A Study on Robustness of PID-type Iterative Learning Controller against Initial State Error

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Abstract

In this paper, the effect of initial state error in the iterative learning control(ILC) system is studied. First, the previous result that the performance of D-type ILC algorithm can be improved by adding a P-term of error in the algorithm is generalized to PID-type algorithm. Then robustness is investigated against initial state error of the generalized ILC algorithm. It is also shown that the trend of error reduction can be effectively controlled by tuning gains of the proposed controller. In order to confirm validity of the proposed ILC algorithm, several examples are presented.

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1 Introduction

In recent years, there has been a great deal of study to overcome limitations of conventional controllers against uncertainty due to inaccurate modelling and/or parameter variations. As one of alternatives, the iterative learning control(ILC) method has been developed(Arimoto et al. 1984) by which complete tracking performance can be achieved as the given task is imposed iteratively(Bondi et al. 1988, Hwang et al. 1993, Togai and Yamano 1985, Bien and Huh 1989).

In most ILC algorithms, it is assumed that the initial state value of the plant is equal to that of the desired trajectory for perfect tracking. It is practically impossible, however, to set the initial state value of the plant to that of the desired trajectory exactly.

Heinzinger et al. (1989) reported some robustness result for a class of nonlinear systems. They showed that output error is bounded when initial state error is bounded and that upper bound of output error can be determined. Arimoto et al. (Arimoto 1990, Arimoto et al. 1990) studied the robustness of the P-type ILC algorithm for serial-link robot manipulators with revolute-type joints. In the paper, the fluctuation of dynamics is also considered in addition to nonzero initial state error and measurement noise. Arimoto and Naniwa(1994) studied the robustness of the ILC algorithm in a more systematic way using the concept of quasi-natural potential and passivity. Saab(1994) studied robustness of the discrete time ILC algorithm and showed that the output error is bounded by norm of initial state error, fluctuations of dynamics and measurement noises. Lee and Bien(1991) reported the possibility of divergence of control input due to the initial state error. Later, Lee and Bien(1996) showed that the trajectory errors can be estimated in terms of initial state error and parameters of the ILC algorithm when the PD-type ILC algorithm is applied.

To be more specific, consider the linear system described by Eqn. (1).

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

Here, $x \in R^n$, $u \in R^r$ and $y \in R^q$ denote the state, the input and the output, respectively. A, B and C are matrices with appropriate dimensions and it is assumed that CB is a full rank matrix. Let $x_d(\cdot)$ be the desired state trajectory and $y_d(\cdot)$ be the corresponding output trajectory. Assume that $y_d(\cdot)$ and $x_d(\cdot)$ are continuously differentiable on $[0, T]$. It is shown in (Lee and Bien 1996) that when the ILC algorithm Eqn. (2) is applied to Eqn. (1), the output trajectory converges to the form in Eqn. (3).

$$u_{k+1}(t) = u_k(t) + \Gamma \left(\dot{e}_k(t) - Re_k(t) \right) \tag{2}$$

$$\lim_{k \to \infty} y_k(t) = y_d(t) + e^{Rt} C \left(x_0 - x_d(0) \right) \tag{3}$$

Here, $y_k(\cdot), u_k(\cdot)$, and $e_k(\cdot)$ are output trajectory, control input trajectory, and output error trajectory at the k-th iteration, respectively. Eqn. (3) shows that the effect of initial state error can be controlled by gain R of the ILC algorithm, and the error asymptotically converges to zero if R is chosen such that all eigenvalues have negative real parts. Lee and Bien also showed that error is bounded and the bound of error can be adjusted by R even if initial state error may be randomly varying as the iteration is repeated.

Sometimes an intentional overshoot may be positively solicited to get a better performance against nonlinear frictions in case the error decreases only exponentially and the frictions cause steady-state error. It is difficult, however, to get an overshoot performance of the existing ILC algorithms (See Eqn. (3)). In this paper, a PID-type ILC algorithm is proposed for more flexibility of the output trajectory and it is shown that the effect of initial state error can be controlled by the parameters of the proposed ILC algorithm as in Lee and Bien(1996). In addition, a design guideline is presented for selection of the parameters. The result is also extended for a class of nonlinear dynamic systems.

The remainder of this paper is organized as follows. In Section 2, effect of initial state error for a linear time invariant(LTI) system is shown. In Section 3, effect of initial state error for a class of nonlinear time varying systems is shown. In Section 4, several numerical examples are presented to show effectiveness of the proposed ILC algorithm, and some concluding remarks follow in Section 5.

In the sequel, for n-dimensional Euclidean space R^n, $\|x\|$ denotes the Euclidean norm of a vector $x = (x_1, \ldots, x_n)^T$. For a matrix A, $\|A\|$ denotes its induced matrix norm. $\|x\|_\infty$ denotes the sup-norm defined by

$$\|x\|_\infty = \sup_{1\le i\le n} |x_i|.$$

For a $n \times r$ matrix A with elements a_{ij}, $\|A\|_\infty$ is defined by

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{r} |a_{ij}|.$$

For a function $h : [0,T] \to R^n$ and a real number $\lambda > 0$, $\|h(\cdot)\|_\lambda$ denotes the λ-norm defined by

$$\|h(\cdot)\|_\lambda = \sup_{t\in[0,T]} e^{-\lambda t}\|h(t)\|_\infty.$$

As in the notation $u_k(t)$, the subscript k denotes the iteration number.

2 Iterative Learning Controller for LTI Dynamic Systems

In this section, the effect of initial state error for LTI systems is shown. To this end, consider the following problem.

Problem *Let there be given a desired trajectory* $y_d(t)$, $t \in [0, T]$ *for a LTI system described by Eqn. (1). Suppose there exists the same initial state error at each iteration, i.e.,* $x_k(0) = x_0 \neq x_d(0), k = 0, 1, 2, \cdots$. *The problem is to investigate effect of the initial state error when the following PID-type ILC algorithm (Eqn. (4)) is applied to the system (Eqn. (1)):*

$$u_{k+1}(t) = u_k(t) + \Gamma \left[\dot{e}_k(t) + Q_0 e_k(t) + Q_1 \int_0^t e_k(\tau) d\tau\right] \tag{4}$$

Here,

$$e_k(t) = y_d(t) - y_k(t)$$

and Q_0 *and* Q_1 *are* $q \times q$ *constant matrices.*

As in Lee and Bien(1996), we can obtain the following result about effect of the initial state error when the ILC algorithm Eqn. (4) is applied.

Theorem 1 *Suppose that the update law Eqn. (4) is applied to the system Eqn. (1) and the initial state at each iteration can be different from the desired initial state, i.e.,* $x_k(0) = x_0 \neq x_d(0)$ *for* $k = 0, 1, 2, \cdots$. *If*

$$\|I - \Gamma CB\|_\infty \leq \rho < 1, \tag{5}$$

then

$$\lim_{k \to \infty} y_k(t) = y_d(t) + C_R e^{A_R t} \xi_0$$

where

$$\begin{aligned} A_R &= \begin{bmatrix} 0 & I \\ -Q_1 & -Q_0 \end{bmatrix} \\ C_R &= \begin{bmatrix} I & 0 \end{bmatrix} \\ \xi_0 &= \begin{bmatrix} I \\ -Q_0 \end{bmatrix} C\left(x_0 - x_d(0)\right). \end{aligned} \tag{6}$$

Proof

Let $u_a(t)$ and $x_a(t)$ be the control input and the state that satisfy Eqn. (7).

$$\begin{aligned} x_a(t) &= x_0 + \int_0^t \left(Ax_a(\tau) + Bu_a(\tau)\right) d\tau \\ y_d(t) &+ C_R e^{A_R t}\xi_0 = Cx_a(t) \end{aligned} \tag{7}$$

Let

$$\begin{aligned} \Delta u_k(t) &= u_a(t) - u_k(t) \\ \Delta x_k(t) &= x_a(t) - x_k(t). \end{aligned}$$

It follows from Eqns. (4) and (7) that

$$\begin{aligned} \Delta u_{k+1}(t) &= u_a(t) - u_k(t) - \Gamma \left[\dot{e}_k(t) + Q_0 e_k(t) + Q_1 \int_0^t e_k(\tau) d\tau\right] \\ &= (I - \Gamma CB)\, \Delta u_k(t) - \Gamma\, (CA + Q_0 C)\, \Delta x_k(t) \\ &\quad -\Gamma Q_1 C \int_0^t \Delta x_k(\tau) d\tau. \end{aligned} \tag{8}$$

Taking the norm $\|\cdot\|_\lambda$ on both sides of Eqn. (8), we have

$$\begin{aligned} \|\Delta u_{k+1}(\cdot)\|_\lambda &\leq \rho \|\Delta u_k(\cdot)\|_\lambda + h_0 \|\Delta x_k(\cdot)\|_\lambda \\ &\quad + h_1 \frac{1 - e^{-\lambda T}}{\lambda} \|\Delta x_k(\cdot)\|_\lambda \end{aligned} \tag{9}$$

where

$$\begin{aligned} h_0 &= \|\Gamma(CA + Q_0 C)\|_\infty \\ h_1 &= \|\Gamma Q_1 C\|_\infty. \end{aligned}$$

From Eqn. (7), we can obtain

$$\Delta x_k(t) = \int_0^t \left[A \Delta x_k(\tau) + B \Delta u_k(\tau)\right] d\tau. \tag{10}$$

Taking the norm $\|\cdot\|_\infty$ on both sides of Eqn. (10) and applying Grownwall-Bellman inequality (Rugh 1993), we find that

$$\|\Delta x_k(t)\|_\infty \leq \int_0^t e^{a(t-\tau)}\|B\|_\infty \|\Delta u_k(\tau)\|_\infty d\tau \tag{11}$$

where

$$a = \|A\|_\infty.$$

Multiplying both sides of Eqn. (11) by $e^{-\lambda t}$ and substituting to Eqn. (9), we further find that

$$\|\Delta u_{k+1}(\cdot)\|_\lambda \leq \left(\rho + h_2(\lambda)\frac{1-e^{(a-\lambda)T}}{\lambda - a}\right)\|\Delta u_k(\cdot)\|_\lambda \tag{12}$$

where

$$h_2(\lambda) = \left(h_0 + h_1\frac{1-e^{-\lambda T}}{\lambda}\right)\|B\|_\infty.$$

Since $0 \leq \rho < 1$ by assumption, it is possible to choose λ sufficiently large so that

$$\rho_0 = \rho + h_2(\lambda)\frac{1-e^{(a-\lambda)T}}{\lambda - a} < 1.$$

From Eqn. (12), we can show that

$$\lim_{k\to\infty}\|\Delta u_k(\cdot)\|_\lambda = 0.$$

From Eqns. (7) and (11), we can finally conclude the following:

$$\lim_{k\to\infty} y_k(t) = y_d(t) + C_R e^{A_R t}\xi_0$$

This completes the proof.

Theorem 1 implies that if the initial state error is the same and constant at each iteration, the output trajectory can be exactly estimated from the information about the desired output trajectory, the initial output error value, and the learning controller parameters Q_0 and Q_1 as stated in Lee and Bien(1996). Since the performance of the learning controller is estimated from the output error trajectory, the result of Theorem 1 can be utilized in real applications.

Note that Eqn. (4) is a PID-type ILC algorithm. If the gain is chosen as $Q_0 = -R$ and $Q_1 = 0$, the algorithm becomes PD-type as in Eqn. (2) and the output trajectory converges to Eqn. (3). From this observation, the proposed PID-type ILC algorithm is considered as an extension of the PD-type ILC algorithm.

In the case of single-output systems, the effect of the initial state error for the PD-type ILC algorithm in Lee and Bien(1996) decreases only exponentially. In the PID-type ILC algorithm, however, the effect of initial state error is governed according to the characteristics of the 2nd-order system by proper choice of Q_0 and Q_1. This implies that the output trajectory can be controlled in a variety of ways by introducing the integral-term.

Theorem 1 also implies that error dynamics depends on eigenvalues of A_R and, if Q_0 and Q_1 are chosen such that all real parts of eigenvalues of A_R are negative, the error decreases as the time increases.

In the following, we are going to examine random initial state error which is a more realistic situation. Assume that initial state value at each iteration is in a neighborhood of x_0 such that

$$\|x_k(0) - x_0\|_\infty \leq \Delta. \tag{13}$$

As in Lee and Bien(1996), we can obtain the following result that shows robustness of the ILC algorithm Eqn. (4) against random initial state error.

Theorem 2 *Suppose that (5) holds and the update law Eqn. (4) is applied to the system Eqn. (1). Let $u_a(t)$ and $x_a(t)$ be control input and state that satisfy Eqn. (7). If the initial state at each iteration satisfies Eqn. (13), then, as $k \to \infty$,* *the error between $u_a(t)$ and $u_k(t)$ is bounded and this bound directly depends on $\|CA + Q_0C\|_\infty$, $\|(CA + Q_0C)A + Q_1C\|_\infty$, and Δ.*

Proof

As in the proof of Theorem 1, one finds from Eqns. (4) and (7) that

$$\begin{aligned}\Delta u_{k+1}(t) &= (I - \Gamma CB)\,\Delta u_k(t) - \Gamma\,(CA + Q_0C)\,(x_0 - x_k(0)) \\ &\quad -\Gamma\,(CA + Q_0C)\,B\int_0^t \Delta u_k(\tau)d\tau \\ &\quad -\Gamma\,((CA + Q_0C)A + Q_1C)\int_0^t \Delta x_k(\tau)d\tau. \end{aligned} \tag{14}$$

Taking the norm $\|\cdot\|_\lambda$ on both sides of Eqn. (14), we have

$$\begin{aligned}\|\Delta u_{k+1}(\cdot)\|_\lambda &\leq \rho\|\Delta u_k(\cdot)\|_\lambda + h_3\Delta + h_4(\lambda)\|\Delta u_k(\cdot)\|_\lambda \\ &\quad + h_5(\lambda)\|\Delta x_k(\cdot)\|_\lambda \end{aligned} \tag{15}$$

where

$$\begin{aligned} h_3 &= \|\Gamma\|_\infty\|CA + Q_0C\|_\infty \\ h_4(\lambda) &= \|\Gamma\|_\infty\|CA + Q_0C\|_\infty\|B\|_\infty\frac{1 - e^{-\lambda T}}{\lambda} \\ h_5(\lambda) &= \|\Gamma\|_\infty\|(CA + Q_0C)A + Q_1C\|_\infty\frac{1 - e^{-\lambda T}}{\lambda}. \end{aligned}$$

From Eqn. (7), we can obtain

$$\Delta x_k(t) = (x_0 - x_k(0)) + \int_0^t \left[A\Delta x_k(\tau) + B\Delta u_k(\tau)\right]d\tau. \tag{16}$$

Taking the norm $\|\cdot\|_\infty$ on both sides of Eqn. (16) and applying Grownwall-Bellman inequality (Rugh 1993), we find that

$$\|\Delta x_k(t)\|_\infty \leq e^{at}\Delta + \int_0^t e^{a(t-\tau)}\|B\|_\infty\|\Delta u_k(\tau)\|_\infty d\tau \tag{17}$$

where

$$a = \|A\|_\infty.$$

Multiplying both sides of Eqn. (17) by $e^{-\lambda t}$ and substituting to Eqn. (15), we further find that

$$\begin{aligned} \|\Delta u_{k+1}(\cdot)\|_\lambda &\leq \left(\rho + h_4(\lambda) + h_5(\lambda)\frac{1-e^{(a-\lambda)T}}{\lambda - a}\right)\|\Delta u_k(\cdot)\|_\lambda \\ &\quad + \left(h_3 + h_5(\lambda)\right)\Delta. \end{aligned} \tag{18}$$

Since $0 \leq \rho < 1$ by assumption, it is possible to choose λ sufficiently large so that

$$\rho_1 = \rho + h_4(\lambda) + h_5(\lambda)\frac{1-e^{(a-\lambda)T}}{\lambda - a} < 1.$$

From Eqn. (18), we can show that

$$\begin{aligned} \lim_{k\to\infty} \|\Delta u_k(\cdot)\|_\lambda &\leq \frac{1}{1-\rho_1}\left(h_3 + h_5(\lambda)\right)\Delta \\ &= \frac{1}{1-\rho_1}\|\Gamma\|_\infty \left(\|CA + Q_0C\|_\infty\right. \\ &\quad \left. + \|(CA+Q_0C)A + Q_1C\|_\infty \frac{1-e^{-\lambda T}}{\lambda}\right)\Delta. \end{aligned}$$

This completes the proof.

As shown in the proof of Theorem 2, the bound of control error, $\|\Delta u_k(\cdot)\|_\lambda$, depends on $\|CA+Q_0C\|_\infty$ and $\|(CA+Q_0C)A+Q_1C\|_\infty$ as well as on Δ. This implies that the bound of control error can be adjusted by Q_0 and Q_1. As commented in Lee and Bien(1996), we can choose Q_0 and Q_1 as follows:

$$\begin{aligned} Q_0 &= -\tilde{C}\tilde{A}\tilde{C}^T(\tilde{C}\tilde{C}^T)^{-1} \\ Q_1 &= -(\tilde{C}\tilde{A} + Q_0\tilde{C})\tilde{A}\tilde{C}^T(\tilde{C}\tilde{C}^T)^{-1} \\ &= -(\tilde{C}\tilde{A} - \tilde{C}\tilde{A}\tilde{C}^T(\tilde{C}\tilde{C}^T)^{-1}\tilde{C})\tilde{A}\tilde{C}^T, \end{aligned}$$

where $\tilde{A}$ and $\tilde{C}$ are models of A and C of system Eqn. (1), respectively.

3 Iterative Learning Controller for a Class of Nonlinear Dynamic Systems

In this section, the effect of initial state error for a class of nonlinear systems is shown. Consider the nonlinear system described by Eqn. (19).

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), t\right) + B\left(x(t), t\right) u(t) \\ y(t) &= g\left(x(t), t\right) \end{aligned} \tag{19}$$

Here, $x \in R^n, u \in R^r$ and $y \in R^q$ denote the state, the input and the output, respectively. The followings assumptions are made for the system Eqn. (19).

A1 The functions $f : R^n \times [0, T] \to R^n$ and $B : R^n \times [0, T] \to R^{n\times r}$ are piecewise continuous in t, and $g : R^n \times [0, T] \to R^q$ is differentiable in x and t with partial derivatives denoted as $g_x\left(x(t), t\right)$ and $g_t\left(x(t), t\right)$.

A2 The functions $f\left(x(t), t\right), B\left(x(t), t\right), g_x\left(x(t), t\right)$ and $g_t\left(x(t), t\right)$ satisfy the uniformly global Lipschitz condition in x on the interval $[0, T]$. That is, for all pair $(x_1(t), x_2(t)) \in R^n \times R^n$, there exists $k_h(0 < k_h < \infty)$ such that

$$\begin{aligned} \|h\left(x_1(t), t\right) - h\left(x_2(t), t\right)\|_\infty &\leq k_h \|x_1(t) - x_2(t)\|_\infty, \\ &\quad \forall t \in [0, T], h \in \{f, B, g_x, g_t\}. \end{aligned}$$

A3 The functions $f\left(x(t), t\right), B\left(x(t), t\right)$ and $g_x\left(x(t), t\right)$ are bounded on $R^n \times [0, T]$.

A4 $g_x\left(x(t), t\right) B\left(x(t), t\right)$ is a full rank matrix for any $(x, t) \in R^n \times [0, T]$ that can be achieved by the system Eqn. (19).

Consider Problem of Section 2 with the system Eqn. (19) instead of the system Eqn. (1). Then we can obtain the following result for the nonlinear system Eqn. (19).

Theorem 3 *Let the system Eqn. (19) satisfy the assumptions from* ***A1*** *to* ***A4****. Assume that the output trajectory* $y_a(t)$ *defined by Eqn. (20) is achievable for a given desired trajectory* $y_d(t)$*:*

$$y_a(t) = y_d(t) + C_R e^{A_R t} \xi_0 \tag{20}$$

Suppose that the update law Eqn. (4) is applied to the system Eqn. (19) and the initial state at each iteration is different from the desired initial state, i.e., $x_k(0) = x_0 \neq x_d(0)$ *for* $k = 0, 1, 2, \cdots$*. If*

$$\|I - \Gamma(t) g_x(x,t) B(x,t)\|_\infty \leq \rho < 1, \forall (x,t) \in R^n \times [0,T], \tag{21}$$

then

$$\lim_{k \to \infty} y_k(t) = y_d(t) + C_R e^{A_R t} \xi_0$$

where

$$\begin{aligned} A_R &= \begin{bmatrix} 0 & I \\ -Q_1 & -Q_0 \end{bmatrix} \\ C_R &= \begin{bmatrix} I & 0 \end{bmatrix} \\ \xi_0 &= \begin{bmatrix} I \\ -Q_0 \end{bmatrix} \left(g(x_0, 0) - g(x_d(0), 0) \right). \end{aligned} \tag{22}$$

Proof

Let $u_a(t)$ and $x_a(t)$ be the control input and the state that satisfy Eqn. (23).

$$\begin{aligned} x_a(t) &= x_0 + \int_0^t \left[f\left(x_a(\tau), \tau\right) + B\left(x_a(\tau, \tau\right) u_a(\tau) \right] d\tau \\ y_a(t) &= g\left(x_a(t), t\right) \end{aligned} \tag{23}$$

Let

$$\begin{aligned}\Delta u_k(t) &= u_a(t) - u_k(t) \\ \Delta x_k(t) &= x_a(t) - x_k(t).\end{aligned}$$

For notational convenience, function parameters are denoted in subscript notation as follows.

$$\begin{aligned}
g_k = g\left(x_k(t), t\right) \quad &, \quad g_a = g\left(x_a(t), t\right) \\
g_{xk} = \frac{\partial}{\partial x} g(x,t)|_{x=x_k(t)} \quad &, \quad g_{xa} = \frac{\partial}{\partial x} g(x,t)|_{x=x_a(t)} \\
g_{tk} = \frac{\partial}{\partial t} g(x,t)|_{x=x_k(t)} \quad &, \quad g_{ta} = \frac{\partial}{\partial t} g(x,t)|_{x=x_a(t)} \\
f_k = f\left(x_k(t), t\right) \quad &, \quad f_a = f\left(x_a(t), t\right) \\
B_k = B\left(x_k(t), t\right) \quad &, \quad B_a = B\left(x_a(t), t\right)
\end{aligned}$$

k_{gx}, k_{gt}, k_f, k_B and k_g are the Lipschitz constants for $g_x\left(x(t), t\right)$, $g_t\left(x(t), t\right)$, $f\left(x(t), t\right)$, $B\left(x(t), t\right)$ and $g\left(x(t), t\right)$, respectively. From Eqns. (4) and (23), it is observed that

$$\begin{aligned}
\Delta u_{k+1}(t) &= u_a(t) - u_k(t) - \Gamma(t)\left[\dot{e}_k(t) + Q_0 e_k(t) + Q_1 \int_0^t e_k(\tau) d\tau\right] \\
&= \left(I - \Gamma(t) g_{xk} B_k\right) \Delta u_k(t) - \Gamma(t)\left[(g_{xa} - g_{xk})(f_a + B_a u_a)\right. \\
&\quad + g_{xk}(f_a - f_k) + g_{xk}(B_a - B_k)u_a + g_{ta} \\
&\quad \left. - g_{tk} + Q_0(g_a - g_k) + Q_1 \int_0^t (g_a - g_k) d\tau\right]. \qquad (24)
\end{aligned}$$

Taking the norm $\|\cdot\|_\lambda$ on both sides of Eqn. (24), we have

$$\|\Delta u_{k+1}(\cdot)\|_\lambda \leq \rho \|\Delta u_k(\cdot)\|_\lambda + k_1(\lambda)\|\Delta x_k(\cdot)\|_\lambda \qquad (25)$$

where

$$k_1(\lambda) = b_\Gamma \left(k_{gx} b_{fBu} + b_{gx} k_f + b_{gx} k_B b_{ua} + k_{gt} + q_0 k_g + q_1 k_g \frac{1 - e^{-\lambda T}}{\lambda}\right)$$

$$\begin{aligned} b_\Gamma &= \sup_{t\in[0,T]} \|\Gamma(t)\|_\infty, b_{fBu} = \sup_{t\in[0,T]} \|f_a + B_a u_a\|_\infty \\ b_{gx} &= \sup_{t\in[0,T]} \|g_{xk}\|_\infty, b_{ua} = \sup_{t\in[0,T]} \|u_a(t)\|_\infty \\ q_0 &= \|Q_0\|_\infty, q_1 = \|Q_1\|_\infty. \end{aligned}$$

From Eqn. (23), we can obtain

$$\begin{aligned} \Delta x_k(t) &= \int_0^t \left[f\left(x_a(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right] d\tau \\ &\quad + \int_0^t \left[B\left(x_a(\tau),\tau\right) u_a(\tau) - B\left(x_k(\tau),\tau\right) u_k(\tau)\right] d\tau. \end{aligned} \tag{26}$$

Taking the norm $\|\cdot\|_\infty$ on both sides of Eqn. (26) and applying Grownwall-Bellman inequality (Rugh 1993), we find that

$$\|\Delta x_k(t)\|_\infty \le \int_0^t e^{k_2(t-\tau)} \|B\|_\infty \|\Delta u_k(\tau)\|_\infty d\tau \tag{27}$$

where

$$k_2 = k_f + k_B b_{ua}.$$

Multiplying both sides of Eqn. (27) by $e^{-\lambda t}$ and substituting to Eqn. (25), we further find that

$$\|\Delta u_{k+1}(\cdot)\|_\lambda \le \left(\rho + k_1(\lambda) \frac{1 - e^{(k_2-\lambda)T}}{\lambda - k_2} \|B\|_\infty\right) \|\Delta u_k(\cdot)\|_\lambda. \tag{28}$$

Since $0 \le \rho < 1$ by assumption, it is possible to choose λ sufficiently large so that

$$\rho_0 = \rho + k_1(\lambda) \frac{1 - e^{(k_2-\lambda)T}}{\lambda - k_2} \|B\|_\infty < 1.$$

From Eqn. (28), we can show that

$$\lim_{k\to\infty} \|\Delta u_k(\cdot)\|_\lambda = 0.$$

From Eqns. (23) and (27), we can finally conclude the following:

$$\lim_{k\to\infty} y_k(t) = y_d(t) + C_R e^{A_R t}\xi_0$$

This completes the proof.

If Theorem 3 is applied to the linear time-invariant system Eqn. (1), the sufficient condition Eqn. (21) becomes the sufficient condition Eqn. (5) since $g_x(x(t),t) = C, B(x(t),t) = B, \Gamma(t) = \Gamma$. Also, Eqn. (22) of Theorem 3 becomes Eqn. (6) of Theorem 1 since $g(x_0,0) - g(x_d(0),0) = C(x_0 - x_d(0))$. From this, one may say that Theorem 3 is an extention of Theorem 1.

4 Numerical Examples

The following examples are given to illustrate efficiency of the proposed algorithms.

Example 1 : Linear Time-invariant System

Consider the following dynamic system(Lee and Bien 1996).

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

The desired output trajectory and the initial state are given as follows.

$$\begin{aligned} y_d(t) &= 0.03t(20-t), 0 \le t \le 20 \\ x_k(0) &= x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

According to Theorem 1, on can know that the best choice of Γ is 1. Under the assumption of 30% uncertainty of the system parameters, the Γ can be chosen as

1/1.3. Fig. 1 shows the output trajectories and the error trajectories at the 50th iteration. Fig. 1 (a), (b), (c), and (d) are the results for the cases of D-type ILC algorithm($Q_0 = 0, Q_1 = 0$), PD-type ILC algorithm($Q_0 = 2, Q_1 = 0$), PID-type ILC algorithm($Q_0 = 2, Q_1 = 2$), and another PID-type ILC algorithm($Q_0 = 2, Q_1 = 1$), respectively. As mentioned in Theorem 1, the converged output trajectory depends on the learning gains Q_0 and Q_1. In the D-type ILC algorithm(Fig. 1 (a)), the error does not change as time increases. In the PD-type ILC algorithm(Fig. 1 (b)), the error decreases exponentially as time increases. In the PID-type ILC algorithm(Fig. 1 (c),(d)), the error changes according to the learning gains and the behavior posseses the characteristics of a 2nd order dynamic system. Fig. 1 (c) shows the property of the solution of charateristic equation $s^2+2s+2 = 0$, that is, the dynamic characteristics with damping ratio $1/\sqrt{2}$, and undamped natural frequency $\sqrt{2}$. Fig. 1 (d) shows the characteristics with damping ratio 1, and undamped natural frequency 1. Over all, Fig. 1 shows that the effect of the initial state error can be arbitrarily controlled by the choice of Q_0 and Q_1.

Example 2 : Random Initial State Error

Consider the system in Example 1 and assume that initial state error is random but bounded which is modeled as follows:

$$x_k(0) = \begin{bmatrix} 0 \\ \epsilon \end{bmatrix}.$$

Here ϵ is a random number which is uniformly distributed in $[-0.5 \quad 0.5]$. We use the ILC algorithm Eqn. (4) with $\Gamma = 1/1.3$ and examine the following four cases:

Case 1 : $Q_0 = 0, Q_1 = 0$

Table 1: The average values of $\int_0^{20} |e_k(\tau)|d\tau$ and $y_k(t)$

Case 1	Case 2	Case 3	Case 4
152.135	1.660	1.742	1.330

Case 2 : $Q_0 = 2.2, Q_1 = 0$

Case 3 : $Q_0 = 0, Q_1 = 1.1$

Case 4 : $Q_0 = 2.2, Q_1 = 1.1$

Fig. 2 and Fig. 3 show the trends of $\int_0^{20} |e_k(\tau)|d\tau$ and $y_k(t)$, respectively. Noting that $-CAC(CC^T)^{-1} = 2$ and $-(CA - CAC^T(CC^T)^{-1}C)AC^T(CC^T)^{-1} = 1$, we may confirm that Case 4 shows the best result. The average values of $\int_0^{20} |e_k(\tau)|d\tau$ over 150 iterations is given in Table 1

Example 3 : Single-link Robot Manipulator

Consider the following dynamics of a single-link robot manipulator(Chen et al. 1996):

$$\begin{aligned} \ddot{\theta}(t) &= \frac{1}{J}(\frac{1}{2}m_0 + M_0)gl\sin\theta(t) + \frac{1}{J}\tau(t) \\ y(t) &= \dot{\theta}(t). \end{aligned}$$

Here, $\theta(t)$ is the angular position of the manipulator, $\tau(t)$ is the joint torque, and J is the moment of inertia of the joint, i.e., $J = M_0 l^2 + m_0 l^2/3$. The parameters are given in Table 2. Desired output trajectory and the initial state are given as follows.

$$y_d(t) = \dot{\theta}_d(t) = \frac{3}{8}t^2 - \frac{3}{8}t^3 + \frac{3}{32}t^4, 0 \le t \le 2$$

Table 2: Robot Manipulator Parameters

m_0	the mass of the link	2Kg
l	the length of the link	0.5m
M_0	the tip load	4Kg
g	the gravitational acceleration	9.8 m/s^2

$$x_k(0) = x_0 = \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix}$$

From Theorem 2, the best choice of Γ can be determined to be 3.5/3. Under the assumption of 10% uncertainty of system parameters, Γ can be chosen as 3.5/3.3. Fig. 4 shows the output trajectories and error trajectories at the 15th iteration. As mentioned in Theorem 2, the converged output trajectory is also decided by the learning gains Q_0 and Q_1. Similar to Example 1, the error does not change as time increases in the D-type ILC algorithm(Fig. 4 (a), $Q_0 = 0, Q_1 = 0$). In the PD-type ILC algorithm(Fig. 4 (b), $Q_0 = 6, Q_1 = 0$), the error decreases exponentially. In the PID-type ILC algorithm(Fig. 4 (c), $Q_0 = 6, Q_1 = 18$ and Fig. 4 (d), $Q_0 = 6, Q_1 = 9$), the error changes with the characteristics of the 2nd order dynamic system, i.e., $s^2 + 6s + 18 = 0$ and $s^2 + 6s + 9 = 0$.

5 Concluding Remarks

In this paper, robustness of PID-type ILC algorithm was investigated against initial state error for linear dynamic systems and for a class of nonlinear dynamic systems. The proposed PID-type ILC algorithm was shown to be an extended form of PD-type ILC algorithm, and the effect of the initial state error could be controlled in various ways according to choice of the gains.

It is remarked that the PID-type ILC algorithm in Theorem 1 can be generalized to the following form of PMID(Proportional Multiple-Integral Derivative)-type ILC algorithm:

$$\begin{aligned} u_{k+1}(t) &= u_k(t) + \Gamma \Big[\dot{e_k}(t) + Q_0 e_k(t) + Q_1 \int_0^t e_k(\tau_1) d\tau_1 + \cdots \\ &\quad + Q_{N-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{N-2}} e_k(\tau_{N-1}) d\tau_{N-1} \cdots d\tau_2 d\tau_1 \Big]. \end{aligned}$$

If the above form of ILC algorithm is applied to the system (Eqn. (1)), it can be easily shown that, after a sufficient number of iterations, the output trajectory converges to the following function:

$$\lim_{k\to\infty} y_k(t) = y_d(t) + C_R e^{A_R t} \xi_0,$$

where

$$\begin{aligned} A_R &= \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_{N-1} & -Q_{N-2} & -Q_{N-3} & \cdots & -Q_0 \end{bmatrix} \\ C_R &= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix} \\ \xi_0 &= \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ -Q_0 & 0 & 0 & \cdots & 0 \\ -Q_1 & -Q_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -Q_{N-2} & -Q_{N-3} & \cdots & -Q_0 & 0 \end{bmatrix}^{N-1} \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} C(x_0 - x_d(0)). \end{aligned}$$

Since the $N \times q$ numbers of eigenvalues of A_R can be arbitrarily chosen, the converged output trajectory can be determined in many different ways.

Robustness of the ILC method for nonlinear systems of more general form is open to further investigation against random initial state error, noise, and state disturbances.

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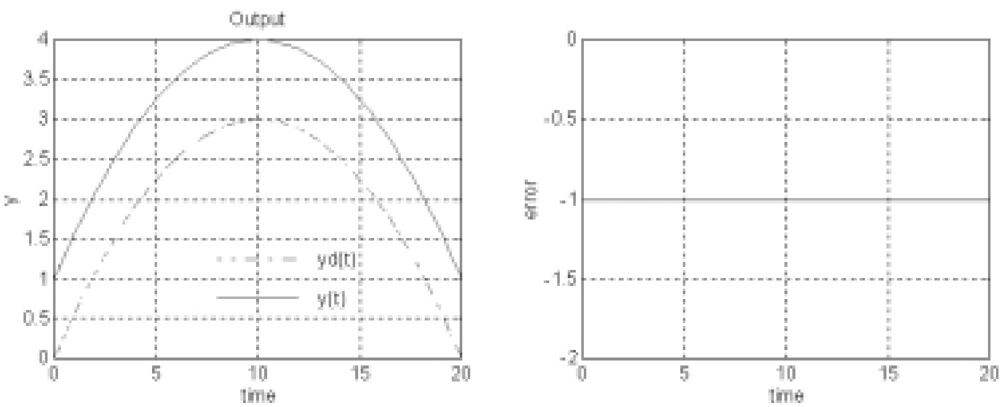
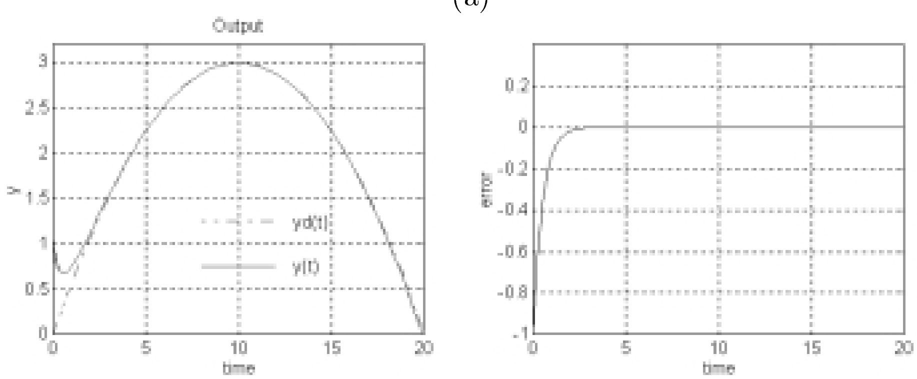


(a)

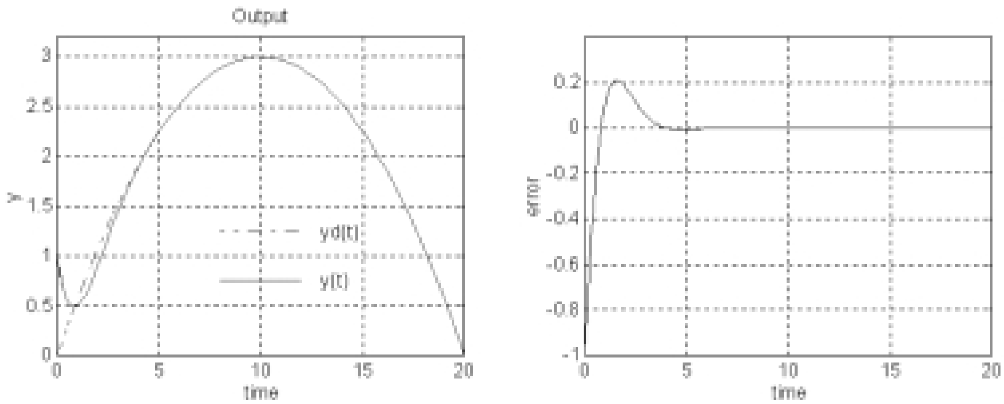


(b)

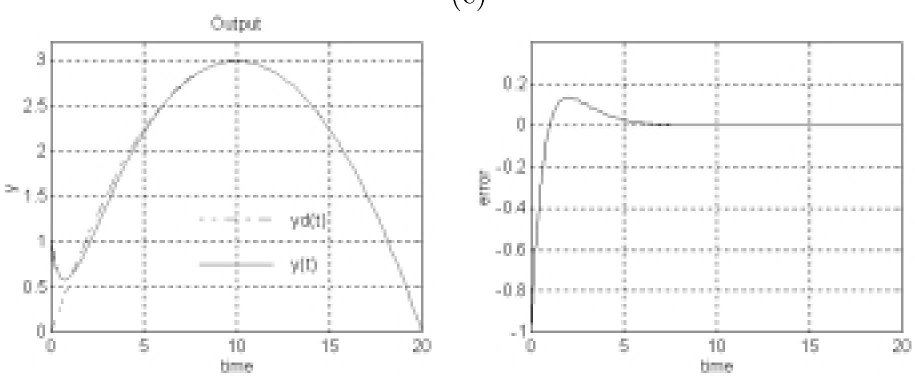
Figure 1: the output and the error for Example 1

(a) the case of $Q_0 = 0, Q_1 = 0$

(b) the case of $Q_0 = 2, Q_1 = 0$



(c)

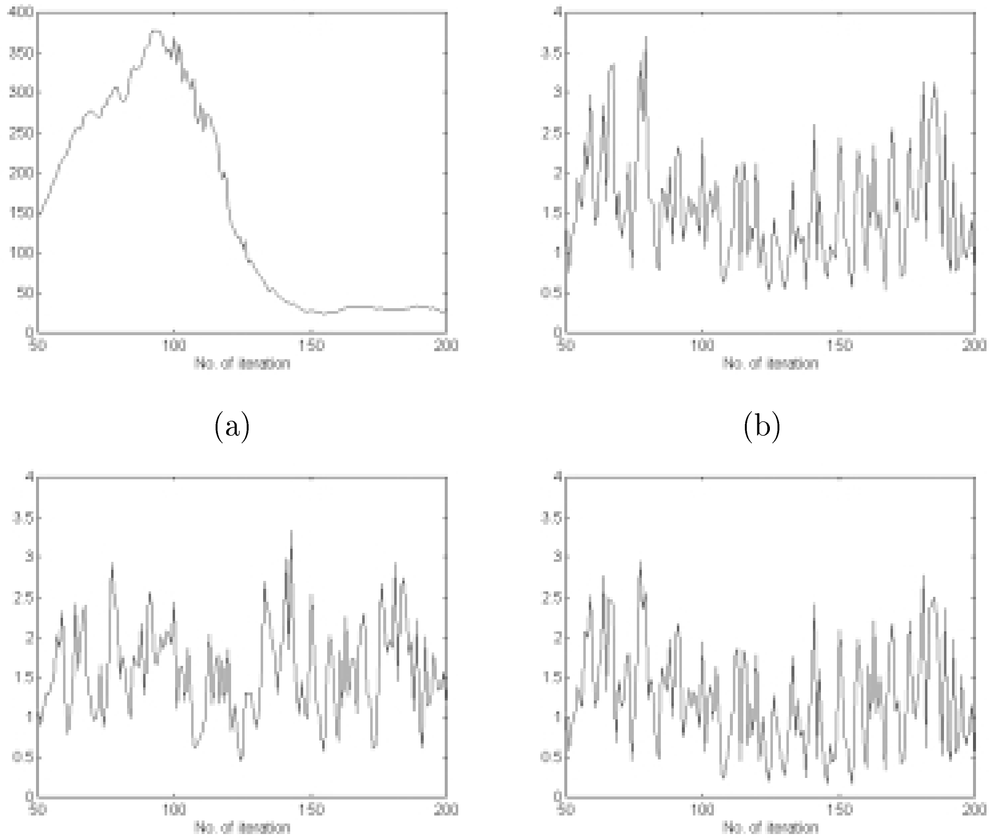


(d)

Figure 1: the output and the error for Example 1 (continued)

(c) the case of $Q_0 = 2, Q_1 = 2$

(d) the case of $Q_0 = 2, Q_1 = 1$



(a) (b) (c) (d)

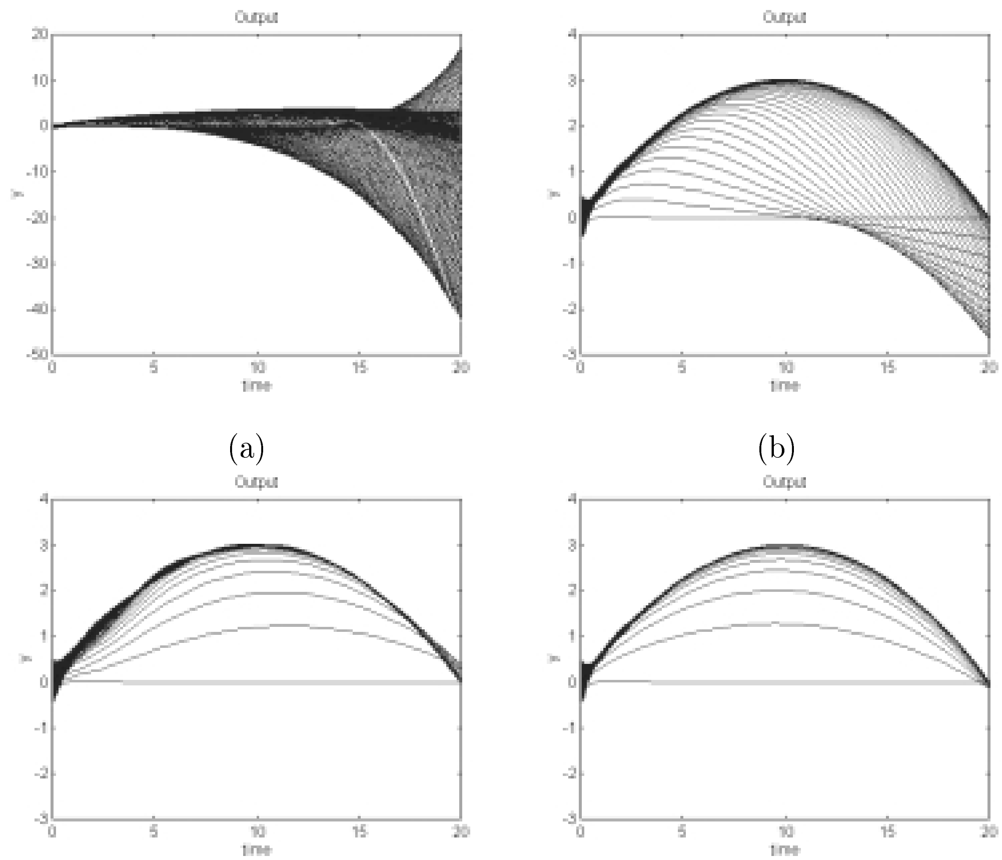
Figure 2: the trend of $\int_0^{20} |e_k(\tau)| d\tau$ for Example 2

(a) the case of $Q_0 = 0, Q_1 = 0$

(b) the case of $Q_0 = 2.2, Q_1 = 0$

(c) the case of $Q_0 = 0, Q_1 = 1.1$

(d) the case of $Q_0 = 2.2, Q_1 = 1.1$



(a) (b)

(c) (d)

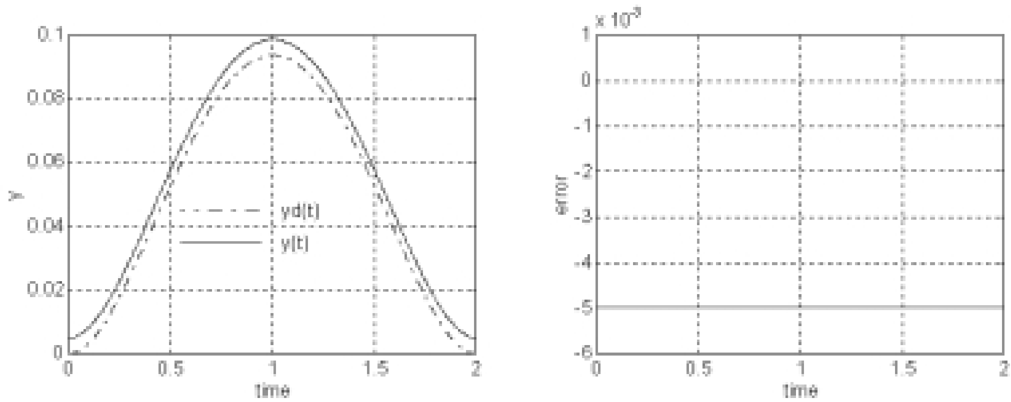
Figure 3: the trend of $y_k(t)$ for Example 2

(a) the case of $Q_0 = 0, Q_1 = 0$

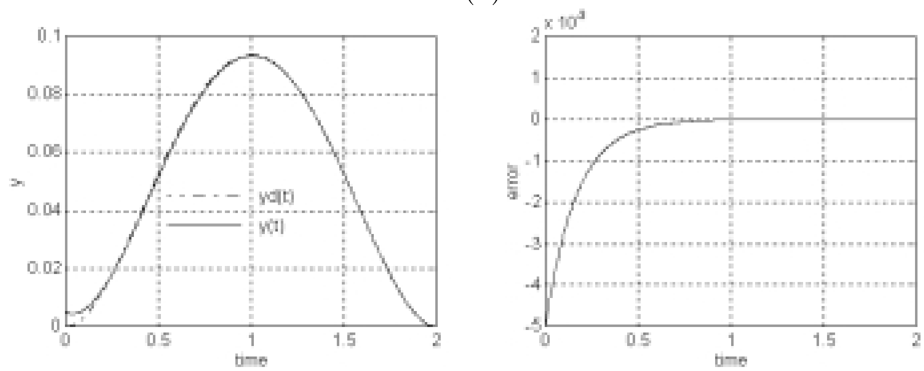
(b) the case of $Q_0 = 2.2, Q_1 = 0$

(c) the case of $Q_0 = 0, Q_1 = 1.1$

(d) the case of $Q_0 = 2.2, Q_1 = 1.1$



(a)

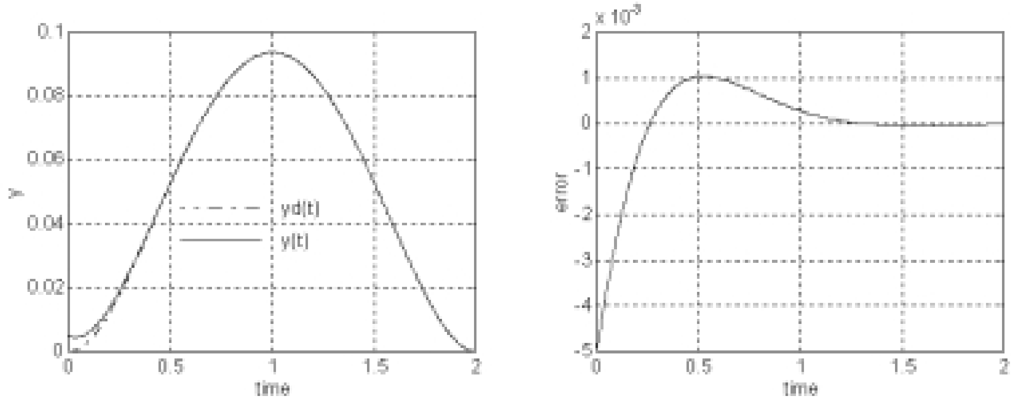


(b)

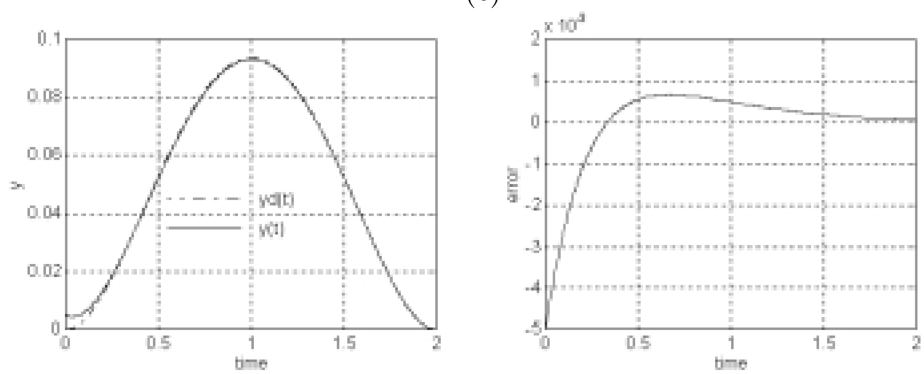
Figure 4: the output and the error for Example 3

(a) the case of $Q_0 = 0, Q_1 = 0$

(b) the case of $Q_0 = 2, Q_1 = 0$



(c)



(d)

Figure 4: the output and the error for Example 3 (continued)

(c) the case of $Q_0 = 2, Q_1 = 2$

(d) the case of $Q_0 = 2, Q_1 = 1$