Feedback, Queueing, and the Reliability of the Ideal Poisson Channel above Capacity

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Abstract — **The ideal Poisson channel, the ideal Poisson channel with feedback, the exponential-server timing channel, and the exponential-server timing channel with feedback are known to have the same capacity. We show that above this capacity they have the same reliability function, which we determine. For the ideal Poisson channel without feedback, this improves upon the strong converse of Burnashev and Kutoyants.**

The *ideal Poisson channel* (IPC) models an optical communication system with direct detection and no dark current: the input is a nonnegative signal ξ_t with a peak constraint, $\xi_t \le \mu$, while the output is a Poisson process with intensity ξ_t. The capacity of the IPC is μ/e nats per unit time, and this capacity is unaffected by the presence of feedback. Capacity-achieving codes for the IPC are explicitly constructed by Wyner [1], who also determines the reliability function of the channel at all rates below capacity. Lapidoth [2] studies the reliability function of the IPC with feedback.

Sundaresan and Verdú [3] observe that the IPC with feedback can be used to emulate an exponential-server queue as follows: if ζ_t is an arrival process and ω_t is the output of the IPC with feedback, then setting the intensity at time t to be $\mu 1(\zeta_t > \omega_{t-})$ causes ω to be distributed as the departure process of an initially-empty exponential-server queue with service rate μ and input ζ. We call the channel in which the encoder submits unmarked packets or jobs to an exponential-server queue while the receiver observes the resulting departure times the *exponential-server timing channel* (ESTC). The capacity of the ESTC with service rate μ is μ/e nats per unit time. This capacity is not increased by feedback.

Arikan [4] shows that the reliability functions of the IPC and ESTC coincide between the critical rate and capacity, and that below the critical rate, the reliability function of the ESTC is no smaller than the reliability function of the IPC. Recently, it has been shown that in the low-rate regime, the ESTC is strictly more reliable than the IPC without feedback, but strictly less reliable than the IPC with feedback [5]. Thus in the low-rate regime, using the IPC with feedback to emulate a queue is strictly suboptimal compared to the best feedback codes, but strictly better than not using feedback at all.

Here we consider the reliability function of these channels above capacity. For $\mu > 0$ and $R > \mu/e$, define the function $E(R,\mu)$ parametrically by

$$E(R,\mu) = \frac{\mu}{(1+\rho)^{(1+\rho)/\rho}}[\rho - \log(1+\rho)]$$

$$R = \frac{\mu}{\rho(1+\rho)^{(1+\rho)/\rho}} \log(1+\rho),$$

where the parameter ρ takes values in $(-1,0)$ (c.f. [4, Proposition 1]). Our main result is that at rate $R > \mu/e$, the optimum probability of correct decoding tends to zero exponentially fast as the blocklength tends to infinity, with exponent $E(R,\mu)$, for the IPC, IPC with feedback, ESTC, and ESTC with feedback. This situation should be compared with that of DMC's, where the reliability function above capacity is unchanged by the introduction of feedback, even though feedback is known to increase the reliability at low rates in some cases.

The proof follows its counterpart for DMC's [6, Ex. 2.5.16], except that we have translated all type-theoretic combinatorics into change-of-measure arguments involving Girsanov's theorem. In particular, the proof avoids approximating the IPC by a DMC, which was the approach taken in early work on the reliability of the IPC. Burnashev and Kutoyants [7] also avoid this reduction in their proof of various error exponent bounds for the IPC without feedback. Their strong converse does not yield the optimum exponent of the probability of correct decoding, however, and their approach requires showing that, without loss of optimality, input signals to the IPC can be assumed to be piecewise constant and to take values in the binary set $\{0,\mu\}$ [7, Proposition 1] [1, Theorem 2.1]. In addition to yielding the optimum exponent, our approach dispenses with the need to show this explicitly. The proof reveals that the Wyner code [1] is optimum for the IPC above capacity, and that it is optimum to operate the ESTC in the stable regime even when attempting to communicate above the queue's Shannon capacity.

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References

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