Lifts, Discrepancy and Nearly Optimal Spectral Gaps

Yonatan Bilu and Nati Linial *

Abstract

Let G be a graph on n vertices. A 2-lift of G is a graph H on $2n$ vertices, with a covering map $\pi : H \to G$. It is not hard to see that all eigenvalues of G are also eigenvalues of H. In addition, H has n "new" eigenvalues. We conjecture that every d-regular graph has a 2-lift such that all new eigenvalues are in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (If true, this is tight , e.g. by the Alon-Boppana bound). Here we show that every graph of maximal degree d has a 2-lift such that all "new" eigenvalues are in the range $[-c\sqrt{d\log^3 d}, c\sqrt{d\log^3 d}]$ for some constant c. This leads to a polynomial time algorithm for constructing arbitrarily large d-regular graphs, with second eigenvalue $O(\sqrt{d\log^3 d})$.
The proof uses the following lemma (Lemma 3.3): Let A be a real symmetric matrix with zeros on the diagonal. Let d be such that the l_1 norm of each row in A is at most d. Let α be such that for every $x, y \in \{0,1\}^n$ with $< x, y >= 0$ it holds that $\frac{|xAy|}{||x||||y||} \leq \alpha$. Then the spectral radius of A is $O(\alpha(\log(d/\alpha)+1))$. An interesting consequence of this lemma is a converse to the Expander Mixing Lemma.

Keywords: Lifts, Lifts of Graphs, Discrepancy, Expander Graphs.

*Institute of Computer Science, Hebrew University Jerusalem 91904 Israel {*johnblue,nati*}*@cs.huji.ac.il*. This research is supported by the Israeli Ministry of Science and the Israel Science Foundation.

1 Introduction

A graph $\hat{G}$ is called a k-lift of a "base graph" G if there is a $k:1$ *covering map* $\pi : V(\hat{G}) \to V(G)$. Namely, if $y_1, \ldots, y_d \in G$ are the neighbors of $x \in G$, then every $x' \in \pi^{-1}(x)$ has exactly one vertex in each of the subsets $\pi^{-1}(y_i)$. See [4] for a general introduction to graph lifts.

The study of lifts of graphs has focused mainly on random lifts [4, 5, 6, 14, 11]. In particular, Amit and Linial show in [5] that w.h.p. a random k-lift has a strictly positive edge expansion. It is not hard to see that the eigenvalues of the base graph are also eigenvalues of the lifted graph. These are called by Joel Friedman the "old" eigenvalues of the lifted graph. In [11] he shows that w.h.p. a random k-lift of a d-regular graph on n vertices is "weakly Ramanujan". Namely, that all eigenvalues but, perhaps, those of the base graph, are, in absolute value, $O(d^{3/4})$. In both cases the probability tends to 1 as k tends to infinity.

Here we study 2-lifts of graphs, and show (in Theorem 3.1) that every graph of maximal degree d has a 2-lift with all new eigenvalues $O(\sqrt{d \log^3 d})$ in absolute value. We conjecture that, at least for regular graphs, this can be improved to $2\sqrt{d-1}$. It is not hard to show (e.g., using the Alon-Boppana bound [15]) that if this conjecture is true, it is tight.

We also show how to use this result to construct expander graphs. Namely, We give a polynomial time algorithm for constructing arbitrarily large d-regular graphs, with second eigenvalue $O(\sqrt{d \log^3 d})$.

A key ingredient in the proof is the following lemma (Lemma 3.3): Let A be a real symmetric matrix such that the l_1 norm of each row of A is at most d. Let $\alpha = \max_{x,y \in \{0,1\}^n, supp(x) \cap supp(y) = \emptyset} \frac{|xAy|}{||x|| ||y||}$, and assume $\alpha < d$. Then the spectral radius of A is bounded by $c\alpha \log(d/\alpha)$, for some universal constant c.

As a corollary, we get the following converse for the Expander Mixing Lemma: Let G be a d-regular graph on n vertices, such that for every two subsets of vertices, A and B, $|e(A,B) - d|A||B|/n| \leq \alpha\sqrt{|A||B|}$ for some $\alpha < d$. Then all eigenvalues of G but the first are, in absolute value, $O(\alpha \log(d/\alpha))$.

2 Definitions

Let $G = (V, E)$ be a graph on n vertices, and let A be its adjacency matrix. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ be the eigenvalues of A. We say that G is an $(n, d, \mu) - expander$ if G is d-regular, and $\max_{i=2,\ldots,n} |\mu_i| \leq \mu$. When $\mu = 2\sqrt{d-1}$ we say that such a graph is *Ramanujan*. When $\mu = \tilde{O}(\sqrt{d})$ we say that such a graph is *Quasi-Ramanujan*.

A *signing* of the edges of G is a function $s : E(G) \to \{-1, 1\}$. The *signed adjacency matrix* of a graph G with a signing s has rows and columns indexed by the vertices of G. The (x, y) entry is $s(x, y)$ if $(x, y) \in E$ and 0 otherwise. A 2-*lift* of G, associated with a signing s, is a graph $\hat{G}$ defined as follows. Associated with every vertex $x \in V$ are two vertices, x_0 and x_1, called the *fiber* of x. If $(x, y) \in E$, and $s(x, y) = 1$ then the corresponding edges in $\hat{G}$ are (x_0, y_0) and (x_1, y_1). If $s(x, y) = -1$, then the corresponding edges in $\hat{G}$ are (x_0, y_1) and (x_1, y_0). The graph G is called the *base graph*, and $\hat{G}$ a 2-lift of G. By the *spectral radius of a signing* we refer to the spectral radius of the corresponding signed adjacency matrix. When the spectral radius of a signing of a d-regular graph is $\tilde{O}(\sqrt{d})$ we say that the signing (or the lift) is *Quasi-Ramanujan*.

For $v, u \in \{-1, 0, 1\}^n$, denote $S(u) = supp(u)$, and $S(u, v) = supp(u) \cup supp(v)$.
It will be convenient to assume that $V(G) = \{1, \ldots, n\}$.

3 Quasi-Ramanujan 2-Lifts and Quasi-Ramanujan Graphs

3.1 Preliminaries

The eigenvalues of a 2-lift of G can be easily characterized in terms of the adjacency matrix and the signed adjacency matrix:

Lemma 3.1 *Let A be the adjacency matrix of a graph G, and A_s the signed adjacency matrix associated with a 2-lift $\hat{G}$. Then every eigenvalue of A and every eigenvalue of A_s are eigenvalues of $\hat{G}$. Furthermore, the multiplicity of each eigenvalue of $\hat{G}$ is the sum of its multiplicities in A and A_s.*

Proof: It is not hard to see that the adjacency matrix of $\hat{G}$ is:

$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$$

Where A_1 is the adjacency matrix of $(V, s^{-1}(1))$ and A_2 the adjacency matrix of $(V, s^{-1}(-1))$. (So $A = A_1 + A_2$, $A_s = A_1 - A_2$). Let v be an eigenvector of A with eigenvalue μ. It is easy to check that $\hat{v} = (v\ v)$ is an eigenvector of $\hat{A}$ with eigenvalue μ.
Similarly, if u is an eigenvector of A_s with eigenvalue λ, then $\hat{u} = (u\ -u)$ is an eigenvector of $\hat{A}$ with eigenvalue λ.
As the $\hat{v}$'s and $\hat{u}$'s are perpendicular and $2n$ in number, they are all the eigenvectors of $\hat{A}$. ∎

Henceforth we follow Friedman's ([11]) nomenclature, and call the eigenvalues of A the *old* eigenvalues of $\hat{G}$, and those of A_s the *new* ones.

Consider the following scheme for constructing (n, d, λ)-expanders. Start with $G_0 = k_{d+1}$, the complete graph on $d+1$ vertices [1] . Its eigenvalues are d, with multiplicity 1, and -1, with multiplicity d. We want to define G_i as a 2-lift of G_{i-1}, such that all new eigenvalues are in the range $[-\lambda, \lambda]$. Assuming such a 2-lifts always exist, the G_i constitute an infinite family of (n, d, λ)-expanders.
It is therefore natural to look for the smallest $\lambda = \lambda(d)$ such that every graph of degree at most d has a 2-lift, with new eigenvalues in the range $[-\lambda, \lambda]$. In other words, a signing with spectral radius $\leq \lambda$.

We note that $\lambda(d) \geq 2\sqrt{d-1}$ follows from the Alon-Boppana bound. We next observe:

Proposition 3.1 *Let G be a d-regular graph which contains a vertex that does not belong to any cycle of bounded length, then no signing of G has spectral radius below $2\sqrt{d-1} - o(1)$.*

To see this, note first that all signing of a tree have the same spectral radius. This follows e.g., from the easy fact that any 2-lift of a tree is a union of two disjoint trees, isomorphic to the base graph. The assumption implies that G

[1] We could start with any small d-regular graph with a large spectral gap. Such graphs are easy to find.

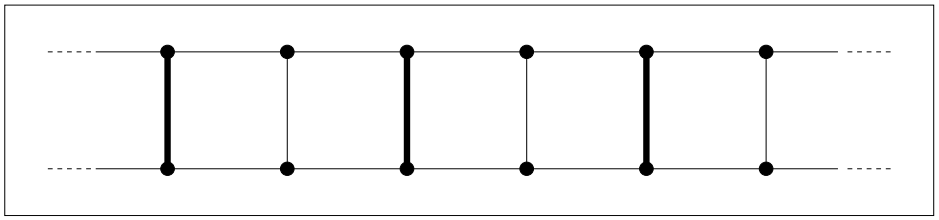

Figure 1: The Railway Graph. Edges where the signing is -1 are bold.

contains an induced subgraph that is a full d-ary tree T of unbounded radius. The spectral radius of T is $2\sqrt{d-1}-o(1)$. The conclusion follows now from the interlacing principle of eigenvalues.

There are several interesting examples of arbitrarily large d-regular graphs for which there is a signing with spectral radius bounded away from $2\sqrt{d-1}$. One such example is the 3-regular graph R defined as follows. $V(R) = \{0,\ldots,2k-1\}\times\{0,1\}$. For $i\in[2k], j\in\{0,1\}$, the neighbors of $(i,j)\in R$ are $((i-1) \bmod 2k, j)$, $((i+1) \bmod 2k, j)$ and $(i, 1-j)$. Define s, a signing of R, to be -1 on the edges $((2i,0),(2i,1))$, for $i\in\{0,\ldots,k-1\}$, and 1 elsewhere (see Figure 1). Let A_s be the signed adjacency matrix. It is easy to see that A_s^2 is a matrix with 3 on the diagonal, and two 1's in each row and column. Thus, its spectral radius is 5, and that of A_s is $\sqrt{5} < 2\sqrt{2}$.

3.2 Quasi-ramanujan 2-lifts for every graph

Based on extensive computer simulations we conjecture that every graph has a signing with small spectral radius:

Conjecture 3.1 *Every d-regular graph has a signing with spectral radius at most $2\sqrt{d-1}$.*

In this subsection we show a close upper bound:

Theorem 3.1 *Every graph of maximal degree d has a signing with spectral radius $O(\sqrt{d\cdot\log^3 d})$.*

The theorem follows immediately from the following two lemmata (along with Lemma 3.1). The first shows that with positive probability the Rayliegh quotient is small for vectors in $v, u \in \{-1, 0, 1\}^n$. The second shows that this is essentially a sufficient condition for all eigenvalues being small.

Lemma 3.2 *For every graph of maximal degree d, there exists a signing s such that for all $v, u \in \{-1, 0, 1\}^n$ the following holds:*

$$\frac{|v^t A_s u|}{||v||||u||} \leq 10\sqrt{d \log d}, \tag{1}$$

where A_s is the signed adjacency matrix.

Lemma 3.3 *Let A be an $n \times n$ real symmetric matrix such that the l_1 norm of each row in A is at most d. Assume that for any two vectors, $u, v \in \{0, 1\}^n$, with $supp(u) \cap supp(v) = \emptyset$:*

$$\frac{|uAv|}{||u||||v||} \leq \alpha,$$

and that all diagonal entries of A are, in absolute value, $O(\alpha(log(d/\alpha) + 1))$. Then the spectral radius of A is $O(\alpha(log(d/\alpha) + 1))$.

Proof: (Lemma 3.2) First note that it's enough to prove this for u's and v's such that $S(u, v)$ is a connected subgraph. Indeed, assume that the claim holds for all connected subgraphs and suppose that $S(u, v)$ is not connected. Split u and v according to the connected components of $S(u, v)$, and apply the claim to each component separately. Summing these up and using the Cauchy-Schwartz inequality, we conclude that the claim for u and v as well. So henceforth we assume that $S(u, v)$ is a connected.

Consider some $u, v \in \{-1, 0, 1\}^n$. Suppose we choose the sign of each edge uniformly at random. Denote the resulting signed adjacency matrix by A_s, and by $E_{u,v}$ the "bad" event that $\frac{|v^t A_s u|}{||v||||u||} > 10\sqrt{d \log d}$. Assume w.l.o.g. that $|S(u)| \geq \frac{1}{2}|S(u, v)|$. By the Chernoff inequality ($v^t A_s u$ is the sum of independent variables, attaining values of either ± 1 or ± 2):

$$\begin{aligned} Pr[E_{u,v}] \leq \quad & 2exp(-\frac{100d \log d|S(u)||S(v)|}{8|e(S(u), S(v))|}) \\ & \leq 2exp(-\frac{100d \log d|S(u)||S(v)|}{8d|S(v)|}) \\ & < d^{(-10|S(u,v)|)} \end{aligned}$$

We want to use the Lovász Local Lemma [8], with the following dependency graph on the $E_{u,v}$: There is an edge between $E_{u,v}$ and $E_{u',v'}$ iff $S(u,v) \cap S(u',v') \neq \emptyset$. Denote $k = |S(u,v)|$. How many neighbors, $E_{u',v'}$, does $E_{u,v}$ have, with $|S(u',v')| = l$?

Since we are interested only in connected subsets, this is clearly bounded by the number of rooted directed subtrees on l vertices, with a root in $S(u,v)$. Fix a labeling on the the vertices of G. We encode each such subtree T by a number in $[k]$ (the root) and a vector $c_T \in \{0,1\}^{(l-1)d}$. We construct c_T by d bits at a time. At each stage we encode which neighbors of the "current" vertex are its children in the subtree. The "current" vertex is initially the root, and afterwards it is the least unencoded vertex in the subtree whose parent was already encoded. Hence the number of 1's in c_T is $l-1$, so there are at most $k\binom{d(l-1)}{l-1} \approx kd^{l-1}$ such trees (a similiar argument appears in [13]. The bound on the number of trees is essentially tight by [2]).

In order to apply the Local Lemma, we need to define for such u and v a numbers $0 \leq X_{u,v} < 1$. It is required that:

$$X_{u,v} \prod_{(u',v'):E_{u,v} \sim E_{u',v'}} (1 - X_{u',v'}) \geq d^{-10|S(u,v)|}. \tag{2}$$

Observe that for $S \subseteq [n]$ there are at most $2^{4|S|}$ distinct pairs $v, u \in \{-1,0,1\}^n$ with such that $S(u,v) = S$.
For all u, v set $X_{u,v} = d^{-3k}$, where $k = |S(u,v)|$. Then in (2) we get:

$$X_{u,v} \cdot \prod_{(u',v'):E_{u,v} \sim E_{u',v'}} (1 - X_{u',v'}) = d^{-3k} \prod_{l=1}^{n} (1 - d^{-3l})^{kd^l 2^{4l}} \approx$$

$$d^{-3k} exp(-k \sum_{l=1}^{n} d^{-3l} d^l 2^{4l}) \geq d^{-3k} e^{-2k} > d^{-10k}$$

as required. ∎

Proof: (Lemma 3.3) For simplicity, assume that all diagonal entries of A are zeros. We explain at the end of the proof how to deal with a general matrix (but in fact it does not matter for the purpose of this paper).

First note that our assumptions imply that for any $u \in \{0,1\}^n$,

$$\frac{|uAu|}{||u||^2} \leq 2\alpha :$$

For any $u_1, u_2 \in \{0,1\}^n$ such $S(u_1) \cap S(u_2) = \emptyset$ we have that

$$|u_1 A u_2| \leq \alpha ||u_1|| ||u_2||. \tag{3}$$

Let $u \in \{0,1\}^n$, and denote $k = |S(u)|$. Set $K = \binom{k}{k/2}$. Summing up inequality (3) over all subsets of $S(u)$ of size $k/2$, we have that:

$$\sum_{u_1 : S(u_1) \subset S(u), |S(u_1)| = k/2} u_1 A (u - u_1) \leq K \alpha k/2.$$

For each $i \neq j \in S(u)$, $a_{i,j}$ is added up $\binom{k-2}{k/2-1}$ times in the sum on the LHS, hence (since diagonal entries are zero):

$$\binom{k-2}{k/2-1} uAu \leq K \alpha k/2,$$

or:

$$uAu \leq \alpha 2k.$$

Next, it follows that for any $u, v \in \{-1,0,1\}^n$, such that $S(u) = S(v)$, or $S(u) \cap S(v) = \emptyset$:

$$\frac{|uAv|}{||u|| ||v||} \leq 4\alpha.$$

Fix $u, v \in \{-1,0,1\}^n$. Denote $u = u^+ - u^-$ and $v = v^+ - v^-$, where $u^+, u^-, v^+, v^- \in \{0,1\}^n$, and $S(u^+) \cap S(u^-) = S(v^+) \cap (v^-) = \emptyset$.

$$\begin{aligned}
&|uAv| = |(u^+ - u^-)A(v^+ - v^-)| \\
&\leq 2\alpha(||u^+|| ||v^+|| + ||u^+|| ||v^-|| + ||u^-|| ||v^+|| + ||u^-|| ||v^-||) \\
&\leq 4\alpha \sqrt{||u^+||^2 ||v^+||^2 + ||u^+||^2 ||v^-||^2 + ||u^-||^2 ||v^+||^2 + ||u^-||^2 ||v^-||^2} \\
&= 4\alpha \sqrt{(||u^+||^2 + ||u^-||^2)(||v^+||^2 + ||v^-||^2)} = 4\alpha ||u|| ||v||
\end{aligned}$$

The first inequality follows from our assumption on vectors in $\{0,1\}^n$, and the second from the l_2 to l_1 norm ratio.

Fix $x \in \mathbb{R}^n$. We need to show that $\frac{|xAx|}{||x||^2} = O(\alpha \log(d/\alpha))$. By losing only a multiplicative factor of 2, we may assume that the absolute value of every non-zero entry in x is a negative powers of 2: Clearly we may assume that $||x||_\infty < \frac{1}{2}$. To bound the effect of rounding the coordinates, denote $x_i =$

$\pm(1+\delta_i)2^{t_i}$, with $0 \le \delta_i \le 1$ and $t_i < -1$, an integer. Now round x to a vector x' by choosing the value of x'_i to be $\text{sign}(x_i) \cdot 2^{t_i+1}$ with probability δ_i and $\text{sign}(x_i) \cdot 2^{t_i}$ with probability $1-\delta_i$. The expectation of x'_i is x_i. As the coordinates of x' are chosen independently, and the diagonal entries of A are 0's, the expectation of $x'Ax'$ is xAx. Thus, there's a rounding, x', of x, such that $|xAx| \le |x'Ax'|$. Clearly $||x'||^2 \le 2||x||^2$, so $\frac{|xAx|}{||x||^2} \le 2\frac{|x'Ax'|}{||x'||^2}$.

Denote $S_i = \{j : x_j = \pm 2^{-i}\}$, $s_i = |S_i|$. Denote by k the maximal index i such that $s_i > 0$. Denote by x^i the sign vector of x restricted to S_i, that is, the vector whose j'th coordinate is the sign of x_j if $j \in S_i$, and zero otherwise. By our assumptions, for all $1 \le i \le j \le k$:

$$|x^i A x^j| \le \alpha\sqrt{s_i s_j}. \tag{4}$$

Also, since the l_1 norm of each row is at most d, for all $1 \le i \le k$:

$$\sum_j |x^i A x^j| \le d s_i. \tag{5}$$

We wish to bound:

$$\frac{|xAx|}{||x||^2} \le \frac{\sum_{i,j=1}^k |x^i A x^j| 2^{-(i+j)}}{\sum_i 2^{-2i} s_i}. \tag{6}$$

Denote $\gamma = \log(d/\alpha)$, $q_i = s_i 2^{-2i}$ and $Q = \sum_i q_i$. Add up inequalities (4) and (5) as follows. For $i = j$ multiply inequality (4) by 2^{-2i}. When $i < j \le i+\gamma$ multiply it by $2^{-(i+j)+1}$. Multiply inequality (5) by $2^{-(2i+\gamma)}$. (We ignore inequalities (4) when $j > i+\gamma$.)

We get that:

$$\begin{aligned}
&\sum_i 2^{-2i}|x^i A x^i| + \sum_i \sum_{i<j\le i+\gamma} 2^{-(i+j)+1}|x^i A x^j| + \sum_i 2^{-(2i+\gamma)} \sum_j |x^i A x^j| \\
&\le \sum_i \alpha q_i + \sum_i \sum_{i<j<i+\gamma} 2\alpha\sqrt{q_i q_j} + \sum_i 2^{-\gamma} d \cdot q_i \\
&\le \alpha \sum_i q_i + \alpha \sum_i \sum_{i<j<i+\gamma} (q_i + q_j) + 2^{-\gamma} d \sum_i \cdot q_i \\
&< (2^{-\gamma} d + 2\gamma\alpha) \sum_i q_i = (\alpha + \alpha \log(d/\alpha)) Q.
\end{aligned}$$

Note that the denominator in (6) is Q, so to prove the lemma it's enough to show that the numerator,

$$\sum_{i<j} 2^{-(i+j+1)}|x^i A x^j| + \sum_i 2^{-2i}|x^i A x^i|, \tag{7}$$

is bounded by

$$\sum_i 2^{-2i}|x^iAx^i| + \sum_i \sum_{i\le j\le i+\gamma} 2^{-(i+j)+1}|x^iAx^j| + \sum_i 2^{-(2i+\gamma)} \sum_j |x^iAx^j|. \quad (8)$$

Indeed, let us compare the coefficients of the terms $|x^iAx^j|$ in both expressions (Since $|x^iAx^j| = |x^jAx^i|$, it's enough to consider $i \le j$). For $i = j$ this coefficient is 2^{-2i} in (7), and $2^{-2i} + 2^{-(2i+\gamma)}$ in (8). For $i < j \le i + \gamma$, it is $2^{-(i+j)+1}$ in (7), and in (8) it is $2^{-(i+j)+1} + 2^{-(2i+\gamma)} + 2^{-(2j+\gamma)}$. For $j > i + \gamma$, in (7) the coefficient is again $2^{-(i+j)+1}$. In (8) it is:

$$2^{-(2i+\gamma)} + 2^{-(2j+\gamma)} > 2^{-(2i+\gamma)} \ge 2^{-(i+j)+1}.$$

It remains to show that the lemma holds when the diagonal entries of A are not zeros, but $O(\alpha(log(d/\alpha)+1))$ in absolute value. Denote $B = A-D$, with D the matrix having the entries of A on the diagonal, and zero elsewhere. We have that for any two vectors, $u, v \in \{0,1\}^n$, with $supp(u) \cap supp(v) = \emptyset$:

$$\frac{|uAv|}{||u||||v||} \le \alpha.$$

For such vectors, $uBv = uAv$, so by applying the lemma to B, we get that its spectral radius is $O(\alpha(log(d/\alpha) + 1))$. By the assumption on the diagonal entries of A, the spectral radius of D is $O(\alpha(log(d/\alpha) + 1))$ as well. The spectral radius of B is at most the sum of these bounds - also $O(\alpha(log(d/\alpha) + 1))$. ∎

3.3 An explicit construction of quasi-ramanujan graphs

For the purpose of building expanders, it is enough to prove a weaker version of Theorem 3.1. Roughly, that every *expander* graph has a 2-lift with small spectral radius. In this sub-section we show that when the base graph is a good expander (in the sense of the definition below), then w.h.p. a random 2-lift has a small spectral radius. We then derandomize the construction to get a deterministic polynomial time algorithm for constructing arbitrarily large expander graphs.

Definition 3.1 *We say that a graph G on n vertices is (β, t)-sparse if for every $u, v \in \{0,1\}^n$, with $|S(u,v)| \le t$,*

$$uAv \le \beta||u||||v||.$$

Lemma 3.4 *Let A be the adjacency matrix of a d-regular $(\gamma(d), \log n)$-sparse G graph on n vertices, where $\gamma(d) = 10\sqrt{d \log d}$. Then for a random signing of G (where the sign of each edge is chosen uniformly at random) the following hold w.h.p.:*

1. $\forall u, v \in \{-1, 0, 1\}^n : |uA_s v| \leq \gamma(d)||u||||v||$.
2. $\hat{G}$ *is* $(\gamma(d), 1 + \log n)$*-sparse*

where A_s is the random signed adjacency matrix, and $\hat{G}$ is the corresponding 2*-lift.*

Proof: Following the same arguments and notations as in the proof of Lemma 3.2, we have that there are at most $n \cdot d^k$ connected subsets of size k. With probability at most d^{-10k} requirement (1) is violated for a given pair u, v such that $|S(u, v)| = k$. Since for each S there are at most $2^{4|S|}$ pairs u, v such that $S(u, v) = S$, by the union bound, w.h.p. no pair u, v such that $|S(u, v)| > \log n$ violates (1). If $|S(u, v)| \leq \log n$ then by (2) there are simply not enough edges between $S(u)$ and $S(v)$ for (1) to be violated.

Next, we show that w.h.p. (2) holds as well. Let s be a signing, and define A_1, A_2 and $\hat{A}$ as in Lemma 3.1. Given $u = (u_1\ u_2), v = (v_1\ v_2) \in \{0, 1\}^n \times \{0, 1\}^n$, we wish to prove that $uAv \leq \gamma(d)||u||||v||$. As in the proof of Lemma 3.2 we may assume that $S(u, v)$ is connected - in fact, that it is connected via the edges between $S(u)$ and $S(v)$. Hence, we may assume that the ratio of the sizes of these subsets is at most d. Define $x = u_1 \vee u_2$, $y = v_1 \vee v_2$, $x' = u_1 \wedge u_2$, and $y' = v_1 \wedge v_2$ (the characteristic vectors of $S(u_1, u_2)$, $S(v_1, v_2)$, $S(u_1) \cap S(u_2)$ and $S(v_1) \cap S(v_2)$). It is not hard to verify that:

$$u\hat{A}v = u_1A_1v_1 + u_1A_2v_2 + u_2A_2v_1 + u_2A_1v_2 \leq xAy + x'Ay'.$$

If $|S(x, y)| \leq \log n$, then clearly $|S(x', y')| \leq \log n$ and from the assumption that G is $(\gamma(d), \log n)$-sparse

$$xAy + x'Ay' \leq \gamma(d)(\sqrt{|S(x)||S(y)|} + \sqrt{|S(x')||S(y')|}).$$

Observe that $|S(u)| = |S(x)| + |S(x')|$ and $|S(v)| = |S(y)| + |S(y')|$, so in particular $u\hat{A}v \leq \gamma(d)\sqrt{|S(u)||S(v)|}$, and requirement (2) holds.
So assume $|S(x, y)| = |S(u, v)| = \log n + 1$. It is not hard to see that

this entails $S(u_1, v_1) \cap S(u_2, v_2) = \emptyset$. In other words, $S(u, v)$ contains at most one vertex from each fiber. Hence, $x' = y' = \vec{0}$ and $|S(u)| = |S(x)|$, $|S(v)| = |S(y)|$. Observe that the edges between $S(u)$ and $S(v)$ in $\hat{G}$ originate from edges between $S(x)$ and $S(y)$ in G in the following way - for each edge between $S(x)$ and $S(y)$ in G there is, with probability $\frac{1}{2}$, an edge between $S(u)$ and $S(v)$ in $\hat{G}$.

Denote $S = S(x, y)$, and assume w.l.o.g. that $|S(y)| > \frac{1}{2}\log n$. As noted above, $|S(x)| \geq \frac{1}{d}|S(y)|$. We now show that in fact $|S(x)| \geq \frac{\log d}{d}|S(y)|$. Assume for contradiction that this is not the case. Consider $i \in S(y) \backslash S(x)$ and denote by δ_i the number of its neighbors in $S(x)$. Denote $y^i = y - e_i$.

$$uAv \leq xAy = xAy^i + \delta_i \leq \gamma(d)\sqrt{|S(x)|(|S(y)| - 1)} + \delta_i.$$

So if $\gamma(d)\sqrt{|S(x)|(|S(y)| - 1)} + \delta_i \leq \gamma(d)\sqrt{|S(x)||S(y)|}$ we are done. Hence we may assume $\delta_i > \gamma(d)\sqrt{\frac{|S(x)|}{2|S(y)|}}$. Since $\frac{|S(x)|}{|S(y)|} > \frac{1}{d}$, $\delta_i > 5\sqrt{\log d}$. A similar argument works for $i \in S(y) \cap S(x)$. Since there are at most $d|S(x)|$ edges emanating from $S(x)$, $\frac{|S(x)|}{|S(y)|} > \frac{5\sqrt{\log d}}{d}$. Repeating the argument above with this better bound on the ratio entails that for all $i \in S(y)$, $\delta_i > (5\sqrt{\log d})^{1.5}$. By continuing this bootstrapping argument we get that $\delta_i > 25 \log d$, and $\frac{|S(x)|}{|S(y)|} > \frac{25 \log d}{d}$.

Next we bound xAy. Averaging over all $S \backslash \{i\}$, for $i \in S(y)$ we have that:

$$(|S(y)| - 2)xAy \leq |S(y)|\gamma(d)\sqrt{(|S(y)| - 1)|S(x)|}.$$

Hence the expectation of $u\hat{A}v$ is at most $\frac{1}{2}c\gamma(d)||u||||v||$, where

$$c = \frac{\sqrt{|S(y)|(|S(y)| - 1)}}{|S(y)| - 2} \leq 1.1$$

(assuming n is not very small). By the Chernoff bound, the probability that $u\hat{A}v > \gamma(d)||u||||v||$ is at most:

$$\begin{aligned} &2exp(-\frac{0.9}{2.2}\gamma(d)||u||||v||) \leq \\ &exp(-0.2\gamma(d)\frac{(\log n + 1)(5\sqrt{\log d})}{\sqrt{d}}) = \\ &exp(-10 \log d(\log n + 1)), \end{aligned}$$

Since $\frac{|S(x)|}{|S(y)|} > \frac{25 \log d}{d}$, and $|S(v)| > \frac{1}{2} \log n$.
There are at most $d^{\log n+1} 4^{\log n+1}$ pairs u, v with $S(u,v)$ connected and of size $\log n + 1$, so by the union bound, w.h.p., requirement (2) holds. ∎

Corollary 3.1 *Let A be the adjacency matrix of a d-regular $(\gamma(d), \log n)$-sparse G graph on n vertices, where $\gamma(d) = 10\sqrt{d \log d}$. Then there is a deterministic polynomial time algorithm for finding a signing s of G such that the following hold:*

1. *The spectral radius of A_s is $O(\sqrt{d \log^3 d})$.*
2. *$\hat{G}$ is $(\gamma(d), 1 + \log n)$-sparse,*

where A_s is the signed adjacency matrix, and $\hat{G}$ is the corresponding 2-lift.

Proof: Consider a random signing s. For each closed path p in G of length $l = 2\lceil \log n \rceil$ define a random variable Y_p equal to the product of the signs of its edges. From lemmas 3.4 and 3.3, the expected value of the trace of A_s^l, which is the expected value of the sum of these variables, is $O(\sqrt{d \log^3 d})^l$ (since l is even the sum is always positive). For each $u, v \in \{0,1\}^n$, with $|S(u,v)| = \log n + 1$, and $S(u,v)$ connected, define $Z_{u,v}$ to be d^l if $u\hat{A}v \leq \gamma(d)||u||||v||$, and 0 otherwise. In the proof of Lemma 3.4 we've seen that the probability that $Z_{u,v}$ is not 0 is at most $d^{-10 \log n}$, thus the expected value of $Z_{u,v}$ is at most $d^{-8 \log n}$. Let Z be the sum of the $Z_{u,v}$'s. Recall that there are at most $n(4d)^{\log n+1}$ pairs (u,v) such that $|S(u,v)| = \log n + 1$, and $S(u,v)$ connected. Hence, the expected value of Z is less than $d^{-6 \log n}$.

Let $X = Y + Z$. Note that the expected value of X is approximately that of Y, $O(\sqrt{d \log^3 d})^l$. The expectation of Y_p and $Z_{u,v}$ can be easily computed even when the sign of some of the edges is fixed, and that of the other is chosen at random. As there is only a polynomial number of variables, using the method of conditional probabilities one can find a signing s such that the value of X is at most its expectation. For this value of X, $tr(A_s^l) = Y = O(\sqrt{d \log^3 d})^l$, and $Z = 0$ since if $Z \neq 0$ then $Z \geq d^l$. Clearly, the spectral radius of A_s is $O(\sqrt{d \log^3 d})$. In the proof of Lemma 3.4 we've seen that if G is $(\gamma(d), \log n)$-sparse then so is $\hat{G}$, for any signing of G. For our choice of s all $Z_{u,v} = 0$, hence $\hat{G}$ is actually $(\gamma(d), \log n + 1)$-sparse. ∎

Recall the construction from the beginning of this section. Start with a d-regular graph G_0 which is an $(n_0, d, \mu) - expander$ expander, for $\mu = 10\sqrt{d \log d}$ and $n_0 > d \log^2 n_0$. From the Expander Mixing Lemma (cf. [3]), G_0 is $(\mu, \log n_0)$-sparse. Iteratively chose G_{i+1} to be a 2-lift of G_i according to Corollary 3.1, for $i = 1, \ldots, \log(n/n_0)$. Clearly this is a polynomial time algorithm that yields a $(n, d, O(\sqrt{d \log^3 d}))$-expander graph.

3.4 Random 2-lifts

Theorem 3.1 states that for every graph there exists a signing such that the spectral radius of the signed matrix is small. The proof shows that for a random signing, this happens with positive, yet exponentially small, probability. The following example shows that this is the best we can hope for. In particular, it is not true that for every graph a random signing yields a small spectral radius with high probability.

Consider a graph composed of $n/(d+1)$ disjoint copies of K_{d+1} (the complete graph on $d+1$ vertices). Call one such copy "bad" for a signing, if all the edges within the copy get the same sign. Suppose that for a signing s a bad copy exists, and consider x, the support vector for the vertices in that copy. $\frac{|xA_sx|}{||x||^2} = d$, and hence the spectral radius of A_s is d. Clearly, for a random signing, the probability that no bad copy exists is $(1 - 2^{-\binom{d+1}{2}})^{n/d}$.

However, for a random d-regular graph, it is true that a random 2-lift will, w.h.p., yield a signed matrix with small spectral radius. This follows from the fact that, w.h.p., a random d-regular graph is an $(n, d, O(\sqrt{d}))$-expander ([12, 10, 9]). In particular, by the Expander Mixing Lemma, it is $(O(\sqrt{d}), \log n)$-sparse. By Lemma 3.4, w.h.p., a random 2-lift yields a signed matrix with small spectral radius.

4 Reflections on Lemma 3.3

4.1 Finding the proof: LP-duality

As the reader might have guessed, the proof for Lemma 3.3 was discovered by formulating the problem as a linear program. Define $\Delta_{i,j} = |x^i A x^j|$. Our

assumptions translate to:

$$\forall 1 \leq i \leq j \leq k : |\Delta_{i,j}| \leq \alpha\sqrt{s_i s_j},$$

$$\forall 1 \leq i \leq k : \sum_j |\Delta_{i,j}| \leq d s_i.$$

We want to deduce an upper bound on $|xAx|$. In other words, we are asking, under these constraints, how big

$$\frac{|xAx|}{||x||^2} \leq \frac{\sum_{i,j=1}^{k} \Delta_{i,j} 2^{-(i+j)}}{\sum_i 2^{-2i} s_i}$$

can be.

The dual program is to minimize:

$$\frac{\alpha \sum_{i \leq j} b_{i,j}\sqrt{s_i s_j} + d\sum_i c_i s_i}{\sum_i 2^{-2i} s_i}$$

under the constraints:

$$\begin{array}{rl} \forall 1 \leq i < j \leq k, & b_{i,j} + c_i + c_j \geq 2^{-(i+j)+1} \\ \forall 1 \leq i \leq k, & b_{i,i} + c_i \geq 2^{-2i} \\ \forall 1 \leq i \leq j \leq k, & b_{i,j} \geq 0 \\ \forall 1 \leq i \leq k, & c_i \geq 0 \end{array}$$

The following choice of b's and c's satisfies the constraints, and gives the desired bound. These indeed appear in the proof above:

$$\begin{array}{rl} \forall 1 \leq i < j \leq k, j < i + \gamma, & b_{i,j} = 2^{-(i+j)+1} \\ \forall 1 \leq i < j \leq k, j \geq i + \gamma, & b_{i,j} = 0 \\ \forall 1 \leq i \leq k, & b_i = 2^{-2i} \\ \forall 1 \leq i \leq k, & c_i = 2^{-2i-\gamma+1} \end{array}$$

Note that we can further constrain the δ's, and perhaps get better bounds this way.

4.2 Tightness

Lemma 3.3 is tight up to constant factors. To see this, consider the n-dimensional vector x whose i'th entry is $1/\sqrt{i}$. Let A be the outer product of x with itself, that is, the matrix whose (i,j)'th entry is $1/\sqrt{i \cdot j}$. Clearly x is an eigenvector of A corresponding to the eigenvalue $||x||^2 = \Theta(\log(n))$. Also, the sum of each row in A is $O(\sqrt{n})$. To prove that the lemma is essentially tight, we need to show that $max_{u,v \in \{0,1\}^n} \frac{uAv}{||u|||v||}$ is constant. Indeed, fix $k, l \in [n]$. Let $u, v \in \{0,1\}^n$ be such that $||u|| = k$ and $||v|| = l$. As the entries of A are decreasing along the rows and the columns, uAv is maximized for such vectors when their support is the first k and l coordinates. For these optimal vectors, $uAv = \Theta(\sqrt{k \cdot l})$. Thus,

$$max_{u,v \in \{0,1\}^n} \frac{uAv}{||u||||v||} = \Theta(1).$$

Clearly, this does not mean that the logarithmic factors in Theorem 3.1 are indeed required.

4.3 A converse to the Expander Mixing Lemma

There are several approaches to expansion in graphs. A combinatorial definition says that a d-regular graph on n vertices is an (n,d,c)-vertex expander if every set of vertices, W, of size at most $n/2$, has at least $c|W|$ neighbors outside itself. An algebraic definition says that such a graph is an (n,d,λ)-expander if all eigenvalues but the largest are, in absolute value, at most λ. The two notions are closely related. For example, it is known (cf. [3]) that an (n,d,λ)-expander is also an $(n,d,\frac{d-\lambda}{2d})$-vertex expander. Conversely, Alon shows in [1] that an (n,d,c)-vertex expander is also an $(n,d,d-\frac{c^2}{4+2c^2})$-expander. Roughly, these results show that one type of expansion implies the other. However, in all such results one implication (from combinatorial to algebraic expansion) is much weaker than the other.

For two subsets of vertices, S and T, let $e(S,T)$ denote the number of edges between them. A very useful property of (n,d,λ)-expanders is known as the Expander Mixing Lemma (cf. [3]): For every two subsets of vertices, A and

B, of an (n, d, λ)-expander:

$$|e(A,B) - d|A||B|/n| \leq \lambda\sqrt{|A||B|}.$$

Lemma 3.3 also implies a converse to this well known fact:

Corollary 4.1 *Let G be a d-regular graph on n vertices. Suppose that for any $S, T \subset V(G)$, with $S \cap T = \emptyset$*

$$|e(S,T) - \frac{|S||T|d}{n}| \leq \alpha\sqrt{|S||T|}$$

Then all but the largest eigenvalue of G are bounded, in absolute value, by $O(\alpha(1 + \log(d/\alpha)))$.

Proof: (Corollary 4.1) Let A be the adjacency matrix of G. Denote $B = A - \frac{d}{n}J$, where J is the all ones $n \times n$ matrix. Clearly B is symmetric, and the sum of the absolute value of the entries in each row is at most $2d$. The first eigenvalue of A is d. The other eigenvalues of A are also eigenvalues of B. Thus, for the corollary to follow from Lemma 3.3 it suffices to show that for any two vectors, $u, v \in \{0,1\}^n$:

$$|uBv| = |u\frac{d}{n}Jv - uAv| \leq \alpha||u||||v||.$$

This is exactly the hypothesis for the sets $S(u)$ and $S(v)$. ∎

It is known that for a random d-regular graph, w.h.p., the condition in Corollary 4.1 holds with $\alpha = O(\sqrt{d})$ (cf. [7]). Hence, it follows from the corollary that w.h.p., such a graph is an $(n, d, O(\sqrt{d}\log d))$-expander. While this result is weaker than previous ones ([12, 10, 9]), the proof here is somewhat shorter and simpler.

4.4 Algorithmic aspect

Lemma 3.3 is algorithmic, in the sense that given a matrix with a large eigenvalue, we can efficiently construct, from its eigenvector, a pair $u, v \in \{0,1\}^n$ such $S(u) \cap S(v) = \emptyset$, and $|uAv| \geq \alpha||u||||v||$ (There is a small caveat - in the proof we used a probabilistic argument for rounding the coordinates.

This can be easily derandomized using the conditional probabilities method). In particular, the largest eigenvalue in a d-regular graph with adjacency matrix A is a $\log d$ approximation for $\alpha = \max_{x,y \in \{0,1\}^n, supp(x) \cap supp(y) = \emptyset} \frac{|xAy|}{||x||||y||}$. In Lemma 3.2 we showed the with positive probablity a random signing has a small spectral radius. It is interesting to find a sample space where this happens with high probability. We conjecture that the following algorithm works: Choose a signing at random. If some eigenvalue of the signed matrix is big, use Lemma 3.3 to find a pair $u, v \in \{0,1\}^n$ such that $|uAv| \geq \alpha ||u|| ||v||$. Choose a signing at random for the edges between $S(u)$ and $S(v)$. Repeat until all eigenvalues are small.

5 Acknowledgments

We thank László Lovász for insightful discussions, and Efrat Daom for help with computer simulations. We thank Eran Ofek for suggesting that Corollary 4.1 might be used to bound the second eigenvalue of random d-regular graphs.
We appreciate the helpful comments we got from Alex Samorodnitsky, Eyal Rozenman and Shlomo Hoory.

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