Gauss-Manin Systems for Isolated Singularities in Singular

Mathias Schulze

October 27, 2006

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$$\mu = \dim_{\mathbb{C}} \mathbb{C}[\underline{x}]/\langle \underline{\partial}(f) \rangle = \sum_{x \in C_f} \mu_x,$$

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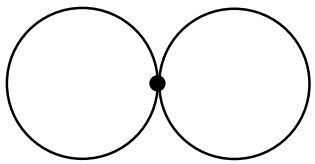
- $f^{-1}(t)$ homotopic to bouquet of $\mu - \sum_{f(x)=t} \mu_x$ n-spheres.



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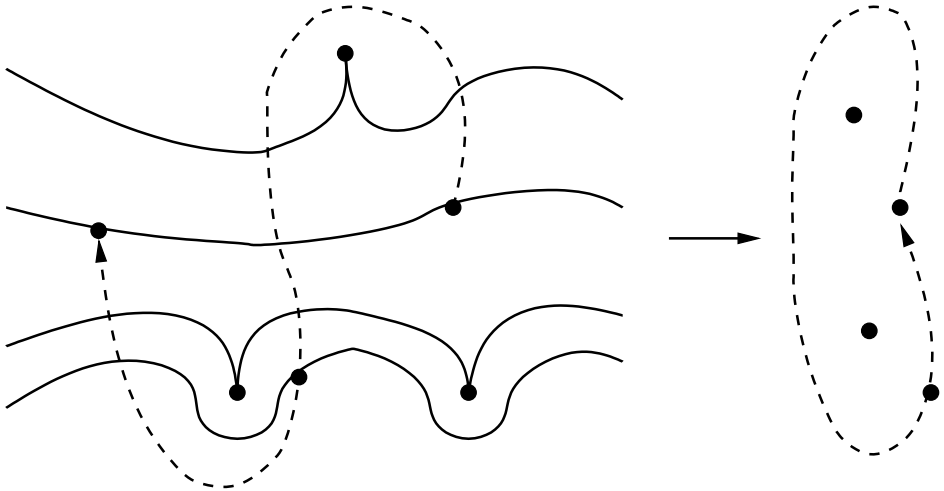

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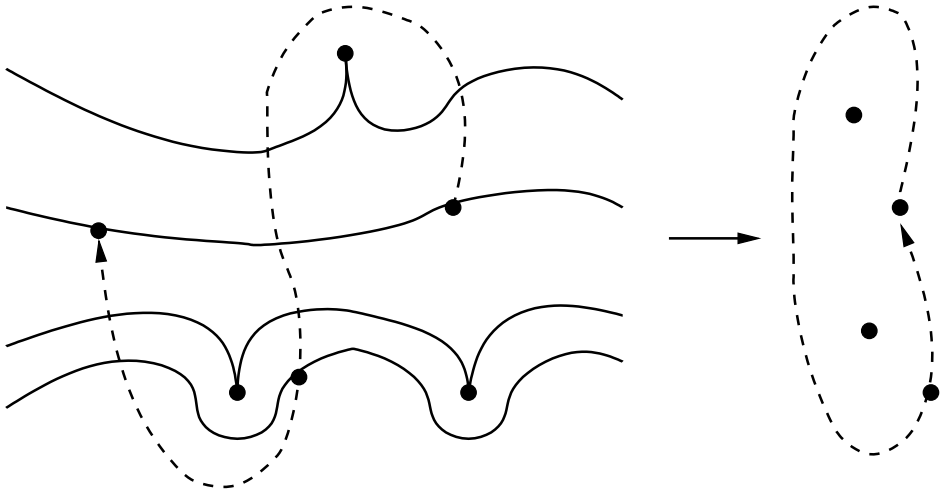

- Sections of S are solutions of a differential system G.

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- ...are specialized but avoid complicated computations:
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- ...are based on deep results by Claude Sabbah.

References

- Claude Sabbah, *Hypergeometric periods for a tame polynomial*, math.AG/9805077 (1998).

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- Mathias Schulze, *Good bases for tame polynomials*, J. Symb. Comp. 39,1 (2005), 103-126.

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- Cohomology of fibres is encoded in Gauss-Manin system:

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- Theorem (Sabbah): H is free $\mathbb{C}[\theta]$-module of rank μ.

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- Base change formula: $\underline{\phi}' = \underline{\phi} U$,

$$A' = U^{-1}(A + \theta^2 \partial_\theta) U.$$

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- Also spectral pairs (mixed Hodge numbers) can be read off.

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- Note: A_i inherit symmetries from mixed Hodge structure.

Naive approach

- Identify $\Omega^{n+1} = \mathbb{C}[\underline{x}]$ to get infinite $\mathbb{C}[\theta]$-presentation:

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$$H^k := \frac{\mathbb{C}[\underline{x}, \theta]_{\leq k}}{(\underline{\partial}(f) - \theta\underline{\partial})\big(\mathbb{C}[\underline{x}, \theta]^{n+1}\big)}, \quad H^k_l := \frac{\mathbb{C}[\underline{x}, \theta]_{\leq k}}{(\underline{\partial}(f) - \theta\underline{\partial})\big(\mathbb{C}[\underline{x}, \theta]^{n+1}_{\leq l}\big)}.$$

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$$H^k := \frac{\mathbb{C}[\underline{x}, \theta]_{\leq k}}{(\underline{\partial}(f) - \theta\underline{\partial})\big(\mathbb{C}[\underline{x}, \theta]^{n+1}\big)}, \quad H^k_l := \frac{\mathbb{C}[\underline{x}, \theta]_{\leq k}}{(\underline{\partial}(f) - \theta\underline{\partial})\big(\mathbb{C}[\underline{x}, \theta]^{n+1}_{\leq l}\big)}.$$

- Canonical maps: $H^k_l \xrightarrow[= \text{ for } l \gg 0]{\pi^k_l} H^k \xhookrightarrow[= \text{ for } k \gg 0]{\iota_k} H.$

t-stable sublattice of H

- Recall the finitely presented $\mathbb{C}[\theta]$-module

$$H_l^k = \frac{\mathbb{C}[\underline{x},\theta]_{\leq k}}{(\underline{\partial}(f)-\theta\underline{\partial})\left(\mathbb{C}[\underline{x},\theta]_{\leq l}^{n+1}\right)\cap\mathbb{C}[\underline{x},\theta]_{\leq k}}.$$

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- Can compute H_l^k from $\mathbb{C}[\theta]$-Gröbner basis $\underline{g}$ of

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w.r.t. monomial ordering on $\mathbb{C}[\underline{x},\theta]$ refining $\underline{x}$-degree.

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- Can compute cyclic generators $\underline{\phi}$ of H_l^k and $\underline{\phi}A = t(\underline{\phi})$.
- Method of increasing k and l successively yields t-stable sublattice $H_l^k \subseteq H$ with equality for $k \gg 0$.

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- Coordinate at infinity: $\tau = \theta^{-1} = \partial_t, \quad \partial_\tau = -\theta^2 \partial_\theta = -t.$

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  - Base change: eigenvalues of $-\tau\partial_\tau$ on $V/\tau V$ in $[\alpha, \alpha - 1)$.

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- Spectrum of $F_\bullet$ on finite $\mathbb{C}$-vector space V:

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- Compute good $\underline{\phi}'$ and

$$A' = P^{-1}(A + \theta^2\partial_\theta)(P)$$

by simultaneous normal form & base change in P.