ON THE EULER CHARACTERISTIC OF FIBRES OF REAL POLYNOMIAL MAPS

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Abstract. Let Y be a real algebraic subset of $\mathbf{R}^m$ and let $F : Y \to \mathbf{R}^n$ be a polynomial map. We show that there exist real polynomial functions $g_1, \dots, g_s$ on $\mathbf{R}^n$ such that the Euler characteristic of fibres of F is the sum of signs of g_i.

The purpose of this paper is to give a new, self-contained and elementary proof of the following result.

Theorem. *Let Y be a real algebraic subset of $\mathbf{R}^m$ and let $F : Y \to \mathbf{R}^n$ be a polynomial map. Then there exist real polynomials $g_1(y), \dots, g_s(y)$ on $\mathbf{R}^n$ such that the Euler characteristic of fibres of F is the sum of signs of g_i, that is*

$$\chi(F^{-1}(y)) = \operatorname{sgn} g_1(y) + \dots + \operatorname{sgn} g_s(y).$$

Our proof is based on a classical and elementary result expressing the number of real roots of a real polynomial of one variable as the signature of an associated quadratic form known already to Hermite [He1, He2] and Sylvester [Syl], see also [B], [BW], [BCR, p. 97]. In the proof we use a modern generalized version of this result presented in [PRS] (note that we need only a one variable case of [PRS], that is precisely [BR, Proposition p. 18]). Our original proof of the theorem [PS] used different means such as the theory of local topological degree of polynomial mappings, Gröbner bases, and the Eisenbud-Levine Theorem and was not so explicit as the one presented below.

The paper is organized as follows. In section 1 we prepare the algebraic part of the proof. In particular we recall Hermite and Sylvester's theorem. In section 2 we present basic properties of Euler characteristic of semialgebraic sets and a useful formalism of Euler integral of constructible functions. The proof of theorem for F proper is presented in section 3. This case is particularly simple and the proof is obtained by an effective elimination procedure. In section 4 we complete the proof in general (non-proper) case.

The main theorem of the paper was inspired by [CK]. For its applications see [MP] and [PS].

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1. Preliminaries I

Let W be an irreducible real algebraic subset of $\mathbf{R}^n$, let $\mathcal{A}$ denote the ring of real polynomials on W, and let $\mathcal{K}$ denote the field of fractions of $\mathcal{A}$.

Let X be a real indeterminate. Take $h^1, \dots, h^k \in \mathcal{K}[X]$ and let I denote the ideal in $\mathcal{K}[X]$ generated by $h^1, \dots, h^k$. The ring $\mathcal{K}[X]$ is a principal ideals domain, and $h = g.c.d(h^1, \dots, h^k)$ is a generator of I. Set $d = \deg h$. Then $Q = \mathcal{K}[X]/I$ is a $\mathcal{K}$-algebra and $d = dim_{\mathcal{K}} Q$. The monomials $1, X, \dots, X^{d-1}$ form a basis in Q. In particular, if $d = 0$, then $I = \mathcal{K}[X]$ and $Q = \{0\}$.

For any $f \in \mathcal{K}[X]$ there are unique $q, r \in \mathcal{K}[X]$ such that $f = qh + r$ and $\deg r < d$. Thus $r = a_0 + a_1 X + \cdots + a_{d-1} X^{d-1}$. Put $r_i(f) = a_i \in \mathcal{K}$. This way $r_i(f)$ is the i-th coordinate of f in Q.

Any $f \in Q$ defines a $\mathcal{K}$-linear endomorphism $A_f : Q \to Q$ by multiplication $A_f(p) = fp$. Let $\mathrm{Tr}(f) \in \mathcal{K}$ be the trace of A_f. Clearly

$$\mathrm{Tr}(f) = r_0(f) + r_1(fX) + \cdots + r_{d-1}(fX^{d-1}).$$

The trace map $\mathrm{Tr} : Q \to \mathcal{K}$ is $\mathcal{K}$-linear.

Fix $g \in \mathcal{K}[X]$. Define a symmetric bilinear form $\Theta^g : Q \times Q \to \mathcal{K}$ by $\Theta^g(a, b) = \mathrm{Tr}(gab)$. For $0 \le i, j \le d-1$ put $T^{ij} = r_0(gX^{i+j}) + \cdots + r_{d-1}(gX^{i+j+d-1}) \in \mathcal{K}$, so that the matrix of Θ^g is $[T^{ij}]$.

Recall that $\mathcal{K}$ is the field of fractions of the polynomial ring $\mathcal{A}$ of an irreducible real algebraic set $W \subset \mathbf{R}^n$. There exists a proper algebraic subset $\Sigma \subset W$ such that the numerators and the denominators of all non-trivial elements of $\mathcal{K}$, which have appeared above, do not vanish on $W \setminus \Sigma$. Given $f \in \mathcal{K}[X]$, $f = a_d X^d + \cdots + a_0$, $a_i \in \mathcal{K}$. For $w \in W$ we denote by f_w the evaluation of f at w that is $f_w = a_d(w) X^d + \cdots + a_0(w)$. Such $f_w$is well-defined provided all $a_i(w)$ exist. Fix $w \in W \setminus \Sigma$. Then $\deg h_w = d$ and h_w is the greatest common divisor of $h^1_w, \dots, h^k_w$. Let I_w denote the ideal generated by $h^1_w, \dots, h^k_w$. Hence $I_w = (h_w)$, $Q_w = \mathbf{R}[X]/I_w$ is an $\mathbf{R}$ algebra, $\dim_{\mathbf{R}} Q_w = d$, and $1, X, \dots, X^{d-1}$ form a basis in Q_w.

Given $g \in \mathcal{K}[X]$. In the same way as above we define the trace map $\mathrm{Tr}_w : Q_w \to \mathbf{R}$, and the symmetric bilinear form $\Theta^g_w : Q_w \times Q_w \to \mathbf{R}$, given by $\Theta^g_w(a, b) = \mathrm{Tr}(g_w ab)$. Then the evaluation $[T^{ij}_w] = [T^{ij}(w)]$ is the matrix of Θ^g_w.

Let $V_w = \{x \in \mathbf{R} | h^1(w, x) = \cdots = h^k(w, x) = 0\} = \{x \in \mathbf{R} | h_w(x) = 0\}$. If $w \in W \setminus \Sigma$ then $I_w \neq \{0\}$ and V_w is finite. Set

$$A_w = \sum_{x \in V_w} \operatorname{sgn} g(w, x). \tag{1}$$

If $g = 1$ then A_w is just the number of real roots of $h_w(x)$. The following result follows from [He1, He2] and [Syl], see [BR, Proposition p. 18] and [PRS, Theorem 2.1].

Proposition 1. *For $w \in W \setminus \Sigma$*

$$A_w = \operatorname{signature} \Theta^g_w = \operatorname{signature} [T^{ij}_w]. \qquad \square$$

Now we apply the lemma of Descartes in order to describe the way A_w depends on w.

Lemma 2. *Let* $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ *be a real polynomial and assume that all roots of* $f(x)$ *are real. Let* p_+ *(resp.* p_-*) denote the number of positive (resp. negative) roots of* f *counted with multiplicities. Let* Λ *denote the set of all pairs* (r,s) *with* $0 \le r < s \le n$ *such that* $a_r \ne 0, a_s \ne 0$*, and* $a_i = 0$ *for* $r < i < s$*. Denote* $\Lambda' = \{(r,s) \in \Lambda \mid r+s \text{ is odd }\}$*. Then*

$$p_+ - p_- = -\sum_{(r,s)\in\Lambda'} \operatorname{sgn}\,(a_r a_s).$$

Proof. We say that the pair of real numbers (a,b) changes sign if $ab < 0$. If this is the case then $(1 - \operatorname{sgn} ab)/2 = 1$, if $ab > 0$ then $(1 - \operatorname{sgn} ab)/2 = 0$.

By Descartes' lemma (see [MS, Theorem 6, p.232], or [BR, Proposition 1.1.10, p.14]), p_+ equals the number of sign changes in the sequence of non-zero coefficients of $f(x)$, that is

$$p_+ = \sum_{(r,s)\in\Lambda} (1 - \operatorname{sgn}\,(a_r a_s))/2.$$

According to the same fact, p_- equals the number of sign changes in the sequence of non-zero coefficients of $f(-x)$, i.e.

$$p_- = \sum_{(r,s)\in\Lambda} (1 - (-1)^{r+s}\operatorname{sgn}\,(a_r a_s))/2.$$

Hence $p_+ - p_- = -\sum_{(r,s)\in\Lambda'} \operatorname{sgn} a_r a_s$ as required. □

Corollary 3. *There exist polynomials* $\varphi_1, \ldots, \varphi_t \in \mathcal{A}$ *and a proper algebraic subset* $\Sigma \subset W$ *such that for every* $w \in W \setminus \Sigma$

$$A_w = \sum_{i=1}^{t} \operatorname{sgn} \varphi_i(w).$$

Proof. Let $T(\lambda) = T_d\lambda^d + \cdots + T_0$, where $T_i \in \mathcal{K}$, be the characteristic polynomial of $[T^{ij}]$. Define Λ' in the same way as above. In particular $T_r T_s \in \mathcal{K} \setminus \{0\}$ for each $(r,s) \in \Lambda'$. Then the evaluation $T_w(\lambda) = T_d(w)\lambda^d + \cdots + T_0(w)$ is the characteristic polynomial of $[T^{ij}_w]$. We enlarge Σ so that $T_r(w)T_s(w) \ne 0$ for $w \in W \setminus \Sigma$ and $(r,s) \in \Lambda'$. Then by Lemma 2

$$\text{signature}\,\Theta^g_w = \text{signature}\,[T^{ij}_w] = -\sum_{(r,s)\in\Lambda'} \operatorname{sgn} T_r(w)T_s(w) \tag{2}$$

as required. □

2. Preliminaries II

In this section we recall the construction and some basic properties of the Euler integral of a constructible function, see e.g. [MP] for more information.

An integer-valued function $\varphi : \mathbf{R}^n \to \mathbf{Z}$ is called *constructible* if it admits a presentation as a finite sum

$$\varphi = \sum m_i \mathbf{1}_{X_i}, \tag{3}$$

where for each i, X_i is a closed semialgebraic subset of $\mathbf{R}^n$, $\mathbf{1}_{X_i}$ is the characteristic function of X_i, and m_i's are integers. The presentation (3) is not unique. If the support of φ is compact, then we may choose all X_i compact. Then the *Euler integral* of φ is defined as

$$\int \varphi = \sum m_i \chi(X_i).$$

For $R > 0$ let Ψ_R^n denote the characteristic function of the ball $\mathbf{B}_R$ centered at the origin and of radius R. For any constructible φ, the product $\varphi \Psi_R^n$ has compact support, and we define the Euler integral of φ as

$$\int \varphi = \int \varphi \Psi_R^n, \qquad \text{for } R \gg 0.$$

This definition makes sense because of the following.

Lemma 4. *Let $X \subset \mathbf{R}^n$ be semialgebraic. Then if $R > 0$ is large enough $X \cap \mathbf{B}_R$ is a deformation retract of X.*

Proof. Let $\rho : X \to \mathbf{R}$ denotes the distance to the origin. By topological triviality of semialgebraic mappings [BCR, Thm. 9.3.1], there is a finite subset $\{y_1, \dots, y_p\} \subset \mathbf{R}$, such that

$$\rho : X \setminus \rho^{-1}(\{y_1, \dots, y_p\}) \to \mathbf{R} \setminus \{y_1, \dots, y_p\}$$

is a locally trivial fibration. In particular, $R > max\{|y_1|, \dots, |y_p|\}$ satisfies the statement of the lemma. □

Corollary 5. *If $X \subset \mathbf{R}^n$ is closed semialgebraic then $\int \mathbf{1}_X = \chi(X)$.*

Proof. If R is large enough then $X \cap \mathbf{B}_R$ is a deformation retract of X. Hence $\int \mathbf{1}_X = \int \mathbf{1}_X \Psi_R^n = \chi(X \cap \mathbf{B}_R) = \chi(X)$, as required. □

Lemma 6. *Let $f(x) = a_d x^d + \cdots + a_0$, $a_d \neq 0$, be a polynomial. Then $\varphi(x) = \operatorname{sgn} f(x)$ is a constructible function. For $2 \le k \le d$ define $V_k = \{x \in \mathbf{R} | f(x) = \cdots = f^{(k-1)}(x) = 0\}$ and put $Z_k = \sum_{x \in V_k} \operatorname{sgn} f^{(k)}(x)$ Then*

$$\int \varphi = \begin{cases} \operatorname{sgn} a_d - (Z_2 + Z_4 + \cdots + Z_d), & \textit{for } d \textit{ even;} \\ -(Z_2 + Z_4 + \cdots + Z_{d-1}), & \textit{for } d \textit{ odd.} \end{cases}$$

Proof. Clearly $\operatorname{sgn} f(x)$ is constructible. Let $x_1 < \ldots < x_p$ be the real roots of f. Take $R > 0$ such that $-R < x_1$ and $x_p < R$. Put $x_0 = -R$ and $x_{p+1} = R$. For $0 \le i \le p+1$, let $k(i) = \min\{j \ge 0 | f^{(j)}(x_i) \neq 0\}$, and let $S_i = \operatorname{sgn} f^{(k(i))}(x_i)$.

The sign of f on (x_i, x_{i+1}) is constant and equals $S_i = (-1)^{k(i+1)} S_{i+1}$. We have

$$\varphi\psi_R^1 = \sum_{i=0}^{p} \frac{1}{2}(S_i + (-1)^{k(i+1)} S_{i+1}) \mathbf{1}_{[x_i, x_{i+1}]} - \sum_{i=1}^{p} ((-1)^{k(i)} + 1) S_i \mathbf{1}_{\{x_i\}}.$$

Hence

$$\int \varphi = \frac{1}{2}(S_0 + S_{p+1}) - \sum_{i=1}^{p} \frac{1}{2}((-1)^{k(i)} + 1) S_i = \frac{1}{2}(S_0 + S_{p+1}) - (Z_2 + Z_4 + \cdots).$$

If d is even then $\frac{1}{2}(S_0 + S_{p+1}) = \operatorname{sgn} a_d$. If d is odd then $\frac{1}{2}(S_0 + S_{p+1}) = 0$. This ends the proof of Lemma 6. $\square$

Proposition 7. *Let W be an irreducible real algebraic subset of $\mathbf{R}^n$ and let $\mathcal{A}$ denote the ring of real polynomials on W. Let $f \in \mathcal{A}[X]$. Then there exist a finite family of polynomials $\psi_i \in \mathcal{A}$ and a proper algebraic subset $\Sigma \subset W$, such that for every $w \in W \setminus \Sigma$*

$$\int_{\mathbf{R}} \operatorname{sgn} f(w, .) = \sum \operatorname{sgn} \psi_i(w).$$

Proof. Let $f = a_r x^r + \cdots + a_0$, where $a_i \in \mathcal{A}$. We may suppose $a_r \neq 0$ in $\mathcal{A}$. Then for $w \in W \setminus a_r^{-1}(0)$, $\int_{\mathbf{R}} \operatorname{sgn} f(w, .)$ is calculated in Lemma 6.

For each $1 \leq k \leq r$, we apply the construction of section 1 to the ideal $I = (h^1, \ldots, h^k)$ generated by $h^1 = f, \ldots, h^k = \partial^{k-1} f / \partial X^{k-1}$ and $g = \partial^k f / \partial X^k$. Then, by Proposition 1 and Corollary 3, each of Z_k of Lemma 6 is a finite sum of signs of polynomials in w as required. This shows the proposition. $\square$

Proposition 8. *Let W be a real algebraic set and let $\mathcal{A}$ denote the ring of polynomials on W. Let $f \in \mathcal{A}[X]$. Then there exists a finite family of polynomials $\gamma_i \in \mathcal{A}$ such that for every $w \in W$*

$$\int_{\mathbf{R}} \operatorname{sgn} f(w, .) = \sum_i \operatorname{sgn} \gamma_i(w). \tag{4}$$

Proof. The proof is by induction on $\dim W$ and the number of irreducible components of W.

If W is irreducible then, by Proposition 7, we may find a finite family of polynomials $\gamma'_k \in \mathcal{A}$ satisfying (4) in the complement of the proper algebraic subset Σ of W. By the inductive assumption there exists a finite family of polynomials γ''_j on Σ satisfying (4) on Σ. We consider γ''_j as the restriction of polynomials on W. Let $P \in \mathcal{A}$ be any non-negative polynomial such that $P^{-1}(0) = \Sigma$. Then

$$\operatorname{sgn} P\gamma'_k = \begin{cases} \operatorname{sgn} \gamma'_k, & \text{on } W \setminus \Sigma; \\ 0, & \text{on } \Sigma. \end{cases}$$

Similarly

$$\operatorname{sgn} \gamma''_j - \operatorname{sgn} P\gamma''_j = \begin{cases} 0, & \text{on } W \setminus \Sigma; \\ \operatorname{sgn} \gamma''_j, & \text{on } \Sigma, \end{cases}$$

and hence the family $\{P\gamma'_k, \gamma''_j, -P\gamma''_j\}$ satisfies the statement.

Suppose $W' \subset W$ is an irreducible component of W and let W'' be the union of the other components. Let γ'_k (resp. γ''_j) be the family satisfying (4) on W' (resp. W''). Let $P \in \mathcal{A}$ be any non-negative polynomial such that $P^{-1}(0) = W'$. Then, by the same argument as above, the family $\{P\gamma'_k, \gamma''_j, -P\gamma''_j\}$ satisfies the statement. This ends the proof. $\square$

3. Proof of Theorem (proper case)

Suppose $F : Y \to \mathbf{R}^n$ is proper. By replacing Y by the graph of F we may assume that $Y \subset \mathbf{R}^n \times \mathbf{R}^m$ and F is induced by the projection on the first factor.

The proof is by induction on m. Suppose $m = 1$ and Y is the zero set of a non-negative polynomial function $f(y, x)$. We apply Proposition 8 to $W = \mathbf{R}^n$ and $\mathcal{A} = \mathbf{R}[y]$. Note that, since F is proper, $F^{-1}(y)$ has to be finite and $\chi(F^{-1}(y)) = 1 - \int_{\mathbf{R}} \operatorname{sgn} f(y, .)$. Hence the result follows from Proposition 8.

Inductive step. Let $p : Y \to \mathbf{R}^n \times \mathbf{R}^1$ denote the projection. By inductive hypothesis we may assume that there exists a finite family of polynomials $f_i(y, x_1)$ such that for each $(y, x_1) \in \mathbf{R}^n \times \mathbf{R}$

$$\chi(p^{-1}((y, x_1))) = \sum_i \operatorname{sgn} f_i. \tag{5}$$

Denote $\varphi(y, x_1) = \chi(p^{-1}((y, x_1))$. We claim that

$$\int_{\mathbf{R}} \varphi(y, .) = \chi(F^{-1}(y)). \tag{6}$$

Indeed, this follows from a "Fubini-type" formula for the Euler integral,

$$\int_{\{y\}\times\mathbf{R}} \varphi(y, .) = \int_{\{y\}\times\mathbf{R}} p_* \mathbf{1}_Y = \int_{F^{-1}(y)} \mathbf{1}_Y = \chi(F^{-1}(y)), \tag{7}$$

see e.g. [MP, A.4.2]. Then, by (5) and (6)

$$\chi(F^{-1}(y)) = \sum_i \int_{\mathbf{R}} \operatorname{sgn} f_i(y, .),$$

and the theorem follows again from Proposition 8. $\square$

4. Proof of Theorem (general case)

The proof presented in the previous sections does not work in general since the "Fubini type" formula (7) fails for non-proper maps. To complete the proof in the general case we use the projective compactification and the link at infinity. We shall also need the following corollary of the theorem.

Proposition 9. *Let $Y \subset \mathbf{R}^m$ be algebraic and let $F : Y \to \mathbf{R} \times \mathbf{R}^n$ be a proper polynomial map. Then there exists a finite family of polynomials $\gamma_i(y)$, where $(t, y) \in \mathbf{R} \times \mathbf{R}^n$, such that*

$$\lim_{t \to 0^+} \chi(F^{-1}(t, y)) = \sum \operatorname{sgn} \gamma_i(y).$$

Proof. The proposition follows easily from the theorem and the following lemma.

Lemma 10. *Let W be real algebraic and let $g(t, w)$ be a polynomial on $\mathbf{R} \times W$. Then there exists a finite family of polynomials $\gamma_i(w)$ on W such that*

$$\lim_{t \to 0^+} \operatorname{sgn} g(t, w) = \sum \operatorname{sgn} \gamma_i(w). \tag{9}$$

Proof. Denote $\psi(w) = \lim_{t \to 0^+} \operatorname{sgn} g(t, w)$. We proceed by induction on $\dim W$. Without loss of generality we may assume that that W is irreducible. We shall show that the statement of lemma holds generically on W, that is to say there exist a proper algebraic subset W' of W and the polynomials γ_i such that (9) holds in the complement of W'. Then the lemma will follow from the inductive assumption.

We may also assume that g does not vanish identically, and then there exists a non-negative integer k such that $g(t, w) = t^k h(t, w)$, where $h(t, w)$ is a polynomial which does not vanish identically on $\{0\} \times W$. Then, in the complement of $W' = \{w | h(0, w) = 0\}$, $\psi(w) = \operatorname{sgn} h(0, w)$ as required. This ends the proof of lemma and proposition. □

Remark 11. The assumption of properness of F in Proposition 9 is not necessary. We have made it since we are going to use Proposition 9 in the proof of theorem in general case, and it is only the proper case of theorem which has been proven till now.

Let X be a real algebraic subset of $\mathbf{R}^m$ and denote by $\widetilde{X}$ the one point compactification of X (If X is compact then $\widetilde{X} = X \amalg \{\text{point}\}$). Let $L_R = X \cap S_R^{m-1}$, where S_R^{m-1} denote the sphere centered at origin and of radius $R > 0$. We call such L_R, for R sufficiently large, *the link at infinity of* X, and denote by $L_\infty(X)$. By an argument similar to the proof of Lemma 4 it is easy to see that

$$\chi(X) = \chi(\widetilde{X}) + \chi(L_\infty(X)) - 1. \tag{10}$$

Let us recall the standard construction of $\widetilde{X}$ as an algebraic set (we will need it below in a parametrized case). Suppose X is given by a finite number of equations $f_i(x) = 0, i = 1, \ldots, s$. Set $h(x, x_{m+1}) = (f_1^2(x) + \cdots + f_s^2(x)) + (x_{m+1} - 1)^2$, so that $h^{-1}(0)$ is homeomorphic to X and h is a non-negative polynomial of degree, say, bounded by $2p$. Put $x' = (x, x_{m+1}) \in \mathbf{R}^{m+1}$ and let $H(s, x') = (s\|x'\|)^{4p} h(x'/(s\|x'\|)^2)$. Then, it is easy to see that H extends to a non-negative polynomial on $\mathbf{R} \times \mathbf{R}^{m+1}$ such that for $s \neq 0$, $\widetilde{X}_s = \{(x' \in \mathbf{R}^{m+1} | H(s, x') = 0\}$ is homeomorphic to the single point compactification of X. Also $L_s = \{(x' \in \mathbf{R}^{m+1} | H(s, x') = 0, \|x'\| = 1\}$, for $s \neq 0$ and small enough, is homeomorphic to the link at infinity of X.

Now we are ready to prove the theorem. Firstly we assume that $Y \subset \mathbf{R}^n \times \mathbf{R}^m$ and F is induced by the projection onto the first factor. Proceeding exactly in the same way as above we may compactify simultaneously the fibres of F. In particular the following statement holds.

Proposition 12. *Let $Y \subset \mathbf{R}^n \times \mathbf{R}^m$ be algebraic and let $F : Y \to \mathbf{R}^n$ be the projection on the first factor. Then there is a non-negative polynomial $H : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{m+1} \to \mathbf{R}$ such that for every $y \in \mathbf{R}^n$*

(i) $\{x' \in \mathbf{R}^{m+1} \mid H(s, y, x') = 0, s = 1\}$ *is homeomorphic to the single point compactification of $F^{-1}(y)$;*

(ii) $\{x' \in \mathbf{R}^{m+1} \mid H(s, y, x') = 0, \|x'\| = 1\}$, *for $s = s(y) \neq 0$ small enough, is homeomorphic to the link at infinity of $F^{-1}(y)$.* □

Note that the identity (10) holds for every $X = F^{-1}(y)$. Hence to prove the theorem it suffices to consider the families of compactifications and links at infinity of fibres $F^{-1}(y)$ parametrized by $y \in \mathbf{R}^n$. For the first one the statement holds by Proposition 12 (i) and the proper case. For the family of links consider $L \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{m+1}$ given by $H(s, y, x') = 0, \|x'\| = 1$. Then, clearly the projection $L \to \mathbf{R} \times \mathbf{R}^n$, $(s, y, x') \to (s, y)$, is proper and we apply to it Proposition 9. Now Proposition 12 (ii) gives the statement for the family of links at infinity. This, in virtue of (10), shows the theorem. □

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