中 原 大 學
應用數學系
碩士學位論文

變形的蜂窩環輪面
Deformed Honeycomb Tori

指導教授: 高欣欣

研 究 生: 石圜鋼

中華民國九十五年六月

Deformed Honeycomb Tori

Yuan-Kang Shih[†], Shin-Shin Kao[†1], and Lih-Hsing Hsu[‡]
[†]Department of Applied Mathematics,
Chung-Yuan Christian University, Chong-Li, Taiwan 320.
[‡]Department of Computer Science and Information Engineering,
Providence University, Tai-Chung, Taiwan 433.

[1]This work was supported in part by the National Science Council of the Republic of China under Contract NSC 94-2115-M-033-006. Correspondence to: Associate Professor S.S. Kao, e-mail: skao@math.cycu.edu.tw.

Contents

List of Figures

摘 要

在平行和分散式的網際網路中, 有一種名爲一般化蜂窩環輪面 (Generalized Honeycomb Tori $GHT(m,n,s)$) 的圖形被廣泛應用。在一般化蜂窩環輪面家族中, 有一種被稱爲六角形蜂窩環輪面 (Honeycomb Hexagonal Torus $HT(n)$) 的圖形, 其結構是非常對稱的, 因而吸引衆多學者去深入研究及探討。而六角形蜂窩環輪面 圖形是由一個參數n所決定, 故其圖形在擴張時是非常規則的同時向外延伸。在這篇文章中, 我們提出一個 新的圖形概念, 這個新圖形是由六角形蜂窩環輪面所衍生出來的, 我們將其命名爲變形的蜂窩環輪面 (Deformed Honeycomb Tori $DHT(h,l,r)$), 此圖形是由三個參數所決定出來的, 而這三個參數彼此都是互相獨立的。最後我們將會給一個明確的對應方式, 藉以說明我們所提出之變形的蜂窩環輪面是一般化蜂窩環輪面的一個 子集。

Abstract

Assume that m, n and s are integers with $m \geq 2, n \geq 4, 0 \leq s < n$ and s is of the same parity of m. The generalized honeycomb tori $GHT(m, n, s)$ have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications. Among the various families of graphs of $GHT(m, n, s)$, numerous studies are devoted to honeycomb hexagonal torus $HT(n)$ due to its nice symmetrical structure. Although each vertex of $HT(n)$ is described by a three-dimensional coordinate (x, y, z), the graph grows uniformly in the three directions. In this article, we propose a new class of graphs extended from $HT(n)$, namely, *deformed honeycomb torus* $DHT(h, l, r)$. $DHT(h, l, r)$ is defined to allow the graph to grow in the three independent dimensions. We prove that this more general class of graphs still remains a subset of the generalized honeycomb torus. Furthermore, for any given $DHT(h, l, r)$, we have a concrete correspondence between its vertex set and the associated $GHT(m, n, s)$.

Keywords: Honeycomb torus, interconnection, network.

Chapter 1

Introduction

Network topology is a crucial factor for interconnection network since it determines the performance of the network [6]. Stojmenovic [8] proposed three variations of honeycomb tori by adding wraparound edges on honeycomb meshes, namely honeycomb rectangular torus, honeycomb rhombic torus, and honeycomb hexagonal torus. These honeycomb tori have been recognized as an attractive architecture to existing torus interconnection networks in parallel and distributed applications and numerous studies are devoted to honeycomb hexagonal torus due to its nice symmetrical structure [2, 4, 5, 7, 8]. Cho and Hsu [3] proved that all these honeycomb torus networks can be characterized in a unified way, and hence they proposed a class of interconnection networks, the *generalized honeycomb torus*.

In this article, we are interested in a class of graphs extended from honeycomb hexagonal torus which we call the *deformed honeycomb torus*. In [8], each vertex of the honeycomb hexagonal torus $HT(n)$ is described by a three-dimensional coordinate (x_0, y_0, z_0) even though the graph grows uniformly in the three directions. (See Figure 1 for an illustration.) In contrast to $HT(n)$, the deformed honeycomb torus $DHT(h, l, r)$ is defined to allow the graph to grow in the three independent dimensions. We prove that this more general class of graphs still remains a subset of the generalized honeycomb torus $GHT(m, n, s)$. Furthermore, for any given $DHT(h, l, r)$, we have a concrete correspondence between its vertex set and the associated $GHT(m, n, s)$.

In the following section, we give some graph terms that are used in this paper, a formal definition of the generalized honeycomb torus $GHT(m, n, s)$ and that of the deformed honeycomb torus $DHT(h, l, r)$. In Section 3, we show that every $DHT(h, l, r)$ is a $GHT(m, n, s)$ by constructing an isomorphism between $DHT(h, l, r)$ and $GHT(m, n, s)$ and hence every deformed honeycomb torus is a generalized honeycomb torus.

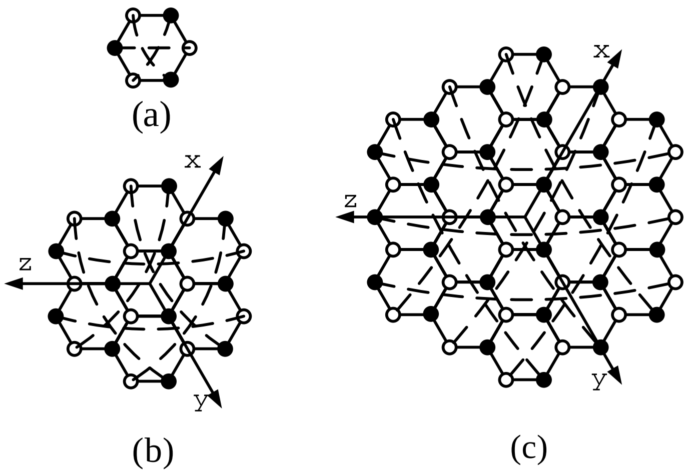

Figure 1.1: The graphs (a)$HT(1)$ (b)$HT(2)$ (c)$HT(3)$.

Chapter 2

Preliminary

Usually, computer and communication networks are represented by graphs where vertices represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For the graph definition and notation we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \cdots, v_k\rangle$. We also write the path $\langle v_0, v_1, v_2, \cdots, v_k\rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \cdots, v_j, P_2, v_t, \cdots, v_k\rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_{i-1}, v_i\rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_{t-1}, v_t\rangle$. Hence, it is possible to write a path $\langle v_0, v_1, P, v_1, v_2, \cdots, v_k\rangle$ if the length of P is zero. If a path $Q = \langle v_0, v_1, v_2, \cdots, v_k\rangle$, then Q^{-1} denotes the path $\langle v_k, v_{k-1}, \cdots, v_1, v_0\rangle$. A *cycle* is a path of at least three vertices such that the first vertex and the last vertex are identical. Let C be a cycle and P be a path. We use $|C|$ to denote the total number of distinct vertices/deges on C and $|P|$ to denote the total number of distinct edges of P.

For any two positive integers r and d, we use $[r]_d$ to denote r (**mod** d). Let m, n and s be positive integers with $m \geq 2, n \geq 4$, n and $m + s$ are even. The *generalized honeycomb torus* $\mathrm{GHT}(m, n, s)$ is the graph with the vertex set $\{(i, j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that (i, j) and (k, l) with $i \leq k$ are adjacent if they satisfy one of the following conditions:

1. $(k, l) = (i, [j \pm 1]_n)$;
2. $0 \leq i \leq m - 2$, $i + j$ is odd and $(k, l) = (i + 1, j)$;
3. $i = 0$, j is even, and $(k, l) = (m - 1, [j + s]_n)$.

For example, the graph $\mathrm{GHT}(8, 6, 2)$ is shown in Figure 2. Obviously, any generalized honeycomb torus is a 3-regular bipartite graph. Moreover, any generalized honeycomb torus is vertex transitive.

Let h, l, and r be positive integers. Let $G = DHT(h, l, r)$ be a *deformed honeycomb torus* and $G = G(V, E)$. The vertex set is defined by $V(G) = \bigcup_{1 \leq k \leq l}\{(i, k - 1) \mid r - k \leq i \leq r + 2h + k - 2\} \cup \bigcup_{1 \leq k \leq r-l}\{(i, l + k - 1) \mid r - l - k \leq i \leq r + l + 2h - k - 1\} \cup \bigcup_{0 \leq k \leq l-1}\{(i, r + k) \mid$

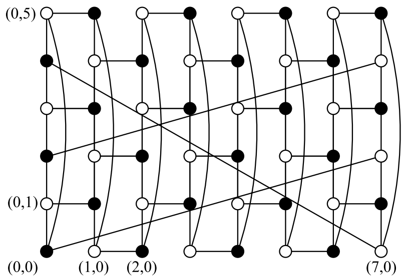

Figure 2.1: The graph $GHT(8, 6, 2)$.

$k \le i \le 2l+2h-k-2\}$. Let (i, j) and (x, y) be two vertices of $V(G)$. Then $e = ((i, j), (x, y))$ is an edge of $E(G)$ if and only if any of the following conditions holds:

1. $j = y$ and $|i - x| = 1$.
2. j is even, $y = j + 1$, $i = x$ and $(i - r \textbf{ mod } 2) \equiv 1$.
3. j is odd, $y = j + 1$, $i = x$ and $(i - r \textbf{ mod } 2) \equiv 0$.
4. $(i, j) = (r + 2h - 1 + n, n)$ and $(x, y) = (n, r + n)$ for $0 \le n \le l - 1$.
5. $(i, j) = (r - 1 - n, n)$ and $(x, y) = (l + r + 2h - n - 2, l + n)$ for $0 \le n \le r - 1$.
6. $(i, j) = (r - 1 + n, 0)$ and $(x, y) = (l - 1 + n, l + r - 1)$ for n is odd and $1 \le n \le 2h - 1$.

For example, the graph $DHT(2, 3, 6)$ is shown Figure 4(a)(neglecting the numbers in the square brace). It is obvious that $HT(n) = DHT(n, n, n)$ for $n \ge 1$.

Chapter 3

Every $DHT(h, l, r)$ is a generalized honeycomb torus

To prove that $DHT(h, l, r)$ is a generalized honeycomb torus for any combination of h, l and r, we consider the cases where $l \leq r$ without loss of generality. We will construct an isomorphism between every $DHT(h, l, r)$ and its associated $GHT(m, n, s)$ by mapping each vertex of $DHT(h, l, r)$ to its corresponding vertex of the $GHT(m, n, s)$. Let *g.c.d.(a,b)* denote the greatest common divisor of two positive integers a and b and define g_0 and t_0 by

$$\begin{aligned} g_0 &= g.c.d.(l, r); \\ t_0 &= 2(hl + hr + lr)/g_0. \end{aligned}$$

For example, $g_0 = 3$ and $t_0 = 24$ for the graph $DHT(2, 3, 6)$. We will use the symbols g_0 and t_0 with above definition in the rest of the paper.

Lemma 1. *If* $l = mg_0$, $r = ng_0$ *and* $n \geq m$, *then*

$$\left\lfloor \frac{r-1}{l} \right\rfloor = \left\lfloor \frac{r-2}{l} \right\rfloor = \cdots = \left\lfloor \frac{r-g_0}{l} \right\rfloor.$$

Proof. It is obvious that

$$\left\lfloor \frac{r-1}{l} \right\rfloor \geq \left\lfloor \frac{r-2}{l} \right\rfloor \geq \cdots \geq \left\lfloor \frac{r-g_0}{l} \right\rfloor.$$

Case 1: $m = n = 1$. In this case, $r = l = g_0$. Then

$$\begin{aligned} \left\lfloor \frac{r-1}{l} \right\rfloor &= \left\lfloor \frac{r-1}{r} \right\rfloor = \left\lfloor 1 - \frac{1}{r} \right\rfloor = 0; \\ \left\lfloor \frac{r-g_0}{l} \right\rfloor &= \left\lfloor \frac{r-r}{r} \right\rfloor = \left\lfloor \frac{0}{r} \right\rfloor = 0. \end{aligned}$$

Case 2: $m = 1$ and $n > m$.

$$\begin{aligned} \left\lfloor \frac{r-1}{l} \right\rfloor &= \left\lfloor \frac{ng_0 - 1}{mg_0} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} - \frac{1}{mg_0} \right\rfloor = \left\lfloor n - \frac{1}{g_0} \right\rfloor = n - 1; \\ \left\lfloor \frac{r-g_0}{l} \right\rfloor &= \left\lfloor \frac{ng_0 - g_0}{mg_0} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} - \frac{g_0}{mg_0} \right\rfloor = \left\lfloor n - 1 \right\rfloor = n - 1. \end{aligned}$$

Case 3: $m \neq 1$ and $n > m$. The remainder of n/m is $[n]_m$. Since

$$[n]_m = n - \left\lfloor \frac{n}{m} \right\rfloor m \geq 1,$$

we have

$$\begin{aligned} [ng_0]_{mg_0} &= ng_0 - \left\lfloor \frac{ng_0}{mg_0} \right\rfloor mg_0 \\ &= g_0(n - \left\lfloor \frac{n}{m} \right\rfloor m) \\ &= g_0[n]_m \\ &\geq g_0. \end{aligned}$$

Notice that if $[ng_0]_{mg_0} \geq [b]_{mg_0}$ for some $b < mg_0$, then $\left\lfloor \frac{ng_0}{mg_0} - \frac{b}{mg_0} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} \right\rfloor$. We have

$$\left\lfloor \frac{r-1}{l} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} - \frac{1}{mg_0} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} \right\rfloor = \left\lfloor \frac{r}{l} \right\rfloor$$

and

$$\left\lfloor \frac{r-g_0}{l} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} - \frac{g_0}{mg_0} \right\rfloor = \left\lfloor \frac{ng_0}{mg_0} \right\rfloor = \left\lfloor \frac{r}{l} \right\rfloor.$$

That is,

$$\left\lfloor \frac{r-1}{l} \right\rfloor = \left\lfloor \frac{r-g_0}{l} \right\rfloor.$$

Thus,

$$\left\lfloor \frac{r-1}{l} \right\rfloor = \left\lfloor \frac{r-2}{l} \right\rfloor = \cdots = \left\lfloor \frac{r-g_0}{l} \right\rfloor.$$

$\square$

Lemma 2. *In $DHT(h, l, r)$, we define some path patterns as follows.*

$$\begin{aligned} P_1^t(i,j) &= \langle (i,j), (i-1,j), \cdots, (i-t,j) \rangle; \\ P_2(i,j) &= \langle (i,j), (l+r+2h-j-2, l+j) \rangle; \\ P_3(i,j) &= \langle (i,j), (r+2h-1+i, i) \rangle; \\ Q(i,j) &= \langle (i,j), P_2(i,j), (l+r+2h-j-2, l+j), P_1^{2(h+l)-1}(l+r+2h-j-2, l+j), \\ &\quad (r-l-j-1, l+j) \rangle. \end{aligned}$$

We let $Q^2(i,j) = \langle (i,j), Q(i,j), (r-l-j-1, l+j), Q(r-l-j-1, l+j), (r-2l-j-1, 2l+j) \rangle$ and $Q^t(i,j) = \langle (i,j), Q(i,j), (r-l-j-1, l+j), Q^{t-1}(r-l-j-1, l+j), (r-tl-j-1, tl+j) \rangle$ for $t \geq 2$. There are three more path patterns.

$$
\begin{aligned}
R(i,j) &= \langle (i,j), P_1^{2(h+j)}(i,j), (i-2h-2j,j), Q^{\lfloor \frac{r-j-1}{l} \rfloor}(i-2h-2j,j), \\
&\quad (r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j)\rangle; \\
S(i,j) &= \langle (i,j), P_2(i,j), (l+r+2h-j-2, l+j), P_1^{2(r+h-1-j)}(l+r+2h-j-2, l+j), \\
&\quad (l-r+j, l+j), P_3(l-r+j, l+j), (l+2h+j-1, l-r+j)\rangle; \\
T(i,j) &= \langle (i,j), R(i,j), (r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j), \\
&\quad S(r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j), \\
&\quad (l+2h+\left\lfloor \frac{r-j-1}{l} \right\rfloor l+j-1, l-r+\left\lfloor \frac{r-j-1}{l} \right\rfloor l+j)\rangle.
\end{aligned}
$$

(See Appendix A for an illustration.)
Then $|T(i,j)| = 4h+2r+2h\left\lfloor \frac{r-j-1}{l} \right\rfloor$ *for any* $(i,j) = (2h+r-1+k, k) \in V(DHT(h,l,r))$, $0 \leq k \leq l-1$.

Proof. It is obvious that $|P_1^t(i,j)| = t$ and $|Q^t(i,j)| = 2t(h+l)$.

$$
\begin{aligned}
|T(i,j)| &= |\langle (i,j), R(i,j), (r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j), \\
&\quad S(r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j), \\
&\quad (l+2h+\left\lfloor \frac{r-j-1}{l} \right\rfloor l+j-1, l-r+\left\lfloor \frac{r-j-1}{l} \right\rfloor l+j)\rangle| \\
&= |R(i,j)| + |S(r-\left\lfloor \frac{r-j-1}{l} \right\rfloor l-j-1, \left\lfloor \frac{r-j-1}{l} \right\rfloor l+j)| \\
&= (2(h+j) + \left\lfloor \frac{r-j-1}{l} \right\rfloor (1+2(h+l)-1)) + \\
&\quad (1+2(r+h-1-(\left\lfloor \frac{r-j-1}{l} \right\rfloor l+j))+1) \\
&= 2h+2j+\left\lfloor \frac{r-j-1}{l} \right\rfloor (2h+2l)+2r+2h-2\left\lfloor \frac{r-j-1}{l} \right\rfloor l-2j \\
&= 4h+2r+2h\left\lfloor \frac{r-j-1}{l} \right\rfloor.
\end{aligned}
$$

□

Lemma 3. *For any* $g.c.d(n,m) = 1$ *and* $n > m$,

$$
\begin{aligned}
&\left\lfloor \frac{n-1}{m} \right\rfloor + \left\lfloor \frac{n-2}{m} \right\rfloor + \cdots + \left\lfloor \frac{n-m}{m} \right\rfloor \\
= \;& (\frac{n-1}{m} + \frac{n-2}{m} + \cdots + \frac{n-m}{m}) - (\frac{0}{m} + \frac{1}{m} + \cdots + \frac{m-1}{m}).
\end{aligned}
$$

Proof. We know that the remainder of n/m is $[n]_m$, then

$$\left\lfloor \frac{n}{m} \right\rfloor = \frac{n-[n]_m}{m}.$$

Thus,

$$\begin{aligned}
& \left\lfloor \frac{n-1}{m} \right\rfloor + \left\lfloor \frac{n-2}{m} \right\rfloor + \cdots + \left\lfloor \frac{n-m}{m} \right\rfloor \\
= & \frac{n-1-[n-1]_m}{m} + \frac{n-2-[n-2]_m}{m} + \cdots + \frac{n-m-[n-m]_m}{m} \\
= & (\frac{n-1}{m} + \frac{n-2}{m} + \cdots + \frac{n-m}{m}) - (\frac{[n-1]_m}{m} + \frac{[n-2]_m}{m} + \cdots + \frac{[n-m]_m}{m}) \\
= & (\frac{n-1}{m} + \frac{n-2}{m} + \cdots + \frac{n-m}{m}) - (\frac{0}{m} + \frac{1}{m} + \cdots + \frac{m-1}{m})
\end{aligned}$$

□

Lemma 4. *There exist g_0 disjoint cycles, C_k's with $0 \le k \le g_0 - 1$, of $DHT(h,l,r)$ such that $|C_k| = \dfrac{2(hl+lr+hr)}{g_0}$ for any k and $\bigcup_{k=0}^{g_0-1} V(C_k) = V(DHT(h,l,r))$.*

Proof. We define $C_k = \Big(T(2h+r-1+k,k)\Big)^{l/g_0}$ for $0 \le k \le g_0 - 1$. Figure 3 gives an illustration for the cycles on $DHT(2,3,6)$ and the details of these cycles are given in Appendix B. Now, we want to show that $|C_k| = \dfrac{2(hl+lr+hr)}{g_0}$, $0 \le k \le g_0 - 1$.
Let $i = 2h + r - 1$.

$$\begin{aligned}
|C_k| &= |T(i+k,k)| + |T(i+k,k+g_0)| + \cdots + |T(i+k,k+(\frac{l}{g_0}-1)g_0)| \\
&= |T(i+k,k)| + |T(i+k,k+g_0)| + \cdots + |T(i+k,k+l-g_0)| \\
&\overset{Lemma\ ??}{=} (4h+2r+2h\left\lfloor\frac{r-k-1}{l}\right\rfloor) + (4h+2r+2h\left\lfloor\frac{r-(k+g_0)-1}{l}\right\rfloor) + \cdots \\
&\quad (+4h+2r+2h\left\lfloor\frac{r-(k+l-g_0)-1}{l}\right\rfloor) \\
&\overset{Lemma\ ??}{=} (4h+2r+2h\left\lfloor\frac{r-g_0}{l}\right\rfloor) + (4h+2r+2h\left\lfloor\frac{r-2g_0}{l}\right\rfloor) + \cdots \\
&\quad +(4h+2r+2h\left\lfloor\frac{r-l}{l}\right\rfloor)
\end{aligned}$$

$$
\begin{aligned}
& (4h+2r+2h\left\lfloor\frac{r-g_0}{l}\right\rfloor)+(4h+2r+2h\left\lfloor\frac{r-2g_0}{l}\right\rfloor)+\cdots+(4h+2r+2h\left\lfloor\frac{r-l}{l}\right\rfloor) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(\left\lfloor\frac{r-g_0}{l}\right\rfloor+\left\lfloor\frac{r-2g_0}{l}\right\rfloor+\cdots+\left\lfloor\frac{r-l}{l}\right\rfloor) \\
\overset{l=mg_0,\ r=ng_0}{=} \quad & \frac{l}{g_0}(4h+2r)+2h(\left\lfloor\frac{ng_0-g_0}{mg_0}\right\rfloor+\left\lfloor\frac{ng_0-2g_0}{mg}\right\rfloor+\cdots+\left\lfloor\frac{ng_0-mg_0}{mg_0}\right\rfloor) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(\left\lfloor\frac{n-1}{m}\right\rfloor+\left\lfloor\frac{n-2}{m}\right\rfloor+\cdots+\left\lfloor\frac{n-m}{m}\right\rfloor) \\
\overset{Lemma\ ??}{=} \quad & \frac{l}{g_0}(4h+2r)+2h((\frac{n-1}{m}+\frac{n-2}{m}+\cdots+\frac{n-m}{m})-(\frac{0}{m}+\frac{1}{m}+\cdots+\frac{m-1}{m})) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(\frac{((n-1)+(n-m))\cdot m\cdot\frac{1}{2}}{m}-\frac{0+(m-1)\cdot m\cdot\frac{1}{2}}{m}) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(n-m) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(\frac{ng_0-mg_0}{g_0}) \\
= \quad & \frac{l}{g_0}(4h+2r)+2h(\frac{r-l}{g_0}) \\
= \quad & \frac{2(hl+lr+hr)}{g_0}.
\end{aligned}
$$

Hence $|C_k| = \dfrac{2(hl+lr+hr)}{g_0}$ for all $0 \le k \le g_0-1$.

It is easy to see that each cycle C_k starts with distinct vertex $(2h+r-1+k, k)$ for $0 \le k \le g_0-1$, and hence they are disjoint. Since $|\bigcup_{k=0}^{g_0-1} C_k| = 2(hl+lr+hr) = |V(DHT(h,l,r))|$, the union of C_k with $0 \le k \le g_0-1$ spans $V(DHT(h,l,r))$. $\square$

From the construction of C_k in Lemma 1, it is obvious that each C_k consists of $\frac{l}{g_0}$ T's. We shall call them the 1-st T, the 2-nd T, $\cdots$, and the $\frac{l}{g_0}$-th T of C_k in the next lemma.

Since $GHT(g_0,t_0,s)$ contains g_0 disjoint cycles $\{C_k\}_0^{g-1}$ with equal length t_0 and $\bigcup_{k=0}^{g_0-1} C_k = V(GHT(g_0,t_0,s))$, the result of Lemma 1 suggests a possible isomorphism from $DHT(h,l,r)$ to $GHT(g_0,t_0,s)$ for some positive integer s with g_0+s an even integer. To show that there is a one-to-one correspondence between $V(DHT(h,l,r))$ and $V(GHT(g_0,t_0,s))$, we give each vertex of $DHT(h,l,r)$ a new label in Definition **??** as follows.

Definition 1. *Let k be an integer with $0 \le k \le g_0-1$ and C_k be the cycle obtained by Lemma 1. Suppose that $C_k = \langle v_0, v_1, \cdots, v_{t_0-1}, v_0\rangle$, where $t_0 = \dfrac{2(hl+lr+hr)}{g_0}$, $v_0 =$*

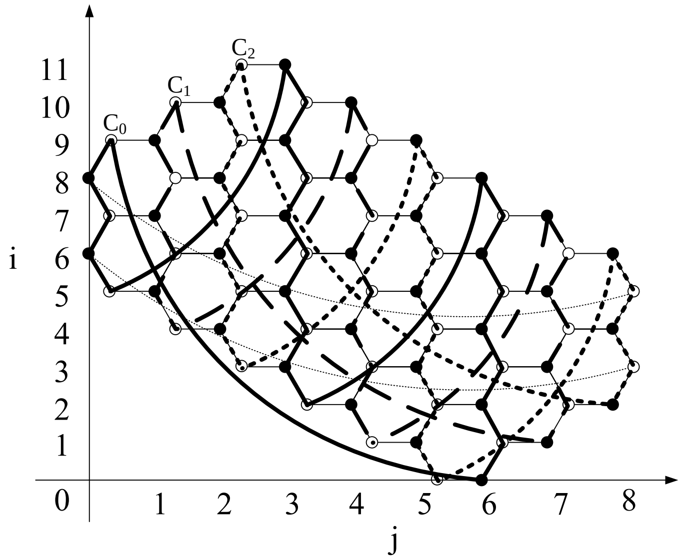

Figure 3.1: $DHT(2,3,6)$ for Lemma 1.

$(2h+r-1+k,k)$, $v_1 = (2h+r-1+k-1,k)$ *and* $v_{t_0-1} = (2l-r+\left\lfloor\frac{r-k-l+g_0-1}{l}\right\rfloor l+k-g_0, 2l+\left\lfloor\frac{r-k-l+g_0-1}{l}\right\rfloor l+k-g_0)$. *We define a new coordinate* $[x,y]_i$ *for each vertex* v_i*, where* $0 \le i \le t_0-1$*, on* C_k *as follows:*

$$\begin{cases} x = k \\ y = [i-1-k]_{t_0} \end{cases}.$$

Figure 4 (a) gives an illustration about the new labeling method of $DHT(2,3,6)$. By aligning $DHT(h,l,r)$ according to its new coordinates such that each vertex labeled $[x,y]$ is located at the intersection of the xth column and the yth row, we recognize that it is a candidate of a generalized honeycomb torus $GHT(g_0,t_0,s)$. More specifically, let $(i,j) \in V(DHT(h,l,r))$ and let its corresponding new label be $[x,y]$. Then the vertex (i,j) of $DHT(h,l,r)$ is associated with a unique vertex $[x,y]$ of $GHT(g_0,t_0,s)$, where $g_0 = g.c.d(l,r)$, $t_0 = \dfrac{2(hl+lr+hr)}{g_0}$ and s is some integer to be determined. In the following paragraphs, we derive the integer s.

Lemma 5. *Let* C_k *be a cycle of* $DHT(h,l,r)$*, where* $0 \le k \le g_0-1$*. Suppose that* $v=(i,j)$*, where* $0 \le j \le l-1$*, is a vertex on the p-th path* T *of* C_k*. Define* $w = j - [j]_{g_0}$*. Then p can be determined by*

$$\begin{cases} \textit{If } w = 0, & \textit{then } p = 1; \\ \textit{If } w \ne 0, & \begin{cases} \textit{(i) if } (r-l)|(r-w), & \textit{then } p = \dfrac{r-w}{r-l}; \\ \textit{(ii) if } (r-l)\nmid(r-w), & \textit{let } \alpha \textit{ be the least positive integer such that} \\ & (r-l)|(r-w+\alpha l), \textit{then } p = \dfrac{r-w+\alpha l}{r-l}. \end{cases} \end{cases}$$

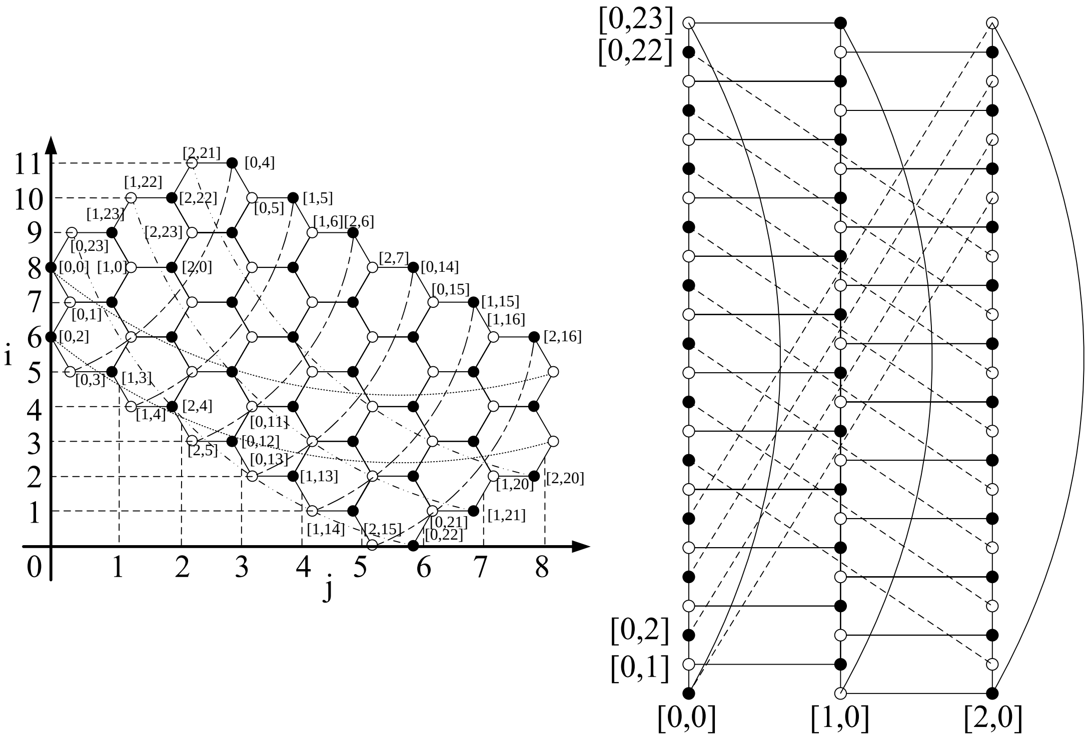

Figure 3.2: The isomorphic graphs (a)$DHT(2,3,6)$ and (b)$GHT(3,24,17)$.

Proof. Let $C_k = \langle v_0^k, v_1^k, \cdots, v_{t_0-1}^k, v_0^k \rangle$ where $v_0^k = (2h+r-1+k, k)$ for $0 \leq k \leq g_0 - 1$. According to Lemma 1, each cycle C_k starts from the vertex $(2h+r-1+k, k)$ and follows the path T. Thus we can determine the p-th path T of C_k that passes the given vertex $v_\beta^k = (i, j)$ with $0 \leq j \leq l-1$ by first finding the p-th path T of C_0 that passes v_β^0. Therefore $w = j - [j]_{g_0}$.

Case 1: $w = 0$. That is, $w = j - [j]_{g_0} = 0$. It implies $0 \leq j \leq g_0 - 1$. Since vertex $v = (i, j)$, $0 \leq j \leq g_0 - 1$ lies on the first T, we know $p = 1$.

Case 2: $w \neq 0$. A similar but tedious counting leads to the above result. The proof is skipped. $\square$

Example 1. *If $G = DHT(1, 10, 14)$, $v_1 = (10, 1)$, $v_2 = (14, 7)$ and $v_3 = (24, 9)$, we have $g_0 = 2$.*

Case 1: *$v_1 = (10, 1)$, by Lemma **??**, $w_1 = 1 - [1]_2 = 0$. Therefore, $p = 1$.*

Case 2: *$v_2 = (14, 7)$, by Lemma **??**, $w_2 = 7 - [7]_2 = 6 \neq 0$. And*

$$(r - l) = (14 - 10) | (14 - 6) = (r - w).$$

Therefore,

$$p = \frac{r - w}{r - l} = \frac{8}{4} = 2.$$

Case 3: $v_3 = (24, 9)$*, by Lemma* **??***,* $w_3 = 9 - [9]_2 = 8 \neq 0$ *and*

$$(r - l) = (14 - 10) \nmid (14 - 8) = (r - w).$$

We can find the least positive integer $\alpha = 1$*, such that*

$$(r - l) = (14 - 10)|(14 - 8 + 1 \cdot 10) = (r - w + \alpha l).$$

Then

$$p = \frac{r - w + \alpha l}{r - l} = \frac{16}{4} = 4.$$

Theorem 1. *Consider* $G = DHT(h, l, r)$ *and the cycle* C_{g_0-1} *on* G *defined by Lemma 1. Let the vertex* $v = (2h + r + l - 2, l - 1)$ *lie on the* p*-th path* T *on* C_{g_0-1}*, where* p *is obtained with Lemma* **??***. There is an isomorphism between* $DHT(h, l, r)$ *and* $GHT(g_0, t_0, s)$ *where*

$$s = \begin{cases} t_0 - g_0 - 2h & \text{if } l = g_0; \\ (p-1)[4h + 2r] + 2h \sum_{u=0}^{p-2} \left(\left\lfloor \frac{r - [l - u(r-l)]_l - 1}{l} \right\rfloor \right) - g_0 - 2h. & \text{if } l \neq g_0. \end{cases}$$

Proof. From the construction of C_k in Lemma **??**, we know that $C_k = \langle v_0^k, v_1^k, \cdots, v_{t_0-1}^k, v_0^k \rangle$ for $0 \leq k \leq g_0 - 1$. Moreover, $v_0^0 = (2h + r - 2, 0)$ and $v_0^{g_0-1} = (2h + r - 2, g_0 - 1)$. Since the vertex v_i^k of C_k is relabeled $[k, [i - 1 - k]_{t_0}]$, we can compute s by finding the row difference of the two endvertices of an edge (a, b) where $a \in C_{g_0-1}$ and $b \in C_0$. We also notice that $a = (2h + r + l - 2, l - 1) \in C_{g_0-1}$, $b = (2h + r + l - 2, l) \in C_0$ and $(a, b) \in E(DHT(h, l, r))$. Suppose that a is the n-th vertex on C_{g_0-1} and b is the m-th vertex on C_0. It is easy to see that $s = [n - m]_{t_0}$. We prove the theorem by the following two cases.

Case 1: $l = g_0$. Now that

$$a = (2h + r + l - 2, l - 1) = (2h + r + l - 2, g_0 - 1)$$

Hence if we count from $v_0^{g_0-1}$, the position of a is

$$[(2h + r - 2) - (2h + r + l - 2)]_{t_0} + 1 = [-l]_{t_0} + 1 = [-g_0]_{t_0} + 1 = t_0 - g_0 + 1 = t_0 - g_0 + 1.$$

Thus,

$$n = t_0 - (g_0 - 1).$$

By using $P_1^{2h-1}(2h+r-2, 0) = \langle (2h+r-2, 0), (2h+r-3, 0), \cdots, (r-1, 0) \rangle$ and $P_3(r-1, 0) = \langle (r-1, 0), (2h + r + l - 2, l) \rangle$, we know that $m = (2h - 1) + 1 + 1 = 2h + 1$. Therefore,

$$s = n - m = t_0 - g_0 + 1 - (2h + 1) = t_0 - g_0 - 2h.$$

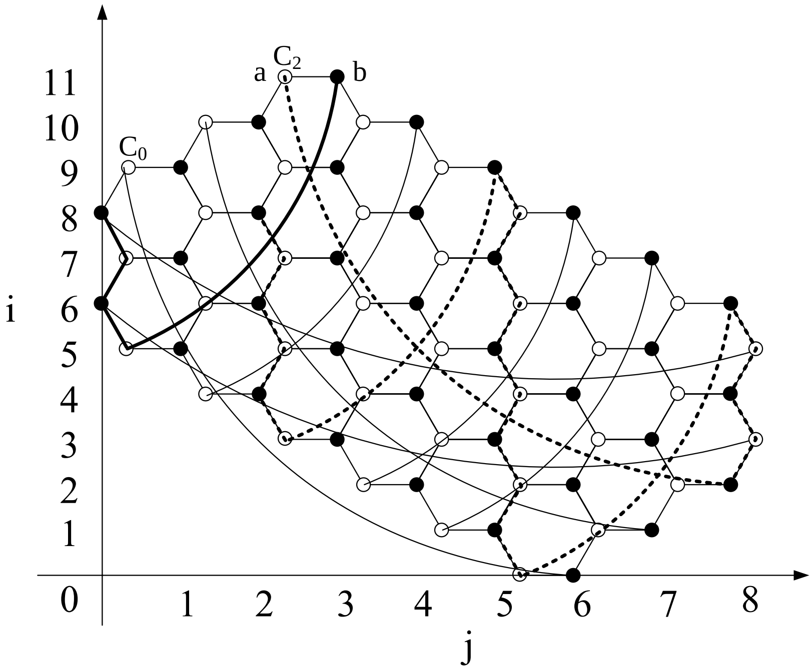

Figure 3.3: A example for case 1 of Theorem ??.

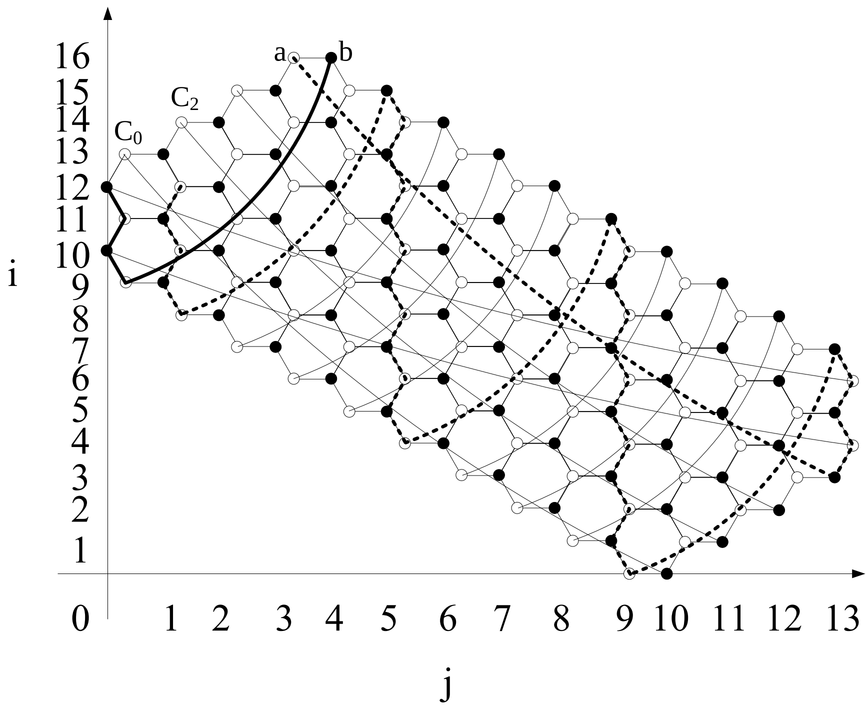

Figure 3.4: A example for case 2 of Theorem ??.

Case 2: $l \neq g_0$. By Lemma **??** and Lemma **??**, we know that the vertex $(2h+r+l-2, l-1)$ is the first vertex of the p-th path T in C_{g_0-1}. It is easy to know that $|T(2h+r-1+k, k)| =$

$|T(2h+r-1+k+g_0-1, k+g_0-1)|$ by Lemma **??** and Lemma **??**.

$$\begin{aligned}
n &= |T(2h+r-1+[l-0\cdot(r-l)]_l+(g_0-1),[l-0\cdot(r-l)]_l+(g_0-1))|-g_0\\
&\quad +|T(2h+r-1+[l-1\cdot(r-l)]_l+(g_0-1),[l-1\cdot(r-l)]_l+(g_0-1))|\\
&\quad +|T(2h+r-1+[l-2\cdot(r-l)]_l+(g_0-1),[l-2\cdot(r-l)]_l+(g_0-1))|\\
&\quad +\cdots\\
&\quad +|T(2h+r-1+[l-(p-2)\cdot(r-l)]_l+(g_0-1),[l-(p-2)\cdot(r-l)]_l+(g_0-1))|+1\\
&= |T(2h+r-1,0)|+|T(2h+r-1+[l-1\cdot(r-l)]_l,[l-1\cdot(r-l)]_l)|\\
&\quad +|T(2h+r-1+[l-2\cdot(r-l)]_l,[l-2\cdot(r-l)]_l)|\\
&\quad +\cdots+|T(2h+r-1+[l-(p-2)\cdot(r-l)]_l,[l-(p-2)\cdot(r-l)]_l)|-g_0+1\\
&= \sum_{u=0}^{p-2}|T(2h+r-1+[l-u\cdot(r-l)]_l,[l-u\cdot(r-l)]_l)|-g_0+1\\
&\overset{Lemma\ ??}{=} (4h+2r+2h\left\lfloor\frac{r-[l-0\cdot(r-l)]_l-1}{l}\right\rfloor)+(4h+2r+2h\left\lfloor\frac{r-[l-1\cdot(r-l)]_l-1}{l}\right\rfloor)\\
&\quad +\cdots+(4h+2r+2h\left\lfloor\frac{r-[l-(p-2)\cdot(r-l)]_l-1}{l}\right\rfloor)-g_0+1\\
&= (p-1)[4h+2r]+2h(\left\lfloor\frac{r-0-1}{l}\right\rfloor+\left\lfloor\frac{r-[l-1\cdot(r-l)]_l-1}{l}\right\rfloor+\cdots\\
&\quad +\left\lfloor\frac{r-[l-(p-2)\cdot(r-l)]_l-1}{l}\right\rfloor)-g_0+1\\
&= (p-1)[4h+2r]+2h\sum_{u=0}^{p-2}(\left\lfloor\frac{r-[l-u\cdot(r-l)]_l-1}{l}\right\rfloor)-g_0+1.
\end{aligned}$$

And

$$m=(2h-1)+1+1=2h+1$$

Hence

$$\begin{aligned}
s &= n-m\\
&= (p-1)[4h+2r]+2h\sum_{u=0}^{p-2}(\left\lfloor\frac{r-[l-u\cdot(r-l)]_l-1}{m}\right\rfloor)-g_0-2h.
\end{aligned}$$

□

Now we illustrate how Theorem **??** is used to find $GHT(m,n,s)$ that is isomorphic to the given $DHT(h,l,r)$ by the following examples.

Example 2. *Let* $G=DHT(2,3,6)$. *Here*

$$g_0=g.c.d(3,6)=3$$

and

$$\begin{aligned} t_0 &= 2(hl + hr + lr)/g_0 \\ &= 2(2 \cdot 3 + 2 \cdot 6 + 3 \cdot 6)/3 \\ &= 24. \end{aligned}$$

Given a vertex $v = (2h + r + l - 2, l - 1) = (11, 2)$*, by Lemma* **??**, $w = 2 - [2]_3 = 0$*, then* $p = 1$*, so the vertex* v *lies on the* 1*-st path* T *on* $C_{g_0 - 1} = C_2$*. By Theorem* **??**,

$$s = t_0 - g_0 - 2h = 24 - 3 - 4 = 17.$$

Thus, $DHT(2, 3, 6) \cong GHT(3, 24, 17)$ *as shown in Figure 4*

Example 3. *Let* $G = DHT(1, 10, 14)$*. Here*

$$g_0 = g.c.d(10, 14) = 2$$

and

$$\begin{aligned} t_0 &= 2(hl + hr + lr)/g_0 \\ &= 2(1 \cdot 10 + 1 \cdot 14 + 10 \cdot 14)/2 \\ &= 164. \end{aligned}$$

Given a vertex $v = (2h + r + l - 2, l - 1) = (24, 9)$*. By Case 3 of Example 1,* $p = 4$*. Hence the vertex* v *lies on the* 4*-th path* T *on* $C_{g_0 - 1} = C_1$*. By Theorem* **??**

$$\begin{aligned} s &= (p-1)[4h + 2r] + 2h \sum_{u=0}^{p-2} (\Big\lfloor \frac{r - [l - u(r - l)]_l - 1}{l} \Big\rfloor) - g_0 - 2h \\ &= 3(4 + 28) + 2(\sum_{u=0}^{2} \Big\lfloor \frac{14 - [10 - u(-4)]_{10} - 1}{10} \Big\rfloor) - 2 - 2 \\ &= 3 \cdot 32 + 2(\Big\lfloor \frac{14 - [10 - 0(-4)]_{10} - 1}{10} \Big\rfloor + \Big\lfloor \frac{14 - [10 - 1(-4)]_{10} - 1}{10} \Big\rfloor + \\ &\quad \Big\lfloor \frac{14 - [10 - 2(-4)]_{10} - 1}{10} \Big\rfloor) - 4 \\ &= 96. \end{aligned}$$

Thus, $DHT(1, 10, 14) \cong GHT(2, 164, 96)$*.*

Bibliography

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, North-Holland, New York, (1980).

[2] J. Carle, J-F. Myoupo, and D. Seme, "All-to-All Broadcasting Algorithms on Honeycomb Networks and Applications", *Parallel Processing Letters*, vol. 9, pp. 539–550, 1999.

[3] H. Cho, and L. Hsu, "Generalized Honeycomb Torus", *Information Processing Letters*, vol. 86, pp. 185–190, 2003.

[4] G.M. Megson, X. Yang, and X. Liu, "Honeycomb Tori are Hamiltonian", *Information Processing Letters*, vol. 72, pp. 99–103, 1999.

[5] G.M. Megson, X. Liu, and X. Yang, "Fault-Tolerant Ring Embedding in a Honeycomb Torus with Nodes Failures", *Parallel Processing Letters*, vol. 9, pp. 551–561, 1999.

[6] B. Parhami, "A Introduction to Parallel Processing: Algorithms and Architectures", *Plenum Press*, New York, 1999.

[7] B. Parhami and D.M. Kwai, "A Unified Formulation of Honeycomb and Diamond Networks", *IEEE Trans. Parallel and Distributed Systems*, vol. 12, pp. 74–80, 2001.

[8] I. Stojmenovic, "Honeycomb Networks: Topological Properties and Communication Algorithms", *IEEE Trans. Parallel and Distributed Systems*, vol. 8, pp. 1036–1042, 1997.

[9] X. Yang, D. J. Evans, H. Lai, and G. M. Megson, "Generalized honeycomb torus is hamiltonian", *Information Processing Letters*, vol. 92, pp. 31–37, 2004.

Appendix A

Examples For Lemma ??

Example 1: The path patterns $P_1^t(i,j)$, $P_2(i,j)$ and $P_3(i,j)$ for $DHT(2,3,6)$.

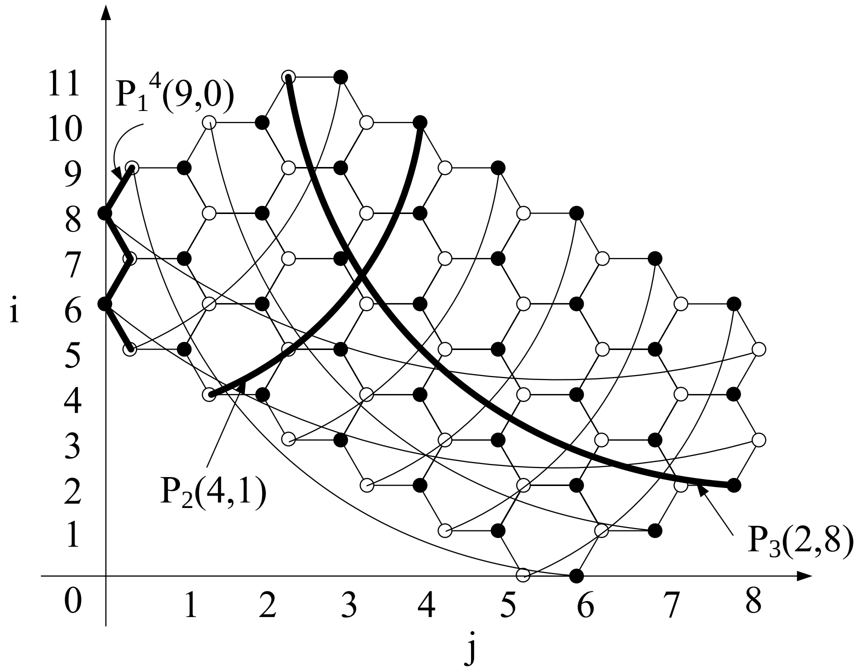

Figure A.1: The path patterns $P_1^t(i,j)$, $P_2(i,j)$ and $P_3(i,j)$ for $DHT(2,3,6)$.

Example 2: The path pattern $Q(i,j)$ for $DHT(2,3,6)$.

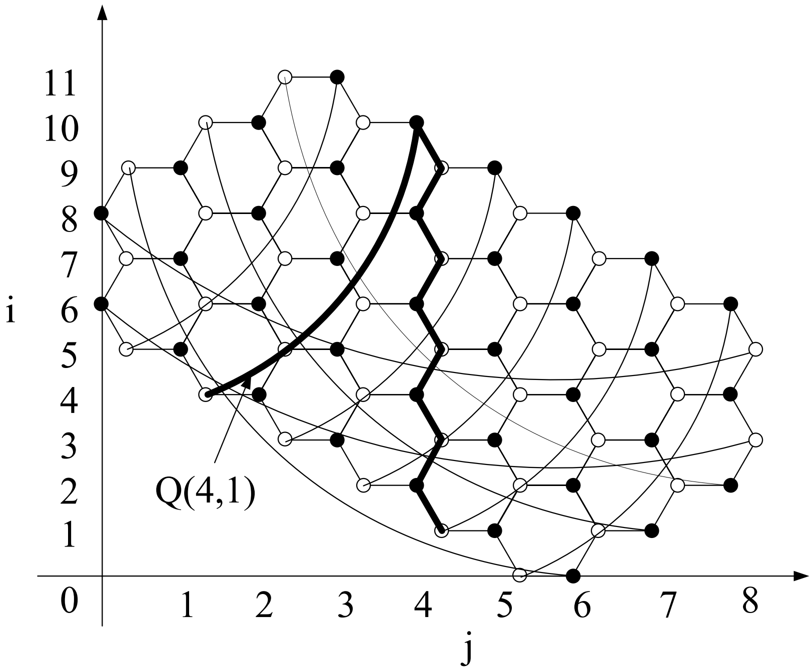

Figure A.2: The path pattern $Q(i,j)$ for $DHT(2,3,6)$.

Example 3: The path pattern $R(i,j)$ for $DHT(2,3,6)$.

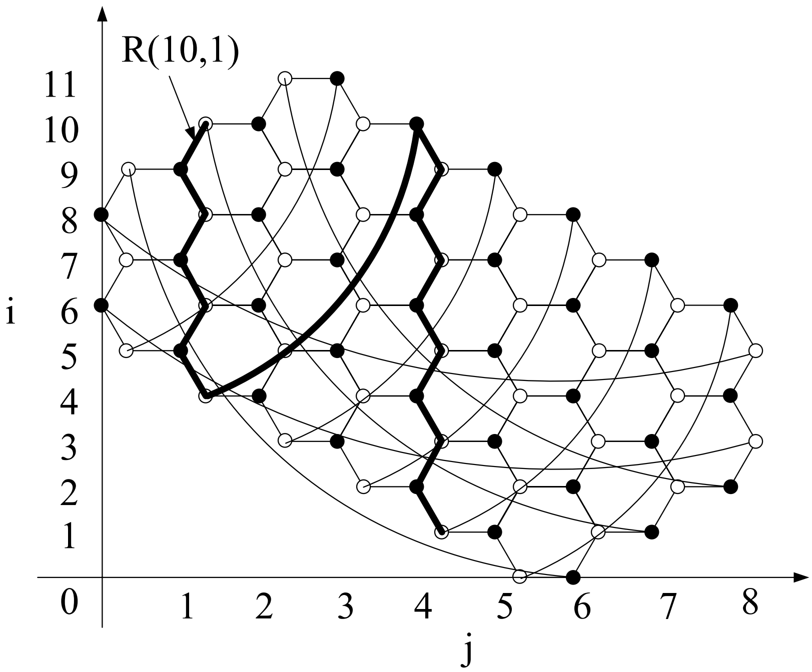

Figure A.3: The path pattern $R(i,j)$ for $DHT(2,3,6)$.

Example 4: The path pattern $S(i, j)$ for $DHT(2, 3, 6)$.

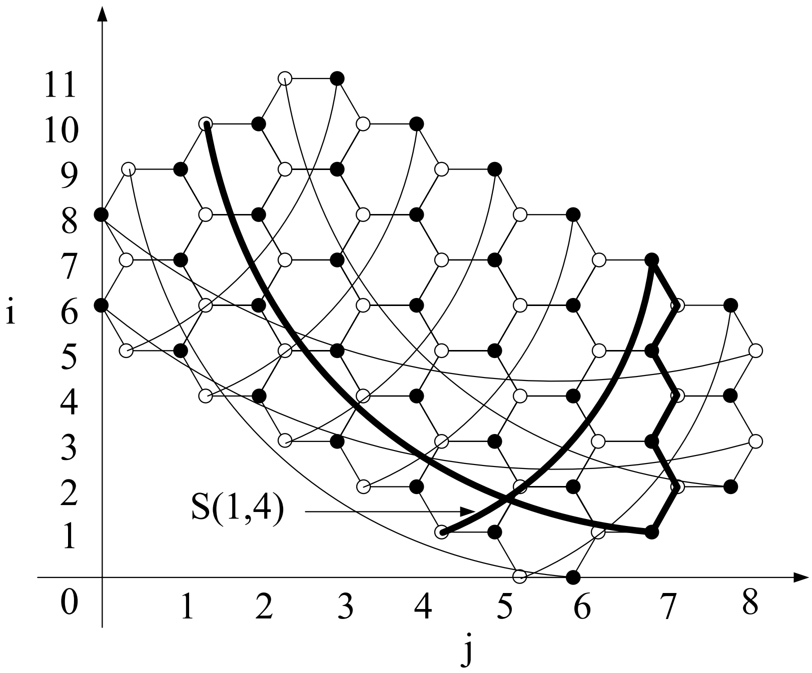

Figure A.4: The path pattern $S(i, j)$ for $DHT(2, 3, 6)$.

Example 5: The path pattern $T(i, j)$ for $DHT(2, 3, 6)$ when $l = g_0$.

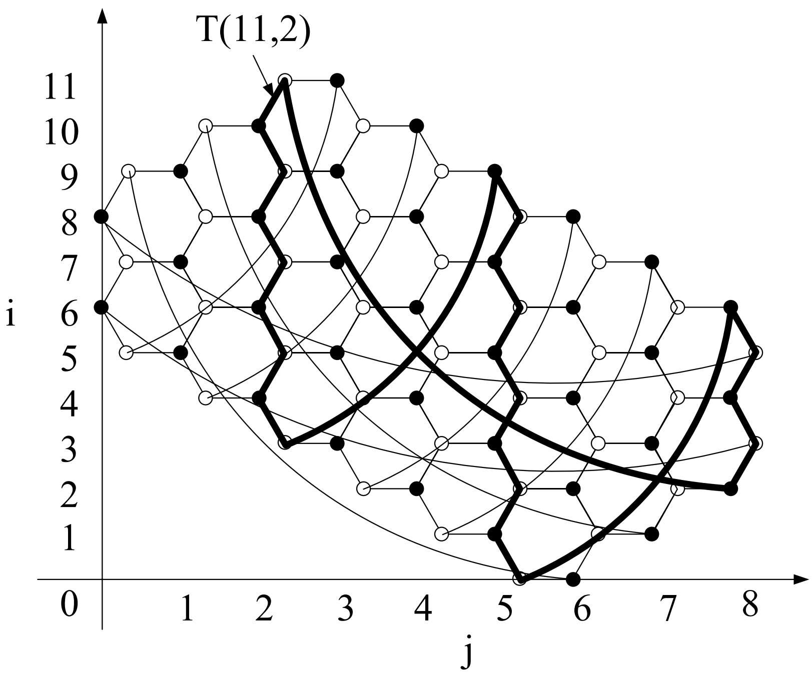

Figure A.5: The path pattern $T(i, j)$ for $DHT(2, 3, 6)$ when $l = g_0$.

Example 6: The path pattern $T(i,j)$ for $DHT(2,4,10)$ when $l \neq g_0$.

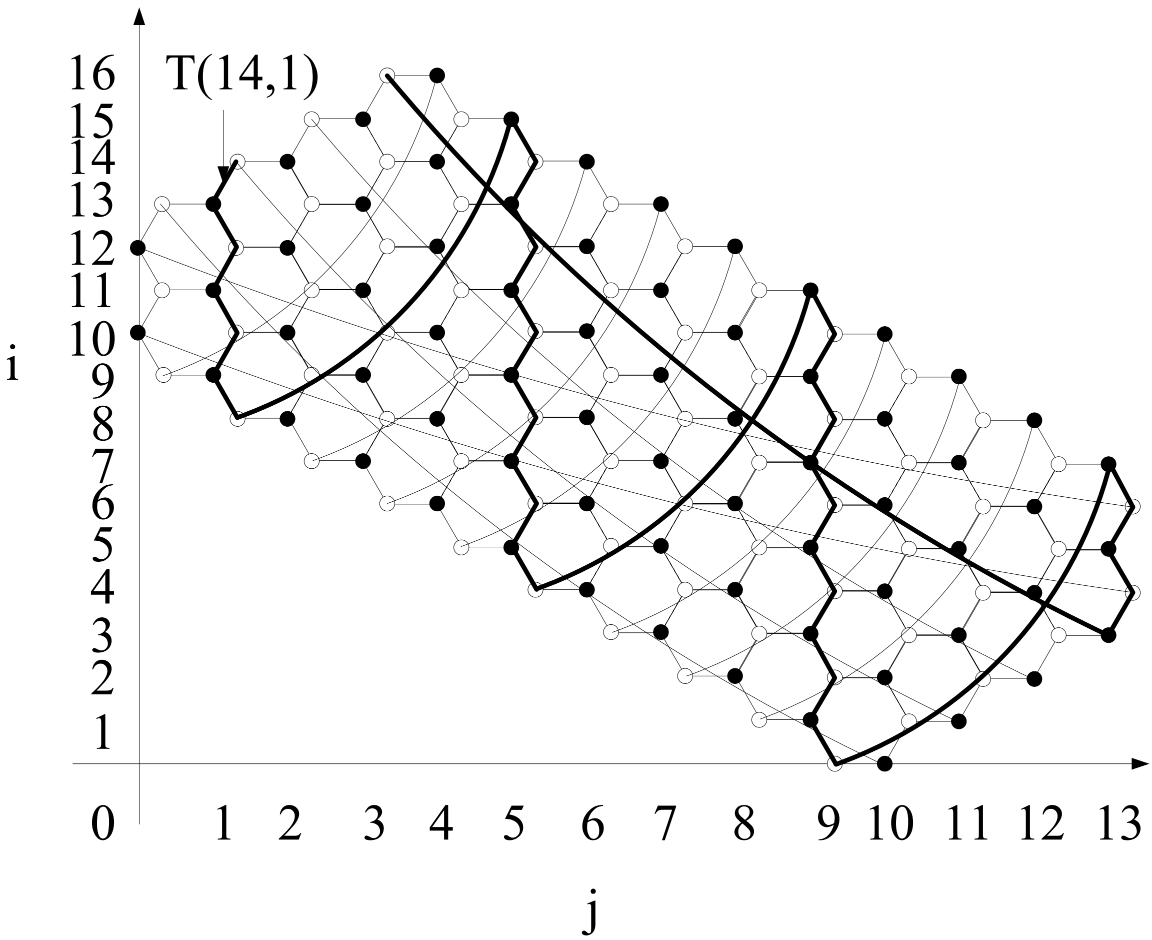

Figure A.6: The path pattern $T(i,j)$ for $DHT(2,4,10)$ when $l \neq g_0$.

Appendix B

Example For Lemma 1

Example: We use Lemma 1 to obtain the three disjoint cycles $\{C_0, C_1, C_2\}$ of $DHT(2,3,6)$ such that $\cup_{0\leq k\leq 2} V(C_k) = V(DHT(2,3,6))$ as follows.

$$
\begin{aligned}
C_0 &= \langle T(2\cdot 2+6-1+0,0)\rangle^{3/3} \\
&= \langle T(9,0)\rangle \\
&= \langle (9,0), R(9,0), (6-\left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3-0-1, \left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3+0), \\
&\quad S(6-\left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3-0-1, \left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3+0), \\
&\quad (3+2\cdot 2+\left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3+0-1, 3-6+\left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3+0)\rangle \\
&= \langle (9,0), R(9,0), (2,3), S(2,3), (9,0)\rangle \\
&= \langle (9,0), P_1^{2(2+0)}(9,0), (9-2\cdot 2-2\cdot 0, 0), Q^{\lfloor\frac{6-0-1}{3}\rfloor}(9-2\cdot 2-2\cdot 0, 0), \\
&\quad (6-\left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3-0-1, \left\lfloor\frac{6-0-1}{3}\right\rfloor\cdot 3+0), P_2(2,3), (3+6+2\cdot 2-3-2, 3+3), \\
&\quad P_1^{2(6+2-1-3)}(3+6+2\cdot 2-3-2, 3+3), (3-6+3, 3+3), P_3(3-6+3, 3+3), \\
&\quad (3+2\cdot 2+3-1, 3-6+3)\rangle \\
&= \langle (9,0), P_1^4(9,0), (5,0), Q^1(5,0), (2,3), P_2(2,3), (8,6), P_1^8(8,6), (0,6), P_3(0,6), (9,0)\rangle \\
&= \langle (9,0), P_1^4(9,0), (5,0), P_2(5,0), (3+6+2\cdot 2-0-2, 3+0), \\
&\quad P_1^{2(2+3)-1}(3+6+2\cdot 2-0-2, 3+0), (6-3-0-1, 3+0), P_2(2,3), (8,6), P_1^8(8,6), \\
&\quad (0,6), P_3(0,6), (9,0)\rangle \\
&= \langle (9,0), P_1^4(9,0), (5,0), P_2(5,0), (11,3), P_1^9(11,3), (2,3), P_2(2,3), (8,6), P_1^8(8,6), (0,6), \\
&\quad P_3(0,6), (9,0)\rangle \\
&= \langle (9,0), (8,0), (7,0), (6,0), (5,0), (11,3), (10,3), (9,3), (8,3), (7,3), (6,3), (5,3), (4,3), \\
&\quad (3,3), (2,3), (8,6), (7,6), (6,6), (5,6), (4,6), (3,6), (2,6), (1,6), (0,6), (9,0)\rangle
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \langle T(2\cdot 2+6-1+1,1)\rangle^{3/3}\\
&= \langle T(10,1)\rangle\\
&= \langle (10,1), R(10,1), (6-\left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3-1-1, \left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3+1),\\
&\quad S(6-\left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3-1-1, \left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3+1),\\
&\quad (3+2\cdot 2+\left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3+1-1, 3-6+\left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3+1)\rangle\\
&= \langle (10,1), R(10,1), (1,4), S(1,4), (10,1)\rangle\\
&= \langle (10,1), P_1^{2(2+1)}(10,1), (10-2\cdot 2-2\cdot 1,1), Q^{\lfloor\frac{6-1-1}{3}\rfloor}(10-2\cdot 2-2\cdot 1,1),\\
&\quad (6-\left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3-1-1, \left\lfloor\frac{6-1-1}{3}\right\rfloor\cdot 3+1), P_2(1,4), (3+6+2\cdot 2-4-2, 3+4),\\
&\quad P_1^{2(6+2-1-4)}(3+6+2\cdot 2-4-2,3+4), (3-6+4,3+4), P_3(3-6+4,3+4),\\
&\quad (3+2\cdot 2+4-1, 3-6+4)\rangle\\
&= \langle (10,1), P_1^6(10,1), (4,1), Q^1(4,1), (1,4), P_2(1,4), (7,7), P_1^6(7,7), (1,7), P_3(1,7), (10,1)\rangle\\
&= \langle (10,1), P_1^6(10,1), (4,1), P_2(4,1), (3+6+2\cdot 2-1-2, 3+1),\\
&\quad P_1^{2(2+3)-1}(3+6+2\cdot 2-1-2,3+1), (6-3-1-1,3+1), P_2(1,4), (7,7), P_1^6(7,7),\\
&\quad (1,7), P_3(1,7), (10,1)\rangle\\
&= \langle (10,1), P_1^6(10,1), (4,1), P_2(4,1), (10,4), P_1^9(10,4), (1,4), P_2(1,4), (7,7), P_1^6(7,7),\\
&\quad (1,7), P_3(1,7), (10,1)\rangle\\
&= \langle (10,1), (9,1), (8,1), (7,1), (6,1), (5,1), (4,1), (10,4), (9,4), (8,4), (7,4), (6,4), (5,4),\\
&\quad (4,4), (3,4), (2,4), (1,4), (7,7), (6,7), (5,7), (4,7), (3,7), (2,7), (1,7), (10,1)\rangle
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= \langle T(2\cdot 2+6-1+2,2)\rangle^{3/3} \\
&= \langle T(11,2)\rangle \\
&= \langle (11,2), R(11,2), (6-\left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3-2-1, \left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3+2), \\
&\quad S(6-\left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3-2-1, \left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3+2), \\
&\quad (3+2\cdot 2+\left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3+2-1, 3-6+\left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3+2)\rangle \\
&= \langle (11,2), R(11,2), (0,5), S(0,5), (11,2)\rangle \\
&= \langle (11,2), P_1^{2(2+2)}(11,2), (11-2\cdot 2-2\cdot 2,2), Q^{\lfloor \frac{6-2-1}{3} \rfloor}(11-2\cdot 2-2\cdot 2,2), \\
&\quad (6-\left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3-2-1, \left\lfloor \frac{6-2-1}{3} \right\rfloor \cdot 3+2), P_2(0,5), (3+6+2\cdot 2-5-2, 3+5), \\
&\quad P_1^{2(6+2-1-5)}(3+6+2\cdot 2-5-2,3+5), (3-6+5,3+5), P_3(3-6+5,3+5), \\
&\quad (3+2\cdot 2+5-1, 3-6+5)\rangle \\
&= \langle (11,2), P_1^8(11,2), (3,2), Q^1(3,2), (0,5), P_2(0,5), (6,8), P_1^4(6,8), (2,8), P_3(2,8), (11,2)\rangle \\
&= \langle (11,2), P_1^8(11,2), (3,2), P_2(3,2), (3+6+2\cdot 2-2-2, 3+2), \\
&\quad P_1^{2(2+3)-1}(3+6+2\cdot 2-2-2,3+2), (6-3-2-1,3+2), P_2(0,5), (6,8), P_1^4(6,8), \\
&\quad (2,8), P_3(2,8), (11,2)\rangle \\
&= \langle (11,2), P_1^8(11,2), (3,2), P_2(3,2), (9,5), P_1^9(9,5), (0,5), P_2(0,5), (6,8), P_1^4(6,8), (2,8), \\
&\quad P_3(2,8), (11,2)\rangle \\
&= \langle (11,2), (10,2), (9,2), (8,2), (7,2), (6,2), (5,2), (4,2), (3,2), (9,5), (8,5), (7,5), (6,5), \\
&\quad (5,5), (4,5), (3,5), (2,5), (1,5), (0,5), (6,8), (5,8), (4,8), (3,8), (2,8), (11,2)\rangle
\end{aligned}
$$