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Finite deformation of a class of rubber cylindrical tubes subjected to axial loads

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Abstract. In this paper, the problem of finite deformation of a cylindrical rubber tube with an internal rigid tube is examined, where the rubber tube is subjected to axial loads on its two ends. A reasonable mathematic model is formulated by using the nonlinear field theory. Then, the implicit solutions that describe the cases of tension and compression are derived. The influence of end loads, thickness and length on the finite deformation is discussed in detail. Numerical simulations are given simultaneously. It also shows that for a large domain of the middle part of the tube, the deformation is very close to a uniform case, but near the two ends of the tube, the change is very fast, which coincides with the mechanical backgrounds.

Introduction

In recent years, many experts and scholars are concerned with the problems of finite deformation of various kinds of structures which are composed of hyperelastic materials, such as the cavitation in the interior of solid spheres and cylinders, the growth of pre-existing micro-voids, the torsion and tension of cylinders, and so on. The detail overviews are given in [1, 2]. As we all know, the structures composed of hyperelastic materials, where the typical represents are rubber and rubber-like materials, have caused broad concerns for their widely use, such as rubber circles, seal rings, andi-skidding rings and rubber tubes, and so on. Klingbeil and Shield [3] investigated a deformable circular pad between two rigid end-plates subjected to vertical loads. Under reasonable assumptions, Dai [4] derived the analytical solutions of the problem of finite deformation of a neo-Hookean cylinder rod which is subjected to an end load. For hyperelastic tubes, Chadwick [5] proved the existence and uniqueness of the overturn solutions of a Mooney-Rivlin circular tube. Sierakowski [6] examined the stability of a rubber material tube with the initial pressure. Haughton and Orr [7] gave the stability analysis about the overturn of a tube composed of incompressible isotropic materials. Recently, Pamplona [8] derived the experimental and numerical results of a rubber tube under internal pressure, and the tube also has initial stresses. Fu and Ogden [9] gave the monograph on these sides. In this paper, we present some researches on the problem of finite deformation of a cylindrical rubber tube composed of the neo-Hookean material, where the inner surface of the tube is supported by a rigid tube. The rubber tube is subjected to axis loads on its two ends. Firstly, we formulate the reasonable mathematic model by the nonlinear field theory. Then, the implicit solutions of the cases of tension and compression are derived. Finally, numerical simulations are given and the effect of structure parameters and end loads on axial deformation of the tube is discussed in detail.

Mathematic description of the problem

For a rubber cylindrical tube composed of incompressible neo-Hookean hyperelastic materials, assume that its inner surface is supported by a rigid tube, which is sufficiently contacted and lubricated with the rubber tube. We are concerned with the finite deformation of the rubber tube under the axial loads q_0 on its two ends. Let (R,Θ,Z) and (r,θ,z) be the cylinder polar coordinates in the reference and the current configurations, respectively. Under the assumption of axially symmetric deformation, the general deformation field can be given by

$$r = r(R,Z), 0 < b < R \le a; \theta = \Theta, z = z(Z), -L \le Z \le L; \quad (1)$$

where $r(R,Z)$ and $z(Z)$ are the deformation functions to be determined; B and A are radii of the inner and the outer surfaces of the tube in the undeformed configuration; and the initial length is $2L$.

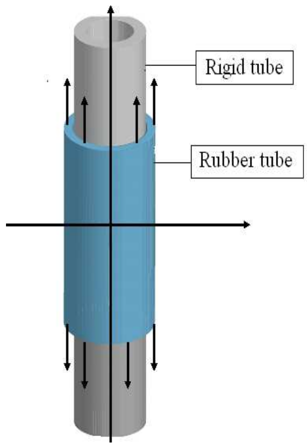

Fig. 1 Structure diagram

The orthogonal bases associated with the cylindrical polar coordinates are denoted by $\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z$ and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ in the reference and current placements, respectively. According to the nonlinear field theory [10], the deformation gradient tensor $\mathbf{F}$ and the left Cauchy-Green strain tensor $\mathbf{B}$ are respectively given by

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} = r_R \mathbf{e}_r \otimes \mathbf{E}_R + r_Z \mathbf{e}_r \otimes \mathbf{E}_Z + (r/R)\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + z_Z \mathbf{e}_z \otimes \mathbf{E}_Z, \quad (2)$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (r_R^2 + r_Z^2)\mathbf{e}_r \otimes \mathbf{e}_r + (r/R)^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + z_Z^2 \mathbf{e}_z \otimes \mathbf{e}_z + z_Z r_Z (\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r). \quad (3)$$

The three invariants of $\mathbf{B}$ are easily calculated from Eq. (3), which are given by

$$I_1 = (r/R)^2 + r_R^2 + r_Z^2 + z_Z^2, \; I_2 = (r/R)^2 (r_R^2 + r_Z^2 + z_Z^2) + r_R^2 z_Z^2, \; I_3 = z_Z^2 (r/R)^2 r_R^2. \quad (4)$$

In this paper, assuming that the tube is composed of the incompressible neo-Hookean material and the strain-energy function [11] is given by

$$W = \frac{\mu}{2}(I_1 - 3), \quad (5)$$

where μ is a material parameter, which denotes the usual infinitesimal shear modulus.

The Cauchy stress tensor $\boldsymbol{\sigma}$ and Piola-Kirchhoff tensor $\mathbf{S}$ associated with the incompressible neo-Hookean material are given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{B}, \; \mathbf{S} = \boldsymbol{\sigma}\mathbf{F}^{-T}, \quad (6)$$

and the nonzero components are as follows

$$\sigma_{rr} = \mu(r_R^2 + r_Z^2) - p, \; \sigma_{rz} = \sigma_{zr} = \mu z_Z r_Z, \; \sigma_{\theta\theta} = \mu(r/R)^2 - p, \; \sigma_{zz} = \mu z_Z^2 - p, \quad (7)$$

$$S_{rR} = \mu r_R - \frac{z_Z r}{R} p, \; S_{rZ} = \mu r_Z, \; S_{\theta\Theta} = \mu \frac{r}{R} - \frac{R}{r} p, \; S_{ZR} = \frac{r_Z r}{R} p, \; S_{zZ} = \mu z_Z - \frac{r_R r}{R} p, \quad (8)$$

where p is the hydrostatic pressure.

Under the assumption of axially symmetric deformation, the equilibrium equation $\mathbf{div}\,\mathbf{S} = 0$ in the absence of body force can be reduced to

$$(S_{rR})_R + (S_{rZ})_Z + R^{-1}(S_{rR} - S_{\theta\Theta}) = 0, \quad (9)$$

$$(S_{zR})_R + (S_{zZ})_Z + R^{-1} S_{zR} = 0. \quad (10)$$

Then, substituting Eq. (8) into Eqs. (9) and (10), using the condition of incompressibility $I_3 = 1$, we obtain

$$\mu[r_{RR} + r_{ZZ} + R^{-2}(r_R R - r)] - z_Z r R^{-1} p_R = 0, \quad (11)$$

$$\mu z_{ZZ} + r R^{-1}(r_Z p_R - r_R p_Z) = 0. \quad (12)$$

Since the inner surface and the outer surface of the tube are traction free, the following equations are valid, i.e.,

$$S_{rR}(B)=0\,,\ S_{rR}(A)=0\,,\ S_{ZR}(B)=0\,,\ S_{ZR}(A)=0\,. \tag{13}$$

The tube is subjected to axial loads q_0 on its two ends, the force condition becomes

$$\frac{1}{\pi(A^2-B^2)}\int_0^{2\pi}\int_B^A[\sigma_{ZZ}]_{Z=\pm L}RdRd\Theta=-q_0\,. \tag{14}$$

Assume that the two ends of the tube are fixed on rigid bodies and that there is no lateral deformation on the ends,

$$r(R,L)=r(R,-L)=R\,. \tag{15}$$

Since the deformation is finite and the inner surface of the rubber tube is supported by a rigid tube, the inner radius of the rubber tube is invariable, namely,

$$r(B,Z)=B\,. \tag{16}$$

So far, the mathematic model of the problem is formulated by governing equations (9) ~ (12) and the boundary conditions (13) ~ (16).

Solution of the problem

Since the material is assumed to be incompressible, from the incompressibility condition $I_3=1$ we obtain

$$r(R,Z)=z_Z^{-1/2}(R^2+C^2)^{1/2}\,. \tag{17}$$

It is easy to show that the following equations are valid,

$$r_R(R,Z)=R/(rz_Z)\,,\ r_{RR}(R,Z)=\frac{1}{z_Z^2}\frac{C^2}{r^3}\,,\ r_{ZZ}(R,Z)=\frac{3}{4}z_{ZZ}^2z_Z^{-2}r-\frac{1}{2}z_{ZZZ}z_Z^{-1}r\,. \tag{18}$$

Integrating Eq. (11) with respect to R with Eq. (18) yields

$$p(R,Z)=\frac{\mu}{z_Z}\int[r_{RR}+r_{ZZ}+R^{-2}(r_RR-r)]\frac{R}{r}dR=\frac{1}{2}\mu\left(-\frac{C^2}{z_Z^2r^2}+\frac{3}{4}\frac{z_{ZZ}^2}{z_Z^2}r^2-\frac{1}{2}\frac{z_{ZZZ}}{z_Z}r^2+\frac{1}{z_Z}\ln r^2\right)-\frac{\mu}{z_Z}\ln R+D_1\,. \tag{19}$$

Using Eq. $(8)_1$ and the boundary condition $(13)_2$, we have

$$p(A,Z)=\mu r_R^2(A,Z)=\frac{\mu}{z_Z^2}\frac{A^2}{r_A^2}=\frac{\mu}{z_Z}\frac{A^2}{A^2+C^2}\,. \tag{20}$$

So that

$$D_1=\frac{\mu}{z_Z}\frac{A^2}{A^2+C^2}-\frac{1}{2}\mu\left(-\frac{C^2}{z_Z^2r_A^2}+\frac{3}{4}\frac{z_{ZZ}^2}{z_Z^2}r_A^2-\frac{1}{2}\frac{z_{ZZZ}}{z_Z}r_A^2+\frac{1}{z_Z}\ln r_A^2\right)+\frac{\mu}{z_Z}\ln A\,, \tag{21}$$

where $r_A=r(A,Z)=z_Z^{-1/2}(A^2+C^2)^{1/2}$.

Multiplying Eq. (10) by R, and integrating it with respect to R from B to A yields

$$\left(\int_B^A S_{zZ}RdR\right)_Z=0. \tag{22}$$

Integrating Eq. (22) with respect to Z, and using Eq. (19), we obtain

$$\frac{1}{2}z_Z(A^2-B^2)-\frac{1}{4z_Z^2}\frac{2A^2+C^2}{A^2+C^2}(A^2-B^2)+\frac{3}{32}\frac{z_{ZZ}^2}{z_Z^4}(A^2-B^2)^2-\frac{1}{16}\frac{z_{ZZZ}}{z_Z^3}(A^2-B^2)^2-\frac{1}{4}\frac{B^2}{z_Z^2}\ln\frac{(A^2+C^2)B^2}{(B^2+C^2)A^2}=D_2\,. \tag{23}$$

Upon using Eqs. $(7)_4$, $(8)_5$ and the boundary condition (14), we have

$$D_2 = -\frac{q_0(A^2 - B^2)}{2}. \tag{24}$$

Multiplying Eq. (24) by z_{ZZ}, and integrating it with respect to Z yields

$$\frac{1}{4}z_Z^2(A^2-B^2)+\frac{1}{4z_Z}\frac{2A^2+C^2}{A^2+C^2}(A^2-B^2)-\frac{1}{32}\frac{z_{ZZ}^2}{z_Z^3}(A^2-B^2)^2+\frac{1}{4}\frac{B^2}{z_Z}\ln\frac{(A^2+C^2)B^2}{(B^2+C^2)A^2}=-\frac{q_0(A^2-B^2)}{2\mu}z_Z+D_3, \tag{25}$$

where D_3 is an integration constant.

Using the boundary condition (16), we have

$$C^2 = (z_Z - 1)B^2. \tag{26}$$

Let

$$\alpha = B/L,\ \delta = A/B,\ G = Z/L,\ g = z/L,\ Q = \frac{q_0(\delta^2-1)}{2\mu},\ h = g_G = z_Z. \tag{27}$$

Substituting Eqs.(26) and (28) into Eq. (25) yields

$$\pm\alpha(\delta^2-1)h_G\left(\frac{1}{H(h(w),\delta)}\right)^{1/2} = 1, \tag{28}$$

where

$$H(h(G),\delta) = 8h^5(\delta^2-1) + 8h^2\frac{2\delta^2+h-1}{\delta^2+h-1}(\delta^2-1) + 8h^2\ln\frac{\delta^2+h-1}{h\delta^2} + 32Qh^4 - 32D_3h^3. \tag{29}$$

For the deformation is axially symmetric, the function $z(Z)$ must be symmetric at $Z=0$ ($G=0$), so the values of $h(G)$, namely, z_Z, are equal at $G=\pm Z/L$. It means that there must be an extreme value of $h(G)$ at $G=0$, therefore,

$$H(h(0),\delta)=8h(0)^5(\delta^2-1)+8h(0)^2\frac{2\delta^2+h(0)-1}{\delta^2+h(0)-1}(\delta^2-1)+8h(0)^2\ln\frac{\delta^2+h(0)-1}{h(0)\delta^2}+32Qh(0)^4-32D_3h(0)^3=0 \tag{30}$$

Next, we discuss how to choose the sign of Eq. (28), which corresponds to the cases of tension and compression, respectively.

The tension case

In this part, the tension case is discussed. It means that $q_0<0$ （$Q<0$） and $h(G)\ge 1$. As $0\le G\le 1$, integrating Eq. (28) from G to 1 by variable separation yields

$$-\alpha(\delta^2-1)\int_{h(G)}^{1}\left(\frac{1}{H(h(w),\delta)}\right)^{1/2}dh(w) = 1-G. \tag{31}$$

As $-1\le G\le 0$, integrating Eq. (28) from G to -1 by variable separation yields

$$\alpha(\delta^2-1)\int_{h(G)}^{1}\left(\frac{1}{H(h(w),\delta)}\right)^{1/2}dh(w) = -1-G. \tag{32}$$

Eqs. (31) and (32) contain the integration constant D_3, and they all satisfy

$$\alpha(\delta^2-1)\int_{h(0)}^{1}\left(\frac{1}{H(h(w),\delta)}\right)^{1/2}dh(w)=-1\,. \tag{33}$$

The integration constant D_3 can be determined by Eqs. (30) and (33). Then, in the tension case, the implicit solutions (31) and (32) are obtained.

Now, the influences of the dimensionless load Q, the thickness δ and the length α of the rubber tube on the solutions are considered.

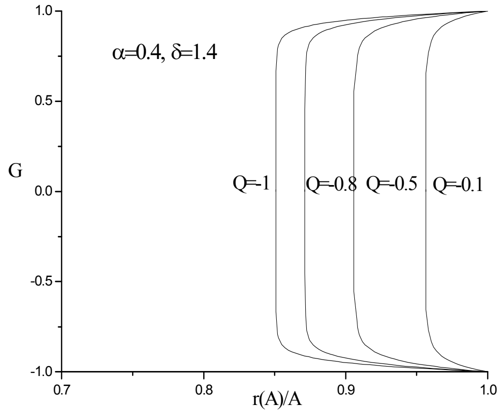

Fig. 2 ($r(A)/A$, G)-curve as $\alpha=0.4$, $\delta=1.4$

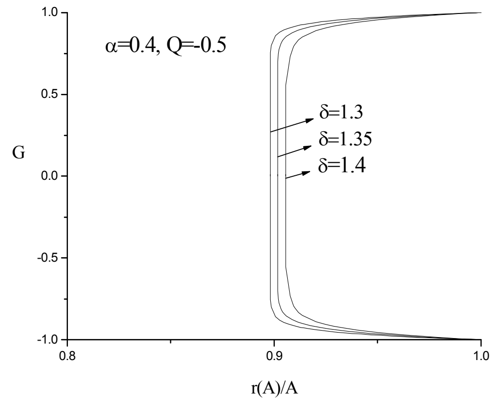

Fig. 3 ($r(A)/A$, G)-curve as $\alpha=0.4$, $Q=-0.5$

For the fixed values of δ and α, we have plotted the solution curves of the lateral surface $r(A)/A$ in Fig. 2 for different dimensionless loads Q. It can be seen that as Q increases, the lateral surface of the tube shrinks more and more, which is as expected. However, in the middle part, the variation is relatively small. Near the two ends of the tube, the change is very fast.

In Fig. 3, the change of the lateral surface about the relative thickness δ is plotted for fixed Q and α. We find from the diagram that as the tube is relatively thicker and thicker (δ is bigger and bigger), the change of the wall is smaller and smaller. And the middle part of the tube varies small along G ~axis, the two ends change very fast also.

The change of the lateral surface of the tube about the relative length α is plotted in Fig. 4. It can be found that as the tube is relatively longer and longer (α is bigger and bigger), the change of the wall is smaller and smaller. And the whole change of the tube is similar to the two front situations.

The compression case

In this part, we consider the compression case, it means that $q_0>0$ ($Q>0$) and $h(G)\le 1$. Then

$$\alpha(\delta^2-1)\int_{h(G)}^{1}\left(\frac{1}{H(h(w),\delta)}\right)^{1/2}dh(w)=1-G\,,\ \ 0\le G\le 1\,. \tag{34}$$

$$-\alpha(\delta^2-1)\int_{h(G)}^{1}\left(\frac{1}{H(h(w),\delta)}\right)^{1/2}dh(w)=-1-G\,,\ \ -1\le G\le 0\,. \tag{35}$$

Eqs. (34) and (35) are the implicit solutions of the compression case with an integration constant D_3. Using the same methods as the tension case to determine D_3, then, the solutions of the problem are obtained.

The change of the lateral surface $r(A)/A$ with the dimensionless load Q is shown in Fig.5. It can be found that as Q increases, the lateral surface of the tube expands more and more, and in the middle part, the variation is relatively small. Near the two ends of the tube, the change is very fast.

The influence of the thickness δ and the length α of the rubber tube on the lateral surface $r(A)/A$ is similar to those of the tension case.

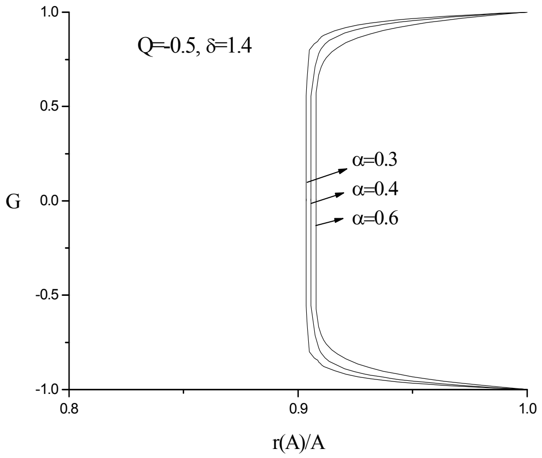

Fig. 4 ($r(A)/A$, G)-curve as $\delta = 1.4$, $Q = -0.5$

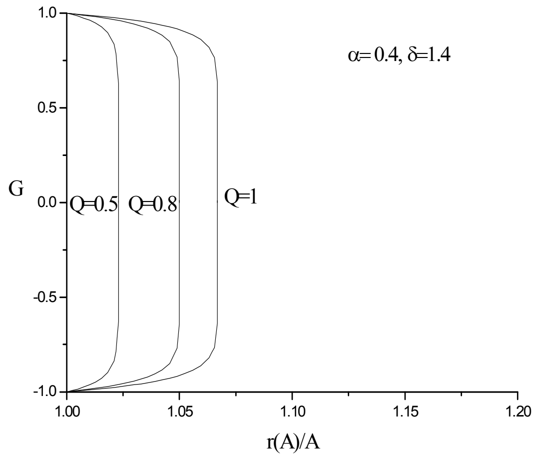

Fig. 5 ($r(A)/A$, G)-curve as $\alpha = 0.4$, $\delta = 1.4$

Conclusions

In this paper, the finite deformation problem of a rubber circular cylinder tube with an internal tube is examined, where the rubber tube is subjected to axial loads on its two ends. The implicit solutions of the tension and compression cases are derived respectively. Some conclusions are obtained as follows.

(i) For the tension case, as the axis load increases, the lateral surface of the tube shrinks more and more. For the middle part of the tube along with the axial direction, the deformation tends to a uniform case, but near the two ends of the tube, the change is very fast. The surface changes smaller and smaller when the initial tube is relatively thicker and longer.

(ii) For the compression case, the lateral surface of the tube expands more and more as the axial load increases, and the variation is relatively small in the middle part. Near the two ends, the change is very fast. The influences of thickness and length of the rubber tube on the lateral surface are similar to those of the tension case.

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