constructed parallel line and the arc of the sector. (Determine distance between X and X_L.) Construct a square using that length as side, contained within the sector.

Domain: Set of sectors and vertices of squares contained in sectors, lines.

Range: Set of squares with two vertices on arcs of the sectors and two vertices on segments of sectors.

r_{ST} — Similar triangle rule

Construct a triangle with a pair of given angles with parts corresponding to given segments.

Domain: Set of pairs of angles, and other given parts of goal triangle.

Range: Set of triangles with parts corresponding to given segments.

r_{GT} — Goal triangle rule

Construct a triangle having an integral part a given length similar to a given triangle with a corresponding part.

Domain: Pairs of triangles with a labeled part and lengths of part of desired triangle corresponding to labeled part.

Range: Set of triangles having a part with given length.

r_{PB} — See Table 2.

r

r_{AB} — See Table 2.

r_{SC} — Similar circle rule

Construct an arbitrary circle with center on a line tangent to another line.

Domain: Pairs of lines.

Range: Circles with center on one line and tangent to another.

r_{LL} — Parallel lines rule

From a predetermined point on a circle draw a line through the center of the circle. Construct from a given point a line that is parallel to the drawn line and that intersects another line containing the center of the circle.

Domain: Circles, points on circles, lines containing centers of circles.

Range: Points (intersection of lines).

$r_{LL'}$ — (2nd) parallel lines rule

From a predetermined point on a circle draw a chord to a point where the circle is tangent to a given line. Construct from a given point a line parallel to the chord, intersecting the given line. Apply rule r_{LL} to the intersection point.

Domain: Circles, points on circles, points, tangent lines.

Range: Points.

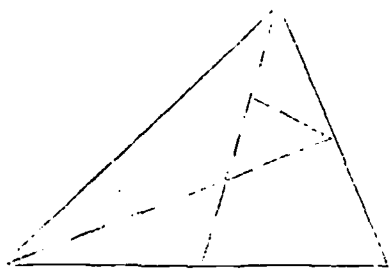
r_C — See Table 2.

APPENDIX C

Table 5

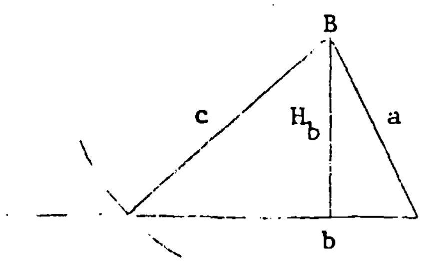
Auxiliary Figures Tasks and Solution Rules

Task 1. Given the three medians of a triangle, construct the triangle.



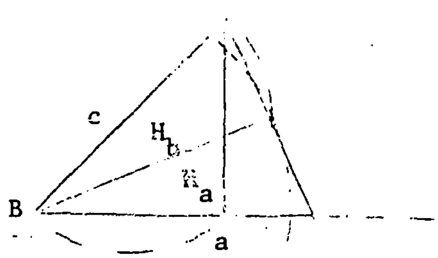
Rule 1. Trisect the three medians; construct a triangle using segments one-third the lengths of the medians as sides; extend one side of the triangle twice its length in one direction; extend another side of the triangle its own length in both directions; draw segments between all pairs of endpoints that are on extended parts of the medians; extend these segments until they intersect.

Task 2. Given sides a and c of a triangle, and the altitude H_b to side b, construct the triangle.



Rule 2. Construct a right triangle with side a as hypotenuse, and segment H_b as leg, (that is, draw an arbitrary line b; construct H_b perpendicular to b; apply r_C using the other endpoint B of H_b as center with the point of intersection of the r_C locus and line b).[1] Apply r_C using B as center and c as radius; connect B and the intersection of the r_C locus and b.

Task 3. Given angle B, the altitude H_b to side b, and the altitude H_a to side a, construct the triangle.

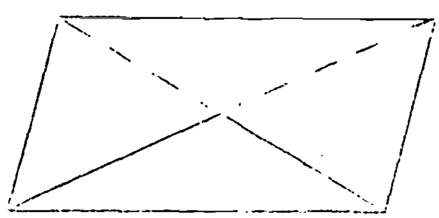


Rule 3. Construct a right triangle using angle B (or 180 - B if B > 90°) as an acute angle, and H_a as the opposite leg; apply r_C using B as center and H_a as radius; apply r_{AV} to find the locus of possible vertices of a 90° angle subtending side c, the hypotenuse of the constructed right triangle;

1. Rule subscripts refer to Table 6, Appendix C.

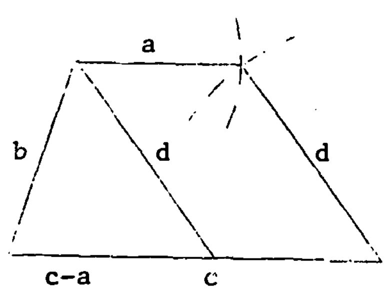
connect the point of intersection of the two loci with the endpoint of c opposite b and extend this segment until it intersects the line containing the other leg of the constructed right triangle.

Task 4. Given a side of a parallelogram, and its two diagonals, construct the parallelogram.



Rule 4. Bisect each diagonal; construct a triangle using the given side of the parallelogram and sides one-half the length of each diagonal. Extend each diagonal to its length from their point of intersection. Draw segments connecting these obtained points and the endpoints of the given side.

Task 5. Given the four sides a, b, c, d of a trapezoid (a and c are parallel, c > a), construct the trapezoid.



Rule 5. Subtract side a from side c yielding side c-a; construct a triangle from the sides c-a, b, d; apply r_C using the vertex opposite side c-a as center and side a as radius; apply r_C using the endpoint of c opposite the constructed triangle as center and side d as radius. From the point of intersection of the two loci, draw segments to the endpoints of b and c.

Table 6

Component Rules for Auxiliary Figures Tasks

r_{EG}	Extend-auxiliary-figure-to-goal rule	Extend constructed segments of auxiliary figure which are part of the goal figure to their given lengths, and draw lines through the endpoints of the extended segments to obtain the goal figure. Domain: Sets of constructed auxiliary figures and given lengths of parts of goal figure. Range: Goal figures.
r_{AXP}	Auxiliary-figure-and-point-to-goal rule	Through corner points of an auxiliary figure, and through another point not in the auxiliary figure, draw segments to complete a goal figure. Domain: Sets of constructed auxiliary figures and points which are not elements of the auxiliary figure. Range: Goal figures.
r_{MT}	Median triangle rule	Construct a triangle from segments one-third the lengths of three given medians. Domain: Sets of triples of medians. Range: Triangles.
r_{HLRT}	Hypotenuse-leg right triangle rule	Construct a right triangle using a given line segment as hypotenuse and a given altitude as leg. Domain: Set of pairs of segments (hypotenuse and leg). Range: Right triangles.
r_{ALRT}	Angle-leg right triangle rule	Construct a right triangle using a given acute angle (or the supplement of a given obtuse angle) and a given altitude as leg. Domain: Set of pairs consisting of one angle and one leg. Range: Right triangles.
r_{DPST}	Diagonals of parallelogram-side triangle rule	Construct a triangle from segments one-half the lengths of given diagonals of a parallelogram and a given side of the parallelogram. Domain: Set of triples consisting of two diagonals and a side of parallelograms. Range: Triangles.
r_{TT}	Trapezoid triangle rule	Construct a triangle from four sides of a trapezoid using sides b, d, and the difference of a and c, where a is parallel to c. Domain: Sets of sides of trapezoids. Range: Triangles.
r_C		See Table 2.
r_{AV}		See Table 2.

MERG
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University of Pennsylvania

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