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AN EFFICIENT METHOD FOR SIMULATING FRACTIONAL STABLE MOTION

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ABSTRACT

An efficient methodology for simulating paths of fractional stable motion is presented. The proposed approach is based on invariance principles for linear processes. A detailed analysis of the error terms involved is given and the performance of the method is assessed through an extensive simulation study.

1 INTRODUCTION

The extreme complexity of modern communication and computer networks, coupled with their traffic characteristics -heavy tails, self-similarity and long range dependence (Paxon and Floyd 1995); Willinger et al. 1995) makes the characterization of their performance through analytical models an extremely difficult task. Under such circumstances, simulations become one of the most promising tools for understanding the behavior of such networks (Park and Willinger 1995; Yuksel et al. 2000). One then must be able to generate traffic that exhibits the necessary temporal behavior over large time scales (Norros 1995; Norros, Mannersalo and Wang 1999).

One of the simplest models exhibiting long-range dependence is fractional Brownian motion (fBM) introduced by Kolmogorov (1940) and further developed by Mandelbrot and Van Ness (1969). It is a Gaussian, non-stationary, self-similar process indexed by a parameter H. The self-similar nature of fBM has made it particularly attractive for using it as an input process when simulating queueing networks (Paxon 1997). However, several traffic measurement studies do not show an agreement with the Gaussian marginal distribution assumption. There exists empirical evidence supporting a heavy tailed assumption (Paxson and Floyd 1995) backed by theoretical work that explains how the former assumption induces through an appropriate mechanism long range dependence in the aggregate traffic (Konstantopoulos and Lin 1998). Therefore, researchers have focused their attention on a more general process exhibiting in a natural way both scaling behavior and extreme local irregularity. Fractional stable motion (also known as fractional Levy motion) is a process with such characteristics. Fractional stable motion can be thought of as the generalization of fBM (Samorodnitsky and Taqqu 1994) and is characterized by two parameters: the Hurst parameter H that measures the degree of the long-range dependence of the process and the Levy parameter α that measures the heaviness of the tails of the marginal distributions. When $\alpha = 2$ (i.e. the marginal distributions are Gaussian) we recover fBM. Over the last few years there have appeared studies of queuing performance under a self-similar stable motion input (Gallardo, Makrakis and Orozco-Bardosa 2000; Giordano, Porcarelli and Procissi 2000; Harmantzis, Hatzinakos and Katzela 2001; Karasaridis and Hatzinakos 2000).

The objective of the present study is to introduce an efficient method for simulating fractional stable motion, which is also of extreme theoretical interest given the underlying intricacies of the process involved. The literature over the last decade has focused on methods for simulating sample paths for fBM (our special case for $\alpha = 2$). The main difficulty in simulating sample paths of fBM lies in the non-stationary nature of the process, and the various proposed methods approach the problem from different viewpoints. For example, there are methods that are based on the properties of fBM, such as through its stochastic representation (Mandelbrot and Van Ness 1969) or the fractional integration of Gaussian white noise (Abry and Sellan 1996), or through matching its covariance function; other methods attempt to first synthesize the increments process and then generate a realization of fBM by calculating the cumulative sums process of fractional Gaussian noise, such as those of Levinson (Coeurjolly 2000) and of Wood and Chan (Dietrich and Newsam 1997; Wood and Chan 1994). These methods take advantage of the fact that the increment process is a stationary one and its covariance matrix is a Toeplitz matrix. Finally, there are methods that rely on approximations of fBM; for example, the method of Flandrin (1992) involves computing the wavelet coefficients corresponding to wavelet transform of fBM and then synthesize fBM through the inverse wavelet transformation, while the random midpoint displacement method (Leland et al. 1994) progressively subdivides the interval over which a sample path is generated and at each subinterval a Gaussian displacement is used to determine the value of the process at the midpoint. Other approximate methods use queueing models and renewal processes to generate fBM at the limit (Willinger et al. 1995). Unlike fBM, there have been hardly any proposals for efficiently generating sample paths from a fractional stable motion.

The main shortcomings of most of these methods are either their time and memory complexity (e.g. $\mathcal{O}(n^2)$, where n is the number of points being generated, see Coeurjolly (2000), or the fact that due to the approximate nature of the method, the quality of the generated process can not be accurately assessed (e.g. Paxson 1997). However, the simulation study of Coeurjolly (2000) suggests that the method of Wood and Chan (1994) with a time complexity of $\mathcal{O}(n \log n)$ and memory complexity of $\mathcal{O}(n)$ performs satisfactorily.

Our approach allows one to generate a fractional stable process efficiently, since it has linear memory requirements and a competitive time complexity (for details see Section 2). The paper is organized as follows: in Section 2, the theoretical development and the proposed algorithm are given. In Section 3, a simulation study assesses the performance of the proposed methodology, while in Section 4 some concluding remarks are drawn.

2 THE ALGORITHM AND A LIMIT THEOREM

Let $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ be a sequence of independent and identically distributed (iid) symmetric α-stable (SaS) (see Samorodnitsky and Taqqu 1994) random variables, with $\alpha \in (0,2]$. Define the sequence $\{a_n\}_{n\in\mathbb{Z}_+}$ with $a_1 = 1$ and $a_n = n^\beta - (n-1)^\beta$, for $n \geq 2$ and $\beta < 1/2$. Also, define $s_n = n^\beta$ for all $\beta \geq 1$.

The one-sided linear process (or MA(∞) process, see Avram and Taqqu 1986) is given by

$$X_n = a_1\varepsilon_{n-1} + a_2\varepsilon_{n-2} + a_3\varepsilon_{n-3} + \ldots = \sum_{i\in\mathbb{Z}_+} a_i\varepsilon_{n-i}. \quad (1)$$

Under the condition $\sum_{i=1}^{\infty} |a_i|^\alpha < \infty$, namely $H := 1/\alpha + \beta < 1$, the process exists almost surely (Avram and Taqqu 1986).

Let $S_n(t) = \sum_{j=1}^{\lceil nt\rceil} X_j = \sum_{i=0}^{\infty} X_i(\sum_{j=1-i}^{\lceil nt\rceil - i} a_j)$, $0 < t \leq 1$ denote the partial sum process of X_n. Under an appropriate set of conditions on the coefficients a_i, a properly normalized version of $S_n(t)$ converges in the sense of finite dimensional distributions to a self-similar process $\xi_{H,\alpha}(t)$ that is called *fractional stable process* or *fractional Levy motion* (Samorodnitsky and Taqqu(1994)). The limiting process has continuous paths and its self-similarity parameter is given by $H = 1/\alpha + \beta$. In the special case where the innovations $\{\varepsilon_i\}$ have finite variance, then a similar type of result established in Davydov (1970) shows that the limiting process corresponds to *fractional Brownian motion*. In the following graphs, sample paths of fractional stable motion for different values of the parameters α and H are shown. In Figure 1, sample paths of fractional stable motion for different values of the Hurst and Levy parameters (H and α) are shown. It can be seen that for $H = .1$ the sample paths for fBM are less regular than those of ordinary Brownian motion, while for $H = .9$ the sample paths are more regular, since in both cases they correspond to a fractal curve with dimension $D = 2 - H$ (Flandrin 1992). A similar feature can be observed for the paths of the fractional stable motion, as well. Moreover, the heaviness of the tails of the marginal distribution induces very large bursts, especially for $\alpha < 1$.

In what follows we will make use of these two results and the representation provided in (1) in order to give an efficient algorithm for simulating the sample paths of a fractional stable process. However, in principle the linear process has an infinite number of terms. Therefore, in practice we are forced to use a finite number of terms, which leads to using a truncated version of the linear process. Moreover, in order to speed up calculations we embed the coefficients a_i in a circulant matrix (an idea also used in Dietrich and Newsam (1997) as shown below. The main issue then becomes to decide on the number of terms to be used in order to achieve a satisfactory approximation and to provide an estimate of the error term.

Define for fixed $n,\ m \in \mathbb{Z}_+$

$$\begin{pmatrix} Y_{m,1} \\ Y_{m,2} \\ \ldots \\ Y_{m,m} \end{pmatrix} = A \begin{pmatrix} \varepsilon_0 \\ \varepsilon_{-1} \\ \ldots \\ \varepsilon_{1-m} \end{pmatrix}, \quad (2)$$

where

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{m-1} & a_m \\ a_2 & a_3 & \ldots & a_m & a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_m & a_1 & \ldots & a_{m-2} & a_{m-1} \end{pmatrix} \quad (3)$$

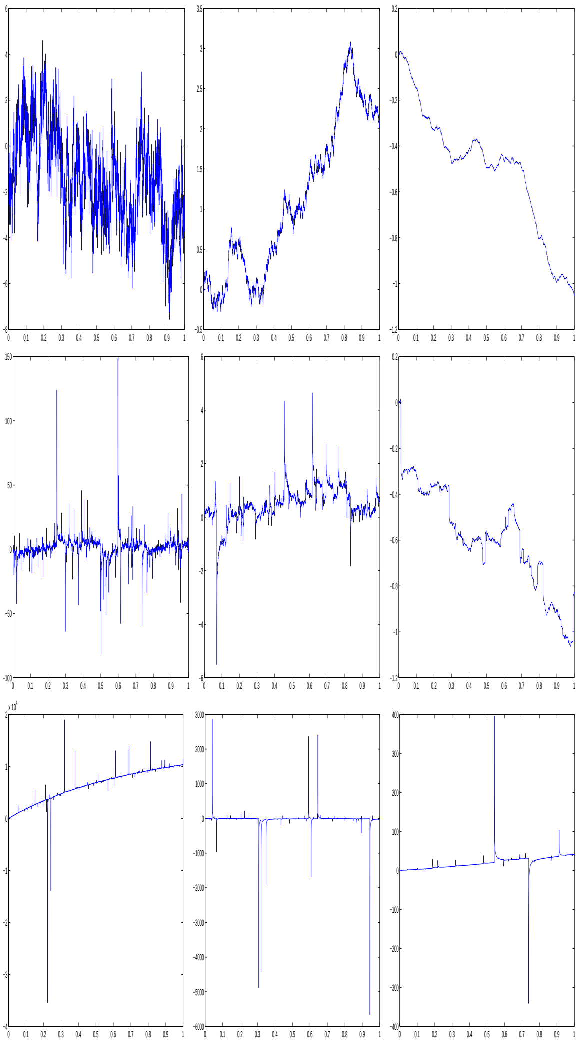

Figure 1: Top Panel: Realizations of fBM. Middle and Bottom Panels: Realizations of Fractional Stable Motions for $\alpha = 1.2$ and $\alpha = 0.6$. In all cases, Left Panel $H = .1$, Middle Panel: $H = .5$ and Right Panel $H = .9$

is a $m \times m$ circulant matrix (Dietrich and Newsam 1997). Let $S_n^Y(t)$, $0 \le t \le 1$ be a stepwise constant function such that $S_n^Y(k/n) = \sum_{i=1}^{k} Y_{m,i}$, $0 \le k \le n$. In the sequel $S_n^{\xi}(t)$, $S_n^X(t)$ and $S_n^U(t)$ etc are similarly defined. Let also

$$\sigma = [(\alpha\beta + 1)^{-1} + \int_0^{\infty} |(u+1)^{\beta} - u^{\beta}|^{\alpha} du]^{1/\alpha}.$$

Theorem 1. *Suppose that n and m are chosen so that $n/m \to 0$. Then,*

$$\frac{S_n^Y(t)}{n^H \sigma} \overset{f.d.d}{\Longrightarrow} \xi_{H,\alpha}(t), \ \ 0 \le t \le 1 \tag{4}$$

where $\overset{f.d.d}{\Longrightarrow}$ denotes convergence of the underlying finite dimensional distributions.

Proof. Since $\{\varepsilon_i\}_{i \in \mathbb{Z}}$ are iid symmetric α-stable, $a_1 = 1$ and $a_n = n^{\beta} - (n-1)^{\beta}$, it follows easily from classical results on functional limit theorems of moving averages of stable random variables that

$$\frac{S_n^X(t)}{n^H \sigma} \overset{\text{f.d.d}}{\Longrightarrow} \xi_{H,\alpha}(t), \ \ 0 \le t \le 1. \tag{5}$$

(see, for example, Astrauskas 1983 and Avram and Taqqu 1986) where more general situations were discussed). In the sequel we shall show that $S_n^Y(\cdot)$ asymptotically approximates $S_n^X(\cdot)$ in distribution. This approximation will be done in several steps. The details are included in Wu, Michailidis and Zhang (2002) and omitted here. ◇

2.1 Approximation Errors

From the proof of Theorem 1, it can be that the convergence (4) involves certain errors that converge to 0 in distribution. We can further establish that an upper bound for the errors is

$$\mathcal{O}[\frac{1}{n} + \frac{1}{n^{\alpha H}} + (\frac{n}{m})^{1-\beta}]. \tag{6}$$

Details of the derivation are given in Wu, Michailidis and Zhang (2002). This upper bound shows that, with all other parameters held constant, it is a decreasing function of the embedding dimension m, of the heaviness of the tails of the marginal distribution α, and of the Hurst parameter H.

2.2 The Algorithm

Based on the above discussion the following algorithm will generate sample paths of a fractional stable motion. The $m \times 1$ vector Y in (2) can be easily computed by fast Fourier transforms (FFT). Let

$$\mathbf{a} = (a_1, \ldots a_m) \quad \text{and} \quad \mathbf{e} = (\varepsilon_0, \varepsilon_{1-m}, \varepsilon_{2-m}, \ldots, \varepsilon_{-1})$$

and let $\bar{\mathbf{a}} = \texttt{FFT}(\mathbf{a})$ and $\bar{\mathbf{e}} = \texttt{FFT}(\mathbf{e})$ be their corresponding discrete Fourier transforms. Then due to the fact that the $m \times m$ matrix A is circulant (Dietrich and Newsam 1997) we have

$$Y = \texttt{IFFT}(\mathbf{v}), \ \text{where } \mathbf{v} = (v_1, \ldots, v_m) \text{ and } v_j = \bar{\mathbf{a}}_j \bar{\mathbf{e}}_j. \tag{7}$$

The proposed algorithm can be summarized as follows:

Step 1: Compute FFT_A; i.e. the FFT of the vector $[a_i, 1 \le i \le m]$

Step 2: Compute FFT_B, the FFT of the vector $[\varepsilon_i, 1 \le i \le m]$

Step 3: Let $\text{FFT}_\text{C} = \text{FFT}_\text{A} \times \text{FFT}_\text{B}$
Step 4: Apply the Inverse `IFFT` on FFT_C to obtain the coefficients $Y_{m,j}$, $j = 1,\ldots,m$.

Then, by calculating the cumulative sum of the $Y_{m,j}$, $1 \leq j \leq n$ and rescaling it by $n^{\beta+1/\alpha}\sigma$ we obtain the desired sample paths. For generating the SaS variates one can use the algorithm of Chambers et al. in 1976. It is important to note that this algorithm requires $\mathcal{O}(m)$ memory space and has $\mathcal{O}(m \log m)$ time complexity (as a result of the use of the `FFT`).

Another distinguishing feature of this algorithm is that it allows one to simultaneously obtain $L = \lfloor m/n \rfloor$ fractional stable motions, thus significantly reducing the cost for producing input traces for network simulations. Actually, the L vectors $[Y_{m,j},\ 1 \leq j \leq n]$, $[Y_{m,j},\ n+1 \leq j \leq 2n]$, $[Y_{m,j},\ 1+(L-1)n \leq j \leq Ln]$ are identically distributed by appropriate permutations on ε_i. Then L identically distributed fractional stable motions are obtained by taking cumulative sums of each $Y_{m,j}$ vector.

3 PERFORMANCE ASSESSMENT

In this section we provide empirical evidence about the performance of the proposed method. Some special attention is paid to the case of $\alpha = 2$ (corresponding to fBM), since comparisons can be made with other existing methods. In our simulation study 100 samples of size $n = 5,000,\ 50,000$ and 100,000 were generated using two different values of $m = 2^{20}$ and 2^{22} for the following values of the Hurst parameter $H = .1, .2, .3, .4, .5, .6, .7, .8$ and $.9$ and for $\alpha = .5, 1, 1.5$ and 2. Overall 216 combinations of the four parameters involved in the method were examined. Notice that for the combination $\alpha = 2$ and $H = .5$ the resulting process corresponds to ordinary Brownian motion, for which many other methods for generating sample paths exist. The Hurst index of the generated traces was estimated based on the wavelet method Abry and Veitch (1998) that has been shown to be unbiased and robust against deterministic trends. For the case $\alpha = 2$, the normality and stationarity of the increments process was also examined and the estimation of the Hurst index was also confirmed using the discrete variations method (Istas and Lang 1997; Kent and Wood 1997; Coeurjolly 2000). For the case $\alpha < 2$, the heaviness of the tails of the increments process was also assessed using various graphical methods, such as the Hill, QQ and sum-plots (Sousa and Michailidis 2001).

Boxplots for some indicative cases are given in Figure 2. It is worth noting the following facts: (i) the variability is considerably smaller for larger values of the Levy parameter, (ii) the variability is also smaller for values of the Hurst parameter around .5, (iii) the resulting estimates for $H < .4$ are upwards biased, while they are downwards biased for $H > .6$ and for smaller values of α and (iv) the effect of

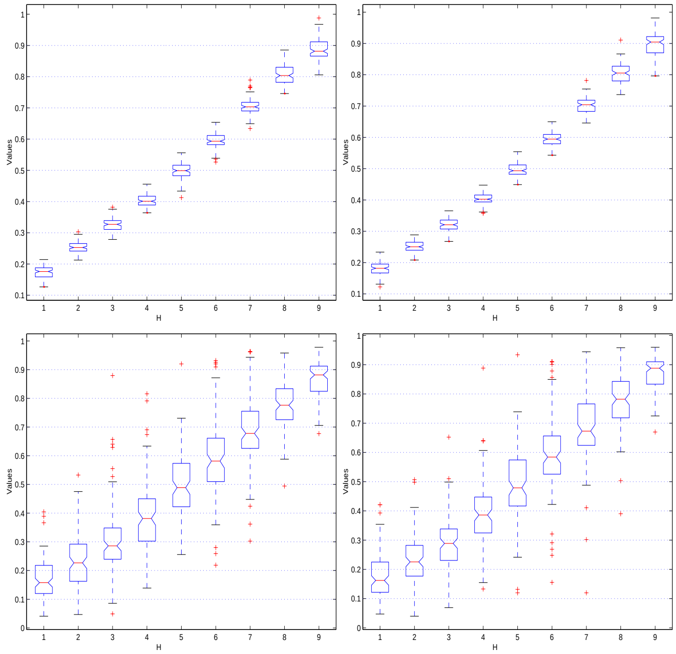

Figure 2: Boxplots for Different Values of H for $n = 5000$, Left Panel: $m = 2^{20}$, Right Panel: $m = 2^{22}$. Top Panel: $\alpha = 2$, Bottom Panel: $\alpha = 1.5$

m is rather marginal on the results. It should be noted that the embedding dimension m considered in the simulations is fairly large (i.e. $m \geq 2^{2}0$); this would not be the case for values of m close to number of generated points n, as our error bound shows (see (6)). The first three findings are consistent with our estimate of the error rate (see also Remark in the previous section).

The mean squared error (MSE) of the estimates for $n = 5000$ and for $m = 2^{20}$ and $m = 2^{22}$ are given in Figure 3. It can be seen that the MSEs are ordered according to the value of α (higher MSEs for smaller values of α) as expected. Moreover, it can be seen that for smaller values of H the higher MSEs are due to higher biases (see also Figure 2), while for values of H in the neighborhood of .5 the MSE is mainly due to the variance of the estimates, while for higher values of H both the bias and the variance make a contribution. Analogous qualitative results are obtained for $n = 50,000$ and $n = 100,000$.

4 CONCLUDING REMARKS

An efficient method both in terms of time and memory complexity based on invariance principles for linear processes for generating sample paths of fractional stable motion is presented. Extensive simulation results confirm its good performance, especially for values of the Hurst parameter larger than .5, which corresponds to the case of interest

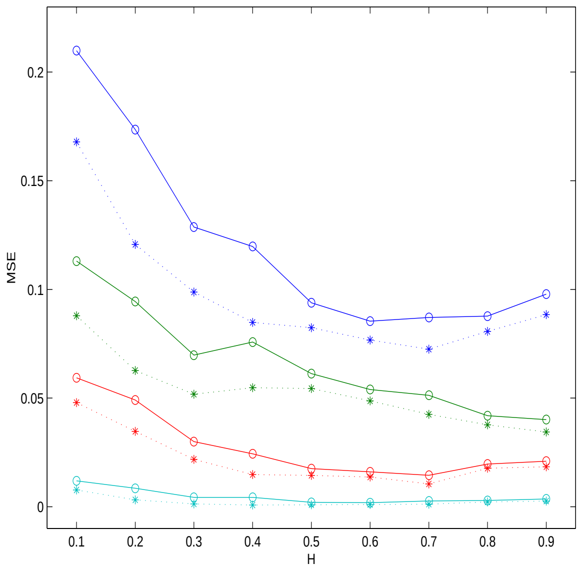

Figure 3: MSE for $n = 5000$. Dashed lines correspond to $m = 2^{22}$, while solid lines to $m = 2^{20}$

when it comes to network simulations. The numerical results indicate that the embedding dimension m should be large enough, but after a certain point the gains from using larger values are fairly small. Finally, the proposed method allows one to generate multiple identically distributed traces, which leads to additional savings.

Several additional research threads are currently being pursued, including the use of similar ideas in the generation of higher dimensional processes which are of interest to hydrologists and environmental scientists and more extensive simulation studies and comparisons with competing methods especially for the fBM case.

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