The cost of cutting out convex n-gons

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Abstract

Given a convex n-gon P drawn on a piece of paper Q of unit diameter we prove that it can be cut with a total cost of $O(\log n)$. This bound is shown to be asymptotically tight: a regular n-gon (whose circumscribed circle has radius, say, 1/3) drawn on a square piece of paper of unit diameter requires a cut cost of $\Omega(\log n)$.

Keywords: Combinatorial geometry, Complexity of cutting, Convex polygons.

1 Introduction

Overmars and Welzl considered the following problem [3].

> Given a polygonal piece of paper Q with a polygon P drawn on it, cut P out of the piece of paper in the cheapest possible way.

We are in the same framework as in [3]. A *cut* is a line which divides the piece of paper into a number of pieces, those that lie left of the line and those that lie right of the line (see Fig. 1). A cut is not allowed to intersect the interior of P. If the piece of paper is a concave polygon, one has to cut along all intervals where the cutting line intersects the paper. The pieces of paper resulting from a cut are the connected components of the paper minus the cutting line.

After a cut is made we continue with the piece of paper containing P. It is thus understood that if a cut generates multiple pieces, those but the one which contains P are discarded. A *cutting sequence* is a sequence of cuts such that, after the last one, the piece of paper is the polygon P, as illustrated in Fig. 2. The cost of a cut is the length of the intersection of the cutting line with the (current) piece of paper containing P. The problem asks for a cutting sequence whose total cost is minimum. Such a sequence is called

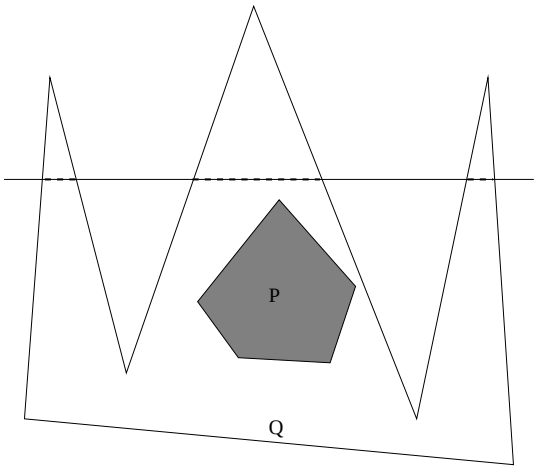

Fig. 1. A straight-line cut which generates four pieces of paper.

an *optimal cutting sequence*. The total cost of an optimal cutting sequence, denoted $h(P, Q)$, is called the *cut cost*. Note that $h(P, Q)$ not only depends on P and Q, but also on the position of P inside Q.

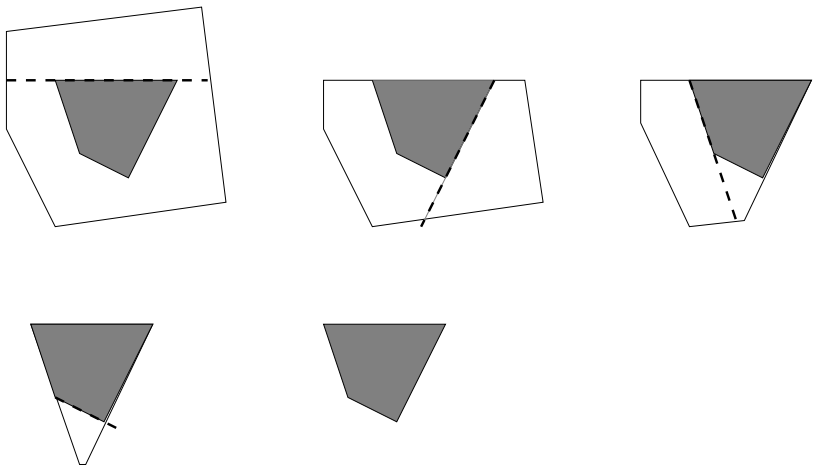

Fig. 2. A cutting sequence.

It is clear that the problem is solvable only if the polygon P is convex, which we will assume. If the piece of paper Q is convex, Overmars and Welzl have shown that there exists an optimal cutting sequence for P with $O(n)$ cuts, in which each cut touches polygon P, where n is the number of edges of P [3]. For the case of non-convex piece of paper Q, if Q is considered a closed set (as opposed to open), there are examples in which no optimal cutting sequence exists. On the other hand, when Q is non-convex and considered an open set, it is unknown whether an optimal cutting sequence exists. A more detailed discussion of these matters appears in [3].

In this note we estimate the maximum cutting cost of a convex polygon with n vertices drawn on a convex piece of paper of roughly the same size. Let

$$h(n) = \sup_{P \subset Q,\ diam(Q)=1} \{h(P,Q) \mid P \text{ is a convex } n\text{−gon},\ Q \text{ is convex}\},$$

where $diam(Q)$ stands for the diameter of Q. (Clearly $h(n) \leq n$ exists, since cutting along each edge of P, in any order, gives a cutting sequence whose cost is not more than n.)

Theorem 1 *$h(n) = \Theta(\log n)$.*

The proof of the upper bound $h(n) = O(\log n)$ in the theorem yields also the following result. If $diam(Q)/diam(P) = O(1)$, there exists an $O(n \log n)$-time algorithm which cuts out P through a sequence of cuts of cost $O(|P| \log n)$, where $|P|$ denotes the perimeter of P. This fact has already been used to obtain a $O(\log n)$-approximation algorithm which runs in $O(Nn + n \log n)$ time, for cutting out a convex n-gon P out of a convex polygon Q with N sides [2].

We briefly mention two related problems. Pach and Tardos have studied the problem of separating a large subfamily from a given family of pairwise disjoint compact convex sets on a sheet of glass, using the same type of line cuts [4]. Recently, Demaine et al. have given a characterization of the class of polygons that can be cut from a piece of material using a sufficiently small cutting segment (which models a circular saw) [1].

2 Proof of Theorem 1

We first prove the upper bound using the following two-stage cutting algorithm.

Stage 1: Perform two parallel horizontal cuts touching P from below and from above. Then perform two parallel vertical cuts touching P from the left and from the right. We can thus assume without loss of generality that Q is the minimum axis-aligned rectangle containing P. Perform at most eight additional edge cuts (along edges of P) at each vertex of P contained in ∂Q (the boundary of Q). The cost of the cuts done in Stage 1 is clearly $O(1)$.

Stage 2: In this stage, each cut is along an edge of P (i.e. we perform an edge cutting). Consider a set $\mathcal{T}$ of triangles, each corresponding to a convex arc of P which remains to be cut. Initially $|\mathcal{T}| \leq 4$. For each triangle, two vertices are midpoints of edges of P supporting consecutive edge cuts along P and the third vertex is the intersection point of those cutting lines (see Fig. 3).

We only include in $\mathcal{T}$ such triangles corresponding to pairs of edges which are themselves not adjacent in P. For each $\Delta \in \mathcal{T}$, we call the edge connecting the two midpoints of edges of P the *base edge* of Δ (see Fig. 3). Each triangle in $\mathcal{T}$ is contained in the current piece of paper. Stage 2 proceeds in rounds, and the set $\mathcal{T}$ is updated during each round. At each round, the triangles in $\mathcal{T}$ satisfy the following properties.

- Each triangle corresponds to two consecutive cuts along P (i.e. none of the edges of the convex arc between these edges have been cut).
- The interior angle of the triangle corresponding to the base edge is at least $90°$.

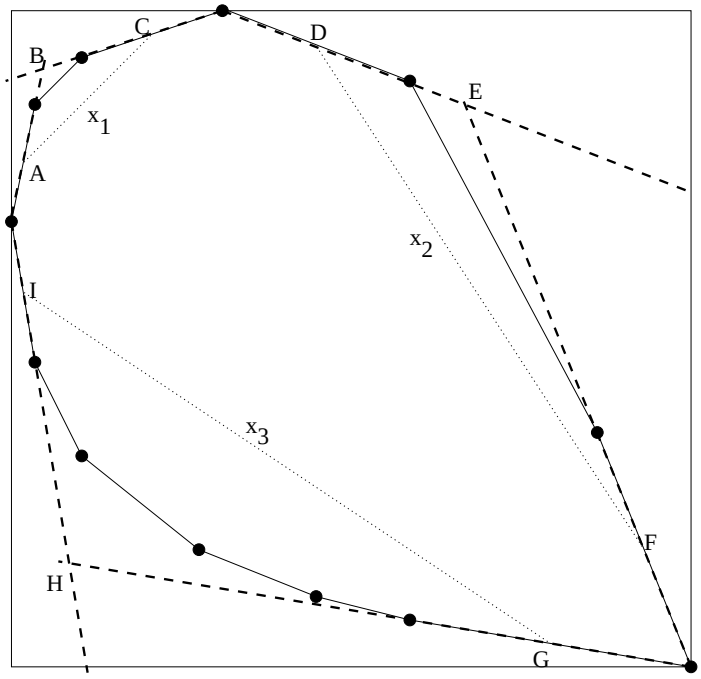

Fig. 3. Proof of the upper bound; triangles ABC, DEF and GHI are initially in $\mathcal{T}$; their base edges are x_1, x_2, x_3.

Consider one such round. For each triangle $\Delta \in T$, perform a cut along the middle edge of the convex arc of P contained in Δ (i.e. if the convex arc has k edges numbered $1, 2, \ldots, k$, cut along edge $\lceil \frac{k+1}{2} \rceil$ counting from one of its ends). Since the interior angle corresponding to the base edge of Δ is at least $90°$, the length of the cut in Δ is bounded by the length of its base edge. We note that the endpoints of the base edges of the triangles in $\mathcal{T}$ form a convex polygon contained in P, so its perimeter is $O(1)$. This implies that the total cost of the cuts in one round is also $O(1)$. Each $\Delta \in T$ generates zero, one or two triangles to be included in $\mathcal{T}$ for the next round, while Δ is removed from $\mathcal{T}$. Clearly, the process terminates after at most $\log n$ rounds (when cuts along all edges of P have been done and $\mathcal{T} = \emptyset$), so the cost of the cuts done in Stage 2 is $O(\log n)$.

Since the cost of Stage 1 is $O(1)$, the total cost of the cutting sequence produced by the algorithm is $O(\log n)$. As each round takes $O(n)$ time, the total time complexity is $O(n \log n)$.

Next we prove the lower bound. Consider a regular n-gon P inscribed in a circle of radius $r = 1/3$ drawn on a square piece of paper Q of unit diameter (i.e. the diagonal of the square has unit length), so that P and Q have the same center O. We may assume that $n \geq 8$. First make two *free* cuts (whose cost is not counted) along two edges e, f of P so that the convex arc C between e and f has $n/8$ uncut edges. Note that the angle between these cut-lines at their point of intersection and facing the polygon is larger than $90°$. It is enough to prove a lower bound of $\Omega(\log n)$ on the cut cost of C.

Let $\alpha = \frac{2\pi}{n}$ be the angle at O between two adjacent points of P, and let $T(k)$ be the cut cost of a convex arc of P consisting of k (uncut) edges of P (which we may assume is contained in a triangular piece of paper), see Fig. 4. We will prove by induction on k that $T(k) \geq c \cdot \alpha \cdot (k+1) \cdot \log(k+1)$ for $1 \leq k \leq n/8$. The lower bound then follows setting $k = n/8$:

$$T(\frac{n}{8}) \geq c \cdot \frac{2\pi}{n} \cdot \frac{n}{8} \cdot \log \frac{n}{8} = \Omega(\log n).$$

Denote by l the side length of P: $l = 2r \sin \frac{\alpha}{2}$. The basis of the induction

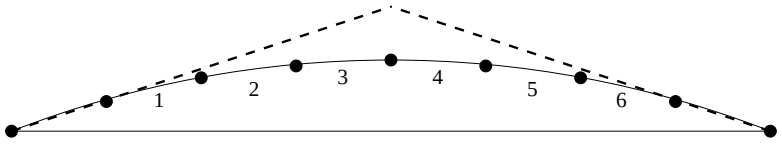

Fig. 4. A convex arc of $k = 6$ edges of P.

is satisfied by choosing c, as $T(i) \geq i \cdot l$ for $i = 1, 2, 3, 4$, and $\sin \frac{\alpha}{2} > \frac{\alpha}{4}$ for $\alpha \leq \pi/4$. We use the following result on optimal cutting sequences.

Theorem 2 (Overmars and Welzl [3]) *If the piece of paper is convex, then there exists an optimal cutting sequence with $O(n)$ cuts in which each cut touches polygon P, where n is the number of edges of P.*

There are two possible cases, as the first cut touching the arc may be an *edge cut* (i.e. along an edge of P), or a *vertex cut* (i.e. through a vertex of P).

Case 1: First cut touching the convex arc of k edges is an edge cut. Refer to Fig. 5 for notation. Put $h = |ON| = |OM| = |OP| = r \cos \frac{\alpha}{2}$. The length of the cut is

$$z = |ST| = |SM| + |MT| = h(\tan \frac{i\alpha}{2} + \tan \frac{j\alpha}{2}),$$

for some $i, j \geq 0$ with $i + j = k + 1$. We have $\cos \frac{\alpha}{2} > \frac{1}{2}$ for $\alpha \leq \pi/4$. Using the inequality $\tan x > x$, for $x \in (0, \pi/2)$, we get that

$$z \geq \frac{r}{2} \min_{i+j=k+1} (\tan \frac{i\alpha}{2} + \tan \frac{j\alpha}{2})$$
$$\geq \frac{r}{2} \min_{i+j=k+1} (\frac{i\alpha}{2} + \frac{j\alpha}{2}) = \frac{(k+1)\alpha}{12}.$$

After performing the first cut, we are left with two smaller convex arcs to be cut. This yields a cutting cost of at least

$$\frac{(k+1)\alpha}{12} + \min_{i+j=k-1} \{T(i) + T(j)\}.$$

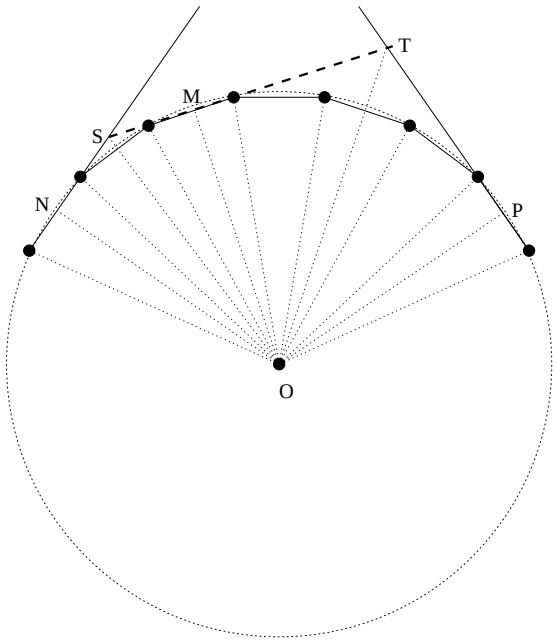

Fig. 5. Case 1 in the proof of the lower bound: an edge cut.

Case 2: First cut touching the convex arc of k edges is a vertex cut (through vertex M). Refer to Fig. 6 for notation. The length of the cut is

$$z = |ST| = |SM| + |MT| \geq |MU| + |MY|$$
$$= h(\tan \frac{i\alpha}{2} - \tan \frac{\alpha}{2}) + h(\tan \frac{j\alpha}{2} - \tan \frac{\alpha}{2}),$$

for some $i, j \geq 0$ with $i + j = k$. A similar calculation to the previous case (using the inequality $\tan x/2 < x$, for $x \in (0, \pi/2)$), gives

$$z \geq \frac{r}{2} \min_{i+j=k+2} (\frac{i\alpha}{2} - \alpha + \frac{j\alpha}{2} - \alpha) = \frac{(k-2)\alpha}{12}.$$

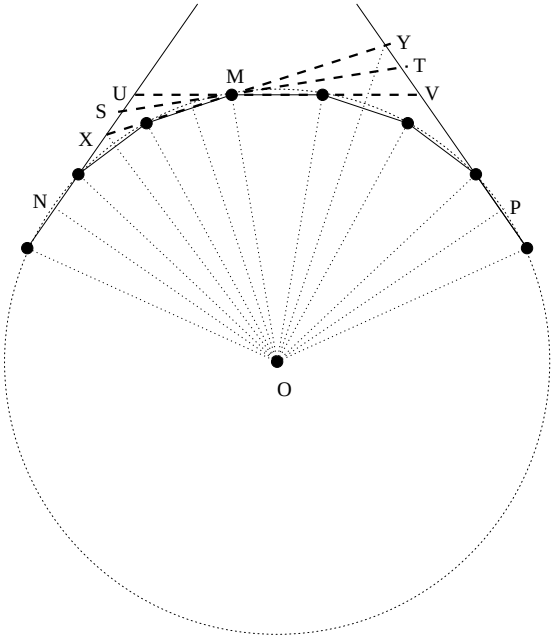

Fig. 6. Case 2 in the proof of the lower bound: a vertex cut.

After performing the first cut, we do two more free cuts (whose cost is not counted) along the two edges of P adjacent to M. We are left again with two smaller convex arcs to be cut. The corresponding cutting cost is at least

$$\frac{(k-2)\alpha}{12} + \min_{i+j=k-2}\{T(i)+T(j)\}.$$

Putting together *Case 1* and *Case 2* we obtain the recurrence

$$T(k) \geq \frac{(k-2)\alpha}{12} + \min_{i+j=k-2}\{T(i)+T(j)\},$$

whose solution is

$$T(k) \geq c \cdot \alpha \cdot (k+1)\log(k+1), \text{ for some } c > 0.$$

This completes the proof of the theorem.

3 Conclusion

It is natural to consider the following variant of the above problem in three dimensions.

> Given a polyhedral piece of material Q, and a convex polyhedron P contained in it, cut P out of the piece of material in the cheapest possible way.

At this time, a cut is a plane which divides the piece of material into a number of pieces, and which is not allowed to intersect the interior of P. A cutting sequence is defined similarly to the planar case, so that after the last cut, the piece of material is the polyhedron P.

The cost of a cut is the *area* of the intersection of the cutting plane with the (current) piece of material containing P, and the problem asks for an optimal cutting sequence (i.e one whose total cost is minimum).

Similarly to the planar convex case, let

$$g(n) = \sup_{\substack{P \subset Q \\ d(Q)=1}} \{g(P,Q) \mid P \text{ is a convex polyhedron with } n \text{ vertices, } Q \text{ is convex}\},$$

where $d(Q)$ stands for the diameter of Q, and $g(P,Q)$ is the cost of cutting P out of Q. What is the asymptotic growth rate of $g(n)$?

References

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