Progress of Theoretical Physics, Vol. 17, No. 1, January 1957

Relation between the Nuclear Collective Model and the Individual Particle Model with Configurational Mixing

Satio HAYAKAWA and Toshio MARUMORI

Research Institute for Fundamental Physics, Kyoto University, Kyoto

(Received September 10, 1956)

By using the method of the quantum mechanical description of collective motion, the relation between the collective model and the individual particle model with configurational mixing is discussed on the basis of the following postulates: (I) The wave function of the nucleus, as a system composed of actual nucleons, implies strong correlations, so that the Hartree approximation makes little sense. Consequently, the method of configurational mixing currently employed as well as the shell model may not be usable without altering the representation: (II) The strong correlation gives rise to collective motion, chiefly the surface oscillation at low energies. The individual motion of nucleons which remains after subtraction of the collective motion is only weakly correlated, so that the shell model is a good approximation to the individual motion.

Leaving aside the essential problem of proving this statement, we reach the following conclusions concerning the relationship between the collective and the shell models of nuclei. i) The individual particle model with configurational mixing does in fact hold in the case of small nuclear deformation, provided that the individual particle motion in the "collective represention" can be treated by means of the adiabatic approximation. ii) The effective inter-particle force responsible for the configurational mixing consists of the following two parts, one arising from the exchange of surfons between an extra-particle and a particle in the core and the other arising from a direct inter-particle interaction. iii) As far as the quadrupole moment is concerned, the collective model and the method of configurational mixing do not show any essential difference. However, there exists an essential difference in these two methods for the interpretation of the magnetic moments.

§ 1. Introduction

There have been proposed a number of nuclear models that can, at least in part, explain various nuclear properties. Among them the success of the shell model is well known.[1),2)] The shell model is, however, to be modified in the following two ways in order to interpret the detailed behaviour of nuclear moments.[3)] One is the method of configurational mixing,[4),5)] in which single particle states in the shell structure of a core are mixed through the perturbation caused by extra-particles.*) The other is the collective model, in which the particles forming the core undergo a collective motion which may couple with the motion of extra-particles. The purpose of this paper is to make clear the relation between these two points of view.

*) Of course, the mixing of states of extra-particles may be caused. In this paper, however, we are rather interested in the relation between the method of configurational mixing and that of the collective model so we shall focus the discussion on the behaviour of the core in these two methods.

Table I. Calculated and observed values of quadrupole moments in 10^{-24} cm^2

Nucleus	Proton (neutron) configuration	Single particle model	Hydrodynamical model	Configurational mixing	Collective model	Observed value
$^{27}_{13}Al_{14}$	$(1d_{5/2})^{-1}$	0.062	0.24	0.16	0.37	0.16
$^{37}_{17}Cl_{20}$	$1d_{3/2}$	−0.051	−0.25	−0.08	−0.087	−0.062
$^{59}_{27}Co_{32}$	$(1f_{7/2})^{-1}$	0.12	0.90	0.19	0.30	0.5
$^{69}_{31}Ga_{38}$	$(2p_{3/2})^{-1}$	0.08	0.52	0.15	0.20	0.23
$^{71}_{31}Ga_{40}$	$(2p_{3/2})^{-1}$	0.08	0.52	0.15	0.14	0.14
$^{141}_{59}Pr_{82}$	$2d_{5/2}$	−0.18	−1.7	−0.30	−0.35	−0.05
$^{209}_{83}Bi_{126}$	$1h_{9/2}$	−0.30	−5.2	−0.53	−0.55	−0.4
$^{17}_{8}O_{9}$	$1d_{5/2}$	−0.0013*	−0.14	−0.04	−0.031	−0.004
$^{33}_{16}S_{17}$	$1d_{3/2}$	0	−0.20	−0.09	−0.054	−0.064
$^{35}_{16}S_{19}$	$(1d_{3/2})^{-1}$	0	0.20	0.09	0.036	0.045
$^{73}_{32}Ge_{41}$	$1g_{9/2}$	0	−1.1	−0.43	−0.11	−0.2

*) The motion of the center of mass is taken into account.

The method of configurational mixing worked out by Arima and Horie[5] and the collective model calculated by Marumori et al.[6] are found to have a common origin in accounting for nuclear quadrupole moments in the sense that the nuclear core is perturbed through interactions with extra-nucleons; the interaction in the former method is ascribed to a nuclear force between a nucleon in the core and an extra-nucleon, while that in the latter is due to the deformation of the core induced by an extra-nucleon. The fluctuation in the values of quadrupole moments from nucleus to nucleus essentially depends upon how strongly the excited states of the core particles are admixed to their ground state. As this situation holds for both methods, one can expect similar values of quadrupole moments to result. Indeed, as shown in Table I, these values are quite similar as far as comparison can be made, and reproduce observed quadrupole moments fairly well in most cases, except those in which the deformation and hence the quadrupole moment is large.

Now we will ask why these two methods predict nearly the same result, although their starting points seem to be different. First of all, we notice that in the method of configurational mixing an inter-nucleon force is introduced which is assumed to be nothing but the one giving rise to the pairing energy.[5] It does not seem to be justified that such a force is equal to the nuclear force acting between two free nucleons, but should be rather regarded as an "effective" inter-nucleon force acting in a nucleus.

The origin of the effective inter-nucleon force seems to have an important bearing on the validity of the shell model. The nuclear force observed in the two nucleon system is so strong that a single particle orbit can hardly persist for a sufficiently long time, if the same force exists between nucleons in a nucleus. In this connection, one might emphasize the importance of the Pauli principle which could effectively diminish the inter-nucleon force.[7] This is certainly correct qualitatively, but quantitatively the free nucleon force is a little too strong to account correctly for the absorption of a nucleon in nuclear

matter.[8),9),10)] However, there is some indication that the effective force is actually weaker than the free force. Vogt[11)] has shown that the imaginary part of the optical potential may be reduced when the nucleus undergoes deformation in correlation with the motion of an impinging nucleon, otherwise the imaginary part is found to be too large[12)] to account for the shape elastic scattering of neutrons.[13)]

Such an analysis together with the partial validity of the shell and collective models leads us to the following view of nuclear structure. Although the nuclear force is so strong as to give rise to a strong correlation between nucleons, the strongly correlated part contributes mainly to the collective motion and the remainig part is indeed weak enough for the shell structure to hold. We have not been able to prove this statement, but in the present paper regard this as a postulate. On the basis of this postulate our supposition may be described as follows.

(I) The wave function of the nucleus as a system composed of actual nucleons implies the strong correlation, so that the Hartree approximation makes little sense. Consequently, the method of configurational mixing as well as the shell model may not be usable without altering the representation.

(II) The strong correlation gives rise to collective motion, chiefly the surface oscillation at low energies, the individual motion of the nucleons which remains after subtraction of the collective motion being only weakly correlated. Thus the shell model is a good approximation to the individual motion.

The picture of (II) is described in a representation which results by means of a unitary transformation from the representatation used in (I).[6)] Hence the "nucleon" described in (II) is different from the actual nucleon described in (I) and is subject to a kind of constraint conditions. However, owing to the transformation function being symmetric with respect to the nucleon coordinates, an essential feature of the nucleon, namely the Fermi statistics, is maintained. The symmetric character results in the irrotational flow of the collective motion. Thus the "nucleons" can be regarded as very similar to actual nucleons and, as a first approximation, may form a shell structure.

As discussed in our previous paper,[6)] cited as M-S-Y, the separation of the collective motion by means of the adiabatic approximation is carried out for the system of "nucleons" forming a core. The "nucleons" which form the core are then described by a product wave function in the Hartree approximation. An extra-"nucleon" perturbs the shell structure of the core through collective deformation. This can be described in such a way that a collective quantum, a surfon, is exchanged virtually between the extra-"nucleon" and one of the individual "nucleons" in the core. This results in a force between them that may be responsible for the configurational mixing of the shell states In addition to this force there may exist a weak direct force between "nucleons".

Thus it becomes evident why nearly identical values of quadrupole moments are obtained by the configurational mixing method and the collective model. The inter-nucleon force assumed in the former is nothing but the force caused by the exchange of a surfon and the additional direct force. Moreover, as will be shown in a separate paper, the pairing energy, which is assumed due to the inter-nucleon force, can be accounted for in

terms of the exchange of a surfon between two extra-nucleons and also of the direct force between them.

As far as the quadrupole moment is concerned, it is not easy to distinguish between the surfon exchange force and the direct force, because the quadrupole moment arises from the second order deformation caused directly by the collective motion. The situation is different in the case of the magnetic dipole moment. Here, the direct force seems more effective than the surfon exchange force because only the former contributes directly to the magnetic moment, while the contribution of the latter appears only in higher orders. In analogous ways, one can estimate the respective contributions of these two kinds of forces when various nuclear properties are to be considered. In the present paper however, we shall be concerned only with the nuclear moments mentioned below and shall leave other cases for later consideration.

§ 2. Method of configurational mixing and its underlying assumptions

In this section we shall first give a review of the method of configurational mixing[4),5)] and try to clarify our supposition (I).

According to the method of configurational mixing, the wave function of a nucleus, Φ_{mixed}, is expressed by a superposition of those shell states which have a common total angular momentum and the same z-component of angular momentum. Thus,

$$\Phi_{mixed}=\Phi_0+\sum_n \beta_n \Phi_n \tag{2·1}$$

where Φ_0 corresponds to the ground state configuration and Φ_n to the excited states. The amplitude for a mixed state, β_n, is given by means of the perturbation theory as

$$\beta_n=(\Phi_n, \sum_{i<k} V_{ik}\Phi_0)/(E_0-E_n), \tag{2·2}$$

where E_0 and E_n indicate the energy levels of the ground and excited shell states respectively. V_{ik} is the interaction potential between the i^{th} and k^{th} nucleons.

The expectation value of any physical operator, O, is given by

$$\begin{aligned}(\Phi_{mixed}, O \Phi_{mixed}) &= (\Phi_0, O \Phi_0) \\ &+ \{\sum_n \beta_n^* (\Phi_n, O \Phi_0) + \text{comp. conj.}\},\end{aligned} \tag{2·3}$$

in which the second term on the right hand side is added, as an effect of configurational mixing, to the first one which is the expectation value in the perfect shell model. It has been suggested on an intuitive basis that the correction term corresponds to the effect of the core deformation in the collective model.[14)] Leaving this problem to later sections, we shall recall the assumptions employed in the method of configurational mixing.

We start from the Schrödinger equation for a nucleus composed of A nucleons, whose positions are $\boldsymbol{x}_1 \cdots, \boldsymbol{x}_A$, respectively,

$$(E-\mathcal{H}_A)\Phi(\boldsymbol{x}_1, \cdots \boldsymbol{x}_A)=0, \tag{2·4}$$

with

$$\mathcal{H}_A=\sum_{i=1}^{A} T_i+V_A(x_1\cdots x_A). \tag{2·5}$$

In this representation the kinetic energy of the system is expressed as the sum of that of an individual nucleon, T_i, and the potential energy, V_A, is a function of the positions of A nucleons. By introducing a Hartree potential $U(x_i)$, (2·4) may be rewritten as

$$(\mathcal{H}_0+\mathcal{H}_{int})\,\Phi=E\Phi, \tag{2·4′}$$

where

$$\mathcal{H}_0=\sum_{i=1}^{A}(T_i+U(x_i)),\quad \mathcal{H}_{int}=V_A(x_1\cdots x_A)-\sum_{i=1}^{A}U(x_i). \tag{2·6}$$

$U(x_i)$ is chosen in such a way that $\mathcal{H}_{int}$ is ineffective in the ground state:

$$(\Phi_0,\ \mathcal{H}_{int}\ \Phi_0)=0. \tag{2·7}$$

The validity of the configurational mixing lies in the fact that *not only the diagonal elements but also the non-diagonal elements of* $\mathcal{H}_{int}$ *are small enough to justify the expansion*, (2·1). Actually, in view of the strong nuclear forces, the non-diagonal elements are expected to be considerable.

This difficulty may be overcome, if we follow the argument given by Eden and Francis.[15] According to them, the shell model is valid in a representation, which is different from Φ that obeys (2·4). The new representation is attained by a model operator M which leaves quantum numbers unchanged:

$$\Psi=M\Phi. \tag{2·8}$$

In this representation the nucleon no longer maintains its original nature, but is transformed into something which may be called the "particle". Interactions between the "particles" are now weak enough to ensure the validity of the shell structure of the "particle" system.

However, the physical content of this way of thinking is not specified unless a particular form of the model operator be chosen. We should prefer that model operator which results in a representation in which the expectation values of as many as possible physical operators defined in the original representation are as close as possible to the experimental values. As will be shown in what follows, if the choice of the model operator be inadequate, the expectation values of some of the operators defined in the original representation may differ from the experimental values.

Suppose a nuclear state of configurational mixing, like the one in (2·1), be given by a model operator defined as in (2·8). Then the expectation value of an operator in this representation is

$$(\Psi,\ M\,O\,M^{-1}\,\Psi)=(\Psi,\ O\,\Psi)+(\Psi,[M,\ O]M^{-1}\,\Psi). \tag{2·9}$$

On the other hand, the expectation value in the configurational mixing method is usually calculated by $(\Psi,\ O\ \Psi)$, similar to (2·3). These two expressions are not in general identical; if (2·3) can give agreement with experiments, (2·9) may fail. Keeping this

fact in mind, we shall in a later section introduce the model operator that gives the collective representation.

§ 3. Collective model

Our supposition, (II), will be discussed along the line developed in our previous work, M-S-Y.[6] Firstly we consider the case where A nucleons form a core. The Schrödinger equation, (2·4), is transformed by a model operator,

$$M=\exp[i(m/\hbar)\sum_{i=1}^{A}\phi(\boldsymbol{x}_i)]=\exp[i(m/\hbar)\int\rho(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x}], \tag{3·1}$$

which is U given in (2·3) of M-S-Y. Here m is the nucleon mass and $\rho(\boldsymbol{x})$ the density operator; $\rho(\boldsymbol{x})=\sum_{i=1}^{A}\delta(\boldsymbol{x}-\boldsymbol{x}_i)$. $\phi(\boldsymbol{x})$ corresponds to the velocity potential of an irrotational and incompressible carrier fluid and is related to the collective momenta, π^{l}_{m}, through

$$\phi(\boldsymbol{x})=\sum_{lm}\pi^{*}_{lm}F_{lm}(\boldsymbol{x}). \tag{3·2}$$

The model transformation, M, yields the "collective" state,

$$\Psi(\boldsymbol{x}_1,\cdots\boldsymbol{x}_A,\alpha)=M\Phi(\boldsymbol{x}_1\cdots\boldsymbol{x}_A). \tag{3·3}$$

The extra-degrees of freedom, α_{lm}*), thus introduced require the subsidiary conditions,

$$(\alpha_{lm}-\sum_{i=1}^{A}F_{lm}(\boldsymbol{x}_i))\Psi(\boldsymbol{x}_1,\cdots\boldsymbol{x}_A,\alpha)=0. \tag{3·4}$$

Since the transformation operator M does not involve the momenta of the nucleons, only the kinetic energy $\sum_i T_i$ in (2·5) is transformed, yielding the kinetic energy of the collective motion, $m/2\cdot\int\rho(\boldsymbol{x})\,(grad\,\phi(\boldsymbol{x}))^2d\boldsymbol{x}$, and the interaction between the individual particles' and collective velocities, $m\int\boldsymbol{v}(\boldsymbol{x})\cdot grad\,\phi(\boldsymbol{x})d\boldsymbol{x}$, as additional terms. On the other hand, the potential energy in (2·5), which is for the time being assumed to be a function of position alone, remains unchanged. In the collective representation however, owing to the subsidiary condition (3·4), there arises a dependence of V on α. As a rule, the α-dependence of V is given uniquely by the method developed by Tomonaga,[16] but we shall not touch on this point in this paper. Thus we are able to write down the transformed Hamiltonian as

$$H=H_0+H_{int}, \tag{3·5}$$

$$H_0=\sum_{i=1}^{A}(T_i+U(\boldsymbol{x}_i,\alpha))+\frac{m}{2}\int\rho(\boldsymbol{x})\,(grad\,\phi(\boldsymbol{x}))^2d\boldsymbol{x}$$

---

*) α_{lm} are the expansion parameters of the nuclear surface defined by

$$R(\theta,\varphi)=R_0[1+\sum_{lm}\alpha_{lm}Y_{lm}(\theta,\varphi)]$$

where R_0 is the equilibrium radius.

$$+m\int \boldsymbol{v}(\boldsymbol{x})\, grad\, \phi(\boldsymbol{x})\, d\boldsymbol{x}, \tag{3·5a}$$

$$H_{int}=V(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)-\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha). \tag{3·5b}$$ *)

With the above expression for the Hamiltonian, the Schrödinger equation in the collective representation is given by

$$(H-E)\,\Psi=0. \tag{3·6}$$

In order to solve this equation we use the adiabatic approximation**) as in M-S-Y:

$$\Psi(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)=\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)\,\chi(\alpha). \tag{3·7}$$

ψ^{core} describes the individual motion of the core for a given collective deformation α, and is the solution of

$$\Big[\sum_{i=1}^{A}(T_i+U(\boldsymbol{x}_i,\ \alpha))+H_{int}-E^{core}(\alpha)\Big]\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)=0. \tag{3·8}$$

Moreover, because of the condition (3·4), ψ^{core} must satisfy the condition:

$$(\psi^{core},\ \sum_{i=1}^{A}F_{lm}(\boldsymbol{x}_i)\,\psi^{core})=\alpha_{lm}. \tag{3·9}$$

With the aid of (3·8) and (3·9), the Schrödinger equation, (3·6), with (3·4) is reduced to the equation of motion of collective modes:

$$\Big[m/2\int(\psi^{core},\ \rho(\boldsymbol{x})\,\psi^{core})\,(grad\,\phi(\boldsymbol{x}))^2 d\boldsymbol{x}+E^{core}(\alpha)\Big]\chi(\alpha)=E\chi(\alpha). \tag{3·10}$$

Since we are interested in the case in which the deformation of the core is small, we expand $\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha)$ and $H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)$ in (3·8) around $\alpha_{lm}=0$:

$$\begin{aligned}\sum_{i=1}^{A}U(\boldsymbol{x}_i,\ \alpha)&=\sum_{i=1}^{A}(U^{(0)}(\boldsymbol{x}_i)+\sum_{lm}U^{(1)}_{lm}(\boldsymbol{x}_i)\,\alpha_{lm}+\sum_{lm}U^{(2)}_{lm}(\boldsymbol{x}_i)\,|\alpha_{lm}|^2)\\ H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)&=H^{(0)}_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)+\sum_{lm}H^{(1)}_{int;lm}\cdot\alpha_{lm}+\sum_{lm}H^{(2)}_{int;lm}\,|\alpha_{lm}|^2.\end{aligned} \tag{3·11}$$

Regarding $\sum_i\sum_{lm}U^{(1)}_{lm}\,\alpha_{lm}$, $\sum_i\sum_{lm}U^{(2)}|\alpha_{lm}|^2$ and $H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\ \alpha)$ as small perturbation terms we can solve eq. (3·8). We obtain

$$\begin{aligned}\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A;\ \alpha)=\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)+\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}(\psi_n,[H^{(1)}_{int}+\sum_i\sum_{lm}U^{(1)}_{lm}\,\alpha_{lm}\\ +\sum_{lm}H^{(1)}_{int;lm}\,\alpha_{lm}]\psi_0)\times\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A),\end{aligned} \tag{3·12}$$

$$E^{core}(\alpha)=E^{(0)}+\sum_{lm}E^{(1)}_{lm}\,\alpha_{lm}+\sum_{lm}E^{(2)}_{lm}|\alpha_{lm}|^2. \tag{3·13}$$

---

*) As is shown in M-S-Y, A. Bohr's model corresponds to neglecting H_{int}.

**) As will be discussed later, if the suppositions (I) and (II) in § 1 be correct, the tenability of this approximation may be significant for the shell model.

Here $\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)$ is the solution of the equation

$$\sum_{i=1}^{A}(T_i+U^{(0)}(\boldsymbol{x}_i))\,\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)=E_n^{(0)}\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A), \tag{3·14}$$

and

$$E^{(0)}=E_0^{(0)}+\sum_n|(\psi_n,\ H_{int}^{(0)}\ \psi_0)|^2/(E_0^{(0)}-E_n^{(0)}), \tag{3·15a}$$

$$\sum_{lm}E_{lm}^{(1)}\ \alpha_{lm}=\sum_{lm}(\psi_0,\ \sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)\,\alpha_{lm}\ \psi_0)+2\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}(\psi_n,\ H_{int}^{(0)}\psi_0)\times$$
$$\times(\psi_0,\ \sum_{lm}[\sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)+H_{int;lm}^{(1)}]\ \alpha_{lm}\ \psi_n), \tag{3·15b}$$

$$\sum_{lm}E_{lm}^{(2)}|\alpha_{lm}|^2=\sum_{lm}(\psi_0,\ \sum_i U_{lm}^{(2)}(\boldsymbol{x}_i)\,|\alpha_{lm}|^2\psi_0)+\sum_n\frac{1}{E_0^{(0)}-E_n^{(0)}}\times$$
$$\times|(\psi_n,\ \sum_{lm}[\sum_i U_{lm}^{(1)}(\boldsymbol{x}_i)+H_{int;lm}^{(1)}]\cdot\alpha_{lm}\ \psi_0)|^2, \tag{3·15c}$$

provided that the Hartree potential $\sum_{i=1}^{A}U(\boldsymbol{x}_i,\alpha)$ is chosen so as to satisfy the condition

$$(\psi_0,\ H_{int}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A,\alpha)\ \psi_0)=0.$$

It should be noted that ψ_0 has a vanishing angular momentum because we consider the core to be a closed shell, and that the quantities α_{lm} lose their meaning if l becomes smaller than 2. Therefore, if we neglect the second term in (3·15b), which is a higher order correction to the first term, it is readily seen that the term linear in α_{lm} does not appear in $E^{core}(\alpha)$.

Thus we have

$$E^{core}(\alpha)=E^{(0)}+\sum_{lm}E_{lm}^{(2)}|\alpha_{lm}|^2. \tag{3·16}$$

By using (3·16) and (3·12), (3·10) becomes

$$\sum_{lm}\left[\frac{1}{2B_{lm}}|\pi_{lm}|^2+\frac{1}{2}\ C_{lm}|\alpha_{lm}|^2\right]\chi(\alpha)=(E-E^{(0)})\,\chi(\alpha), \tag{3·17}$$

where $1/2\cdot C_{lm}\equiv E_{lm}^{(2)}$ and

$$1/B_{lm}=m\int(\psi_0,\ \rho(\boldsymbol{x})\,\psi_0)\,|grad\,F_{lm}(\boldsymbol{x})|^2d\boldsymbol{x}.$$

B is constant in the adiabatic approximation, whereas the non-adiabatic contributions give rise to the dependendce of B on the core structure, as discussed by Inglis[18] and by Bohr and Mottelson.[19] In the present paper we shall not consider the non-adiabatic contributions.

The collective surface motion, which plays a role in the deformation of the closed shell nuclei, is described by the Schrödinger equation, (3·17). This surface motion gives rise to energy $\sum n\hbar\omega_l$, where $\omega_l=(C_{lm}/B_{lm})^{1/2}$ and n is the number of surfons excited. Thus the energy eigenvalue in the ground state of eq. (3·6) is equal to $E^{(0)}$, provided

that the zero-point energy of the collective motion is omitted.

It should be noted that $E^{(0)}$ is the energy eigenvalue of the equation

$$[\sum_i (T_i+U^{(0)}(\boldsymbol{x}_i))+H_{int}^{(0)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)]\ \Psi^s(\boldsymbol{x}_1\cdots\boldsymbol{x}_A) =E^{(0)}\Psi^s(\boldsymbol{x}_1\cdots\boldsymbol{x}_A), \qquad (3\cdot18)$$

where

$$\Psi^s(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)=\psi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_A)+\sum_n \frac{(\psi_n, H_{int}^{(0)}\psi_0)}{E_0^{(0)}-E_n^{(0)}}\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_A).$$

In the above discussion we have shown that the shell model should be a good approximation, provided the collective motion be separated so as to diminish the inter-nucleon forces, although we have not been able to prove the ineffectiveness of H_{int}. This way of approach should be distinguished from that in Section 2, where no possibility exists for making $\mathcal{H}_{int}$ ineffective. On the other hand, we have found a device for making H_{int} small, so that the shell model is indeed a good approximation. Thus we are allowed to start from the Schrödinger equation for the shell model, (3·18), and to treat the non-diagonal parts as perturbation. This will be the basis for the following discussion.

§ 4. Interactions between the core and an extra-particle*)

We shall now discuss the behaviour of the nucleus consisting of the core and one extra-particle, the latter being the cause of the core deformation. The coordinates of nucleons forming the core are designated by the suffices $1,\cdots N$ and that of the extra-particle is by the suffix 0. Since we restrict ourselves to the case of only one extra-particle, the approximations we adopt, such as the perturbation or weak coupling treatment, may be valid and unnecessary complication can be avoided.**)

In this case our Schrödinger equation in the collective representation is given by

$$[H_{core}+H_{int}+H_{0.c}+T_0+U_0(\boldsymbol{x}_0,\ \alpha)]\ \Psi(\boldsymbol{x}_1\cdots\boldsymbol{x}_N,\ \boldsymbol{x}_0,\ \alpha) =E\Psi(\boldsymbol{x}_1\cdots\boldsymbol{x}_N,\ \boldsymbol{x}_0,\ \alpha). \qquad (4\cdot1)$$

with the subsidiary conditions

$$[\alpha_{lm}-\sum_{i=1}^{N}F_{lm}(\boldsymbol{x}_i)]\Psi(\boldsymbol{x}_1\cdots\boldsymbol{x}_N,\ \boldsymbol{x}_0,\ \alpha)=0, \qquad (4\cdot2)$$

*) For the sake of simplicity, we do not take into account the Pauli principle explicitly.

**) The method of configurational mixing in the shell model is based on taking into account the effect of the core particles as a small perturbation. Therefore, it is clear intuitively that the weak coupling scheme in the collective model, in which the effect of the core deformation is treated as a perturbation, corresponds to the method of configurational mixing. This is one of the reasons why we consider the "core+one extra-particle" type of nucleus. In the case of the "core+several extra-particles" type of nucleus, the situation may be more complicated; we must take into account not only the collective motion of the core but also that of the extra-particles and the subsidiary condition (4·2) may have to be modified suitably. Furthermore, the weak coupling scheme may be unsatisfactory in many cases.

For nuclei for which the configurational mixing method is successful however, the situation may, as a rule, be the same as that discussed in this section.

where $H_{0.c}$ represents the direct interaction between the extra-particle and the core particles and is expressed by

$$H_{0.c}=V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0,\ \alpha)-U_0(\boldsymbol{x}_0,\ \alpha). \tag{4·3}$$

In the original representation, $V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)$ corresponds to the interaction potential between the extra-nucleon and the core nucleons, and the α-dependence of $V_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0,\ \alpha)$ in the collective representation can also, as a rule, be given by Tomonaga's procedure. H_{core} and H_{int} are the same as those given by (3·5a) and (3·5b), except for the change of the number of particles from A to N.

Since our interest in this section will be directed to the interaction between the core and the extra-particle, we shall hereafter neglect the effect of H_{int} having discussed it in the preceding section.

As discussed in M-S-Y, the correspondence to A. Bohr's model is made clear, if we neglect $H_{0.c}$ and expand $U_0(\boldsymbol{x}_0,\ \alpha)$ around $\alpha_{lm}=0$;

$$U_0(\boldsymbol{x}_0,\ \alpha)=U_0^{(0)}(\boldsymbol{x}_0)+\sum_{lm}U_{0;lm}^{(1)}(\boldsymbol{x}_0)\,\alpha_{lm}+\sum_{lm}U_{0;lm}^{(2)}(\boldsymbol{x}_0)\,|\alpha_{lm}|^2. \tag{4·4}$$

The parts depending on α are to be regarded as an interaction between the extra-particle and the collective motion of the core.

Now, according to our supposition, (II), we regard $H_{0.c}$ as a perturbation term, and start from the unperturbed equation in which the interaction between an extra-particle and a core takes place only through the collective oscillation. Using the adiabatic approximation for the core part, the solution of the unperturbed equation,

$$(H_0-E_0)\,\Psi^0=0, \tag{4·5}$$

where

$$H_0=H_{core}+T_0+U_0(\boldsymbol{x}_0,\ \alpha), \tag{4·6}$$

is expressed by

$$\Psi^0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0\ \alpha)=\phi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \alpha)\,\chi(\boldsymbol{x}_0,\ \alpha). \tag{4·7}$$

Using the same procedure as in deriving (3·10), we obtain the equation for $\chi(\boldsymbol{x}_0,\ \alpha)$, namely

$$[m/2\cdot\int(\phi^{core},\ \rho(\boldsymbol{x})\,\phi^{core})\,(grad\ \phi(\boldsymbol{x}))^2 d\boldsymbol{x}+E^{core}(\alpha)+T_0+U_0(\boldsymbol{x}_0,\ \alpha)]\times$$
$$\times\chi(\boldsymbol{x}_0,\ \alpha)=E_0\,\chi(\boldsymbol{x}_0,\ \alpha). \tag{4·8}$$

Assuming that the core deformation is small, (4·8) may be rewritten in the form

$$(H_{surf.}+H_{part.}+\bar{H}_{int})\,\chi(\boldsymbol{x}_0,\ \alpha)=(E_0-E^{(0)})\,\chi(\boldsymbol{x}_0,\ \alpha), \tag{4·9}$$

where

$$\left.\begin{aligned} H_{surf.} &= \sum_{lm}\left(\frac{1}{2B_{lm}}|\pi_{lm}|^2+E^{(2)}_{lm}|\alpha_{lm}|^2\right), \\ H_{part.} &= T_0+U^{(0)}_0(\boldsymbol{x}_0), \\ \text{and} \\ \bar{H}_{int} &= \sum_{lm}(E^{(1)}_{lm}+U^{(1)}_{0;lm}(\boldsymbol{x}_0))\alpha_{lm}+\sum U^{(2)}_{0;lm}(\boldsymbol{x}_0)|\alpha_{lm}|^2. \end{aligned}\right\} \quad (4\cdot10)^{*)}$$

Here $E^{(0)}$, $E^{(1)}$, and $E^{(2)}$ are given by (3·15a), (3·15b) and (3·15c) respectively. It should be noted that $\sum E^{(1)}_{lm}\,\alpha_{lm}$ vanishes actually as discussed in the preceding section.

Now we shall solve eq. (4·9) regarding $\bar{H}_{int}$ as a perturbation term. Then we obtain

$$\chi(\boldsymbol{x}_0,\ \alpha)=\varphi(\boldsymbol{x}_0)\chi_0(\alpha)+\sum_{lm}\frac{(\varphi,\ U^{(1)}_{0;lm}\ \varphi)<\chi_{lm},\ \alpha_{lm}\ \chi_0>}{-\hbar\underline{\omega}_l}\varphi(\boldsymbol{x}_0)\chi_{lm}(\alpha), \quad (4\cdot11)$$

where $\underline{\omega}_l=(2E^{(2)}_{lm}/B_{lm})^{1/2}$. Here, the extra-particle is assumed to stay in the ground state, $\varphi(\boldsymbol{x}_0)$, thus preserving the correspondence to the method of configurational mixing. The energy eigenvalue is

$$\begin{aligned}(E_0-E^{(0)})=E_p+\sum_{lm}&\frac{|(\varphi,\ U^{(1)}_{0;lm}\varphi)\langle\chi_{lm},\ \alpha_{lm}\ \chi_0\rangle|^2}{-\hbar\,\underline{\omega}_l} \\ &+(\varphi,\ U^{(2)}_{0;lm}\ \varphi)<\chi_0,|\alpha_{lm}|^2\chi_0>, \end{aligned}\quad (4\cdot12)$$

where E_p is the energy eigenvalue of the equation

$$H_{part.}\ \varphi(\boldsymbol{x}_0)=E_p\ \varphi(\boldsymbol{x}_0).$$

We have omitted the zero point energy of the surface oscillation.

The second and third terms of (4·12) arising from the coupling with the extra-particle may also be amalgamated into $\sum_{lm} E^{(2)}_{lm}|\alpha_{lm}|^2$ in $H_{surf.}$ provided that the extra-particle is not excited, and should contribute to the surface rigidity even though this effect is neglected in M-S-Y as well as in A. Bohr's model. Indeed, if we define the quantities

$$\frac{1}{2}\sum_{lm}C_{lm}|\alpha_{lm}|^2=\sum_{lm}\left[E^{(2)}_{lm}+\frac{|(\varphi,\ U^{(1)}_{0;lm}\ \varphi)|^2}{-\hbar\underline{\omega}_l}+(\varphi,\ U^{(1)}_{0;lm}\ \varphi)\right]\cdot|\alpha_{lm}|^2 \quad (4\cdot13)$$

and replace $\bar{H}_{surf.}$ and H_{int} in (4·10) by

$$\underline{H_{surf.}}=\sum_{lm}\left\{\frac{1}{2B_{lm}}|\pi_{lm}|^2+\frac{1}{2}C_{lm}|\alpha_{lm}|^2\right\}, \quad (4\cdot14a)$$

$$\begin{aligned}\underline{\bar{H}_{int}}=&\sum_{lm}(E^{(1)}_{lm}+U^{(1)}_{0;lm}(\boldsymbol{x}_0))\alpha_{lm} \\ &+\sum_{lm}(U^{(2)}_{0;lm}(\boldsymbol{x}_0)+E^{(2)}_{lm}-1/2\cdot C_{lm})\times|\alpha_{lm}|^2. \end{aligned}\quad (4\cdot14b)$$

respectively, the second and third terms of (4·12) vanish. This corresponds to the re normalization of surfon frequencies,

---

*) This is nothing but the Hamiltonian of the collective model currently employed.

$$\omega_l \longrightarrow \omega_l, \tag{4·15}$$

where ω_l is given by $(C_{lm}/B_{lm})^{1/2}$, C_{lm} being defined in (4·13). Hereafter we shall use the renormalized surfon frequencies, ω_l, so that $H_{surf.}$ and $\bar{H}_{int}$ of (4·9) should be read as (4·14).

Up to this point, we have dropped the term, $\sum E^{(1)}_{lm}\alpha_{lm}$ in (4·10) or (4·14b) because this term vanishes as discussed in the preceding section. If we write formally the effect of this term to the first order of $E^{(1)}_{lm}$, (4·12) is expressed, using the renormalized surfon frequency, as

$$(E_0-E^{(0)})=E_p+2\sum_{lm}\frac{(\varphi,\ U^{(1)}_{0;lm}(\boldsymbol{x}_0)\varphi)\cdot E^{(1)}_{lm}}{-\hbar\omega_l}<\chi_0,\ \alpha^2_{lm}\chi_0>. \tag{4·16}$$

Using

$$E^{(1)}_{lm}=(\phi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N),\sum_{i=1}^{N}U^{(1)}_{lm}(\boldsymbol{x}_i)\phi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N)),$$

the second term of (4·16) may be rewritten as

$$(\Psi^s_0,\ 2\sum_{lm}\frac{\sum_{i=1}^{N}U^{(1)}_{lm}(\boldsymbol{x}_i)\cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0)}{-\hbar\omega_l}<\chi_0,\ \alpha^2_{lm}\chi_0>\Psi^s_0), \tag{4·17}$$

where Ψ^s_0 means

$$\Psi^s_0=\varphi(\boldsymbol{x}_0)\phi_0(\boldsymbol{x}_1\cdots\boldsymbol{x}_N). \tag{4·18}$$

As is readily seen from its form this arises from the exchange of surfons between an extra-particle and particles in the core. If the collective motion is eliminated, we obtain the interaction,

$$V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)=2\sum_{lm}\frac{(\sum_{i=1}^{N}U^{(1)}_{lm}(\boldsymbol{x}_i)\cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0))}{-\hbar\omega_l}<\chi_0,\ \alpha^2_{lm}\chi_0>. \tag{4·19}$$

In fact, (4·16) with (4·17) shows that in so far as the energy eigenvalue of the ground state is concerned, we can reduce the Schrödinger equation, (4·5), to that for the individual motions of the particles:

$$\{\sum_{i=1}^{N}(T_i+U^{(0)}(\boldsymbol{x}_i))+T_0+U_0{}^{(0)}(\boldsymbol{x}_0)+V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\}\Psi^s(\boldsymbol{x}_0,\boldsymbol{x}_1\cdots\boldsymbol{x}_N)$$
$$=E_0\Psi^s(\boldsymbol{x}_0,\boldsymbol{x}_1\cdots\boldsymbol{x}_N). \tag{4·20}$$

Although $V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\boldsymbol{x}_0)$ has no effect upon the energy calculation, it may become important for the calculation of the expectation values of physical operators such as the nuclear moments.

Thus we are led to an approximate solution of unperturbed equation (4·5). For the real case we must take account of $H_{0.c}$ which represents a very complicated interaction owing to the fact that all variables are contained. In the decomposition:

$$H_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N:\ \boldsymbol{x}_0,\ \alpha)=H^{(0)}_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)+\sum_{lm}H^{(1)}_{0.c;lm}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\alpha_{lm}$$
$$+\cdots\cdots\cdots,$$

we presume that $H^{(0)}_{0.c}$ is the most important one, because the nuclear deformation, α, is supposed to be small in our weak coupling case. Neglecting the effect of the α-dependent part of $H_{0.c}$, the Schrödinger equation (4·1) can be reduced effectively into

$$\{\sum_{i=1}^{N}(T_i+U^{(0)}(\boldsymbol{x}_i))+T_0+U_0{}^{(0)}(\boldsymbol{x}_0)+H^{(0)}_{0.c}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)$$
$$+V^{(ex)}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\ \boldsymbol{x}_0)\}\,\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)=E\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N),\qquad(4\cdot21)$$

provided that the adiabatic and weak coupling approximations are valid. This is nothing but the Schrödinger equation currently employed for the method of configurational mixing, and $\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)$ is given by

$$\Psi^s(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N)=\Psi^s_0(\boldsymbol{x}_0,\boldsymbol{x}_1\cdots\boldsymbol{x}_N)+$$
$$+\sum_n\frac{(\Psi^s_n,\{H^{(0)}_{0.c}+V^{(ex)}\}\Psi^s_0)}{E_0-E_n}\Psi^s_n(\boldsymbol{x}_0,\ \boldsymbol{x}_1\cdots\boldsymbol{x}_N),\quad(4\cdot22)$$

where

$$\Psi^s_n=\varphi(\boldsymbol{x}_0)\psi_n(\boldsymbol{x}_1\cdots\boldsymbol{x}_N)\ *).$$

The two interactions, $V^{(ex)}$ and $H^{(0)}_{0.c}$, correspond to the "effective inter-particle interaction" which cause the configurational mixing. In M-S-Y, only the effect corresponding to $V^{(ex)}$ is taken into account, while in the current method of configurational mixing the sum of these two effects is regarded as though it were identical with the force between two nucleons.

Thus one can see the correspondence between the method of configurational mixing and the collective model.

§ 5. Expectation values of operators

As mentioned in Section 2 an operator, O, in the original representation undergoes a transformation

$$MOM^{-1}=O+[M,\ O]M^{-1}\qquad(5\cdot1)$$

in the collective representation. M is given by (3·1). Since M depends only on the positions of the nucleons, but not on their momenta, the operator involving only positions will commute with M and only the first term in the right hand side of (5·1) will remain. This is the case for the quadrupole moment, but not the case for the magnetic

*) It should be noted that the averaged potential $U_0(\boldsymbol{x}_0,\ \alpha)$ in (4·1) and (4·3) must be chosen so as to satisfy

$$(\Psi_0{}^s,\ H_{0.c}\ \Psi_0{}^s)=0,$$

moreover because of the condition (4·2), the shell model wave function $\Psi^s{}_n$ must satisfy a condition, which will be discussed in a later section.

moment.

For use in later applications we shall calculate the commutator. In most cases the operators of interest are expressed as a sum of those, each of which involves coordinates of each nucleon.

$$O=\sum_i O_i\ (\boldsymbol{x}_i,\ \boldsymbol{p}_i). \tag{5·2}$$

We assume that in O_i the momentum $\boldsymbol{p}_i$ appears linearly. Hence we have only to know the commutators for $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$:

$$[M,\ \boldsymbol{x}_i]M^{-1}=0$$

$$[M,\ \boldsymbol{p}_i]M^{-1}=m\int\delta(\boldsymbol{x}-\boldsymbol{x}_i)\,grad\,\phi(\boldsymbol{x})\,d\boldsymbol{x}=m\ grad\ \phi(\boldsymbol{x}_i). \tag{5·3}$$

Thus we obtain the commutator for O_i as

$$[M,\ O_i(\boldsymbol{x}_i,\ \boldsymbol{p}_i)\,]M^{-1}=O_i(\boldsymbol{x}_i,\ m\ grad\,\phi(\boldsymbol{x}_i))=\bar{O}_i(\boldsymbol{x}_i,\ \pi^*_{lm}). \tag{5·4}$$

Thus, one has only to replace $\boldsymbol{p}_i$ by $m\,grad\,\phi(\boldsymbol{x}_i)$ or $m\sum_{lm}\pi^*_{lm}\cdot grad\,F_{lm}(\boldsymbol{x})$.

Thus we arrive at

$$[M,\ O]M^{-1}=\sum_i O_i(\boldsymbol{x}_i,\ m\ grad\,\phi(\boldsymbol{x}_i))=\int\rho(\boldsymbol{x})\,\bar{O}(\boldsymbol{x},\ \pi^*_{lm})\,d\boldsymbol{x}. \tag{5·5}$$

Since π_{lm} are the collective momenta conjugate to α_{lm}, this part can be regarded as representing the dynamical effect of the collective motion. This effect cannot arise from the shell model, in which the core is assumed to behave as if static. Therefore, the correspondence between the method of configurational mixing and the collective model is strictly true only for the case

$$MOM^{-1}=O. \tag{5·6}$$

As discussed in Section 2, in the method of configurational mixing currently employed the expectation value of an operator, O, is given by

$$\begin{aligned}(\Psi_{mixed},\ O\ \Psi_{mixed})&=(\Psi^s_0,\ O\ \Psi^s_0)\\&\quad+[\sum_n\beta_n(\Psi^s_n,\ O\ \Psi^s_0)+\text{comp. conj.}],\end{aligned} \tag{5·7}$$

$$\beta_n=(\Psi^s_n,\ V_{eff}(\boldsymbol{x}_0,\ \boldsymbol{x}_1,\cdots\boldsymbol{x}_N)\,\Psi^s_0)/(E_0-E_n).$$

On the other hand, using the adiabatic approximation, the expectation value in the collective representation is given by

$$(\chi(\boldsymbol{x}_0,\ \alpha),<\psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\alpha),\ MOM^{-1}\ \psi^{core}(\boldsymbol{x}_1\cdots\boldsymbol{x}_N;\alpha)>\chi(\boldsymbol{x}_0,\ \alpha)). \tag{5·8}$$

If O satisfies (5·6), and if we drop the direct interactions H_{int} and $H_{0,c}$ discussed in Sections 3 and 4, (5·8) is expressed as

$$(\chi,<\psi^{core},\ MOM^{-1}\ \psi^{core}>\chi)=(\chi,<\psi^{core},\ O\psi^{core}>\chi)$$

$$= (\Psi^s_0,\ O\ \Psi^s_0) + \Big[\sum_n \frac{1}{E_0 - E_n} \Big\{ \Big(\Psi^s_n,\ 2\sum_{lm} \frac{\sum_{i=1}^{N} U^{(1)}_{lm}(\boldsymbol{x}_i) \cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0)}{-\hbar\omega_l} \times$$

$$\times (<\chi_{lm},\ \alpha_{lm}\ \chi_0>)^2\ \Psi^s_0 \Big) \Big\} (\Psi^s_n\ O\ \Psi^s_0) + \text{comp. conj.} \Big] \qquad (5\cdot 9)$$

by using (3·12), (4·11) and (4·18). Comparing (5·9) with (5·7) these two are equivalent if one chooses the effective interaction to be

$$V_{eff}(\boldsymbol{x}_0,\ \boldsymbol{x}_1 \cdots \boldsymbol{x}_N) = 2\sum_{lm} \frac{\sum_{i=1}^{N} U^{(1)}_{lm}(\boldsymbol{x}_i) \cdot U^{(1)}_{0;lm}(\boldsymbol{x}_0)}{-\hbar\omega_l} < \chi_0,\ \alpha^2_{lm}\ \chi_0 > . \qquad (5\cdot 10)$$

(5·10) is nothing but $V^{(ex)}(\boldsymbol{x}_0\,;\ \boldsymbol{x}_1 \cdots \boldsymbol{x}_N)$ given by (4·19).

§ 6. Electric quadrupole moments

In the original representation, the electric quadrupole moment operator is expressed by

$$Q = \sum_{i=0}^{A} g_i (3z_i^2 - r_i^2) = \sum_{i=1}^{N} g_i (3z_i^2 - r_i^2) + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 1)$$

where

$$g_i = 0 \quad \text{for a neutron,}$$

and

$$g_i = 1 \quad \text{for a proton.}$$

Since this operator does not involve momenta, the last term of (5·1) vanishes and formulas (5·7) and (5·9) can be used.

In the collective model of Bohr and Mottelson,[17] the quadrupole moment operator is given by

$$Q^{B\text{-}M} = Q_{core} + Q_0 = (3/\sqrt{5\pi})\, \mathcal{Z} R_0^2\, \alpha_{20} + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 2)$$

and the expectation value by

$$(\chi(\boldsymbol{x}_0, \alpha),\ Q^{B\text{-}M}\chi(\boldsymbol{x}_0, \alpha)) = (3/\sqrt{5\pi})\ \mathcal{Z} R_0^2 (\chi(\boldsymbol{x}_0, \alpha),\ \alpha_{20}\ \chi(\boldsymbol{x}_0, \alpha)) + (\chi(\boldsymbol{x}_0, \alpha),\ Q_0\ \chi(\boldsymbol{x}_0, \alpha)). \qquad (6\cdot 3)$$

In order to justify the equality of (5·9) and (6·3), we rewrite (6·1) as

$$Q = (3/\sqrt{5\pi})\, N R_0^2 \sum_{i=1}^{N} g_i\, F_{20}(\boldsymbol{x}_i) + g_0 (3z_0^2 - r_0^2), \qquad (6\cdot 4)$$

by taking into account the fact that

$$\sum_{i=1}^{N} F_{lm}(\boldsymbol{x}_i) = (4\pi/3N) \sum_{i=1}^{N} (r/R_0)^l Y_{lm}(\theta_i\ \varphi_i).$$

According to the condition, (3·9), the expectation value of (6·4) in the core particle state, $\psi^{core}(\boldsymbol{x}_1 \cdots \boldsymbol{x}_N\,;\ \alpha)$, is given by

$$(\psi^{core},\ Q\ \psi^{core}) = (3/\sqrt{5\pi})\ \mathcal{Z} R_0^2\ \alpha_{20} + Q_0 = Q^{B\text{-}M}, \qquad (6\cdot 5)$$

so that

$$(\chi, <\phi^{core}, Q\,\phi^{core}>\chi) = (\chi, Q^{B\text{-}M}\,\chi).$$

This demonstrates the equality of (5·9) and (6·4). In fact, it should be noted that if we use the expression (3·12) for ϕ^{core}, but neglecting H_{int}, the condition (3·9) becomes

$$\sum_n \frac{(\psi_n, \sum_{i=1}^{N} U_{lm}^{(1)}(\boldsymbol{x}_i)\,\psi_0)}{E_0 - E_n}(\psi_n, \sum_{i=1}^{N} F_{lm}(\boldsymbol{x}_i)\,\psi_0) + \text{comp. conj.} = 1. \tag{6·6}$$

Inserting (6·6) into (5·9) and using the expression (4·11) for $\chi(\boldsymbol{x}_0, \alpha)$, we can easily see that (5·9) reduces to (6·3).

Using the expression on the right hand side of (5·9), the expectation value in the collective model is found to correspond to that in the method of configurational mixing as given by (5·7), provided that V_{eff} used in (5·7) is taken to be that given by (5·10). The similarity which is shown in Table I between the values of the quadrupole moment calculated by the configurational mixing method[5] and by the collective model[6] is based upon such a procedure.

Although these two ways of calculation are similar in forms, there exist some discrepancies between them. Firstly, in the configurational mixing method the perturbation interaction, V_{eff}, is assumed to be independent of the structure of the core, the shell structure being implied only in the energy denominator of β_n. In the collective model however, V_{eff} is dependent on the structure of the core through the collective eigenfrequencies, ω_l. This provides us with an explanation for the differences in the quadrupole moments of some isotopes.[6]

Secondly, the summation over n in (5·7) is usually carried out only for the main terms, and the applicability of configurational mixing is restricted to the case of small perturbation. The same restriction holds in the collective model, in so far as the weak coupling approximation is employed. However, the adiabatic and weak coupling approximations we have used in the foregoing discussions are only for the purpose of showing the correspondence to the configurational mixing method. One may forego such approximations and choose suitable ways for the consideration of real nuclei. In this respect the collective model has wider applicability than the method of configurational mixing. In fact, we have shown that the configurational mixing is valid as a special case of the collective model.

Thirdly, our method is different from the current collective model of A. Bohr.[17] In the latter case ω_l is determined on the basis of the hydrodynamical model, whereas in our case ω_l is obtained from the collective motion for a given rigidity which depends strongly upon the structure of the core.[6]

§ 7. Magnetic dipole moments

In the original representation the magnetic dipole moment operator is expressed as

$$\mu_z=(e\hbar/2mc)\sum_{i=0}^{A}(g_i\,\boldsymbol{l}_i+\gamma_i\,\boldsymbol{s}_i)_z, \tag{7·1}$$

where

$$\gamma_i=-3.826 \quad \text{for neutron,}$$
$$\gamma_i=\ 5.587 \quad \text{for proton.}$$

In the collective representation this is transformed into

$$\begin{aligned} M\mu_z M^{-1}&=\mu_z+[M,\ \mu_z]M^{-1}\\ &=\mu_z+\frac{e\hbar}{2mc}\int\rho_p(\boldsymbol{x})[\boldsymbol{r}\times m\ grad\ \phi(\boldsymbol{x})]_z\ d\boldsymbol{x} \qquad (7\cdot2)\\ &=\mu_z+\frac{e\hbar}{2mc}\int\rho_p(\boldsymbol{x})\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}, \end{aligned}$$

where $\rho_p(\boldsymbol{x})$ is the density of protons and

$$\boldsymbol{R}(\boldsymbol{x})=\boldsymbol{r}\times m\ grad\ \phi(\boldsymbol{x}) \tag{7·3}$$

represents the angular momentum arising from the collective motion of the core.

The first term in the right hand side of (7·2) represents the magnetic moment due to individual nucleons, so that its expectation value can be evaluated by (5·9). However, the Ψ_n^s which give non-vanishing matrix elements of $(\Psi_n^s,\ \mu_z\ \Psi_0^s)$, do not satisfy the collective deformation condition; if V_{eff} in $(\Psi_n^s,\ V_{eff}\ \Psi_0^s)$ is expressed by (5·10), these matrix elements are found to vanish for such Ψ_n^s. The non-vanishing elements appear in the next order, if Ψ_n^s are restricted to that owing to the collective deformation. Nevertheless, the first order term in (5·7) has an effect owing to the direct interaction $H_{0.c}$. This is what is considered in the current method of configurational mixing.

We shall now discuss the dynamical effect implied in the last term of (7·2), which can not be taken into account in the configurational mixing method. In order to obtain the expectation value of the last term of (7·2), we first note that by using the density operator of the core particles the condition (3·9) may be rewritten as follows:

$$(\psi^{core},\ \rho(\boldsymbol{x})\psi^{core})=(1/2)\rho_0\{1+\epsilon(R(\theta,\varphi)-r)\}. \tag{7·4}$$

Here $\rho_0=3N/4\pi R_0^3$ and

$$R(\theta,\ \varphi)=R_0[1+\sum_{lm}\alpha_{lm}\ Y_{lm}(\theta,\ \varphi)]$$

represents the nuclear surface. $\epsilon(x)$ is expressed by

$$\epsilon(x)=\frac{x}{|x|}=\begin{cases}+1, & x>0\\ -1, & x<0.\end{cases}$$

Therefore, the dynamical part is reduced to

$$(\chi, <\psi^{core}, \frac{e\hbar}{2mc}\int \rho_p(\boldsymbol{x})\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}\ \psi^{core}>\chi)$$

$$=\frac{e\hbar}{2mc}(\chi,\ (1/2)\rho_{0.p}\int R_0\,\delta(R_0-r)\sum_{lm}\alpha_{lm}\,Y_{lm}\boldsymbol{R}_z(\boldsymbol{x})\,d\boldsymbol{x}\,\chi). \qquad (7\cdot6)$$

This is simply the expression derived by Bohr and Mottelson.[17] If this is to be compared with the method of configurational mixing, it will be necessary to retain at least the second order terms in $V^{(ex)}$. Therefore, the equivalence between the collective model and the configurational mixing holds only to first order in $V^{(ex)}$. In the collective model currently employed, if higher oder effects are neglected, not only the dynamical part, but also the static part arising from the mixing of single particle states will disappear, leaving only the magnetic moment due to the unperturbed extra-nucleon. The magnetic moment thus calculated is the same as in the single particle model.

The above circumstances are in contrast to the case of the quadrupole moment discussed in the preceding section. In the latter case the angular momentum states of the core particles to be mixed are the $J=2$ states, both in the configurational mixing method and the collective model, while in the case of magnetic moments the states to be mixed in first order are of $J=1$, which cannot occur by the collective deformation. Thus the configurational mixing due to the direct interaction, $H_{0.c}$, seems to play a significant role in accounting for magnetic moments. Therefore, we may infer the possibility of analyzing the contributions from $V^{(ex)}$ and $H_{0.c}$ by the quantitative studies of quadrupole moments and magnetic moments. For this purpose it is advisable to refer to as many nuclear properties as possible, such as the pairing energy, the optical potential and so forth. Studies along this line will be made in subsequent papers.

§ 8. Concluding remarks

In the above discussion we have considered the relation between the collective model and the individual particle model with configurational mixing on the basis of the supposition (II). The effective inter-particle interaction is, according to our supposition, shown to be of collective origin in part. Such an effective interaction is presumed to be weak enough to justify the success of the shell model. Leaving this essential problem unsolved, we reach the following conclusions concerning the intimate relationship between the collective and the shell models of nuclei.

i) The individual particle model with configurational mixing does in fact hold in the case of small nuclear deformation, provided that the single particle motion in the collective representation can be treated by means of the adiabatic approximation.

ii) The effective inter-particle force responsible for the configurational mixing consists of two parts, one arising from the exchange of surfons between an extra-particle and particles in the core and the other from a direct inter-particle interaction.

iii) As far as the quadrupole moment is concerned, the collective model and the configurational mixing method do not show any essential difference under the conditions

adopted in i). The difference lies in the effective interaction responsible for mixing single particle states. In the collective model the interaction is due to the exchange of surfons, while in the method of configurational mixing it is due to the direct internucleon force. The former depends strongly on the structure of the core, but the latter does not.

iv) There exists an essential difference in these two methods concerning the interpretation of the magnetic moment. In the collective model the dynamical effect of the core rotation is important, while in the configurational mixing formalism the direct interaction plays a significant role. The latter effect can also be taken into consideration in the collective model, but the dynamical effect cannot appear in the configurational mixing formalism.

v) For large nuclear deformations the configurational mixing method is no longer valid, but the collective model remains tenable, provided that the strong coupling approximation is used.

The above conclusions seem to provide the basis of the results obtained by Blin-Stoyle[3] through his extensive analysis of empirical evidence. Namely, the configurational mixing method is successful in interpreting the nuclear moments of nuclei except for those in which $150<A<190$ or $A>225$. In the latter case the nuclear deformation is believed to be considerable.

As an example in which the current collective model fails, we mention the magnetic moment of ^{209}Bi. This nucleus is composed of a closed shell and an extra-proton. Hence the collective deformation is not so large that the deviation from the Schmidt line may be accounted for only in terms of the dynamical effect. Thus one must take account of the configurational mixing due to the direct interaction. On the contrary, the magnetic moments of ^{151}Eu and ^{153}Eu cannot be explained in terms of the configurational mixing method, but the collective model predicts the moments in good agreement with the observed values. This is because these nuclei are deformed strongly, so that the dynamical effect alone play a dominant role in the deviation from the Schmidt line.

In conclusion we should like to notice that our method of dealing with the coupling of the collective motion to the shell structure is similar to the method developed by Inglis[18] and Araújo[20]. Inglis treated the collective coordinates as time dependent. This can be obtained from our formalism by transforming our Schrödinger representation of the collective coordinates into the Heisenberg representation. Inglis, Araújo and several workers in Copenhagen noticed that their theory of collective motion can describe the moment of inertia in relation to the shell structure. This effect is dropped in our theory because of the adiabatic approximation.*) In the present paper we are rather interested in the relation between the collective and shell models and leave the discussions of such problem to a forthcoming paper.

---

*) The implication of the adiabatic approximation was carefully examined by Tomonaga,[21] who succeeded in describing the moment of inertia in relation to shell structure by means of a method similar to ours.

References

1) M. G. Mayer, Phys. Rev. **78** (1950), 16, 22.
2) O. Haxel, J. H. D. Jensen and H. E. Suess, ZS. f. Phys. **128** (1950), 295.
3) R. J. Blin-Stoyle, Rev. Mod. Phys. **28** (1956), 75.
4) R. J. Blin-Stoyle, Proc. Phys. Soc. **A66**, (1953), 1158.
   R. J. Blin-Stoyle, and M. A. Perks, Proc. Phys. Soc. **A67** (1954), 885.
   M. A. Perks, Proc. Phys. Soc. **A68** (1955), 1083.
5) A. Arima and H. Horie, Prog. Theor. Phys. **12** (1954), 623.
   H. Horie and A. Arima, Phys. Rev. **99** (1955), 778.
6) T. Marumori, S. Suekane and A. Yamamoto, Prog. Theor. Phys. **16** (1956), 320; cited as M-S-Y.
7) V. F. Weisskopf, Helv. Phys Acta, **23** (1951), 187.
8) C. C. Morrison, H. Muirhead and W. G. V. Rosser, Phil. Mag. **44** (1953), 1326.
9) S. Hayakawa, M. Kawai and K. Kikuchi, Prog. Theor. Phys. **13** (1955), 415.
10) A. M. Lane and C. F. Wandel, Phys. Rev. **98** (1955), 1524.
    E. Clementel and C. Villi, Nuovo Cimento **2** (1955), 176.
11) E. Vogt, Phys. Rev. **101** (1956), 1792.
12) A. M. Lane, R. G. Thomas and E. P. Wigner, Phys. Rev. **98** (1955), 693.
13) H. Feshbach, C. E. Porter and V. F. Weisskopf, Phys. Rev. **96** (1954), 448.
14) M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (1955).
15) R. J. Eden and N. C. Francis, Phys. Rev. **99** (1955), 1326.
16) S. Tomonaga, Prog. Theor. Phys. **13** (1955), 467, especially § 6.
17) A. Bohr and B. R. Mottelson, Dan. Mat. -fys. Medd. **27** (1953), No. 16.
18) D. Inglis, Phys. Rev. **96** (1954), 1059.
    D. Inglis, Phys. Rev. **97** (1955), 701.
19) A. Bohr and B. R. Mottelson, Dan. Mat. -fys. Medd. **30** (1955), No. 1.
20) J. M. Araújo, Nuclear Physics **1** (1956), 259.
21) S. Tomonaga, Lecture at the annual meeting of the Physical Society of Japan, July (1956).