Parametric optimization for the evaluation of investment projects

Jürgen Zimmermann
Institute for Information and Technology Management, Saarland University
jz@itm.uni-sb.de

Christoph Schwindt
Institute for Economic Theory and Operations Research, University of Karlsruhe
schwindt@wior.uni-karlsruhe.de

Abstract

The assessment of investment projects by the net present value method presupposes that the discount rate and the project deadline are known. In literature, however, it is commonly accepted that generally the discount rate cannot be determined with sufficient accuracy. Moreover, the project deadline is generally subject to negotiations. In this paper, we develop a procedure for computing the project net present value as a function of the discount rate and the project deadline. On the basis of this function, investment projects with uncertain discount rate and variable project deadline can be evaluated.

1. Introduction

Project managers are frequently confronted with the problem to decide on whether some investment project should be performed or to select one out of several mutually exclusive investment projects from a given portfolio. For the assessment of investments, the net present value criterion is well-established in research and practice. In classical investment theory, investments are specified by a stream of payments, i.e., a series of payments with associated payment times. For a stream of payments, the net present value is obtained by summing up all payments discounted to time 0 (cf. Fig. 1a, where exogenous parameters are written in italics).

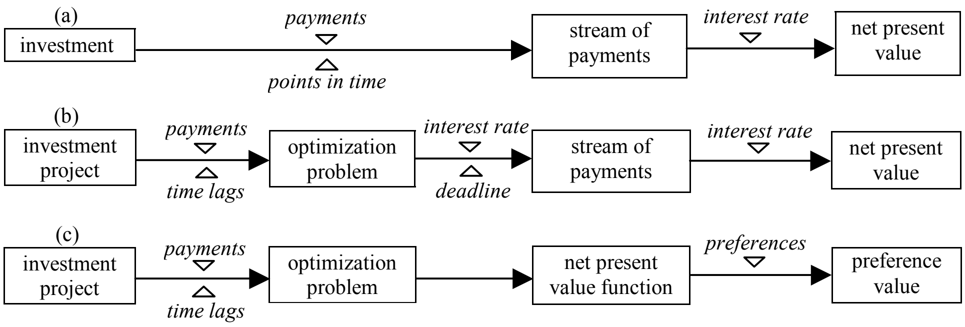

Figure 1. Evaluation of investments and investment projects

In case of investment projects, the payment times are no longer given in advance but are subject to the scheduling process. An investment project consists of a set of events each of which is associated with a paying in or paying out. Moreover, there are prescribed time lags between the occurrence of the events. Thus, the stream of payments results from maximizing the net present value of the project subject to the given time lags (cf. Fig. 1b). The formulation of this optimization problem presupposes the knowledge of the proper interest rate for discounting the payments and the specification of a maximum project duration (project deadline). In practice, however, often neither the exact interest rate nor a project deadline are known. The proper interest rate is a theoretical value and can only be estimated. The project deadline is generally the result of negotiations between the investor and the contractor. The purpose of this paper is to develop a parametric optimization approach which provides the maximum project net present value as a function of the interest rate and project deadline chosen. The resulting net present value curve then serves as a basis for the decision of the investor, which depends on his individual risk preference (cf. Fig. 1c).

2. Project definition

Let $V:=\{0,1,\ldots,n+1\}$ be the set of project events, including the project beginning and the project termination, prescribed milestones for the completion of subprojects, and the start and end times of the project activities. Temporal constraints between the occurrence of two project events are given by minimum and maximum time lags. The project events and the corresponding time lags are represented by an event-on-node network $N=\langle V,E;\delta\rangle$ with node set V, arc set E, and arc weights δ_{ij}, $(i,j)\in E$. For each minimum time lag d_{ij}^{min}, there is an arc (i,j) emanating from node i and terminating at node j weighted by $\delta_{ij}=d_{ij}^{min}$. A maximum time lag d_{ij}^{max} corresponds to an arc (j,i) weighted by $\delta_{ji}=-d_{ij}^{max}$. For each activity there is a negative payment c_i associated with the start event $i\in V$ of that activity. Positive payments arise when a subproject is completed. For each subproject, a milestone associated with the corresponding positive payment is added. Progress payments are often based on the direct cost which is incurred by the activities of the corresponding subproject (cf. Dayanand and Padman, 1997). Let $c^P<0$ be the sum of the negative payments caused by the execution of subproject P and let $j\in V$ be the milestone representing the completion of subproject P. Then we have $c_j:=-\gamma^P c^P$, where γ^P reflects the profitability aspired for subproject P. If the execution of activity i belongs to several subprojects (case of shared activities), the corresponding expense is proportionally assigned to these subprojects as follows. Let V^P be the set of all start events of subproject P, Π_i be the set of subprojects activity i belongs to, and x_{iP} be the fraction of paying out c_i which is to be assigned to subproject P. Then the sum of expenses considered for subproject P is given by

$$c^P := \sum_{i\in V^P} c_i\, x_{iP}$$

The values x_{iP} can be determined by solving the nonlinear equation system

$$x_{iP} = \frac{c^P}{\sum_{Q\in\Pi_i} c^Q} = \frac{\sum_{j\in V^P} c_j x_{jP}}{\sum_{Q\in\Pi_i}\sum_{j\in V^Q} c_j x_{jQ}}$$

3. The project net present value

Let S_i be the occurrence time of event $i\in V$. The vector $S=(S_0,S_1,\ldots,S_{n+1})$ of all occurrence times is called a schedule. Given an interest rate $\alpha>0$ and a project deadline d, which is represented by a maximum time lag between the project beginning and the project termination, the net present value problem NPVP reads as follows.

$$\begin{aligned}
&\text{Max.} \quad C_0(S) = \sum_{i\in V} c_i\, e^{-\alpha S_i} \\
&\text{s.t.} \quad S_j - S_i \ge \delta_{ij} \qquad (i,j)\in E \\
&\qquad\;\; S_0 = 0
\end{aligned}$$

$C_0(S)$ denotes the project net present value for given schedule S. A review of exact solution procedures for NPVP can be found in Kolisch and Padman (2001) and Neumann et al. (2001).

The steepest ascent procedure by Schwindt and Zimmermann (2001) for solving NPVP can be described as follows. In a first step, the earliest occurrence times ES_i of all events $i \in V$ and a corresponding spanning forest G of active temporal constraints are determined. Proceeding from the earliest schedule S=ES, in each of the following iterations a set V' of events is determined whose marginal delaying incurs a maximum increase in project net present value $C_0(S)$. This means that we determine a steepest ascent direction $z \in \{0,1\}^{n+2}$ with $z_i=1$ for $i \in V'$ and $z_i=0$, otherwise. $C_0(\cdot)$ is monotone when moving along a binary direction, and thus the optimal step size is given by slack

$$\sigma = \min\{S_j - S_i - \delta_{ij} \mid (i,j) \in E,\ z_i = 1, \text{ and } z_j = 0\} \qquad (1)$$

The procedure terminates if $V'=\emptyset$. Since for NPVP each local maximum represents a global one, the schedule found leads to a maximum net present value.

4. Parametric optimization

Thus far we have assumed that interest rate α and project deadline d are exogenous problem parameters. As has been mentioned before, project deadline d is generally negotiated between the investor and the contractor and the proper discount rate α can only be estimated. Thus, the investor should know the project net present value for a relevant range of values for α and d. In the sequel, we develop a parametric optimization procedure that determines the maximum project net present value $C_0^*(\alpha, d)$ as a function of interest rate α and project deadline d. The following considerations are based on two observations: first, there always exists an optimal solution S to NPVP that is a vertex of the feasible region $\mathcal{S}_T$ of NPVP and second, each vertex S of $\mathcal{S}_T$ can be represented by a spanning tree $G=\langle V,E_G\rangle$ of project network $N=\langle V,E;\delta\rangle$.

Let V_{ij} be the node set of the tree which results from G by deleting arc (i,j) and does not contain node 0. Arc $(i,j) \in E_G$ is called a forward arc if it is oriented in direction of the unique path from node 0 to node j in G, and a backward arc, otherwise. Moreover, let $C_0^{ij}(S)$ be the net present value of the events from set V_{ij} given schedule S. Then we have the following sufficient and necessary optimality condition.

Proposition 1. Given α and d, vertex S is optimal if and only if for each arc $(i,j) \in E_G$ of corresponding spanning tree G it holds that $C_0^{ij}(S) \geq 0$ if arc (i,j) is a forward arc and $C_0^{ij}(S) \leq 0$ if (i,j) is a backward arc.

Furthermore, it can be shown that

Proposition 2. Given $\alpha > 0$, spanning tree G belonging to optimal schedule S does not change when varying project deadline d until a temporal constraint $S_j - S_i \geq \delta_{ij}$ with $(i,j) \notin E_G$ becomes active.

The set V' of events which are moved when altering d is equal to the set $V_{n+1,0}$. The amount by which d can be increased until a temporal constraint becomes binding is

$$\sigma^+ = \min\{S_j - S_i - \delta_{ij} \mid (i,j) \in E \setminus \{(n+1,0)\}, i \in V', j \notin V'\}$$

Analogously, for the amount by which d can be decreased, we obtain

$$\sigma^- = \min\{S_j - S_i - \delta_{ij} \mid (i,j) \in E \setminus \{(n+1,0)\}, i \notin V', j \in V'\}$$

The spanning tree G' for an optimal schedule S' with respect to a project deadline $d'=d+\sigma^+$ or $d'=d-\sigma^-$ can be constructed by first, adding the arc (i,j) representing the new temporal constraint to G and second, deleting an oppositely directed arc (g,h) with minimum $C_0^{gh}(S')$ in the resulting cycle. In analogy to Proposition 2, we have

Proposition 3. Given deadline d, the spanning tree G of an optimal schedule S does not change when varying interest rate α until for some arc $(i,j)\in E_G$, net present value $C_0^{ij}(S)$ changes in sign.

Let $\alpha_{ij}^1 < \alpha_{ij}^2 < \cdots < \alpha_{ij}^\nu$ be all interest rates $\alpha_{ij} > \alpha$ for which

$$\sum_{h\in V_{ij}} c_h e^{-\alpha_{ij}} = 0 \qquad (2)$$

Each of those interest rates α_{ij} corresponds to an internal rate of return for the payment stream given by payments c_h, $h\in V_{ij}$, and schedule S. Thus, these interest rates can be determined by one of the standard algorithms for the calculation of internal rates of return. With $\Delta\alpha := \min\{\alpha_{ij}^1 - \alpha, \alpha_{ij}^2 - \alpha_{ij}^1, \ldots, \alpha_{ij}^\nu - \alpha_{ij}^{\nu-1}\}$ the following condition is sufficient and necessary for a change in the sign of $C_0^{ij}(S)$ at α_{ij}

$$\text{sign} \sum_{h\in V_{ij}} c_h\, e^{-S_h(\alpha_{ij}-\Delta\alpha/2)} \neq \text{sign} \sum_{h\in V_{ij}} c_h\, e^{-S_h(\alpha_{ij}+\Delta\alpha/2)} \quad (3)$$

For each set V_{ij}, we calculate the smallest interest rate $\alpha_{ij} > \alpha$ for which (2) and (3) are satisfied. Optimal schedule S' for interest rate $\alpha' := \min_{(i,j)\in E_G} \alpha_{ij}$ then results from delaying events $h\in V_{ij}$ by σ^+ if (i,j) is a forward arc or by scheduling the events σ^- time units earlier if (i,j) is a backward arc. The corresponding spanning tree G' can be determined analogously to the case where deadline d is varied. In summary, each spanning tree is valid for a rectangle $[\alpha,\alpha']\times[d,d']$ where α' and d' can be computed as described above. Within those rectangles, the maximum net present value is a closed-form function

$$C_0^*(\alpha,d) = \sum_{i\in V\setminus V'} c_i e^{-\alpha S_i} + \sum_{i\in V'} c_i e^{-\alpha(S_i+d-S_{n+1})} \qquad (4)$$

In particular, function C_0^* is continuous on $\mathrm{IR}^2_{\geq 0}$ and differentiable aside from the transitions from one to another rectangle. The following algorithm shows how to cover $\mathrm{IR}^2_{\geq 0}$ by appropriate rectangles $[\alpha,\alpha']\times[d,d']$ within which the net present value function has the form (4).

Algorithm 1: Determination of function C_0^*

Compute spanning tree G^0 for $d=ES_{n+1}$ and some initial $\alpha > 0$.
Set stack $L:=\{(0, ES_{n+1}, G^0)\}$.
While $L \neq \emptyset$
    Pop (α,d,G) and all other elements $(\alpha,\cdot,G)$ off L.
    Increase stepwise deadline d for fixed α until $\sigma^+(G')=\infty$.
    For all spanning trees G' and corresponding deadlines d
        If there is no triple $(\cdot, d',G')$ in L
            Store $(\cdot, d',G')$ and determine $\alpha':=\min\{\alpha_{ij} | (i,j)\in E_{G'}$ such that α_{ij} meets (2) and (3)$\}$.
            If $\alpha' < \infty$
                Construct spanning tree G'' belonging to α' and d'.
                Push (α', d',G'') onto L.

References

[1] Dayanand, N. and Padman, R. (1997). On modeling payments in projects, Journal of the Operational Research Society 48, 906-918.

[2] Kolisch, R. and Padman, R. (2001). An integrated survey of deterministic project scheduling, Omega 29, 249-272.

[3] Neumann, K., Schwindt, C., and Zimmermann J. (2001). Project Scheduling with Time Windows and Scarce Resources, Springer, Berlin.

[4] Schwindt, C. and Zimmermann, J. (2001). A steepest ascent approach to maximizing the net present value of projects, Mathematical Methods of Operations Research 53, 435-450.