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Translation invariants of Zernike moments

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Abstract

Moment functions defined using a polar coordinate representation of the image space, such as radial moments and Zernike moments, are used in several recognition tasks requiring rotation invariance. However, this coordinate representation does not easily yield translation invariant functions, which are also widely sought after in pattern recognition applications. This paper presents a mathematical framework for the derivation of translation invariants of radial moments defined in polar form. Using a direct application of this framework, translation invariant functions of Zernike moments are derived algebraically from the corresponding central moments. Both derived functions are developed for non-symmetrical as well as symmetrical images. They mitigate the zero-value obtained for odd-order moments of the symmetrical images. Vision applications generally resort to image normalization to achieve translation invariance. The proposed method eliminates this requirement by providing a translation invariance property in a Zernike feature set. The performance of the derived invariant sets is experimentally confirmed using a set of binary Latin and English characters.

Keywords: Radial moments; Zernike moments; Translation invariants; Symmetrical images; Image normalization

1. Introduction

In image analysis, it is of utmost importance to look for pattern features that are invariant with respect to change of size (similitude), translation, and/or rotation. Jan Flusser et al. have classified two different moment approaches to this problem: (i) direct description by moment invariants and (ii) image normalization [1]. The direct description of moment invariants was first introduced by Hu, showing how they can be derived from algebraic invariants in his fundamental theorem of moment invariants [2]. He used geometric moments to generate a set of invariants that were then widely used in pattern recognition by Alt [3], ship identification by Smith [4], aircraft identification by Dudani [5], pattern matching by Dirilten [6] and scene matching by Wong [7].

However, the kernel function of geometric moments is not orthogonal, which makes it computationally expensive to reconstruct an image from the moments. Geometric moments suffer from a high degree of information redundancy, and they are sensitive to noise for higher-order moments [8–10]. Moreover, the kernel function of geometric moments of order $(n+m)$, $\psi_{nm}(x,y) = x^n y^m$ $(n,m \in Z^*)$, often involves power of n and m. Moments computed with the above schemes will therefore have large variation in the dynamic range of values for different orders. Applications involving such moment functions will have to additionally include scale normalization to maintain equal weight for all the components in a set of feature vectors. It may also be necessary to develop methods for avoiding numerical instabilities when the image size is large.

Teague proposed Zernike moments based on the basis set of orthogonal Zernike polynomials [11]. Other orthogonal moments are Legendre and pseudo-Zernike moments which are derived from Legendre and pseudo-Zernike polynomials, respectively. Zernike moments have been proven

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to be more robust in the presence of noise. They are able to achieve a near-zero value of redundancy measure in a set of moment functions where the moments correspond to independent characteristics of the image [8,10]. Since their moment functions are defined using a polar coordinate representation of the image space, Zernike moments are commonly used in recognition tasks requiring rotation invariance. However, this coordinate representation does not easily yield translation invariant functions, which are also sought after in pattern recognition applications. To the authors' knowledge, no report has been published on two-dimensional translation invariants of Zernike moments based on Zernike polynomials. The existing Zernike translation invariants can be indirectly obtained by making use of the existing geometric central moments. This approach is denoted hereafter as the indirect method. For instance, the equations for second and third order of Zernike central moments are given as a series of geometric central moments as follows [10]:

$$\hat{Z}_{20} = \frac{6}{\pi}(\mu_{20} + \mu_{02}) - \frac{3}{\pi}\mu_{00},$$

$$\hat{Z}_{22} = \frac{3}{\pi}(\mu_{20} - \mu_{02} - 2\hat{j}\mu_{11}),$$

$$\hat{Z}_{31} = \frac{12}{\pi}(\mu_{30} + \mu_{12}) - \frac{12}{\pi}\hat{j}(\mu_{03} + \mu_{21}) - \frac{8}{\pi}(\mu_{10} - \hat{j}\mu_{01}),$$

$$\hat{Z}_{33} = \frac{4}{\pi}(\mu_{30} - 3\mu_{12}) + \frac{4}{\pi}\hat{j}(\mu_{03} - 3\mu_{21}),$$

where geometric central moments of order $(n+m)$ are given by

$$\mu_{nm} = \int_x \int_y (x - x_0)^n (y - y_0)^m f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Apart from the indirect method, Khotanzad et al. have shown that translation invariants of Zernike moments can also be achieved using image normalization method [10,12]. They are achieved by transforming the image into a new one whose first-order moments, M_{10} and M_{01} are both equal to zero. This is done by transforming the translated image intensity distribution, $f(x, y)$ into $f(x + x', y + y')$, where x' and y' are the centroids for the respective x and y-axis of the original image. They are computed as follows:

$$x' = \frac{M_{10}}{M_{00}} \quad \text{and} \quad y' = \frac{M_{01}}{M_{00}},$$

where geometric moments of order $(n+m)$ can be evaluated as $M_{nm} = \int_x \int_y x^n y^m f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

In this paper, we introduce the direct description of translation invariants of Zernike moments based on Zernike polynomials. We present a mathematical framework for the derivation of translation invariants of radial moments defined in polar form. Using a direct application of this framework, translation invariant functions of Zernike moments are derived algebraically from the corresponding central moments. Both derived functions are developed for non-symmetrical as well as symmetrical images. They mitigate the zero value obtained for odd-order moments of symmetrical images. Vision applications generally resort to image normalization or the indirect method to achieve translation invariance. The proposed method eliminates this requirement by providing a translation invariance property in a Zernike feature set. The performance of the derived invariant sets is experimentally confirmed using a set of binary Latin and English characters.

The organization of this paper is given as follows: In Section 2, we give the mathematical background of radial and Zernike moments. Section 3 provides the derivation steps of Zernike central moments via radial central moments, and discusses the problems of the proposed invariant sets when they are used in the case of symmetrical images. Section 4 examines the performance of the proposed radial central moments and Zernike central moments. Section 5 concludes the study. The invariant sets used in the experimental study for radial moments and Zernike moments are listed in Appendices A and B, respectively.

2. Mathematical background

This section introduces the theories of radial and Zernike moments. The radial moments of order p with repetition q are defined as

$$D_{pq} = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} r^p \mathrm{e}^{-\hat{j}q\theta} f(r, \theta) r\,\mathrm{d}r\,\mathrm{d}\theta, \quad \hat{j} = \sqrt{-1}, \tag{2.1}$$

where $p = 0, 1, 2, \ldots, \infty$ and q takes on any positive or negative integer values.

The kernel of Zernike moments is a set of orthogonal Zernike polynomials defined over the polar coordinate space inside a unit circle. The two-dimensional Zernike moments of order p with repetition q of an image intensity function $f(r, \theta)$ are defined as [10]

$$Z_{pq} = \frac{p+1}{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} V_{pq}^*(r, \theta) f(r, \theta) r\,\mathrm{d}r\,\mathrm{d}\theta, \quad |r| \leqslant 1, \tag{2.2}$$

where Zernike polynomials of order p with repetition q, $V_{pq}(r, \theta)$, are defined as

$$V_{pq}(r, \theta) = R_{pq}(r)\mathrm{e}^{\hat{j}q\theta} \tag{2.3}$$

and the real-valued radial polynomial, $R_{pq}(r)$, is given as follows:

$$R_{pq}(r) = \sum_{k=0}^{(p-|q|/2)} (-1)^k \frac{(p-k)!}{k!((p+|q|)/2-k)!((p-|q|)/2-k)!} r^{p-2k}, \tag{2.4}$$

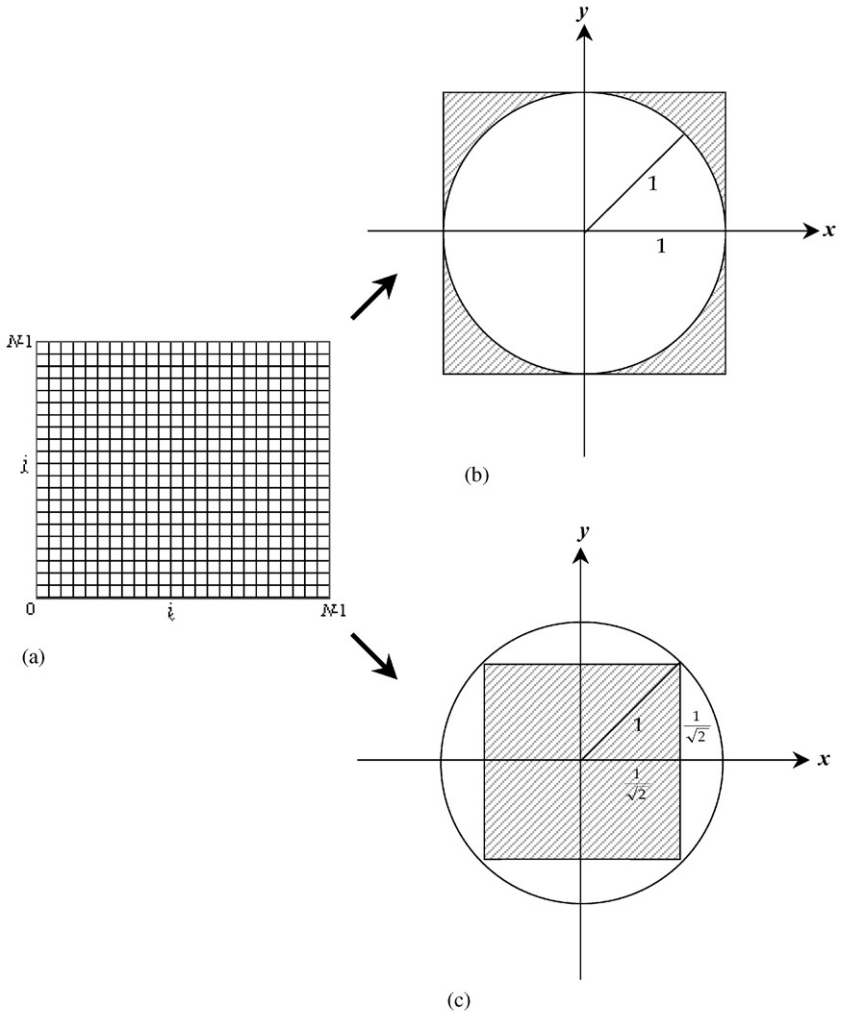


Fig. 1. Two coordinate normalization schemes for radial and Zernike moments: (a) Discrete image coordinate space of size $(N \times N)$; (b) Image coordinate normalization using the mapping $(0, N-1) \rightarrow (-1, +1)$; and (c) Image coordinate normalization using the mapping $(0, N-1) \rightarrow (-1/\sqrt{2}, +1/\sqrt{2})$.

where $0 \leqslant |q| \leqslant p$ and $p - |q|$ is even. The relationship between Zernike and radial moments is given by [10]

$$Z_{pq} = \lambda_p \sum_{k=q}^{p} B_{pqk} D_{kq}, \qquad \lambda_p = \frac{p+1}{\pi}, \tag{2.5}$$

where the polynomial coefficient, B_{pqk}, is defined as

$$B_{pqk} = \frac{(-1)^{(p-k)/2}((p+k)/2)!}{((p-k)/2)!((k+q)/2)!((k-q)/2)!}. \tag{2.6}$$

Since radial moments and Zernike moments are defined in terms of polar coordinates (r, θ) with $|r| \leqslant 1$, their computation requires a linear transformation of the image coordinates (i, j), $i, j = 0, 1, 2, \ldots, N-1$ to a suitable domain $(x, y) \in R^2$ inside a unit circle. Two commonly used cases of the transformations are shown in Fig. 1(b) and (c). Based on these figures, we have the following discrete approximation to their respective continuous moments' integral in (2.1) and (2.2)

radial moments:

$$D_{pq} = \lambda_d(N) \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} r_{ij}^{p} \, \mathrm{e}^{-\hat{j} q \theta_{ij}} f(i,j), \tag{2.7}$$

Zernike moments:

$$Z_{pq} = \lambda_z(p,N) \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} R_{pq}(r_{ij}) \mathrm{e}^{-\hat{j} q \theta_{ij}} f(i,j), \tag{2.8}$$

where the most general image coordinate transformation to the interior of the unit circle is given by

$$r_{ij} = \sqrt{x_i^2 + y_j^2}, \quad \theta_{ij} = \tan^{-1}\left(\frac{y_j}{x_i}\right), \quad x_i = c_1 i + c_2,$$

$$y_j = c_1 j + c_2. \tag{2.9}$$

In particular for

Fig. 1(b): $\lambda_d(N) = \dfrac{\pi}{(N-1)^2}$,

$$\lambda_z(p,N) = \frac{p+1}{(N-1)^2}, \tag{2.10}$$

$$c_1 = \frac{2}{N-1}, \quad c_2 = -1, \tag{2.11}$$

Fig. 1(c): $\lambda_d(N) = \dfrac{2}{(N-1)^2}$,

$$\lambda_z(p,N) = \frac{2(p+1)}{\pi(N-1)^2}, \tag{2.12}$$

$$c_1 = \frac{\sqrt{2}}{N-1}, \quad c_2 = -\frac{1}{\sqrt{2}}. \tag{2.13}$$

3. Derivation of translation invariants of Zernike moments

Zernike moments are powerful feature descriptors from which rotational invariance can be easily constructed. Adding translation invariance will considerably enhance the utility of Zernike moments in many recognition and classification tasks. This section provides a framework for deriving translation invariants for both radial and Zernike moments, enabling them to be used as robust classifiers.

To derive the Zernike central moments, we first need to solve the radial central moments. In view of the condition ($p - q$ = even) in Zernike moments, radial moments of ($p - q$ = even) will only be considered. The radial central moments of order p with repetition q, $\hat{D}_{pq}$, are defined as

$$\hat{D}_{pq} = \int_x \int_y [(x - x_0) - \hat{j}(y - y_0)]^{(p+q)/2}[(x - x_0) + \hat{j}(y - y_0)]^{(p-q)/2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.1}$$

where $p - q$ = even. The centroids of the x- and y-coordinates, x_0 and y_0, respectively, are given by

$$x_0 = \frac{\sum\sum x\, f(x,y)}{\sum\sum f(x,y)}, \quad y_0 = \frac{\sum\sum y\, f(x,y)}{\sum\sum f(x,y)}. \tag{3.2}$$

Eq. (3.1) can be rewritten as

$$\hat{D}_{pq} = \int_x \int_y (K - A)^u (K^* - A^*)^v f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.3}$$

where $K = x - \hat{j}y$, $A = x_0 - \hat{j}y_0 = D_{11}/D_{00}$, $u = (p+q)/2$ and $v = (p-q)/2$. It is known that

$$(K - A)^u = \sum_{m=0}^{u} \binom{u}{m} (K)^{u-m}(-A)^m \tag{3.4}$$

and

$$(K^* - A^*)^v = \sum_{n=0}^{v} \binom{v}{n} (K^*)^{v-n}(-A^*)^n \tag{3.5}$$

hence Eq. (3.3) can be expressed in the form of

$$\hat{D}_{pq} = \sum_{m=0}^{u}\sum_{n=0}^{v} \binom{u}{m}\binom{v}{n} (-A)^m(-A^*)^n \int_x \int_y (K)^{u-m}(K^*)^{v-n} f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6}$$

Based on Eq. (3.6), the radial central moments of order p with repetition q can now be obtained as a series of original radial moments:

$$\hat{D}_{pq} = \sum_{m=0}^{u}\sum_{n=0}^{v} \binom{u}{m}\binom{v}{n} (-A)^m(-A^*)^n D_{p'q'}, \tag{3.7}$$

where $p' = (u - m) + (v - n)$ and $q' = (u - m) - (v - n)$.

The translation invariants of radial moments derived in Eq. (3.7) are then used to derive the translation invariants of Zernike moments. Based on Eq. (2.5), we define Zernike central moments as follows:

$$\hat{Z}_{pq} = \lambda_p \sum_{k=q}^{p} B_{pqk} \int_x \int_y (K - A)^{(k+q)/2} (K^* - A^*)^{(k-q)/2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.8}$$

where $A = (\lambda_0/\lambda_1)Z_{11}/Z_{00}$. The Zernike central moments can be expressed in terms of radial central moments:

$$\hat{Z}_{pq} = \lambda_p \sum_{k=q}^{p} B_{pqk}\hat{D}_{kq} = \lambda_p \sum_{k=q}^{p} B_{pqk} \sum_{m=0}^{u'}\sum_{n=0}^{v'} \binom{u'}{m}\binom{v'}{n} (-A)^m(-A^*)^n D_{p''q''}, \tag{3.9}$$

where $p'' = (u' - m) + (v' - n)$ and $q'' = (u' - m) - (v' - n)$, $u' = (k+q)/2$ and $v' = (k-q)/2$. Eq. (2.5) is rearranged to express radial moments of order p with repetition q as a series of Zernike moments:

$$D_{pq} = \sum_{s=0}^{a} \gamma_{(q+s)}\delta_{p(q+s)}Z_{(q+s)q}, \tag{3.10}$$

where

$$\gamma_{q+s} = \frac{1}{\lambda_{q+s}B_{(q+s)q(q+s)}},$$

$$\delta_{pp} = 1, \quad \delta_{p(q+s)} = -C_{p(q+s)} + \sum_{b=2}^{a-s-2} - C_{(p-b)(q+s)}\delta_{p(p-b)}$$

for $p \neq (q+s)$,

$$C_{p(q+s)} = \frac{B_{pq(q+s)}}{B_{pqp}}$$

and $a = p - q$, $a - s =$ even.

With the help of Eq. (3.10), the Zernike central moments in Eq. (3.9) can now be expressed as a series of Zernike moments:

$$\hat{Z}_{pq} = \lambda_p \sum_{k=q}^{p} \sum_{m=0}^{u'} \sum_{n=0}^{v'} \binom{u'}{m} \binom{v'}{n} B_{pqk} (-A)^m (-A^*)^n \sum_{s=0}^{a'} \gamma_{(q''+s)} \delta_{p''(q''+s)} Z_{(q''+s)q''}, \tag{3.11}$$

where $a' = p'' - q''$ and $a' - s =$ even.

3.1. The computation of symmetrical images

The respective translation invariants of radial moments in Eq. (3.7) and Zernike moments in Eq. (3.11) work well for non-symmetrical images. However, the entire set of odd orders of radial and Zernike central moments yield a zero value when they are applied to images which are symmetrical along the x- and y-axes. This problem is due to the image centroid used in the calculation of both central moments. Since a symmetrical image has pairs of pixels that are equidistant (in opposite directions) from its centroid, the sum of an odd-order moment calculation produces a net value of zero. The total number of odd-order moments involved can be computed as follows:

$$T(p) = \begin{cases} \dfrac{(p+1)(p+3)}{8} & \text{for } p = \text{odd}, \\ \dfrac{p(p+2)}{8} & \text{for } p = \text{even}. \end{cases} \tag{3.1.1}$$

It can be seen from Eq. (3.1.1) that the number of odd-order moments which give a zero value for symmetrical images is $O(p^2)$. If we compute the first 40 orders of radial and Zernike central moments, there are only approximately 50% of them which produce non-zero values! These limited features may cause difficulties in pattern classification.

In this section, we introduce an improved version of radial and Zernike central moments that are computed from a center other than the centroid of the translated image. This reference point is selected such that the improved central moments produce a non-zero value when they are applied to symmetrical images, and yet they remain invariant to any image translation. The modified radial and Zernike central moments for the symmetrical image, $\tilde{D}_{pq}$ and $\tilde{Z}_{pq}$, respectively, are given as follows:

Modified Radial Central Moments:

$$\tilde{D}_{pq} = \sum_{m=0}^{u} \sum_{n=0}^{v} (-1)^{m+n} \binom{u}{m} \binom{v}{n} (A - A_d)^m (A^* - A_d^*)^n D_{p'q'}, \tag{3.1.2}$$

Modified Zernike Central Moments:

$$\tilde{Z}_{pq} = \lambda_p \sum_{k=q}^{p} \sum_{m=0}^{u'} \sum_{n=0}^{v'} (-1)^{m+n} \binom{u'}{m} \binom{v'}{n} B_{pqk} (A - A_z)^m (A^* - A_z^*)^n \times \sum_{s=0}^{a'} \gamma_{(q''+s)} \delta_{p''(q''+s)} Z_{(q''+s)q''} \tag{3.1.3}$$

and the shift terms for radial and Zernike central moments, A_d and A_z, respectively, are calculated as below

$$A_d = c \left[\frac{\hat{D}_{20}}{\hat{D}_{00}} \right]^{1/2}, \quad A_z = d \left[\frac{\eta_{202} \hat{Z}_{20} + \eta_{200} (\lambda_2/\lambda_0) \hat{Z}_{00}}{\hat{Z}_{00}} \right]^{1/2}, \tag{3.1.4}$$

where the shift factors, c and d, can take any non-zero values. The coefficient η_{pqk} is given as

$$\eta_{pqp} = 1, \quad \eta_{pqh} = \sum_{s=1}^{(p-h)/2} \frac{-\eta_{pq(p-2s+2)} B_{(p-2s+2)qh}}{B_{hqh}}, \quad q \leqslant h \leqslant (p-2). \tag{3.1.5}$$

The new central moments are derived in such a way that they do not only preserve the translation invariants of radial and Zernike moments, they can also be normalized to become invariant to any uniform scaling of the examined image when they are used with scale invariants of Zernike moments.

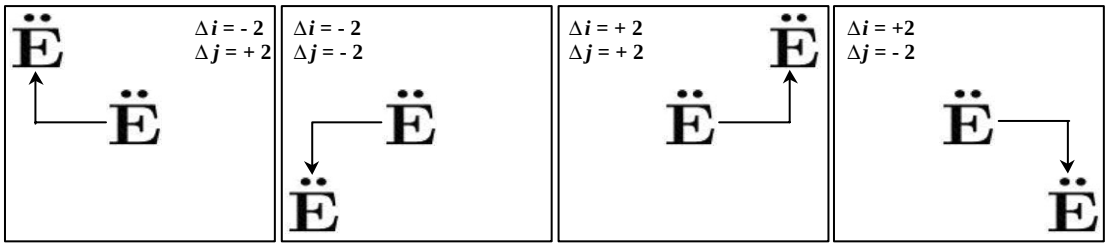
4. Experimental results

In this section, two experiments are carried out. The first experiment verifies the proposed radial central moments of $p - q =$ even. The same experiment is repeated to verify the performance of the proposed Zernike central moments. The second experiment aims to verify the performance of the modified radial and Zernike central moments when they are used in the case of symmetrical images.

Fig. 2 shows the binary image of a Latin character of size 50 × 50 pixels, used for experimental analysis of the moment equations presented in the previous section.

Fig. 2. A 50 × 50 binary latin character.

Table 1
Selected orders of radial central moments, $\hat{D}_{pq}$ for each translation of the latin character



								σ/μ (%)
22	8.6812E-03	22	8.6812E-03	22	8.6812E-03	22	8.6812E-03	0.00
20	1.8952E-02	20	1.8952E-02	20	1.8952E-02	20	1.8952E-02	0.00
33	1.6821E-03	33	1.6821E-03	33	1.6821E-03	33	1.6821E-03	0.00
31	8.5845E-04	31	8.5845E-04	31	8.5845E-04	31	8.5845E-04	0.00
44	4.0790E-04	44	4.0790E-04	44	4.0790E-04	44	4.0790E-04	0.00
42	2.2523E-03	42	2.2523E-03	42	2.2523E-03	42	2.2523E-03	0.00
40	4.0260E-03	40	4.0260E-03	40	4.0260E-03	40	4.0260E-03	0.00

Table 2
Selected orders of Zernike central moments, $\hat{Z}_{pq}$ for each translation of the latin character

								σ/μ (%)
22	8.2899E-03	22	8.2899E-03	22	8.2899E-03	22	8.2899E-03	0.00
20	9.1384E-02	20	9.1384E-02	20	9.1384E-02	20	9.1384E-02	0.00
33	2.1418E-03	33	2.1418E-03	33	2.1418E-03	33	2.1418E-03	0.00
31	3.2790E-03	31	3.2790E-03	31	3.2790E-03	31	3.2790E-03	0.00
44	6.4919E-04	44	6.4919E-04	44	6.4919E-04	44	6.4919E-04	0.00
42	2.7132E-02	42	2.7132E-02	42	2.7132E-02	42	2.7132E-02	0.00
40	7.0102E-02	40	7.0102E-02	40	7.0102E-02	40	7.0102E-02	0.00

Table 3
Selected orders of radial central moments, $\hat{D}_{pq}$ for a symmetrical binary character without center relocation

Image	Symmetry	Translation	22	20	33	31	44	42	40	55	53	51
H	x- and y-axis	Original	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = -2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = 0$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = +2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = 0$, $\Delta_j = -2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = 0$, $\Delta_j = +2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = -2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = 0$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = +2$	0.00499	0.06731	0.00000	0.00000	0.01123	0.00054	0.01834	0.00000	0.00000	0.00000

Several shifted versions of the image are created, and the corresponding selected orders of radial central moments as listed in Appendix A are computed, as shown in Table 1.Table 1 shows that the values of the selected orders of central moments remain unchanged for all translations. The deviation of the central moments, indicated by the percentage spread from the corresponding means of central moments, $\sigma/\mu\%$ is zero. Here σ and μ denote the standard deviation

Table 4
Selected orders of Zernike central moments, $\hat{Z}_{pq}$ for a symmetrical binary character without center relocation

Image	Symmetry	Translation	22	20	33	31	44	42	40	55	53	51
H	x- and y-axis	Original	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = -2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = 0$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = -2$, $\Delta_j = +2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = 0$, $\Delta_j = -2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = 0$, $\Delta_j = +2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = -2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = 0$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000
		$\Delta_i = +2$, $\Delta_j = +2$	0.00191	0.15470	0.00000	0.00000	0.00450	0.01041	0.12997	0.00000	0.00000	0.00000

Table 5
Selected orders of modified radial central moments, $\tilde{D}_{pq}$ for a symmetrical binary character

Image	Symmetry	Translation	22	20	33	31	44	42	40	55	53	51
H	x- and y-axis	Original	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = -2$, $\Delta_j = -2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = -2$, $\Delta_j = 0$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = -2$, $\Delta_j = +2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = 0$, $\Delta_j = -2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = 0$, $\Delta_j = +2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = +2$, $\Delta_j = -2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = +2$, $\Delta_j = 0$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144
		$\Delta_i = +2$, $\Delta_j = +2$	0.03864	0.10097	0.01535	0.05472	0.00489	0.02445	0.04954	0.01511	0.00626	0.03144

Table 6
Selected orders of modified Zernike central moments, $\tilde{Z}_{pq}$ for a symmetric binary character

Image	Symmetry	Translation	22	20	33	31	44	42	40	55	53	51
H	x- and y-axis	Original	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = -2$, $\Delta_j = -2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = -2$, $\Delta_j = 0$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = -2$, $\Delta_j = +2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = 0$, $\Delta_j = -2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = 0$, $\Delta_j = +2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = +2$, $\Delta_j = -2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = +2$, $\Delta_j = 0$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654
		$\Delta_i = +2$, $\Delta_j = +2$	0.00655	0.14542	0.00207	0.04764	0.00389	0.02045	0.10867	0.00389	0.01324	0.02654

and mean of the central moments, respectively. The results establish the invariance of the proposed radial central moments.

The same experiment is repeated for the proposed Zernike central moments. Selected orders of Zernike central moments as listed in Appendix B are computed for each translation, and they are recorded in Table 2. Table 2 shows that the values of the selected orders of central moments remain unchanged for all translations. The percentage spread from the corresponding means of central moments, $\sigma/\mu\%$ is zero. The results establish the invariance of the proposed Zernike central moments.

In the second experiment, a 25×25 binary English character, **H** as shown in Table 3 is used. This character is symmetrical along the x- and y-axes. The character is shifted up, down, left, right and diagonally. The radial and Zernike central moments of orders 2–5 are computed for each translation. Tables 3 and 4 show that the respective central moments of the examined image produce zero values when order p is odd. These results are due to the image centroid used in the calculation of the central moments. It can be seen from Tables 3 and 4 that 50% of the features used in this experiment produce zero values! The number of pattern features is significantly reduced.

To resolve this problem, the modified radial and Zernike central moments as defined in Eqs. (3.1.2) and (3.1.3), respectively, are used. They are computed from a center other than the centroid of the translated image. The second experiment is then repeated to compute the modified central moments of orders 2–5. The values of c and d are fixed arbitrarily at 1 and 0.1 for the respective modified radial and Zernike central moments. Tables 5 and 6 show that the proposed modified central moments preserve the translation invariance achieved by the respective Eqs. (3.7) and (3.11), and they also maintain a non-zero value for all odd orders of central moments when they are applied to the same symmetrical image. The number of useful pattern features is considerably increased.

5. Concluding remarks

In this paper, we have introduced the translation invariants of radial moments for $p-q=$ even. They have then been used to develop the translation invariants of Zernike moments based on orthogonal Zernike polynomials. Both descriptors have been proven to remain invariant under any translations of the original image. The results of simulation have demonstrated the invariant capability of the proposed invariants. In addition to that, we have also proposed modified radial and Zernike central moments, which will produce non-zero values and yet maintain the translation invariance when they are applied to images which are symmetrical along the x- and y-axes.

Appendix A

Translation Invariants of Radial Moments for $p-q=$even

Order 0:

$$\hat{D}_{00} = D_{00}.$$

Order 1:

$$\hat{D}_{11} = D_{11} - AD_{00}.$$

Order 2:

$$\hat{D}_{22} = D_{22} - 2AD_{11} + A^2 D_{00},$$

$$\hat{D}_{20} = D_{20} - A^* D_{11} - AD_{11}^* + AA^* D_{00}.$$

Order 3:

$$\hat{D}_{33} = D_{33} - 3AD_{22} + 3A^2 D_{11} - A^3 D_{00},$$

$$\hat{D}_{31} = D_{31} - A^* D_{22} - 2AD_{20} + 2AA^* D_{11} + A^2 D_{11}^* - A^2 A^* D_{00}.$$

Order 4:

$$\hat{D}_{44} = D_{44} - 4AD_{33} + 6A^2 D_{22} - 4A^3 D_{11} + A^4 D_{00},$$

$$\hat{D}_{42} = D_{42} - A^* D_{33} - 3AD_{31} + 3AA^* D_{22} + 3A^2 D_{20} - 3A^2 A^* D_{11} - A^3 D_{11}^* + A^3 A^* D_{00},$$

$$\hat{D}_{40} = D_{40} - 2A^* D_{31} + A^{*^2} D_{22} - 2AD_{31}^* + 4AA^* D_{20} - 2AA^{*^2} D_{11} + A^2 D_{22}^* - 2A^2 A^* D_{11}^* + A^2 A^{*^2} D_{00}.$$

Appendix B

Translation Invariants of Zernike Moments

Order 0:

$$\hat{Z}_{00} = Z_{00}.$$

Order 1:

$$\hat{Z}_{11} = Z_{11} - 2AZ_{00}.$$

Order 2:

$$\hat{Z}_{22} = Z_{22} - 3AZ_{11} + 3A^2 Z_{00},$$

$$\hat{Z}_{20} = Z_{20} - 3A^* Z_{11} - 3AZ_{11}^* + 6AA^* Z_{00}.$$

Order 3:

$$\hat{Z}_{33} = Z_{33} - 4AZ_{22} + 6A^2 Z_{11} - 4A^3 Z_{00},$$

$$\hat{Z}_{31} = Z_{31} - 4A^* Z_{22} - 4AZ_{20} + 12AA^* Z_{11} - 4AZ_{00}.$$

Order 4:

$$\hat{Z}_{44} = Z_{44} - 5AZ_{33} + 10A^2 Z_{22} - 10A^3 Z_{11} + 5A^4 Z_{00},$$

$$\hat{Z}_{42} = Z_{42} - 5A^* Z_{33} - 5AZ_{31} + 20AA^* Z_{22} + 10A^2 Z_{20} - 5AZ_{11} - 30A^3 Z_{11}^* + 15A^2 Z_{00},$$

$$\hat{Z}_{40} = Z_{40} - 5A^* Z_{31} - 5AZ_{31}^* + 10A^{*^2} Z_{22} + 10A^2 Z_{22}^* + 20AA^* Z_{20} + 5AZ_{11}^* - 90A^2 A^{*^2} Z_{00}.$$

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