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Robustness of control systems designed by polynomial methods*

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The polynomial approach to controller design has been applied to systems with double-integrator plants in continuous time. The trade-off between relative stability, costs, and controller complexity is discussed for a selection of disturbance models and cost functions. Comparisons are made with systems designed by the minimax frequency-domain optimization method.

1. Introduction

The essential features of an error-actuated feedback control system are easily stated.

(1) The system must be stable.
(2) The output must differ from that demanded by an appropriately small amount.

There are further features which may be considered desirable, including the following.

(3) Disturbances entering the system are to have minimum effect on the system error.
(4) The system should respect constraints, arising from practical limitations, on the power or amplitude of signals at various points.
(5) The system should remain stable when operating conditions differ moderately from the design assumptions.

Fuller (1976) has investigated optimal controllers for the regulation of scalar plants of complexity up to double-integrator, subject to stochastic disturbances at the plant input and at the controller input. Fuller's design approach was that of Weiner, and his treatment of the individual cases was algebraic. The Weiner process required the partial-fraction expansion of terms, and this step can cause problems with numerical ill-conditioning. The polynomial method of Kučera (1980) avoids this step by solving linear diophantine equations. The polynomial method of design guarantees absolute stability of the resulting closed-loop system when suitable spectral factors D and E exist. Newmann & Roberts (1990) have shown how to test for the existence of such spectral factors. Grimble (1986) has shown that absolute stability is maintained even when there are modelling errors in the disturbances.

Both Weiner and the polynomial design methods minimize a weighted sum of square measures of system-error and control-effort signals. Thus point 3 above may be satisfied and, by appropriate choice of weights, point 4 also.

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Robustness of systems with respect to changes in the plant is defined quantitatively in terms of phase margin and gain margin, and is studied in a qualitative manner using Nyquist diagrams. The minimax frequency-domain optimization method given by Kwakernaak (1985) has robustness of the closed-loop system as one of its main objectives. Thus systems may be designed which satisfy point 5 as well.

The state-space design approach has been used by Lin & Mingori (1992), who define performance robustness in terms of the sensitivity of system-error and control-effort variances to plant parameter variations. They achieved robustness enhancement by introducing additional terms reflecting the structure of the parameter errors into the cost function.

The paper is ordered as follows. Section 2 describes the system design and analysis model and the cost functions. Section 3 presents Nyquist diagrams, relative stability margins, and total cost evaluations for the chosen systems, which are then discussed in Section 4. The concluding Section 5 contains some general comments. Worked examples of controller design by the polynomial method and by minimax frequency-domain optimization are given in the appendix.

2. System model and cost function

Figure 1 shows the general structure of the systems to be studied. R and S are transfer functions and S_d, S_n, and S_r are the shaping transfer functions for independent sources of white noise.

The cost function which is minimized by the design procedure in the first four of the examples and in the first example in the appendix is given by

$$V^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (K W_u + K u_r u_{r*} + W_e + e_r e_{r*}) \, ds, \tag{2.1}$$

where K is a positive weight, W_u and W_e are the power-spectral densities of the stochastic components of the plant input and error signal, respectively, and u_r and e_r represent the Laplace transforms of their deterministic components. The $*$ operation replaces the variable s by $-s$. Note that, when evaluated, the stochastic part of the cost function is a variance while the deterministic part is an integral square measure with respect to time from zero to infinity.

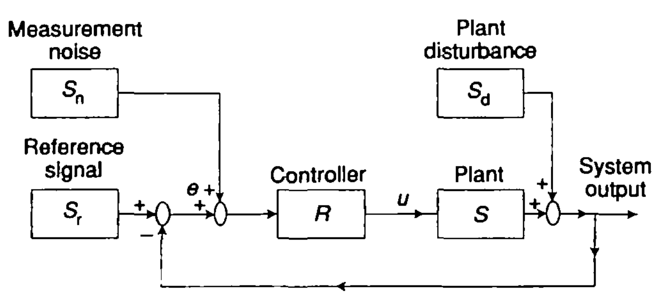

FIG. 1. System structure.

In all the examples considered in the main body of the text, the plant is a double-integrator, and the output disturbance includes the plant as a factor and therefore may be considered as entering the system at the plant input. The weighting between system error and control effort may be adjusted by the factor K in the cost function (2.1). The value $K = 1$ has been chosen in this particular study. Some of the features of the systems studied are now listed.

Stochastic optimal regulator The measurement noise is white gaussian with constant power-spectral density N^2. The plant disturbance is white gaussian noise of unit power-spectral density applied at the plant input, while $S_n = N$, $S_d = 1/s^2$, and $S_r = 0$.

Stochastic optimal tracker Roberts *et al.* (1991) have shown how to extend the design process to take into account a deterministic input to be followed by the system output. In this case, the input is a step of magnitude L, with $S_n = 1$, $S_d = 1/s^2$, and $S_r = L/s$.

Frequency-band-weighted system A frequency-restricted stochastic reference signal is generated by passing white noise through a first-order Butterworth filter with lower and upper corner frequencies of 0.5 and 1.0 rad/s. For this example, $S_n = 1$, $S_d = 1/s^2$, and $S_r = Fs/(1 + s + 2s^2)$.

Fuller-type system Fuller (1976) studied a system in which the measurement noise tends to constant power-spectral density at high frequencies and is considered to be 'nearly' white. For this system, $S_n = Nbs/(1 + bs)$, $S_d = 1/s^2(1 + bs)$, and $S_r = 0$, where the parameter b was taken as 10.

Minimax frequency-domain example This is based on an example given by Kwakernaak (1985). In this case, steps had to be taken to ensure that the controller was strictly proper. This was necessary to maintain a finite value for cost function (2.1) in the presence of white measurement noise. The cost function minimized by this design procedure is

$$\sup_{\omega}[|V(j\omega)\Sigma(j\omega)|^2 + |W(j\omega)T(j\omega)|^2], \tag{2.2}$$

where $\Sigma = 1/(1 + RS)$, the sensitivity function, and $T = RS/(1 + RS)$, the complementary sensitivity function. V and W are suitably chosen weighting functions. Some of the design details are included in the second part of the appendix.

System-evaluation model In all the examples, the model used for cost evaluation of the designed systems is as in Fig. 1 with $S = 1/s^2$, $S_d = 1/s^2$, $S_r = 1/s$, $S_n = 1$.

3. Results

For scalar systems, the polynomial design process is particularly simple and has been implemented in Fortran on an Opus PCVI personal computer. Algorithms for the majority of the design steps have been given in Kučera's earlier work (1979) and that for spectral factorization by Ježek & Kučera (1985). The latter is the only iterative process in the design procedure.

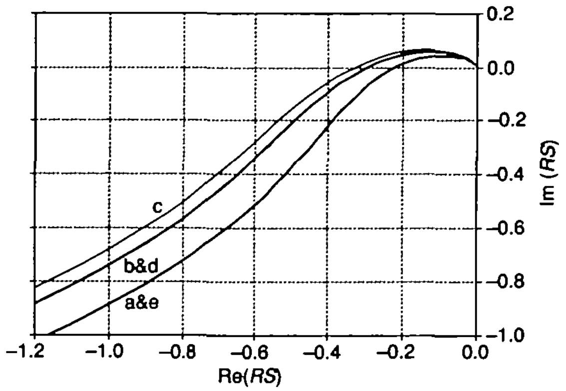

FIG. 2. Nyquist plots for systems designed with different noise intensities: (a) $N = 10$, (b) $N = 3.16$, (c) $N = 1$, (d) $N = 0.316$, and (e) $N = 0.1$.

Cost components are evaluated by an algorithm described by Effertz (1966). Estimates of the gain margin and phase margin are obtained by relatively direct application of their definitions.

Stochastic optimal regulator The Nyquist diagrams for systems optimized for different levels of measurement noise are shown in Fig. 2. The one with the least gain and phase margins corresponds to a noise level in the middle of the range. In fact, it is interesting to note that the plots for $N = 0.1$ and $N = 10$ (and for $N = 0.316$ and $N = 3.16$) appear to coincide even though the characteristic polynomials of the closed-loop systems are different. The controllers are strictly proper of the form degree 1 over degree 2.

Stochastic optimal tracker Figure 3 shows the Nyquist diagrams for systems optimized for different magnitudes of step input. Gain and phase margins increase

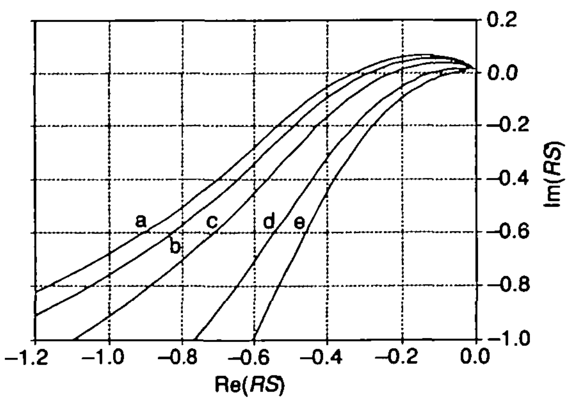

FIG. 3. Nyquist plots for systems indexed by step amplitude L: (a) $L = 0$, (b) $L = 1$, (c) $L = 2$, (d) $L = 5$, and (e) $L = 10$.

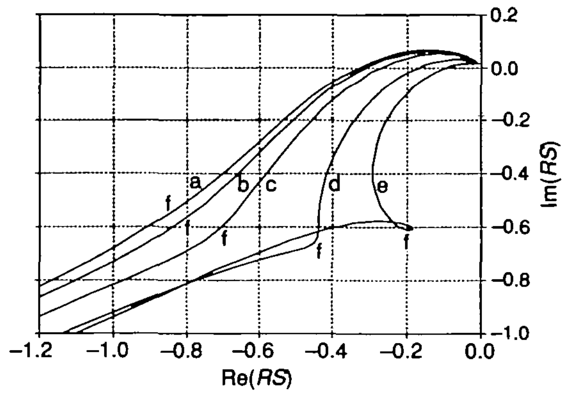

FIG. 4. Nyquist plots for reference frequency band weighted systems indexed by F of Section 2: (a) $F = 0$, (b) $F = 1$, (c) $F = 2$, (d) $F = 5$, and (e) $F = 10$. Points labelled f are at 0.75 rad/s.

along with the step input magnitude. The system designed with $L = 1$ is the optimal system for use with the system evaluation model. The controllers are strictly proper of the form degree 1 over degree 2.

Frequency-band-weighted system Figure 4 shows the Nyquist diagrams for systems optimized for different magnitudes F of frequency-band-weighted reference signal. The controllers are strictly proper of the form degree 3 over degree 4. It should be noted that the region of the plots near to the critical frequency f moves further away from the point $(-1, 0)$ for increasing values of the design parameter F.

Fuller system Figure 5 shows the Nyquist diagrams for Fuller's optimal systems for different measurement noise levels. In this case, gain and phase margins increase along with noise magnitude. The controllers are strictly proper of the form degree 2 over degree 3.

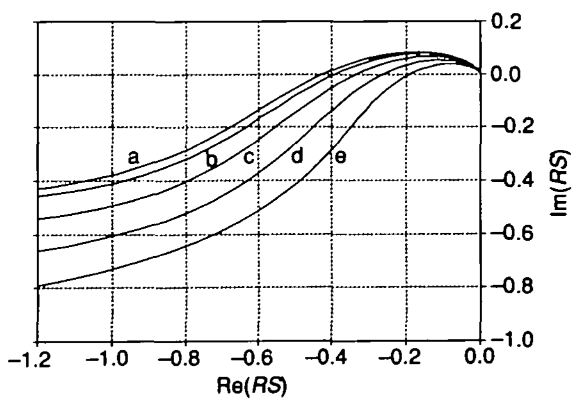

FIG. 5. Nyquist plots for Fuller systems indexed by measurement-noise parameter N: (a) $N = 0.1$, (b) $N = 0.316$, (c) $N = 1$, (d) $N = 3.16$, and (e) $N = 10$.

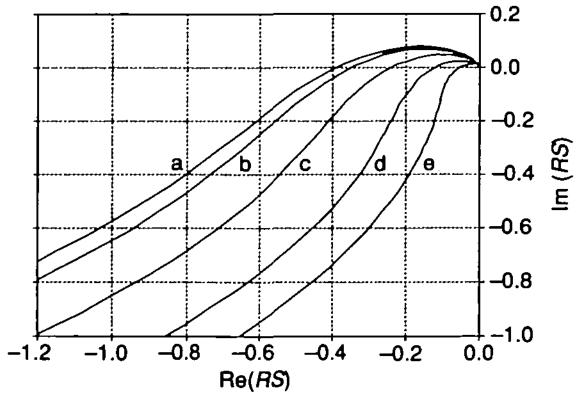

FIG. 6. Nyquist plots for minimax frequency-domain examples indexed by parameter p: (a) $p = 10^2$, (b) $p = 1$, (c) $p = 10^{-2}$, (d) $p = 10^{-4}$, and (e) $p = 10^{-6}$.

Minimax frequency-domain example Figure 6 shows the Nyquist diagrams designed by the minimax frequency-domain method of Kwakernaak. The gain and phase margins are seen to increase along with reduction in parameter p (see Appendix A2). The controllers are strictly proper of the form degree 1 over degree 2.

Variation in cost with phase and gain margin In order to compare the performances of the systems studied, the total cost, V, evaluated using equation (2.1) and the system-evaluation model of Section 2, has been plotted against phase margin in Fig. 7 and against gain margin in Fig. 8. The minimum cost is associated with the optimal tracker design based on the system-evaluation model.

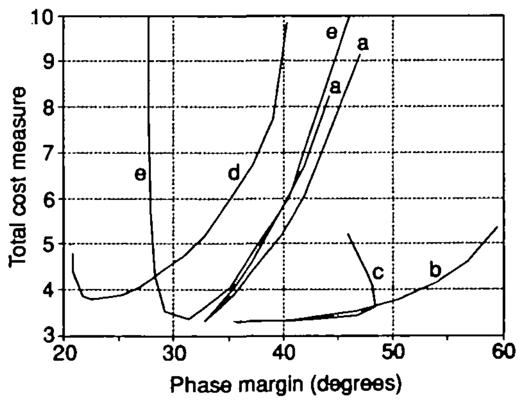

FIG. 7. Variation of total cost with phase margin: (a) stochastic optimal regulator, (b) stochastic optimal tracker, (c) frequency-band-weighted system, (d) Fuller system, and (e) minimax frequency-domain examples.

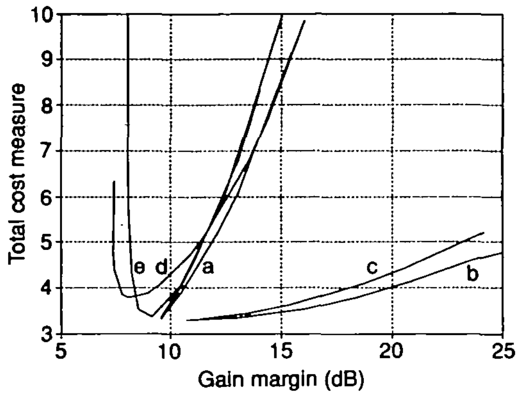

FIG. 8. Variation of total cost with gain margin: (a) stochastic optimal regulator, (b) stochastic optimal tracker, (c) frequency-band-weighted system, (d) Fuller system, and (e) minimax frequency-domain examples.

4. Discussion

Fuller's assertion that system robustness is enhanced by including increased measurement noise at the design stage, even though it may not exist in practice, is verified by Fig. 5 for one of Fuller's systems. However, when the measurement noise is white, Fig. 2 shows that, at least in the measurement-noise range $N = 0.1$ to $N = 1$, the effect of increasing the measurement noise is to reduce the gain and phase margins.

For simple systems, where the closed-loop response is second-order, Phillips & Nagle (1990) point out that step response and phase margin are related, and they give an expression for phase margin in terms of the damping ratio. The inclusion in the cost function of a term relating to step costs has consequences for the relative stability of higher-order systems as shown by Fig. 3. It happens that, with the systems chosen, there are two pairs of complex roots in the system characteristic polynomial. This implies that the systems exhibit underdamped responses and that 'improving' the step response leads to increased damping and therefore to a more stable system. Experiments have shown that the converse is the case when the 'basic' regulator system response is overdamped. The first example worked in the appendix shows how the damping ratios of the system characteristic polynomial factors depend on the design parameters for the optimal tracker system.

Controller complexity is greater in the Fuller system than in the regulator or tracker systems. This is due to the extra mode, $1/(1 + 10s)$, with a time constant of 10 seconds, in the disturbances but not in the plant. The increased complexity introduces a second gain margin at low frequencies. If the extra terms in the Fuller system were introduced solely to allow the relative stability to be adjusted, then the disadvantages in terms of costs and complexity may outweight the advantages. Figure 4 shows that there is some improvement in relative stability when a frequency-dependent term is introduced into the cost function. The lower and upper frequencies were chosen to be close to the critical frequency for phase margin in the 'basic' regulator system. The total costs are of a similar magnitude to those involved in transient weighted systems, while the controllers are rather more complex. The increased complexity in this case may introduce an extra phase margin.

Figure 7 shows that the trade-off of cost against phase margin is similar, for realistic phase margins greater than 30°, for the optimal regulator and for the minimax frequency-domain designs. The plots of cost against gain margin shown in Fig. 8 demonstrate that the optimal-regulator, Fuller-system, and minimax frequency-domain designs have similar cost per extra dB of gain margin above a gain margin of 10 dB.

5. Conclusion

The polynomial approach has been used to design controllers for double-integrator plants when different types of additional terms appear in the cost function. A study of the relative stabilities and costs for the resulting closed-loop systems indicates that, in general, improvements in relative stability are at the expense of increased costs. A similar relationship between enhanced robustness and increased costs has been reported by Lin & Mingori (1992). A useful technique for robustness enhancement is that of transient weighting, and it will not involve a more complex controller if an integrator already exists in the plant.

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Appendix A1: Polynomial optimization

To illustrate the polynomial method, it will be applied to the design of a controller for a double-integrator plant subject to white measurement noise, white plant-disturbance noise at the plant input, and a step signal at the reference input. The design is in the spirit of Fuller, and treated algebraically.

The system transfer functions Plant $S = a/s^2 = B_1/A_1$, plant disturbance $= S_d = d/s^2$, measurement noise $= S_n = n$, step input $= S_r = L/s$.

Least common denominator $A = s^2$, with numerators $B = a$, $C_d = d$, $C_n = ns^2$, $C_r = Ls$.

Spectral factorizations $E_* E = A_{1*} k^2 A_1 + B_{1*} B_1 = a^2 + k^2 s^4$, where k^2 is the control effort weight K in the cost function of Section 2. In this case,

$$E = a + (2ak)^{\frac{1}{2}} s + ks^2, \qquad DD_* = C_d C_{d*} + C_n C_{n*} + C_r C_{r*} = d^2 - L^2 s^2 + n^2 s^4,$$
$$D = d + (L^2 + 2dn)^{\frac{1}{2}} s + ns^2.$$

Generally, $D/A = D_3/A_3$, and $B/D = B_4/D_4$, where (D_3, A_3) and (B_4, D_4) are coprime pairs. However, in this case, assuming $d \neq 0$, we have $D_4 = D_3 = D$, $A_3 = A$, and $B_4 = B$.

Diophantine equation $E_* Y + Z_* A_3 = B_{1*} D_3$ is set up and solved for minimum degree in Z_*. The solution is guaranteed because E_* and A_3 are coprime. In this case, denoting 'degree of' by ∂, we have $\partial Y < \partial A_3 = 2$, $\partial Z_* < \partial E_* = 2$, and

$$[a - (2ak)^{\frac{1}{2}}s + ks^2)][Y_0 + Y_1 s] + [Z_0 - Z_1 s]s^2 = a[d + (L^2 + 2dn)^{\frac{1}{2}}s + ns^2].$$

The solution is $Y = \{ad + [a(L^2 + 2dn)^{\frac{1}{2}} + d(2ak)^{\frac{1}{2}}]s\}/a$.

Controller and closed-loop transfer functions The controller transfer function is given by $R = P/Q = YA_1/(ED_3 - YB_1)$, which is

$$\frac{ad + [a(L^2 + 2dn)^{\frac{1}{2}} + d(2ak)^{\frac{1}{2}}]s}{a\{an + dk + [(2ak)(L^2 + 2dn)]^{\frac{1}{2}} + [k(L^2 + 2dn)^{\frac{1}{2}} + n(2ak)^{\frac{1}{2}}]s + nks^2\}}.$$

The closed-loop transfer function is given by YB_1/ED_3, and it is seen that system absolute stability is guaranteed by the condition that E and D have no zeros on, or to the right of, the imaginary axis. ED_3 is the system characteristic polynomial. The design process minimizes the cost by suitable placement of the closed-loop zeros given by YB_1. The closed-loop transfer function is given by

$$\frac{ad + [a(L^2 + 2dn)^{\frac{1}{2}} + d(2ak)^{\frac{1}{2}}]s}{[d + (L^2 + 2dn)^{\frac{1}{2}}s + ns^2][a + (2ak)^{\frac{1}{2}}s + ks^2]}$$

The damping ratio due to the E spectral factor is 0.7071 while that due to the D spectral factor is given by

$$\zeta = \tfrac{1}{2}(L^2/dn + 2)^{\frac{1}{2}}.$$

Appendix A2: Minimax frequency-domain design

To illustrate the minimax frequency-domain design procedure, example 4 of Kwakernaak (1985) has been used in a slightly modified form. The notation used here follows that of Kwakernaak, and equation numbers prefixed with K refer to equations in that paper.

Weighting functions For disturbance attenuation and rejection, $VV^* = (1 + s^4)/s^4$, so that $V = (1 + s\sqrt{2} + s^2)/s^2$; and for bandwidth limitation, $W = (\tau_2 s)^3$. This choice of W makes sure that the controller transfer function is strictly proper. These are equations (K6.12).

The design equations After the substitution $p = (\tau_2)^6$, equations (K6.17), (K6.18), and (K6.20) are

$$\pi_\lambda \pi_\lambda^* = p(-1 + 1/\lambda^2)s^{10} + (p/\lambda^2)s^6 + s^4 + 1, \qquad \chi = s^2\theta + \zeta,$$
$$\lambda^2 \chi\chi^* = (1 + s^4)\theta\theta^* - ps^6\zeta\zeta^*.$$

TABLE A1

p	λ	ζ_0	ζ_1	θ_0	θ_1
10^{-6}	1.11349	328.18	489.74	365.42	27.042
10^{-4}	1.29880	51.421	82.878	66.786	11.574
10^{-2}	1.83178	8.0169	15.347	14.685	5.4446
1	3.44530	1.2553	3.2970	4.3249	2.9656
10^2	7.99270	0.1982	0.7930	1.5840	1.7974

The first equation requires that $\lambda \geqslant 1$ for the spectral factorization to exist. The second equation is the defining equation for the characteristic polynomial of the closed-loop system when the controller transfer function is ζ/θ. The third equation is required in order to satisfy the minimax condition.

Solutions to the equations Substitution of

$$\chi = \chi_0 + \chi_1 s + \chi_2 s^2 + \chi_3 s^3 + s^4, \qquad \zeta = \zeta_0 + \zeta_1 s, \qquad \theta = \theta_0 + \theta_1 s + s^2,$$

followed by comparison of coefficients and elimination of the χ variables, each of which must be non-negative, leads to five equations in six variables, of which the first four are

$$\lambda\zeta_0 = \theta_0, \qquad \lambda^2(2\zeta_0\theta_0 - \zeta_1^2) = 2\theta_0 - \theta_1^2,$$
$$\lambda^2(2\theta_0 - \theta_1^2) = 2\theta_0 - \theta_1^2 - p\zeta_0^2, \qquad \lambda^2 = 1 + p\zeta_1^2.$$

Substitutions involving these four equations give

$$\zeta_0 = \lambda(\lambda^2 - 1)/\{p[2\lambda^3(\lambda^2 - 1) + p]\}^{\frac{1}{2}}, \qquad \theta_0 = \lambda\zeta_0, \qquad \zeta_1 = [(\lambda^2 - 1)/p]^{\frac{1}{2}},$$
$$\theta_1^2 = \zeta_0[2\lambda(\lambda^2 - 1) + p\zeta_0]/(\lambda^2 - 1),$$

with the fifth equation

$$\lambda^2(2\zeta_0 + \theta_0^2 - 2\zeta_1\theta_1) = 1 + \theta_0^2 \tag{A2.1}$$

to be satisfied by appropriate choice of p and λ.

A spreadsheet was set up to evaluate the numerator and denominator, ζ and θ, of the controller transfer function, for inputs of p and λ, and to display the resulting left-hand and right-hand sides of (A2.1). Various values of p were chosen, and the corresponding values of λ were found. Some of the results, along with the controller parameters, are given in Table A1. The controller is given by

$$R = (\zeta_0 + \zeta_1 s)/(\theta_0 + \theta_1 s + s^2).$$

REFERENCES

EFFERTZ, F. H. 1966 On two coupled matrix algorithms for the evaluation of the rms error criterion of linear systems. *Proc. IEEE* **54**, 879–80.

FULLER, A. T. 1976 Feedback control systems with low-frequency stochastic disturbances. *Int. J. Control* **24**, 165–207.

Grimble, M. J. 1986 Controllers for LQG self-tuning applications: robustness and controller reduction. *Control system design—robustness and polynomial methods* (Johnson, M. A., Grimble, M. J., & Owens, D. H., Eds.). Cameleon Press Ltd. Pp. I1–I10.

Ježek, J., & Kučera, V. 1985 Efficient algorithm for matrix spectral factorization. *Automatica* **21**, 663–9.

Kučera, V. 1979 *Discrete linear control*. Chichester: Wiley.

Kučera, V. 1980 Stochastic multivariable control: a polynomial equation approach. *IEEE Trans* AC **25**, 913–9.

Kwakernaak, H. 1985 Minimax frequency domain performance and robustness optimization of linear feedback systems. *IEEE Trans* AC **30**, 994–1004.

Lin, J. Y., & Mingori, D. L. 1992 Linear-quadratic-gaussian synthesis with reduced parameter sensitivity. *Int. J. Control* **56**, 901–22.

Newmann, M. M., & Roberts, A. P. 1990 Polynomial optimization of discrete-time control for unstable plants. *Int. J. Control* **51**, 1363–80.

Phillips, C. L., & Nagle, H. T. 1990 *Digital control systems: analysis and design*. Englewood Cliffs: Prentice-Hall.

Roberts, A. P., Newmann, M. M., & Parkes, N. E. 1991 Polynomial optimisation of multivariable control systems with both stochastic and deterministic inputs. *Proceedings of the 30th IEEE Conference on Decision and Control*. Pp. 1267–71.