FUNDAÇÃO GETULIO VARGAS
ESCOLA DE PÓS GRADUAÇÃO EM ECONOMIA

PAULO CÉSAR COIMBRA LISBÔA

ON THE CONTAMINATION OF CONFIDENCE

Rio de Janeiro
2009

PAULO CÉSAR COIMBRA LISBÔA

ON THE CONTAMINATION OF CONFIDENCE

Tese para obtenção do grau de doutor apresentada à
Escola de Pós Graduação em Economia

Área de concentração: Economia Matemática

Orientador: Sérgio Ribeiro da Costa Werlang

Rio de Janeiro
2009

PAULO CÉSAR COIMBRA LISBÔA

ON THE CONTAMINATION OF CONFIDENCE

Tese apresentada à Escola de Pós Graduação em Economia para obtenção do grau de doutor. Área de concentração: Economia Matemática.

E aprovada em 30/11/2009.

Pela comissão organizadora:

Sérgio Ribeiro da Costa Werlang
Itaú Unibanco e
Escola de Pós Graduação em Economia da Fundação Getulio Vargas (EPGE/FGV-RJ)

José Heleno Faro
Centro de Desenvolvimento e Planejamento Regional de Minas Gerais da Universidade Federal de Minas Gerais (CEDEPLAR/UFMG)

Victor Filipe Martins-da-Rocha
Escola de Pós Graduação em Economia da Fundação Getulio Vargas (EPGE/FGV-RJ)

Humberto Luiz Athaíde Moreira
Escola de Pós Graduação em Economia da Fundação Getulio Vargas (EPGE/FGV-RJ)

Wilfredo Fernando Leiva Maldonato
Pós-Graduação em Economia da Universidade Católica de Brasília (UCB)

Rio de Janeiro
2009

Dedicatory

To my parents.

Acknowledgements

First of all I wish to express my gratitude to Sérgio Ribeiro da Costa Werlang, who present me the subject and I am also deeply indebted to him, for his friendship, guidance, support, collaboration and encouragement through my education.

I also wish to express my gratitude to José Heleno Faro and I am also deeply indebted to him, for his friendship as well as many discussions, collaboration, encouragement to explore the axiomatization of the contamination of confidence, which is on Chapter 4 of this thesis, and for carefully revised preliminary versions of this thesis with a lot of comments that greatly improved the exposition of this research.

I also wish to express my gratitude to Victor Filipe Martins-da-Rocha for many discussions and for carefully revised preliminary versions of this thesis as well as for comments that greatly improved the exposition of this research.

I also wish to express my gratitude to Humberto Luiz Athaíde Moreira for many discussions as well as encouragement to write about portfolio inertia and contamination, which is on Chapter 6 of this thesis.

I gratefully thank the examiners of my doctoral thesis in Economics: José Heleno Faro, Victor Filipe Martins-da-Rocha, Humberto Luiz Athaíde Moreira, Wilfredo Fernando Leiva Maldonato and, most of all, Sérgio Ribeiro da Costa Werlang for their invaluable comments and suggestions towards this research.

The first part of this thesis has the purpose to provide a simple set of behavioral axioms under which the decision maker's preference is represented by the Choquet expected utility with contamination of confidence.

It starts with a survey on the foundations of Knightian uncertainty theory which emerged from my master dissertation in Economics at EPGE/FGV-RJ (Coimbra (2003)) and from a former version of the article "Contamination of confidence". I also appreciate Aloísio Araújo, Daniel Gottlieb and to many colleagues from EPGE/FGV-RJ for their comments and suggestions.

In the sequel it is presented the characterization of a special subset of convex capacity that can be represented as the contamination of confidence of some probability measure as well as its axiomatization. I also appreciate Aloísio Araújo, Carlos da Costa, Daniel Gottlieb, Luis Braido, Rafael Costa Lima, Walter Novaes and to many colleagues from EPGE/FGV-RJ as well as audiences at academic seminars at EPGE/FGV-RJ (april and july of 2009) for their comments and suggestions.

Early versions of this article were circulated with the titles: "Choquet Integral with contaminations" and "Integral representation with convex capacities that are *squeezes* of (additive) probability measures" and I also appreciate audiences at the following conferences: XXIX Stochastic Process and Applications and VII Brazilian School on Probability, Angra dos Reis, RJ, Brazil (2003); First Brazilian Conference on Statistical Modeling in Insurance and Finance, Ubatuba, SP, Brazil (2003); 2003 Latin American Meeting of the Econometric Society, held in Panama City, Panama (2003); The Second Word Congress of the Game Theory Society, Marseille, France (2004) and XXX Meeting of the Brazilian Econometric Society, Salvador, BA, Brazil (2008) as well as to academic seminars: EPGE/FGV-RJ (2003) and EPGE/FGV-RJ (2006). The axiomatization presented here initially emerged as an article with the title: "An axiomatic approach to generalized ψ-contamination" and I also appreciate feedbacks from participants of the 2007 Latin American Meeting of the Econometric Society, held in Bogota, Colombia (2007) as well as to audiences at XXIX Meeting of the Brazilian Econometric Society, Recife, PE, Brazil (2007). The mathematics around the main theorem was explored isolated in an article with the title: "From probabilities to convex capacities" and I also appreciate audiences at the 2008 Latin American Meeting of the Econometric Society, held in Rio de Janeiro, Brazil (2008).

The second part of this thesis consists of two economic applications of contamination of confidence: the first of them generalizes Dow and Werlang's existence Theorem of Nash equilibrium under uncertainty (which enables to present an explicit solution to the the paradox on which players in a finitely repeated prisoners' dilemma breaks down backward induction) and the other studies the impact of contamination in the portfolio choice.

With respect to the application in game theory, which is based in the article "Nash Equilibrium under uncertainty: an explicit solution to the paradox of breaking down backward induction", I also appreciate to Aloísio Araújo, Carlos da Costa, Daniel Gottlieb, Luis Braido, Rafael Costa Lima and to many colleagues from EPGE/FGV-RJ as well as audiences at academic seminars at EPGE/FGV-RJ (july of 2009) for their comments and suggestions. An early version of this article was circulated with the title "Nash equilibrium under Knightian uncertainty: a generalization of the existence theorem" and I also appreciate audiences at First Brazilian Workshop of the Game Theory Society, São Paulo, SP, Brazil (2002), at XXV Meeting of the Brazilian Econometric Society, Porto Seguro, BA, Brazil (2003), at the Second Word Congress of the Game Theory Society, Marseille, France (2004), at the 2004 Latin American Meeting of the Econometric Society, Santiago, Chile (2004)(special thanks for the comments of Walter Novaes), at the VIII Brazilian School on Probability, Ubatuba, SP, Brazil (2004) and VI Latin American Journey of Economic Theory, Rio de Janeiro (2006) and also at academic seminars: EPGE/FGV-RJ (2003) and EPGE/FGV-RJ (2006).

With respect to the portfolio inertia, which is based in the article "Portfolio inertia and contamination of confidence", I also appreciate Carlos da Costa, Luis Braido, Rafael Costa Lima and to many colleagues from EPGE/FGV-RJ as well as audiences at academic seminars at EPGE/FGV-RJ (july of 2009) for their comments and suggestions.

Needless to say, I am responsible for any remaining mistakes.

I want to acknowledge Getulio Vargas Foundation (FGV/EPGE-RJ), Pontifical Catholic University (PUC-RJ) and IBMEC Business School (IBMEC-RJ) for the opportunity to give lectures of Economics during my studies. These opportunities provides me partial financial support for my studies as a whole and for this research in particular.

I greatly acknowledge partial financial support for this research provided by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (Capes) and for Getulio Vargas Foundation Fellowships.

Obviously the usual disclaimer applies.

"The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself."

Knight (1921, p. 227)

Abstract

[Resumo]

Contaminação da confiança é um caso especial de incerteza Knightiana ou ambiguidade na qual o tomador de decisões está diante de não apenas uma única distribuição de probabilidades, mas sim de um conjunto de distribuições de probabilidades. A primeira parte desta tese tem o propósito de fornecer uma caracterização da contaminação da confiança e então apresentar um conjunto de axiomas comportamentais simples sob os quais as preferências de um tomador de decisões é representada pela utilidade esperada de Choquet com contaminação da confiança. A segunda parte desta tese apresenta duas aplicações econômicas de contaminação da confiança: a primeira delas generaliza o teorema de existência de equilíbrio de Nash de Dow e Werlang (o que permite apresentar uma solução explícita para o paradoxo segundo o qual os jogadores de um jogo do dilema dos prisioneiros com um número finito de repetições não agem de acordo com o esperado pelo procedimento da indução retroativa) e a outra estuda o impacto da contaminação da confiança na escolha de portfolio.

Palavras-chave e frases: *incerteza Knigthiana*, *capacidade (probabilidade não-aditiva)*, *ε-contaminação*, *integral de Choquet*, *maximização da utilidade esperada*, *teoremas de representação*, *jogos não cooperativos*, *conceitos de equilíbrio*, *escolha de portfolio* e *inércia de portfolio*.

Números de classifica ção JEL: $C72$, $D81$, $G11$.

Abstract

Contamination of confidence is a special case of Knightian uncertainty or ambiguity in which the decision maker faces not simple probability measure but a set of probability measures. The first part of this thesis has the purpose to provide a characterization of the contamination of confidence and then present a simple set of behavioral axioms under which the decision maker's preference is represented by the Choquet expected utility with contamination of confidence. The second part of this thesis presents two economic applications of the contamination of confidence: the first of them generalizes Dow and Werlang's existence Theorem of Nash equilibrium under uncertainty (which enables to present an explicit solution to the paradox on which players in a finitely repeated prisoners' dilemma breaks down backward induction) and the other studies the impact of the contamination of confidence in the portfolio choice.

Keywords and phrases: *Knigthian uncertainty*, *capacity (non-additive probability measure)*, *ε-contamination*, *Choquet integral*, *expected utility maximization*, *representation theorems*, *noncooperative games*, *equilibrium concepts*, *portfolio choice* and *portfolio inertia*.

JEL classification numbers: $C72$, $D81$, $G11$.

Contents

Chapter 1

Introduction

The phenomena in the natural sciences, specially the physical phenomena, are governed by laws which are explored by the principles of deduction, induction and experimental methodology (whose base was outlined by Montaigne (1595) and whose original wording is due to Bacon (1620)). According to the experimental methodology, the general conditions that govern the structure of some reality in study establish the basis of a model, and then the phenomenon is isolated to analyze it better (*ceteris paribus*), and after having understood the circumstances are established the laws and principles that are universally valid.

Social phenomena, in turn, are much more complex and difficult to modeling, which complicates the analysis because the results depend on the interactions between humans, which are not easily predictable not only because they differ in their characteristics but also because their reactions are not homogeneous even among those who are part of a group with very similar characteristics. And besides all, the circumstances which are objects of study rarely are repeated and hence the difficulty in establishing laws that are universally valid for all circumstances.

The model that has been proven to be the most suitable for the study of social phenomena whose circumstances rarely repeats itself is one who consider that events are subject not only to *risk* but also the *uncertainty*. The first distinction between *risk* and *uncertainty* was done by Frank Knight.

> *"The* $(\cdots)$ *difference between* $(\cdots)$ *risk and uncertainty, is that in the former the distribution of the outcome in a group of instance is known (either through a calculation* a priori *or from statistics of past experience), while in the case of uncertainty that is not true, the reason being in general that it is impossible to form a group of instances, because the situation dealt with is in high degree unique."*
>
> *Knight (1921, p. 233)*

So, according to Knight (1921) *risk* refers to situations where the decision maker can quantify probabilities whereas *uncertainty* refers to situations where she can not quantify probabilities. Keynes (1921) was among economists who first agreed with this idea.

Although Knight's distinction in the begins of the 20's, many axiomatic models for decision making under uncertainty emerged years late, following the Bayesian interpretation on which uncertainty could be re-interpreted as risk using the notion of subjective probabilities. This idea was first suggested in the work of Ramsey (1931) and was followed by the suggestion of using a subjective expected value by de Finetti (1937), but the most important contribution was due to Savage (1954) who presented an axiomatic approach to the decision making under uncertainty that lead to the criterion of maximization of expected utility on which the expectation operation is carried out with respect to a prior subjective probability derived uniquely from the decision maker's preference over acts, also known as subjective expected utility (SEU). Anscombe and Aumann (1963) proposed a simplification of Savage's axiomatic model on which each act assigns to each state an objectively lottery (objective probability) over deterministic outcomes (roulette lottery) and so the uncertainty concerned on which state would occur, a subjective probability (also called as horse lottery). So, in the Anscombe-Aumann framework, the elements of choice are called lottery-acts.

One of the most important objection to this line of reasoning (the idea that any uncertainty can be reduced to risk using the notion of subjective probabilities) was due to Ellsberg (1961) who proposed well known experiments on which shows that people often fail to behave according with the Bayesian approach.

The following version of Ellsberg Paradox is due to Schmeidler:

> *"There are two urns each containing one hundred balls. Each ball is either red or black. In urn I there are fifty balls of each color and there is no additional information about urn II. One ball is chosen at random from each urn. There are four events, denoted IR, IB, IIR, IIB, where IR denotes the event that the ball chosen from urn I is red, etc. On each of the events a bet is offered:* \$100 *if the event occurs and zero if it does not. According to Ellsberg most decision makers are indifferent between bets on IR and betting on IB and are similarly indifferent between bets on IIR and IIB. It may be that the majority are indifferent among all four bets. However, there is a nonnegible proportion of decision makers who prefer every bet from urn I (IB or IR) to every bet on urn II (IIB or IIR). These decision makers cannot represent their beliefs with respect to the occurrence of uncertain events through an additive probability."*
>
> *Schmeidler (1989, p. 572)*

By the end of 80's two axiomatic approaches for decision making that extends the subjective expected utility and incorporates the difference between *risk* and *uncertainty* emerged: the Choquet expected utility (CEU) which first appeared in a seminal paper of Schmeidler (1989) in a lottery-acts framework à la Anscombe-Aumann and also presented

by Gilboa (1987) in the context of purely subjective probabilities and acts framework à la Savage; and the maxmin expected utility (MEU), due to Gilboa and Schmeidler (1989) in a lottery-acts framework à la Anscombe-Aumann. Recently, Casadesus-Masanell and Ozdenoren (2000) presented the maximin expected utility (MEU) in an environment where the elements of choice are Savage acts and Alon and Schmeidler (2008) in the context of purely subjective probabilities.

We will restrict attention to the axiomatizations in a lottery-acts framework à la Anscombe-Aumann: the Anscombe and Aumann (1963)'s subjective expected utility (SEU) model, the Schmeidler (1989)'s Choquet expected utility (CEU) model and the Gilboa and Schmeidler (1989)'s maximin expected utility (MEU) model.

The contamination of confidence is a special case of Knightian uncertainty or ambiguity in which the decision maker faces not a simple probability measure but a set of probability measures.

The first part of this thesis has the purpose to provide a simple set of behavioral axioms under which the decision maker's preference is represented by the Choquet expected utility with contamination of confidence. The axiomatization presented in this part extends not only the Anscombe and Aumann (1963)'s axiomatization but also the Nishimura and Ozaki (2006)'s axiomatization (on ε-contamination of confidence).

The organization of the chapters of the first part is as follows. In Chapter 2 we present the foundations of Knightian uncertainty theory. It may skipped by the reader who is already familiarized with the literature of Knightian uncertainty theory and capacity integration. Chapter 3 investigate the properties of a special subset of convex capacities, which we will call as generalized simple capacities and in Chapter 4 it is presented an axiomatization of the use of the contamination of confidence. New axioms and a Theorem that extends Anscombe-Aumann and Nishimura-Ozaki´s axiomatizations are presented.

The second part of this thesis presents two economic applications with the use of the contamination of confidence: the generalization of Nash equilibrium under uncertainty through the use of the contamination of confidence and the problem of inertia in the portfolio choice.

The first application consists in present a definition of Nash equilibrium for two-person normal-form games in the presence of Knightian uncertainty that extends the notion which is due to Dow and Werlang (1994). Considering the use of contaminations (instead of ε-contaminations) it is possible to explain precisely the paradox on which players do not backward induct in the finitely repeated prisoners' dilemma. It is also possible to find all pure-strategy Nash equilibria under uncertainty of any two-person finite normal form game and much more equilibria than in Dow and Werlang (1994)'s paper.

The second application extends to the case of contamination of confidence Asano (2008)'s analysis of investors' portfolio selection problem in a two-period dynamic model of Knightian uncertainty (his analysis was based on ε-contamination of confidence).

The organization of the chapters of the second part is as follows. Chapter 5 presents the generalization of Dow and Werlang (1994)'s notion of Nash equilibrium for two-

person normal-form games in the presence of Knightian uncertainty which enables to solve the paradox of breaking down backward induction in the finitely repeated prisoners' dilemma. Finally, Chapter 6 we discuss about portfolio inertia with the contamination of confidence.

Chapter 7 concludes.

The proofs are presented in the Appendix.

Part I

An axiomatic approach to contamination of confidence

Chapter 2

On capacity integration

This Chapter has the purpose to review the foundations of capacity integration, which is the basis of the Knightian uncertainty theory. It is also presented the special case of a simple capacity with parameter ε (also known as ε-contamination).

The organization of this Chapter is as follows. In Section 2.1 we provide basic statements on capacities, including discussions on the support, the uncertainty aversion measure, the core and update rules. In Section 2.2 we review the basic statements on capacity integration, mostly based on Schmeidler (1986). Section 2.3 presents the case of a simple capacity with parameter ε (also known as ε-contamination).

2.1 Basic statements on capacities

Let (S, Σ) be a measurable space, where S denote the set of *states of nature (world)*, which we assume to be nonempty and endowed with an algebra Σ of subsets called *events*.[1]

Let v be a real-valued set-function $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$.

C. 1) v is *normalized* if $v(S) = 1$;

[1]The algebra Σ satisfies:

i) $\emptyset, S \in \Sigma$, where $\emptyset$ denotes the empty set;

ii) $E \in \Sigma$ implies $E^c \in \Sigma$, where $E^c = \{s \in S : s \notin E\}$, for all $E \in \Sigma$;

iii) $\{E_i\}_{i=1}^n \subseteq E$ implies $\bigcup_{i=1}^n E_i \in \Sigma$.

C. 2) v is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in \Sigma$.[2];

C. 3) v is *convex* if for every $E, F \in \Sigma$, $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$; and

C. 4) v is *additive* if $v(E \cup F) = v(E) + v(F)$ for all $E, F \in \Sigma$ with $E \cap F = \emptyset$.

If v is *normalized* and *monotone* then v is a *capacity*. Let V denote the set of all capacities on (S, Σ).[3] [4]

If v is *normalized* and *convex* then v is a *convex capacity*.[5] Let Λ denote the set of all convex capacities on (S, Σ), a subset of the set of capacities (i.e., $\Lambda \subsetneq V$):

$$\Lambda = \{v \in V : \ v(E \cup F) + v(E \cap F) \geq v(E) + v(F), \quad \forall E, F \in \Sigma\}.$$

If v is *normalized* and *additive* then v is a *probability measure*.[6] [7] Let Δ denote the set of all probability measures on (S, Σ), a subset of the set of convex capacities (i.e., $\Delta \subsetneq \Lambda$):

$$\Delta = \{v \in \Lambda \ : v(E \cup F) = v(E) + v(F), \quad \forall E, F \in \Sigma \text{ with } E \cap F = \emptyset\}.$$

Note that $\Delta \subsetneq \Lambda \subsetneq V$.

Example 1. Let $S = \{s_1, s_2\}$, let $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be defined by:

$$v(s_1) = \alpha;\ v(s_2) = \beta,\ \alpha,\ \beta \in [0, 1];\ \text{and}\ v(S) = 1.$$

Note that:

i) if $\alpha + \beta = 1$ then v is a probability measure;

ii) if $\alpha + \beta \leq 1$ then v is a convex capacity; and

iii) if $\alpha,\ \beta \in [0, 1]$ then v is a capacity.

[2]In this thesis the following notation will be used: $E \subseteq F$ means that the set E may not be a proper set of F (i.e., $E = F$ is possible) and $E \subsetneq F$ means that E is a proper set of F (i.e., $E \neq F$).

[3]It is easy to prove that if $v \in V$ then $v(E) \in [0, 1], \quad \forall E \in \Sigma$.

[4]If S is finite then V can be considered as a linear set (over $\mathbb{R}$) with the natural (pairwise) operations.

[5]It is easy to prove that if v is *normalized* and *convex* then v is *monotone*.

[6]If S is finite and if v is *normalized* and *additive* then v is a *finitely probability measure*.

[7]It is easy to prove that if v is *normalized* and *additive* then v is *convex*.

2.1.1 The support of a capacity

The notion of support of a capacity that it will be used here is the same notion used in Dow and Werlang (1994). According to them the interpretation of an event that has an associated capacity equal to zero is very similar to the case of a probability measure: it is an impossible event (a miracle). However, the fact that the capacity associated to an event is equal to zero does not imply that the complementary event has the capacity associated equal to one (which is the case if the capacity is a probability measure). But, if the capacity associated to the complementary event on which the capacity associated is equal to zero is positive, then this imply that this event has relatively infinitely more chance to happening than the original event.

Let v be a capacity. An event $F \in \Sigma$ is said to be a *support* of a capacity v if it satisfies:

i) $v(F^c) = 0$; and

ii) for all $E \subsetneq F$: $v(E^c) > 0$.

Marinacci (2000) shows that the support associated to any capacity $v \in V$ is a nonempty set, but according to this definition it might well not be an unitary set.

It is possible to define a nonempty set of events, denoted by $supp(v)$, which contains all supports of a capacity v:

$$supp(v) = \{F \in \Sigma : \ v(F^c) = 0 \text{ and } \quad \forall E \subsetneq F \Rightarrow v(E^c) > 0\}.$$

The following example is due to Marinacci (see Marinacci (2000, p. 205)) and illustrates that there may exist the possibility that some capacities have more than one event as the support (and more, in the example all events in $supp(v)$ have zero mass).

Example 2. Let $S = \{s_1, s_2, s_3\}$, let the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be defined by:

$v(\{s_i\}) = 0 = v(\{s_2; s_3\})$, $i = 1, 2, 3$; $v(\{s_1, s_2\}) = \alpha = v(\{s_1, s_3\})$, $\alpha \in (0, 1]$; and $v(S) = 1$.

If $\alpha \in (0, \frac{1}{2}]$ then v is a convex capacity and if $\alpha \in (0, 1]$ then v is a capacity.

Note that $supp(v) = \{\{s_1\}, \{s_2, s_3\}\}$ and note also that $v(\{s_1\}) = 0 = v(\{s_2, s_3\})$.

Marinacci (2000, p. 204), presents the conditions that should be satisfied to supp(v) be an unitary set.

Let v be a capacity. The following conditions are equivalent:

i) $\#supp(v) = 1$; and

ii) $v(E \cup \{s\}) = v(E) + v(\{s\})$ for all $E \in \Sigma$ and for all $s \in S$ with $v(E) = v(\{s\}) = 0$.

2.1.2 The uncertainty aversion measure of a capacity

If v is a convex capacity (i.e., $v \in \Lambda$) but is not a probability measure (i.e., $v \notin \Delta$) then there exists at least a pair $E, F \in \Sigma$ such that: $v(E \cup F) + v(E \cap F) > v(E) + v(F)$. In particular, if $F = E^c$ then $v(E) + v(E^c)$ may be less than 1, implying that not all additive mass is allocated to an event and its complement. This is the basis of the definition of the uncertainty aversion measure, as firstly suggested by Schmeidler (1989).

Let v be a capacity and consider an event E (i.e., $E \in \Sigma$). The *uncertainty aversion measure* of a capacity v at event E is defined by:[8]

$$c(v, E) = 1 - v(E) - v(E^c).$$

Marinacci (2000) presents the properties that should be satisfied by the uncertainty aversion measure associated to a convex capacity. Let v be a convex capacity on (S, Σ) and consider events E and F (i.e., E, $F \in \Sigma$), then:

i) $c(v, \emptyset) = 0 = c(v, S)$;

ii) $c(v, E) = c(v, E^c)$, for all $E \in \Sigma$; and

iii) *concavity*: for every $E, F \in \Sigma$, $c(v, E \cup F) + c(v, E \cap F) \leq c(v, E) + c(v, F)$.

As in Marinacci (2000), we can define the Knightian uncertainty level as a set-function $\psi : \Sigma \to [0, 1]$ that satisfies the same properties of the uncertainty aversion measure associated to a convex capacity.

A set-function $\psi : \Sigma \to [0, 1]$ is a *Knightian uncertainty level* if it satisfies:

U. 1) $\psi(\emptyset) = 0 = \psi(S)$;

U. 2) $\psi(E) = \psi(E^c)$, for all $E \in \Sigma$; and

U. 3) *concavity*: for every $E, F \in \Sigma$: $\psi(E \cup F) + \psi(E \cap F) \leq \psi(E) + \psi(F)$.

Let Ψ denote the set of all Knightian uncertainty levels on (S, Σ).

For each $\psi \in \Psi$, let $\Lambda(\psi)$ denote the subset of convex capacities on (S, Σ) that are associated to the Knightian uncertainty level ψ: [9]

$$\Lambda(\psi) = \{v \in \Lambda : c(v, E) = \psi(E), \quad \forall E \in \Sigma\}.$$

[8] It is easy to prove that:

i) if $v \in \Lambda$ then, for all $E \in \Sigma$: $c(v, E) \in [0, 1]$; and

ii) if $v \in \Delta$ then, for all $E \in \Sigma$: $c(v, E) = 0$.

[9] The set $\Lambda(\psi)$ is convex and throughout this thesis we will only consider $\psi \in \Psi$ such that $\Lambda(\psi) \neq \emptyset$.

Note that:

$$\Lambda = \bigcup_{\psi \in \Psi} \Lambda(\psi).$$

It is important to note that there may exist other capacities than convex capacities associated to each $\psi \in \Psi$, as the following example shows.

Example 3. Let $S = \{s_1, s_2, s_3\}$ and let ψ_0 be the Knightian uncertainty level defined by $\psi_0(E) = 0$ for all $E \in \Sigma$.

$$\Lambda(\psi_0) = \{w \in \Lambda : \ c(w, E) = 0, \quad \forall E \in \Sigma\}.$$

Note, firstly, that if p is a probability measure then $p \in \Lambda(\psi_0)$. In fact, it is easy to verify that $\Delta = \Lambda(\psi_0)$.

To illustrate that there may exist other capacities than convex capacities associated to the Knightian uncertainty level ψ_0, let the real-valued set-function $v_0 : \Sigma \to \mathbb{R}_+$ with $v_0(\emptyset) = 0$ be defined by:

$$v_0(\{s_1\}) = \alpha;\ v_0(\{s_2\}) = \beta;\ v_0(\{s_3\}) = \gamma;\ v_0(\{s_1, s_2\}) = 1 - \gamma;$$
$$v_0(\{s_1, s_3\}) = 1 - \beta;\ v_0(\{s_2, s_3\}) = 1 - \alpha;\ \text{and}\ v_0(S) = 1.$$

where $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta \leq 1, \alpha + \gamma \leq 1, \beta + \gamma \leq 1$ and $\alpha + \beta + \gamma \neq 1$.

Then it is immediate to note that v_0 is a capacity associated to this Knightian uncertainty level ψ_0, i.e.,

$$v_0 \in V(\psi_0) = \{w \in V : \ c(w, E) = 0, \quad \forall E \in \Sigma\}.$$

But v_0 is not a convex capacity (in fact v_0 is not a probability measure) associated to this Knightian uncertainty level ψ_0 (i.e., $v \notin \Lambda(\psi_0) = \Delta$).

2.1.3 The core of a capacity

The *core* of a capacity v, denoted by $core(v)$, is a set with all probability measures that majorizes v eventwise:

$$core(v) = \{q \in \Delta : \ q(E) \geq v(E), \quad \forall E \in \Sigma\}.$$

The following example shows that the core may be empty.

Example 4. Let $S = \{s_1, s_2\}$, and let $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be defined by:

$$v(s_1) = \alpha = v(s_2), \ \alpha \in (\frac{1}{2}, 1]; \text{ and } v(S) = 1.$$

Note that v is a capacity. And more, there is not any probability measure, $q \in \Delta$ such that $q(E) \geq v(E)$, for all $E \in \Sigma$. So, $core(v) = \emptyset$.

Shapley (1965, 1972) shows that every *convex capacity* has a nonempty *core* (i.e., if $v \in \Lambda$ then $core(v) \neq \emptyset$). Schmeidler (1972) shows that besides to have a nonempty *core*, every *convex capacity* also has a *convex core* (i.e., if $v \in \Lambda$ then $core(v) \neq \emptyset$ and $core(v)$ is a *convex* set). However the reverse does not hold, in the following sense: given a nonempty and convex set of probability measures, it may be possible that it could not be associated as the core of any convex capacities, as the next example shows.[10]

Example 5. Let $S = \{s_1, s_2, s_3\}$, and let Δ^* be a nonempty and convex set of probability measures, defined by:

$$\Delta^* = \{p \in \Delta : p(s_1) = 0 = p(s_2); \text{ and } p(s_3) = 1\}.$$

Let $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be defined by:

$$v(s_1) = 0 = v(s_2); \ v(s_3) = \alpha, \ \alpha \in (0, 1];$$

$$v(\{s_1, s_2\}) = 0; \ v(\{s_1, s_3\}) = 1 = v(\{s_2, s_3\}); \text{ and } v(S) = 1$$

then, for any $\alpha \in (0, 1]$, $core(v) = \Delta^*$.

However, note that:

[10]The interested reader should also see Schmeidler (1972) for other interesting examples.

i) if $\alpha \in (0,1)$ then v is a capacity, but v it is not a convex capacity; and

ii) if $\alpha = 1$ then v is a convex capacity.

For each $E \in \Sigma$, define:[11]

$$core_E(v) = \{q \in \ core(v) : q(E) = v(E)\}.$$

It is interesting to note that even if a capacity v is such that $core(v)$ is nonempty, it may happen that $core_E(v)$ is empty for some $E \in \Sigma$. However if v is a convex capacity then $core_E(v) \neq \emptyset$ for all $E \in \Sigma$ which implies that:[12]

$$v(E) = min\{q(E) : q \in \ core(v)\}.$$

Shapley (1965, 1972) shows that (when S is finite) a capacity is convex if and only if for any increasing sequence $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq S$, with $E_1, E_2, \cdots, E_n \in \Sigma$ it is true that:

$$core_{E_1}(v) \cap core_{E_2}(v) \cap \cdots \cap core_{E_n}(v) \neq \emptyset.$$

2.1.4 Update rules for a capacity

It is a very known fact that the Bayes update rule for a probability measure p given $F \in \Sigma$ with $p(F) > 0$ is a probability measure p_F defined by:

$$p_F(E) = \frac{p(F \cap E)}{p(F)}, \text{ for all } E \in \Sigma.$$

The Dempster-Shafer update rule (see Dempster (1967) and Shafer (1976)) is one between the possible update rules for capacities.

The Dempster-Shafer update rule for a capacity v given $F \in \Sigma$ with $v(F^c) < 1$ is a capacity defined by:[13]

$$v_F(E) = \frac{v(E \cup F^c) - v(F^c)}{1 - v(F^c)}.$$

[11] Note that $core_S(v) = core_\emptyset(v) = core(v)$.

[12] The reverse is not true.

[13] Note that if p is a probability measure, then the Dempster-Shafer update rule is equivalent to the Bayes update rule:

$$p_F(E) = \frac{p(E \cup F^c) - p(F^c)}{1 - p(F^c)} = \frac{p(F \cap E)}{P(F)}, \text{ for all } E \in \Sigma.$$

On the other hand, the maximum likelihood update rule is one between the possible update rules when considering a set of probability measures.

Let Q be a set of probability measures on (S, Σ). The maximum likelihood update rule for Q given $F \in \Sigma$ with $q(F) > 0$ for some $q \in Q$ is a set Q_F of probability measures defined by:

$$Q_F = \{q_F \in \Delta : \ q \in \arg\max\{q'(F) : q' \in Q\}\}.$$

Gilboa and Schmeidler (1993) shows that if a capacity is convex then the Dempster-Shafer update rule is equivalent to the lower envelope of probability measures obtained by the maximum likelihood update rule, where the set Q is the core of the capacity.

$$v_F(E) = min\{p_F(E) : p \in Q_F\}.$$

Kajii and Ui (2005) shows that that the converse is also true, that is, if there exists equivalence between the Dempster-Shafer update rule for a capacity and the maximum likelihood update rule for the core of the capacity then this implies that the underlying capacity is convex. So, there exists a characterization of convex capacities through update rules.

2.2 Basic statements on capacity integration

Let (S, Σ) be a measurable space, where S denote the set of *states of nature (world)*, which we assume to be nonempty and endowed with an algebra Σ of subsets called *events*. A real-valued function (random variable), bounded on S, $a : S \to \mathbb{R}$ is said to be Σ-measurable if, for all open set $O \subseteq \mathbb{R}$:

$$a^{-1}(O) \in \Sigma$$

where $a^{-1}(O) = \{s \in S : a(s) \in O\}$.

Let $B(S, \Sigma) \equiv B$ denote the set of all real-valued functions, bounded on S, that are Σ-measurable.

Since that capacities can be non-additive probability measures it is not possible to use an integral in the sense of Lebesgue. Choquet (1953) defined an integration operator with respect to a capacity v.

Let B_+ be the set of nonnegative real-valued functions bounded on S and Σ-measurable and let v be a capacity.

For each $a \in B_+$ and $\alpha \in \mathbb{R}$ let $L_\alpha(a)$ be defined by:

$$L_\alpha(a) = \{s \in S : a(s) \geq \alpha\},$$

satisfying the following properties:

i) If $\alpha \leq 0$ then $L_\alpha(a) = S$;

ii) If μ is an upper bound of a then $L_\alpha(a) = \emptyset$ for $\alpha > \mu$;

iii) If $0 < \alpha_1 < \alpha_2 \leq \mu$ (an upper bound of a) then $L_{\alpha_2}(a) \leq L_{\alpha_1}(a)$.

Let I be a norm-continuous functional on the space of nonnegative real-valued functions, bounded on S, that are Σ-measurable (i.e., $I : B_+ \to \mathbb{R}$). The Choquet integral of a w.r.t. (with respect to) the capacity v, denoted by $\int_S a dv$, is defined by:[14]

$$I(a) = \int_S a dv := \int_0^\infty v(L_\alpha(a)) d\alpha$$

where the integral on the right hand side is the extended Riemann integral, which is well defined because a is bounded and v is monotone.

If every event $E \in \Sigma$ is identified with its indicator function $\mathbf{1}_E$, which means:

i) $\mathbf{1}_E = 1$ if $s \in E$; and

ii) $\mathbf{1}_E = 0$ if $s \in E^c$;

then the functional I extends v from Σ to B_+, i.e.,

$$I(\mathbf{1}_E) = \int_S \mathbf{1}_E dv = \int_E dv = v(E).$$

This extension satisfies the following properties:[15]

I. 1) *monotonicity*: if $a(s) \geq b(s)$ for all $s \in S$ then $I(a) \geq I(b)$;

I. 2) *positively homogeneous of degree one*: for every $a \in B_+$ if $\lambda > 0$ then $I(\lambda a) = \lambda I(a)$.

Generally the Choquet functional is not additive, as the following example shows:

Example 6. Let $S = \{s_1, s_2\}$, let $b, c \in B_+ = \mathbb{R}^2_+$ be defined by:

$$b(s_1) = 1 \text{ and } b(s_2) = 2$$

[14]For the special case on which v is an probability measure, this definition coincides with the standard integral.

[15] Denneberger (1994) is a good reference for the proofs of the properties of the Choquet functional for capacities and Simonsen and Werlang (1991) presents other properties that are valid for convex capacities.

and

$$c(s_1) = 2 \text{ and } c(s_2) = 1;$$

and let $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be defined by:

$$v(s_1) = \alpha = v(s_2),\ \alpha \in [0, 1]; \text{ and } v(S) = 1.$$

Note that $I(b) = 1 + \alpha = I(c)$, and so $I(b) + I(c) = 2(1 + \alpha)$.

However, if $a \in B_+ = \mathbb{R}^2_+$ is defined by $a = b + c$ then $a(s_1) = 3 = a(s_2)$ which implies that $I(a) = 3 < 2(1 + \alpha) = I(b) + I(c)$, if $\alpha \in (\frac{1}{2}, 1]$.

Dellacherie (1970) investigates the properties of the norm-continuous functional I on a special subset of the real-valued set-functions, bounded on S, that are Σ-measurable: the co-monotonic functions.

Two real-valued functions (random variables), bounded on S, that are Σ-measurable, a and b (i.e., $a, b \in B$) are said to be co-monotonic if for all s and t in S:

$$(a(s) - a(t)) \cdot (b(s) - b(t)) \geq 0$$

or, equivalently, if there is not any s and t in S such that:

$$(a(s) - a(t)) > 0 \text{ and } (b(s) - b(t)) < 0.$$

Dellacherie (1970) shows that if a and b in B are co-monotonic functions then the norm-continuous functional I satisfies co-monotonic additivity:

I. 3) *co-monotonic additivity*: a and b in B imply $I(a + b) = I(a) + I(b)$.[16]

Schmeidler (1986)[17] investigates the converse of this problem, that is, if a norm-continuous functional, say I, which is monotone and additive on co-monotonic functions, can be represented through an integrator as $I(a) = \int_S a dv$ w.r.t. a capacity v, defined by $v(E) = I(\mathbf{1}_E)$ on Σ.

Firstly we will explore the case of simple function and then we will turn to the main result of Schmeidler (1986).

A real-valued function (random variable) bounded on S and Σ-measurable, $a : S \to \mathbb{R}$, is called a *simple function* if the set $\{a(s) : s \in S\}$ is finite.

[16]Note that $I(0) = 0$.

[17]See also the preliminary version of Schmeidler (1982).

Each simple function a admits an unique representation as a finite step function:

$$a = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} \text{ with } \alpha_1 > \alpha_2 > \cdots > \alpha_k > \alpha_{k+1} = 0$$

where: $A_i = a^{-1}(\{a(s_i)\}) \subseteq S$ and the events A_i, $i = 1, \cdots, k$, are pairwise disjoint, nonempty and $\bigcup_{i=1}^{k} A_i = S$.[18]

Let denoted by B_0 the set of all *simple functions*, a subset of the set of the real-valued functions, bounded on S, that are Σ-measurable (i.e., $B_0 \subsetneq B$): [19]

$$B_0 = \{a \in B : \ a \text{ is a simple function}\}.$$

2.2.1 The Schmeidler (1986)'s Theorem

Schmeidler (1986)'s result for simple functions asserts that if I is a norm-continuous functional on the subset of nonnegative simple functions (i.e., $I : B_0^+ \to \mathbb{R}$) which is normalized (i.e., $I(\mathbf{1}_S) = 1$) and also satisfies: [20]

I. 1) *monotonicity*: if $a(s) \geq b(s)$ for all $s \in S$ then $I(a) \geq I(b)$;

I. 3) *co-monotonic additivity*: a and b in B_0^+ imply $I(a + b) = I(a) + I(b)$.

then, defining the capacity $v : \Sigma \to [0, 1]$ such that for all $E \in \Sigma$ is given by $v(E) = I(\mathbf{1}_E)$, we have for each $a \in B_0^+$:[21]

$$I(a) = \int_S a dv := \int_0^{\alpha_1} v(L_\alpha(a)) d\alpha = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) \ v\left(\bigcup_{j=1}^{i} A_j\right)$$

and conversely, if $I(a) = \int_S a dv$, where v is a capacity, then the norm-continuous functional I satisfies $I.1$ and $I.3$.

The problem of defining the Choquet integral on Σ can be understood as the problem of extending the Choquet functional on B. See for instance Marinacci and Montrucchio (2004) which introduces the property of translation invariant as a required property to extend the norm-continuous functional $I : B \to \mathbb{R}$:

[18] That is, $\{A_i\}_{i=1}^{k}$ is a partition of S.

[19] If S is finite then $B_0 = B = \mathbb{R}^S$

[20] It is easy to prove that if a norm-continuous functional $I : B_0^+ \to \mathbb{R}$ satisfies the properties of *monotonicity* and *co-monotonic additivity* then it also satisfies the property of *positively homogeneous of degree one.*

[21] If v is a probability measure then:

$$I(a) = \int_S a dv := \sum_{i=1}^{k} \alpha_i \ v(A_i).$$

I. 4) *translation invariant*: for every $a \in B$ and $\lambda \geq 0$: $I(a + \lambda \mathbf{1}_S) = I(a) + \lambda I(\mathbf{1}_S)$.

Marinacci and Montrucchio (2004) shows that the Choquet functional $I : B \to \mathbb{R}$ induced by a capacity admits a unique translation invariant extension such that for each $a \in B$ is given by:[22]

$$I(a) = \int_S adv := \int_0^\infty v(L_\alpha(a))d\alpha + \int_{-\infty}^0 v(L_\alpha(a) - 1)d\alpha.$$

Schmeidler (1986)'s main result asserts that for a given norm-continuous functional I on the set of real-valued functions, bounded on S, that are Σ-measurable (i.e., $I : B \to \mathbb{R}$) which is normalized, monotone and co-monotonic additive then, defining $v(E) = I(\mathbf{1}_E)$ for all $E \in \Sigma$, we have for all $a \in B$:[23]

$$I(a) = \int_S adv := \int_0^\infty v(L_\alpha(a))d\alpha + \int_{-\infty}^0 v(L_\alpha(a) - 1)d\alpha.$$

The Rosenmüller (1971, 1972) and Schmeidler (1984, 1986)'s Theorem

A theorem due to Rosenmüller (1971, 1972) and Schmeidler (1984, 1986) provides an alternative characterization of a convex capacity. If I is a norm-continuous functional on the set of real-valued functions, bounded on S, that are Σ measurable (i.e., $I : B \to \mathbb{R}$) that satisfies Schmeidler (1986)'s result then v is a convex capacity if and only if it satisfies the following conditions:

i) $core(v) \neq \emptyset$; and

ii) For every $a \in B$: $I(a) = \int_S adv = \min\{\int_S adp : p \in core(v)\}$.

[22]It is easy to prove that if $I : B_0 \to \mathbb{R}$ satisfies *monotonicity* and *co-monotonic additivity* it has a unique extension to all of B which satisfies *monotonicity* and *co-monotonic additivity* on B (which is immediate since B is the norm closure of B_0 in $\mathbb{R}^S$ - see Dunford and Schwartz (1958) - and *monotonicity* implies *norm-continuity*).

[23]The number 1 appearing in the formula stands for $v(S) = v(\mathbf{1}_S)$. It is 1 due to normalization, which can be done w.l.g (without loss of generality) because of the property of *homogeneity of degree one* of the norm-continuous functional I.

2.3 The case of a simple capacity with parameter ε (also known as ε-contamination of confidence)

There exists a subset of the set of *convex capacities* which is analytically tractable, specially for comparative static exercises: the subset of *simple capacities*.

Let (S, Σ) be a measurable space, where S denote the set of *states of nature (world)*, which we assume to be nonempty and endowed with an algebra Σ of subsets called *events*. Barrios (1996) shows that if a capacity v satisfies the *property of uncertainty neutrality*, which means that given an event E (i.e., $E \in \Sigma$), $E \neq S$, there exists an $\varepsilon > 0$, such that for all $F \in \Sigma$ with $v(F) \leq \varepsilon$, then: [24]

C.5) $v(E \cup F) + v(E \cap F) = v(E) + v(F)$, for all $E, F \in \Sigma$ with $E \cup F \neq S$; and

C.6) $v(F) = v(E) + v(E^c \cap F)$, for all $E, F \in \Sigma$ with $E \subseteq F \neq S$.

Barrios (1996) also shows that given a capacity, say v_ε, probability measures p and q and $\varepsilon \in [0, 1)$, if:

i) v_ε satisfies the *property of uncertainty neutrality*; and

ii) $q \in core(v_\varepsilon)$

then, for any $\varepsilon \in [0, 1)$, there exists a real-valued set-function $v_\varepsilon : \Sigma \to \mathbb{R}_+$ that satisfies the above properties and it is defined by:

$$v_\varepsilon(E) \equiv \begin{cases} (1-\varepsilon) \cdot p(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S \end{cases}$$

which it will be called a *simple capacity with parameter* ε on (S, Σ) .

It is easy to verify that *simple capacities* are *convex capacities*. So, we can define the subset of *simple capacities*.

Let Λ^{sc} denote the set of all simple capacities on (S, Σ), a subset of the set of convex capacities (i.e., $\Lambda^{sc} \subsetneq \Lambda$):

$$\Lambda^{sc} = \{v_\varepsilon \in \Lambda : v_\varepsilon \text{ is a } \textit{simple capacity}, \text{ for all } \varepsilon \in [0, 1)\}.$$

Note that:

i) If $\varepsilon = 0$ then $\Delta = \Lambda^{sc}$;

ii) For any $\varepsilon \in [0, 1)$ if $\#S = 2$ then $\Delta \subsetneq \Lambda^{sc} = \Lambda$; and

iii) If $\varepsilon \in (0, 1)$ and if $\#S > 2$ then $\Delta \subsetneq \Lambda^{sc} \subsetneq \Lambda \subsetneq V$.

[24] Note that finitely probability measures trivially satisfy the *property of uncertainty neutrality*.

2.3.1 The support of a simple capacity with parameter ε

It is easy to verify that if v_ε is a *simple capacity with parameter* ε, then $\#supp(v_\varepsilon) = 1$. In fact, for any $\varepsilon \in [0, 1)$, the support of a simple capacity with parameter ε is the same support of the associated probability measure p (i.e., $supp(v_\varepsilon) = supp(p)$).

2.3.2 The uncertainty aversion measure of a simple capacity with parameter ε

For any $\varepsilon \in [0, 1)$, the uncertainty aversion measure of a *simple capacity with parameter* ε v_ε at event E is given by:

$$c(v_\varepsilon, E) = \begin{cases} \varepsilon & \text{if} \quad E \neq \{\emptyset, S\} \\ 0 & \text{if} \quad E = \{\emptyset, S\} \end{cases}$$

For any $\varepsilon \in [0, 1)$ we can define the associated (constant) Knightian uncertainty level as a set-function $\psi_\varepsilon : \Sigma \to [0, 1]$ given by:[25]

$$\psi_\varepsilon(E) \equiv \begin{cases} \varepsilon & \text{if} \quad E \neq \{\emptyset, S\} \\ 0 & \text{if} \quad E = \{\emptyset, S\} \end{cases}$$

Let p be a probability measure and let ψ_ε be a Knightian uncertainty level as above. If the real-valued set-function $v_\varepsilon : \Sigma \to [0, 1]$ is defined by:

$$v_\varepsilon(E) \equiv \begin{cases} (1 - \varepsilon) \cdot p(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S \end{cases}$$

then v_ε is a simple capacity with parameter ε.

It is easy to verify that the reverse is also true in the following sense: if v'_ε is a *simple capacity with parameter* ε associated to $\Lambda(\psi_\varepsilon)$ (i.e., $v'_\varepsilon \in \Lambda(\psi_\varepsilon)$ and if the real-valued set-function $q : \Sigma \to \mathbb{R}_+$ is defined by:

$$q(E) \equiv \begin{cases} \frac{1}{(1-\varepsilon)} \cdot v'_\varepsilon(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S \end{cases}$$

then q is a probability measure.

Note that:

$$\Lambda^{sc} = \bigcup_{\psi_\varepsilon \in \Psi} \Lambda(\psi_\varepsilon).$$

It is important to note that there may exist other capacities than convex capacities associated to each $\psi_\varepsilon \in \Psi$, as in Example 3.

[25] It is easy to verify that $\psi_\varepsilon \in \Psi$, for all $\varepsilon \in [0, 1)$.

2.3.3 The core of a simple capacity with parameter ε and its equivalence with ε-contamination of confidence

Suppose that the beliefs of an economic agent can be represented by some probability measure that associates $p(E) \in [0,1]$ for each possible event E and suppose also that due to some uncertainty her confidence is contaminated in the following sense: she is $(1-\varepsilon)\cdot p(E)$ certain that uncertainty she faces on a certain event E is characterized by a particular probability measure which associates $p(E)$ over this event, but that she has a fear that, with $\varepsilon \cdot p(E)$ chance, her conviction over the event E is completely wrong and she is left perfectly ignorant about the true measure in the present as well in the future (and similarly for all events). We call this as *ε-contamination of confidence* of some probability measure.[26]

Let p be a probability measure and let $\varepsilon \in [0,1)$. The *ε-contamination of confidence of p*, denoted by $cont(p,\varepsilon)$, is a set of probability measures defined by:

$$cont(p,\varepsilon) = \{(1-\varepsilon)\cdot p + \varepsilon \cdot q : q \in \Delta\}$$

where q is a *"contamination of confidence"* of p and ε reflects the amount of error in p that is deemed possible.

It is easy to verify that the *ε-contamination of confidence of some probability measure* can be represented as the core of a *simple capacity* with parameter ε associated. Let p be a probability measure, let $\varepsilon \in [0,1)$ and let the real-valued set-function $v_\varepsilon : \Sigma \to \mathbb{R}_+$ be defined as a simple capacity with parameter ε then:

$$core(v_\varepsilon) = \{(1-\varepsilon)\cdot p + \varepsilon \cdot q : q \in \Delta\} = cont(p,\varepsilon).$$

2.3.4 Update rules for a simple capacity with parameter ε

The Dempster-Shafer update rule for a *simple capacity with parameter ε* v_ε given $F \in \Sigma$ with $v(F^c) < 1$ is a capacity $v_{\varepsilon,F}$ defined by:[27]

$$v_{\varepsilon,F}(E) \equiv \begin{cases} (1 - \frac{\varepsilon}{(1-\varepsilon)p(F)+\varepsilon}) \cdot p_F(E) & \text{if} \quad (E \cap F) \subsetneq F \\ 1 & \text{if} \quad F \subseteq E \end{cases}$$

[26] For further discussions and applications on *simple capacities with parameter ε* (also known as *ε-contaminations of confidence*), see Asano (2008), Berger (1985), Berger and Berliner (1986), Boff and Werlang (1996), Chen and Epstein (2002), Dow and Werlang (1994), Eichberger and Kelsey (2000), Huber (1973), Nishimura and Ozaki (2004, 2006, 2007) and Wasserman and Kadane (1990).

[27] According to the Bayes update rule:

$$p_F(E) = \frac{p(E \cap F)}{p(F)}.$$

and it is easy to verify that this is equivalent to the lower envelope of probability measures obtained by the maximum likelihood update rule, where the set Q is the core of the *simple capacity with parameter* ε, or, equivalently, Q is the ε-contamination of confidence of p (i.e., $Q = cont(p, \varepsilon)$).[28]

2.3.5 The Choquet functional for a simple capacity with parameter ε

If I is a norm-continuous functional on the subset of nonnegative simple functions (i.e., $I : B_0^+ \to \mathbb{R}$) which is normalized, monotone and co-monotonic additive and considering *simple capacities*, which means that there exists a probability measure p, and an $\varepsilon \in [0, 1)$ such that if a real-valued set-function $v_\varepsilon : \Sigma \to \mathbb{R}_+$ is defined by

$$v_\varepsilon(E) \equiv \begin{cases} (1-\varepsilon) \cdot p(E) & \text{if} \quad E \neq S \\ 1 & \text{if} \quad E = S \end{cases}$$

then, defining $v_\varepsilon(E) = I(\mathbf{1}_E)$ for all $E \in \Sigma$, we have for each $a \in B_0^+$:

$$I(a) = \int_S a dv_\varepsilon := \int_0^{\alpha_1} v_\varepsilon(L_\alpha(a)) d\alpha = (1-\varepsilon) \int_S a dp + \varepsilon \alpha_k$$

where $\int_S a dp$ is the integral of a w.r.t. the probability measure p.

Because simple capacities are convex capacities this implies that the Rosenmüller (1971, 1972) and Schmeidler (1984, 1986)'s Theorem is valid, i.e.,

i) $core(v_\varepsilon) \neq \emptyset$ (trivially, note that $p \in core(v_\varepsilon)$); and

ii) For every $a \in B_0^+$: $I(a) = \int_S a dv_\varepsilon = (1-\varepsilon) \int_S a dp + \varepsilon \min\{a(s) : s \in S\}$.

[28] By Gilboa and Schmeidler (1993), because every simple capacity with parameter ε is a convex capacity.

Chapter 3

Contamination of confidence

As seen in Chapter 2, if p is a probability measure and if ψ_ε is a constant Knightian uncertainty level defined by $\psi_\varepsilon(E) = \varepsilon$ if $E \neq \{\emptyset, S\}$ and $\psi_\varepsilon(E) = 0$ if $E = \{\emptyset, S\}$ for any $\varepsilon \in [0, 1)$ then the real-valued set-function $v_\varepsilon : \Sigma \to \mathbb{R}_+$ defined by $v_\varepsilon \equiv (1 - \psi_\varepsilon) \cdot p$ is a convex capacity, also called as simple capacity with parameter ε. Is this true for any Knightian uncertainty level? The answer is no, as it will be illustrated by the first example in this Chapter.

However there exist non-constant Knightian uncertainty levels $\psi \in \Psi$ such that given some probability measures $p \in \Delta$ the real-valued set-functions $v : \Sigma \to \mathbb{R}_+$ defined by $v \equiv (1 - \psi) \cdot p$ are convex capacities. We know that Knightian uncertainty levels are related to convex capacities. So, our start point will be explore the properties of a special set of convex capacities that will enable us to defining new properties that should be satisfied by the desired Knightian uncertainty levels.

The knowing of these new properties that should also be satisfied by these desired Knightian uncertainty levels will enable us to present our main result. The main theorem of this Chapter asserts that given a Knightian uncertainty level satisfying these new properties (and as it will be seen it is associated to some set D) and considering probability measures such that the support is the set (event) D then the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by $v \equiv (1 - \psi) \cdot p$ is a convex capacity. And the converse it is also true in the sense that given a convex capacity such that the support is the set (event) D and it is associated to the Knightian uncertainty level satisfying new properties (which in turn is associated to the set D) then the real-valued set-function $p : \Sigma \to \mathbb{R}_+$ defined by $p \equiv \frac{1}{(1-c(v,\cdot))} \cdot v$ is a probability measure (where $c(v, \cdot)$ is the uncertainty aversion measure of v).

Contamination of confidence is a special case of Knightian uncertainty or ambiguity on which the decision maker faces not a simple probability measure but a set of probability measures. In the sequel it will be presented the conditions under which a subset of the core of the convex capacities studied here is equivalent to the contamination of confidence of some probability measure. Finally it is presented a Choquet functional for simple

functions with the subset of convex capacities studied in this Chapter.

The organization of this Chapter is as follows. In Section 3.1 we start our study on contamination of confidence. Section 3.2 presents some motivating examples. The main theorem is presented in Section 3.3. Section 3.4 discuss on the restricted core of the capacity that we are studding and its equivalence with contamination of confidence and section 3.5 discuss about update rules. Section 3.6 presents the formula of the Choquet functional for simple functions with the capacity that we are studding.

3.1 A first step towards the characterization of contamination of confidence

Let (S, Σ) be a measurable space, where S denote the set of *states of nature (world)*, which we assume to be nonempty, finite and endowed with an algebra Σ of subsets called *events*. If p is a probability measure and if ψ_ε is a (constant) Knightian uncertainty level defined by $\psi_\varepsilon(E) = \varepsilon$ if $E \neq \{\emptyset, S\}$ and $\psi_\varepsilon(E) = 0$ if $E = \{\emptyset, S\}$ for any $\varepsilon \in [0, 1)$ then the real-valued set-function $v_\varepsilon : \Sigma \to \mathbb{R}_+$ defined by $v_\varepsilon \equiv (1 - \psi_\varepsilon) \cdot p$ is a convex capacity, also called as simple capacity with parameter ε.

Is this true for any Knightian uncertainty level? The answer is no, as the next example shows.

Example 7. Let $S = \{s_1, s_2, s_3\}$, let p be a probability measure defined by:

$$p(\{s_1\}) = \frac{1}{2} \text{ and } p(\{s_i\}) = \frac{1}{4},\ i = 2, 3.$$

and let ψ be the Knightian uncertainty level (i.e., $\psi \in \Psi$) defined by:

$$\psi(\{s_1\}) = \frac{1}{2} = \psi(\{s_2, s_3\}) \text{ and } \psi(\{s_i\}) = \frac{1}{4} = \psi(\{s_1, s_j\}),\ i, j = 2, 3 \text{ and } j \neq i$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1 - \psi) \cdot p$, i.e.:

$$v(\{s_1\}) = \frac{1}{4} = v(\{s_2, s_3\});\ v(\{s_i\}) = \frac{3}{16} \text{ and } v(\{s_1, s_i\}) = \frac{9}{16},\ i = 2, 3$$

then v is a capacity associated to ψ (i.e., $v \in V(\psi)$), but v is not a convex capacity associated to ψ (i.e., $v \notin \Lambda(\psi)$).

The following example illustrates the case on which there exists a non-constant Knightian uncertainty level $\psi \in \Psi$ such that given some probability measure $p \in \Delta$ the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by $v \equiv (1 - \psi) \cdot p$ is a convex capacity.

Example 8. Let $S = \{s_1, s_2, s_3\}$, let p be a probability measure with $supp(p) = \{s_1, s_2\}$ defined by:

$$p(\{s_1\}) = \frac{1}{3};\ p(\{s_2\}) = \frac{2}{3} \text{ and } p(\{s_3\}) = 0$$

and let ψ be the Knightian uncertainty level defined by:

$$\psi(\{s_i\}) = \frac{1}{2} = \psi(\{s_j, s_3\}),\ i, j = 1, 2 \text{ and } i \neq j; \text{ and}$$
$$\psi(\{s_3\}) = \frac{1}{4} = \psi(\{s_1, s_2\}).$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1 - \psi) \cdot p$, i.e.:

$$v(\{s_1\}) = \frac{1}{6} = v(\{s_1, s_3\});\ v(\{s_2\}) = \frac{1}{3} = v(\{s_2, s_3\}));\ v(\{s_3\}) = 0 \text{ and } v(\{s_1, s_2\}) = \frac{3}{4}$$

then v is a convex capacity associated to ψ (i.e., $v \in \Lambda(\psi)$).

Considering convex capacities that has only one element in the support, there exists a special subset which also satisfies other properties: additivity in proper subsets of the event in the support, nullity in the complement of the event in the support and not fully uncertainty in every event.

Definition 1 (convex capacities associated to a set D)**.** A real-valued set-function $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ is called a *convex capacity* on (S, Σ) *associated to a nonempty set* D $(\emptyset \neq D \subseteq S)$ if besides properties $C.$ 1 (normalized) and $C.$ 3 (convex) it also satisfies:

C. 7) v is *additive in proper subsets of* D: for all $\emptyset \neq C, C' \subsetneq D$, with $C \cup C' \neq D$ and $C \cap C' = \emptyset$, $v(C \cup C') = v(C) + v(C')$;

C. 8) v is *null in the complement of* D: for all $\emptyset \neq C \subsetneq D$ and all $E \subseteq D^c$, $v(C \cup E) = v(C)$; and

C. 9) v is *not full uncertain in every event*: for all $E \in \Sigma$, $v(E) + v(E^c) > 0$.

Example 9. Let $S = \{s_1, s_2, s_3\}$, let v be a convex capacity associated to the set $\{s_1, s_2\}$ defined by:

$$v(\{s_1\}) = \frac{1}{6} = v(\{s_1, s_3\});\ v(\{s_2\}) = \frac{1}{3} = v(\{s_2, s_3\});\ v(\{s_3\}) = 0 \text{ and } v(\{s_1, s_2\}) = \frac{3}{4}.$$

Note that properties $C.1$ and $C.3$ of Definition 1 are trivially satisfied. So, it remains to verify properties $C.7$ to $C.9$.

Property $C.7$ is also trivially satisfied, because there is no $\emptyset \neq C, C' \subsetneq \{s_1, s_2\}$, with $C \cup C' \neq \{s_1, s_2\}$ and $C \cap C' = \emptyset$.

To verify that property $C.8$ is satisfied, note that: $\emptyset \neq \{s_1\}, \{s_2\} \subsetneq \{s_1, s_2\}$ and $\{s_3\} \subseteq \{s_1, s_2\}^c$. So:

$$v(\{s_1, s_3\}) = \frac{1}{6} = v(\{s_1\}) \text{ and } v(\{s_2, s_3\}) = \frac{1}{3} = v(\{s_2\})$$

Finally property $C.9$ is trivially satisfied.

3.1.1 The support of a convex capacity associated to a set D

The following Proposition shows that the nonempty set D ($\emptyset \neq D \subseteq S$) is, in fact, the support of the convex capacity associated to the set D (i.e., $supp(v) = \{D\}$).

Proposition 1. Consider a nonempty set D (i.e., $\emptyset \neq D \subseteq S$) and let $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ be a convex capacity associated to the set D. Then the support of v is unique and it is given by the set D (i.e., $supp(v) = \{D\}$).

Proof. See Section $A.1$ of Appendix A. □

Example 10. Let $S = \{s_1, s_2, s_3\}$, let v be a convex capacity associated to the set $\{s_1, s_2\}$ defined by:

$$v(\{s_1\}) = \frac{1}{6} = v(\{s_2, s_3\});\ v(\{s_2\}) = \frac{1}{3} = v(\{s_1, s_3\});\ v(\{s_3\}) = 0 \text{ and } v(\{s_1, s_2\}) = \frac{3}{4}.$$

$supp(v) = \{s_1, s_2\}$.

To see this, note that:

i) $v(\{s_1, s_2\}^c) = v(\{s_3\}) = 0$;

ii) $\emptyset \neq \{s_1\}, \{s_2\} \subsetneq \{s_1, s_2\}$: $v(\{s_1\}^c) = v(\{s_2, s_3\}) = \frac{1}{6}$ and $v(\{s_2\}^c) = v(\{s_1, s_3\}) = \frac{1}{3}$.

From now on we will refer to the capacities in Definition 1 as generalized simple capacities with support D. Let $\Lambda|_D$ denote the set of all generalized simple capacities on (S, Σ) with support D, a subset of the set of convex capacities on (S, Σ) (i.e., $\Lambda|_D \subsetneq \Lambda$):

$$\Lambda|_D = \{v \in \Lambda : v \text{ is a generalized simple capacity with support D}\}.$$

3.1.2 The uncertainty aversion measure of a generalized simple capacity with support D

The following proposition presents the extra properties that are satisfied by the uncertainty aversion measure associated to generalized simple capacities with support D.

Proposition 2 (Properties of the uncertainty aversion measure associated to generalized simple capacities with support D)**.** Let v be a generalized simple capacity on (S, Σ) with support D (i.e., $v \in \Lambda|_D$) . Then, besides the usual properties i to iii, the uncertainty aversion measure associated to v also satisfies:

iv) *uniformity in proper subsets of D*: for all $\emptyset \neq C, C' \subsetneq D$, with $C \cup C' \neq D$ and $C \cap C' = \emptyset$, $c(v, C) = c(v, C') = c(v, C \cup C') \geq c(v, D)$;

v) *nullity in the complement of D*: for all $\emptyset \neq C \subsetneq D$ and for all $E \subseteq D^c$, $c(v, C \cup E) = c(v, C)$; and

vi) *monotone decreasing in D*: [1] for all $E_i \subseteq D^c$ for all $i = \{1, 2, \ldots, j\}$ with $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_j = D^c$, $c(v, D) \geq c(v, D \cup E_1) \geq c(v, D \cup E_2) \geq \cdots \geq c(v, D \cup E_j) = c(v, S) = 0$.

Proof. See Section $A.2$ of Appendix A. □

Example 11. Let $S = \{s_1, s_2, s_3\}$, let v be a convex capacity associated to the set $\{s_1, s_2\}$ defined by:

$$v(\{s_1\}) = \frac{1}{6} = v(\{s_2, s_3\});\ v(\{s_2\}) = \frac{1}{3} = v(\{s_1, s_3\});\ v(\{s_3\}) = 0 \text{ and } v(\{s_1, s_2\}) = \frac{3}{4}.$$

[1] In fact property vi is a consequence of the others.

The uncertainty aversion measure associated to v, $c(v,\cdot)$, is given by:

$$c(v,\{s_i\}) = \frac{1}{2} = c(v,\{s_j,s_3\}),\ i,j = 1,2 \text{ and } i \neq j; \text{ and}$$
$$c(v,\{s_3\}) = \frac{1}{4} = c(v,\{s_1,s_2\}).$$

Note that properties i to iii of Proposition 2 are trivially satisfied. So, it remains to verify properties iv and v (because property vi is a consequence of the others).

Property iv is also trivially satisfied, because there is no $\emptyset \neq C, C' \subsetneq \{s_1,s_2\}$, with $C \cup C' \neq \{s_1,s_2\}$ and $C \cap C' = \emptyset$.

To verify that property v is satisfied, note that: $\emptyset \neq \{s_1\},\{s_2\} \subsetneq \{s_1,s_2\}$ and $\{s_3\} \subseteq \{s_1,s_2\}^c$. So:

$$c(v,\{s_i\}) = \frac{1}{2} = c(v,\{s_j,s_3\}),\ i,j = 1,2 \text{ and } i \neq j$$

The following definition, suggested by the properties of the uncertainty aversion measures, is the basis of the characterization of the subsets of convex capacities that represents contamination of confidence.

Definition 2 (Knightian uncertainty level associated to the set D)**.** Fix a nonempty event D ($\emptyset \neq D \subseteq S$). A Knightian uncertainty level is said to be *associated to a set D* if besides the usual properties $U.$ 1 to $U.$ 3 it also satisfies:

U. 4) *uniformity in proper subsets of D*: for all $\emptyset \neq C, C' \subsetneq D$, with $C \cup C' \neq D$ and $C \cap C' = \emptyset$, $\psi(C) = \psi(C') = \psi(C \cup C') \geq \psi(D)$;

U. 5) *nullity in the complement of D*: for all $\emptyset \neq C \subsetneq D$ and for all $E \subseteq D^c$, $\psi(C \cup E) = \psi(C)$; and

U. 6) *monotone decreasing in D*: [2] for all $E_i \subseteq D^c$ for all $i = \{1,2,\ldots,j\}$ with $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_j = D^c$, $\psi(D) \geq \psi(D \cup E_1) \geq \psi(D \cup E_2) \geq \cdots \geq \psi(D \cup E_j) = \psi(S) = 0$.

[2] In fact property $U.6$ is a consequence of the others.

Example 12. Let $S = \{s_1, s_2, s_3\}$ and let ψ be the Knightian uncertainty level associated to $\{s_1, s_2\}$ (i.e., $\psi \in \Psi|_{\{s_1,s_2\}}$) defined by:

$$\psi(\{s_i\}) = \frac{1}{2} = \psi(\{s_j, s_3\}),\ i, j = 1, 2 \text{ and } i \neq j \text{ and } \psi(\{s_3\}) = 0 = \psi(\{s_1, s_2\}).$$

Note that properties $U.1$ to $U.3$ of Definition 2 are trivially satisfied. So, it remains to verify properties $U.4$ and $U.5$ (because property $U.6$ is a consequence of the others).

Property $U.4$ is also trivially satisfied, because there is no $\emptyset \neq C, C' \subsetneq \{s_1, s_2\}$, with $C \cup C' \neq \{s_1, s_2\}$ and $C \cap C' = \emptyset$.

To verify that property $U.5$ is satisfied, note that: $\emptyset \neq \{s_1\}, \{s_2\} \subsetneq \{s_1, s_2\}$ and $\{s_3\} \subseteq \{s_1, s_2\}^c$. So:

$$\psi(\{s_i\}) = \frac{1}{2} = \psi(\{s_j, s_3\}),\ i, j = 1, 2 \text{ and } i \neq j$$

Let $\Psi|_D$ denote the set of all Knightian uncertainty levels on (S, Σ) associated to the set D, a subset of the set of Knightian uncertainty levels on (S, Σ) (i.e., $\Psi|_D \subsetneq \Psi$):

$$\Psi|_D = \{\psi \in \Psi\ :\ \psi \text{ is a Knightian uncertainty level associated to D}\}.$$

For each $\psi \in \Psi|_D$, let $\Lambda|_D(\psi)$ denote the subset of generalized simple capacities on (S, Σ) with support D that are associated to the Knightian uncertainty level associated to D, a subset of the set of convex capacities on (S, Σ) with support D, i.e., for each $\psi \in \Psi|_D$:

$$\Lambda|_D(\psi) = \{c(v, \cdot) = \psi(\cdot),\ \psi \in \Psi|_D : v \in \Lambda|_D\}.$$

It is important to note that there may exist other capacities than generalized simple capacities with support D (including other convex capacities) associated to each $\psi \in \Psi|_D$, as the following example shows.

Example 13. Let $S = \{s_1, s_2, s_3\}$ and let ψ be the Knightian uncertainty level associated to $\{s_1, s_2\}$ (i.e., $\psi \in \Psi|_{\{s_1,s_2\}}$) defined by:

$$\psi(\{s_i\}) = \frac{1}{2} = \psi(\{s_j, s_3\}),\ i, j = 1, 2 \text{ and } i \neq j \text{ and } \psi(\{s_3\}) = 0 = \psi(\{s_1, s_2\}).$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ with $v(\emptyset) = 0$ is defined by:

$$v(\{s_i\}) = 0,\ i = 1, 2, 3;\ v(\{s_1, s_2\}) = 1 \text{ and } v(\{s_j, s_3\}) = \frac{1}{2},\ j = 1, 2$$

then it is immediate to note that v is a capacity associated to this Knightian uncertainty level ψ, i.e.,

$$v \in V(\psi) = \{w \in V : \ c(w, \cdot) = \psi(\cdot), \ \psi \in \Psi|_{\{s_1,s_2\}} \text{ as above}\}.$$

Note firstly that $\{s_1, s_2\}$ is not the unique support of v (in fact $\{s_1, s_3\}$ and $\{s_2, s_3\}$ also belong to $supp(v)$).

Note also that v is not a convex capacity. To check it note that:

$$v(\{s_1, s_2\}) + v(\{s_i, s_3\}) = 1 + \frac{1}{2} > 1 = 1 + v(\{s_i\}), \ i = 1, 2.$$

3.2 Some motivating examples

Before presenting the main theorem we will show through some examples that given a probability measure $p \in \Delta$, with $supp(p) = \{D\}$ and given a Knightian uncertainty level associated to the set D (i.e., $\psi \in \Psi|_D$) if the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1 - \psi) \cdot p$ then v is a convex capacity associated to ψ ($v \in \Lambda|_D(\psi)$).

Firstly, note that if $D = S$ then this requires that the Knightian uncertainty level will be constant (i.e., $\psi(E) = \varepsilon$, if $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$, if $E = \{\emptyset, S\}$) and so it reduces to the case of a simple capacity with parameter ε (also known as ε-contamination of confidence of p).

Example 14. Let $S = \{s_1, s_2, s_3\}$, let p be a probability measure with $supp(p) = S$ defined by:

$$p(\{s_i\}) = \alpha_i, \ \alpha_i \in (0, 1), \ i = 1, 2, 3 \text{ and } \sum_{i=1}^{3} \alpha_i = 1.$$

and let ψ be the Knightian uncertainty level associated to S (i.e., $\psi \in \Psi|_S$) defined by:

$$\psi(E) = \varepsilon \text{ for all } E \neq \{\emptyset, S\}, \text{ where } \varepsilon \in [0, 1).$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1 - \psi) \cdot p$, i.e.:

$$v(\{s_i\}) = (1 - \varepsilon) \cdot \alpha_i, \ i = 1, 2, 3; \ v(\{s_i, s_j\}) = (1 - \varepsilon) \cdot (\alpha_i + \alpha_j), \ i, j = 1, 2, 3 \text{ and } i \neq j$$

then v is a convex capacity.

The next example illustrates the case on which D has more than one element but is different from the whole set (i.e., $D \neq S$).

Example 15. Let $S = \{s_1, s_2, s_3\}$, let p be a probability measure with $supp(p) = \{s_1, s_2\}$ defined by:

$$p(\{s_1\}) = \alpha;\ p(\{s_2\}) = (1-\alpha) \text{ and } p(\{s_3\}) = 0,\ \alpha \in (0,1)$$

and let ψ be the Knightian uncertainty level associated to $\{s_1, s_2\}$ (i.e., $\psi \in \Psi|_{\{s_1,s_2\}}$) defined by:

$$\psi(\{s_i\}) = \varepsilon = \psi(\{s_j, s_3\}),\ i, j = 1, 2 \text{ and } i \neq j; \text{ and}$$

$$\psi(\{s_3\}) = \delta = \psi(\{s_1, s_2\}), \text{ where } 0 \leq \delta \leq \varepsilon < 1.$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1-\psi) \cdot p$, i.e.:

$$v(\{s_1\}) = (1-\varepsilon)\cdot\alpha;\ v(\{s_2\}) = (1-\varepsilon)\cdot(1-\alpha);\ v(\{s_3\}) = 0;$$

$$v(\{s_1, s_3\}) = (1-\varepsilon)\cdot\alpha;\ v(\{s_2, s_3\}) = (1-\varepsilon)\cdot(1-\alpha) \text{ and } v(\{s_1, s_2\}) = (1-\delta)$$

then v is a convex capacity.

Note that if $\delta = \varepsilon$ then it reduces to the case of a simple capacity with parameter ε (also known as ε-contamination of confidence of p).

The last example of this Section illustrates the case on which the support of the probability measure has only one element (i.e., $supp(p) = \{s\}$, $s \in S$). The interesting fact is that the properties $U.4$, $U.5$, and $U.6$ of the Knightian uncertainty level associated to D are innocuous.

Example 16. Let $S = \{s_1, s_2, s_3\}$, let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{s_1\}$ defined by:

$$p(\{s_1\}) = 1;\ p(\{s_2\}) = 0 = p(\{s_3\})$$

and let ψ be the Knightian uncertainty level associated to $\{s_1\}$ (i.e., $\psi \in \Psi|_{\{s_1\}}$) defined by:

$$\psi(\{s_1\}) = \varepsilon = \psi(\{s_2, s_3\}),\ \psi(\{s_2\}) = \gamma = \psi(\{s_1, s_3\}) \text{ and}$$

$$\psi(\{s_3\}) = \delta = \psi(\{s_1, s_2\}) \text{ where } 0 \leq \delta, \gamma \leq \varepsilon < 1.$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by $v \equiv (1 - \psi) \cdot p$, i.e.:

$$v(\{s_1\}) = (1 - \varepsilon);\ v(\{s_2\}) = 0 = v(\{s_3\});$$
$$v(\{s_1, s_2\}) = (1 - \delta);\ v(\{s_1, s_3\}) = (1 - \gamma) \text{ and } v(\{s_2, s_3\}) = 0$$

then v is a convex capacity.

Note that if $\delta = \gamma = \varepsilon$ then it reduces to the case of a simple capacity with parameter ε (also known as ε-contamination of confidence of p).

3.3 The main theorem

Theorem 1. *Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$). If the real-valued set-function $w : \Sigma \to \mathbb{R}_+$ is defined by:*[3]

$$w \equiv (1 - \psi) \cdot p$$

then w is a generalized simple capacity with support D associated to the Knightian uncertainty level ψ associated to D (i.e., $w \in \Lambda|_D(\psi)$, $\psi \in \Psi|_D$).

And, conversely, let v be a generalized simple capacity with support D. If the real-valued set-function $q : \Sigma \to \mathbb{R}_+$ is defined by:[4]

$$q \equiv \frac{1}{(1 - c(v, \cdot))} \cdot v$$

then q is a probability measure (i.e., $q \in \Delta$).

Proof. See Section A.3 of Appendix A. □

[3]Note that because $\psi(E) < 1$ for all $E \in \Sigma$, the support of w is the same of p (i.e., $supp(w) = supp(p) = \{D\}$).

[4]Note that because $v(E) + v(E^c) > 0$ for all $E \in \Sigma$, the support of q is the same of v (i.e., $supp(q) = supp(v) = \{D\}$).

Summing up, the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by

$$v \equiv (1 - \psi) \cdot p$$

can be represented by:

$$v(E) \equiv \begin{cases} (1-\varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

3.4 The restricted core of a generalized simple capacity with support D and its equivalence with contamination of confidence

Suppose that the beliefs of an economic agent can be represented by some probability measure that associates $p(E) \in [0, 1]$ for each possible event E and suppose also that due to some uncertainty (in the Knight sense) her confidence is contaminated in the following sense: she is $(1 - \psi(E)) \cdot p(E)$ certain that uncertainty she faces on a certain event E is characterized by a particular probability measure which associates $p(E)$ over this event (and similarly for all events), but that she has a fear that, with $\psi(E) \cdot p(E)$ chance, her conviction over the event E is completely wrong and she is left perfectly ignorant about the true measure in the present as well in the future. We call this as contamination of confidence of some probability measure.

Contamination of confidence is defined as follows:

Definition 3 (Contamination of confidence of some probability measure)**.** Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

Let the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ be defined by [5]

$$v \equiv (1 - \psi) \cdot p$$

[5]Note that, because $\psi(E) < 1$ for all $E \in \Sigma$, the support of v is the same of p (i.e., $supp(v) = supp(p) = \{D\}$).

that can be represented by:

$$v(E) \equiv \begin{cases} (1-\varepsilon)\cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1-\psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0,1)$.

The contamination of confidence of p, denoted by $cont(p,\psi)|_D$, is a set of probability measures defined in the following way:

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:

$$cont(p,\psi)|_D = \{(1-\varepsilon)\cdot p(E) + \varepsilon \cdot q(E) : q \in \Delta\}.$$

and

ii) If $\exists E \supseteq D$, with $\psi(E) < \varepsilon$, then:

$$cont(p,\psi)|_D = \{(1-\psi(E))\cdot p(E) + \psi(E)\cdot q(E) : q \in \Delta \text{ such that } supp(q) \subseteq D\}.$$

Proposition 3. Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by

$$v \equiv (1-\psi)\cdot p$$

and can be represented by:

$$v(E) \equiv \begin{cases} (1-\varepsilon)\cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1-\psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0,1)$.

Then, $cont(p,\psi)|_D \subseteq core(v)$.

Proof. See Section *A*.4 of Appendix *A*. □

The following example shows that the reverse set inclusion does not hold.

Example 17. Let $S = \{s_1, s_2, s_3\}$, let $p \in \Delta$ with $supp(p) = \{s_2, s_3\}$ be defined by:

$$p(\{s_1\}) = 0 \text{ and } p(\{s_2\}) = \frac{1}{2} = p(\{s_3\})$$

and let $\psi \in \Psi|_{\{s_2,s_3\}}$ be defined by:

$$\psi(\{s_i\}) = \frac{1}{2} = \psi(\{s_1, s_j\}),\ i, j = 2, 3 \text{ and } j \neq i; \text{ and } \psi(\{s_1\}) = \frac{1}{4} = \psi(\{s_2, s_3\}).$$

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by:

$$v \equiv (1 - \psi) \cdot p.$$

then the probability measure q (i.e. $q \in \Delta$) defined by:

$$q(\{s_1\}) = \frac{1}{4},\ q(\{s_2\}) = \frac{3}{8} = q(\{s_3\})$$

belongs to $core(v)$ (i.e., $q \in core(v)$).

However, $q \notin cont(p, \psi)|_{\{s_2,s_3\}}$.

The following definition is the basis of the equivalence between contamination of confidence and the restricted core of a generalized simple capacity with support D.

Definition 4 (The restricted core of a generalized simple capacity with support D)**.** Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

Let the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ be defined by

$$v \equiv (1 - \psi) \cdot p$$

that can be represented by:

$$v(E) \equiv \begin{cases} (1 - \varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

Then, the restricted core of a generalized simple capacity with support D, denoted by $core(v)|_D$, is a set of probability measures defined in the following way:

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:

$$core(v)|_D = \{(1 - \varepsilon) \cdot p(E) + \varepsilon \cdot q(E) : q \in \Delta\}.$$

and

ii) If $\exists$E$\supseteq D$, with $\psi(E) < \varepsilon$, then:

$$core(v)|_D = \{(1 - \psi(E)) \cdot p(E) + \psi(E) \cdot q(E) : q \in \Delta \text{ such that } supp(q) \subseteq D\}.$$

So, the following equivalence is trivial.

Proposition 4. Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by:

$$v \equiv (1 - \psi) \cdot p$$

and can be represented by:

$$v(E) \equiv \begin{cases} (1 - \varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

Then $cont(p, \psi)|_D = core(v)|_D$.

Proof. See Section *A*.5 of Appendix *A*. □

3.5 Update rules for a generalized simple capacity with support D

The Dempster-Shafer update rule (see Dempster (1967) and Shafer (1976)) for a generalized simple capacity with support D is a capacity defined by the following proposition:

Proposition 5 (The Dempster-Shafer update rule for a generalized simple capacity with support D)**.** Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

If the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ is defined by:

$$v \equiv (1 - \psi) \cdot p$$

and can be represented by:

$$v(E) \equiv \begin{cases} (1-\varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

Then given $F \in \Sigma$ with $v(F^c) < 1$ the Dempster-Shafer update rule is a capacity defined by:

$$v_F(E) = \left(1 - \frac{\psi(F)}{\psi(F) + (1 - \psi(F)) \cdot p(F)}\right) \cdot p_F(E) + \frac{(\psi(F^c) - \psi(E \cup F^c)) \cdot p(E \cup F^c)}{\psi(F) + (1 - \psi(F)) \cdot p(F)}$$

where $p_F(E)$ is the Bayes update rule, i.e.:

$$p_F(E) = \frac{p(E \cap F)}{p(F)}.$$

Proof. See Section *A*.6 of Appendix *A*. □

So, it follows that:

i) If $F \subseteq D$ then:

$$v_F(E) \equiv \begin{cases} \left(1 - \frac{\varepsilon}{\varepsilon + (1-\varepsilon)p(F)}\right) \cdot p_F(E) & \text{if} \quad (E \cap F) \subsetneq F \\ 1 & \text{if} \quad F \subseteq E \end{cases}$$

ii) If $D \subsetneq F$ then:

$$v_F(E) \equiv \begin{cases} \left(1 - \frac{\varepsilon}{\varepsilon+(1-\varepsilon)p(F)}\right) \cdot p_F(E) & \text{if} \quad (D \cap E) \subsetneq D \\ (1 - \psi(E \cup F^c)) & \text{if} \quad D \subseteq (E \cap F) \subsetneq F \\ 1 & \text{if} \quad F \subseteq E \end{cases}$$

iii) If $\emptyset \neq (D \cap F) \subsetneq F$ then:

$$v_F(E) \equiv \begin{cases} \left(1 - \frac{\varepsilon}{\varepsilon+(1-\varepsilon)p(F)}\right) \cdot p_F(E) & \text{if} \quad E \cap (D \cap F) \subsetneq (D \cap F) \\ \frac{(1-\psi(E \cup F^c))}{\psi(F)+(1-\psi(F))\cdot p(F)} & \text{if} \quad (D \cap F) \subseteq (E \cap F) \subsetneq F \\ 1 & \text{if} \quad F \subseteq E \end{cases}$$

Let Q be a set of probability measures on (S, Σ). The maximum likelihood update rule for Q given $F \in \Sigma$ with $q(F) > 0$ for some $q \in Q$ is a set Q_F of probability measures defined by:

$$Q_F = \{q_F \in \Delta : q \in \arg\max\{q'(F) : q' \in Q\}\}.$$

The following proposition shows the equivalence between the Dempster-Shafer update rule and the lower envelope of probability measures obtained by maximum likelihood update rule.

Proposition 6. Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

Because the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by:

$$v \equiv (1 - \psi) \cdot p$$

is a convex capacity and then, given $F \in \Sigma$ with $(1 - v(F^c)) > 0$, the Dempster-Shafer update rule, given by:

$$v_F(E) = \left(1 - \frac{\psi(F)}{\psi(F) + (1 - \psi(F)) \cdot p(F)}\right) \cdot p_F(E) + \frac{(\psi(F^c) - \psi(E \cup F^c)) \cdot P(E \cup F^c)}{\psi(F) + (1 - \psi(F)) \cdot p(F)}$$

is equivalent to the lower envelope of probability measures obtained by maximum likelihood update rule for $Q = core(v)$ given $F \in \Sigma$ with $q(F) > 0$ for some $q \in Q = core(v)$ which is a set Q_F of probability measures defined by:

$$Q_F = \{q_F \in \Delta : q \in \arg\max\{q'(F) : q' \in core(v)\}\}.$$

And so

$$v_F(E) = \min\{p_F(E) : p \in Q_F\}.$$

Proof. See Section *A*.7 of Appendix *A*. □

3.6 Choquet functional with contamination of confidence

The following corollary presents the Choquet functional for simple functions with generalized simple capacity with support D, which is very useful for comparative statics exercises.[6]

Corollary 1 (Choquet functional for simple functions with generalized simple capacity with support D)**.** *Let I be a norm-continuous functional on the subset of nonnegative simple functions (i.e., $I : \mathbb{R}^S_+ \to \mathbb{R}$) which is normalized, monotone and co-monotonic additive.*

Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$). Then considering the real-valued set-function $w : \Sigma \to \mathbb{R}_+$ as a generalized simple capacity with support D associated to the Knightian uncertainty level associated to D (i.e., $w \in \Lambda|_D(\psi)$, $\psi \in \Psi|_D$) defined by $w \equiv (1 - \psi) \cdot p$ and defining $w(E) = I(\mathbf{1}_E)$ for all $E \in \Sigma$, we have for all $a \in \mathbb{R}^S_+$:[7]

$$I(a) = \int_S a dw = \alpha_1 \psi(A_1) + (1 - \psi(A_1)) \int_S a dp + \sum_{i=3}^{k} (\alpha_i - \alpha_{i-1}) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \; p\left(\bigcup_{j=i}^{k} A_j\right)$$

[6] Note that because in this Chapter the set S is finite, then $B_0^+ = \mathbb{R}^S_+$.

[7] If the Knightian uncertainty level $\psi \in \Psi|_D$ is defined by $\psi(E) = \varepsilon \in [0, 1)$, for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$, for $E = \{\emptyset, S\}$ then it is easy to prove that this formula collapses to the formula of the Choquet functional with ε-contamination of confidence, i.e.:

$$\int_S a dv \equiv \varepsilon \; \alpha_k + (1 - \varepsilon) \int_S a dp.$$

where $\int_S adp$ *is the integral of a w.r.t. the probability measure p.*

Proof. See Section *A*.8 of Appendix *A*. □

The next proposition presents a new property of the Choquet functional that can be applied for the subset of convex capacities on which for any two restricted Knightian uncertainty levels (i.e., $\psi, \psi' \in \Psi|_D$) such that if one is greater than the other (which means that for all events $E \neq \{\emptyset, S\}$ it is true that $\psi'(E) \geq \psi(E)$, with strictly inequality for some E) then the Choquet functional of a convex capacity that can be represented as a contamination will be so lesser how much bigger it will be Knightian uncertainty level associated.[8]

Proposition 7 (Monotonicity of the Choquet integral with capacity restricted in the (unique) support D)**.** Let a be a simple function. Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = D$ and let ψ and ψ' be Knightian uncertainty levels restricted in D (i.e., $\psi, \psi' \in \Psi|_D$).

Let v and v' be generalized simple capacities with support D, respectively associated to the Knightian uncertainty levels restricted in D, ψ and ψ' (i.e., $v \in \Lambda|_D(\psi)$ and $v' \in \Lambda|_D(\psi)$). So it follows that: [9]

$$\text{If } \psi' \geq \psi \text{ then } \int_S adv' \leq \int_S adv.$$

Proof. See Section *A*.9 of Appendix *A*. □

It remains to note that because generalized simple capacities with support D are convex capacities then the Rosenmüller (1971, 1972) and Schmeidler (1984, 1986) is valid. This suggests the following Corollary:

Corollary 2 (Rosenmüller (1971, 1972) and Schmeidler (1984, 1986)'s Theorem)**.** *Let p be a probability measure (i.e.,* $p \in \Delta$*) with* $supp(v) = \{D\}$ *and let* ψ *be a Knightian uncertainty level associated to D (i.e.,* $\psi \in \Psi|_D$*).*

Because the real-valued set-function $w : \Sigma \to \mathbb{R}_+$ *defined by* $w \equiv (1 - \psi) \cdot p$ *is a convex capacity the following conditions are satisfied:*

[8]The interest reader should check some properties of the Choquet integral in Denneberger (1994) for capacities and Simonsen and Werlang (1991) for convex capacities.

[9]That is, for all events $E \neq \{\emptyset, S\}$ it is true that $\psi'(E) \geq \psi(E)$, with strictly inequality for some E.

i) *$core(w) \neq \emptyset$ (trivially, note that $p \in core(w)$); and*

ii) *For every $a \in \mathbb{R}_+^S$:*

$$I(a) = \int_S a dw = \alpha_1 \psi(A_1) + (1 - \psi(A_1)) \int_S a dp + \sum_{i=3}^{k} (\alpha_i - \alpha_{i-1}) \left[\psi(A_1) - \psi \left(\bigcup_{j=i}^{k} A_j \right) \right] \; p \left(\bigcup_{j=i}^{k} A_j \right).$$

Proof. See Section $A.10$ of Appendix A. □

Chapter 4

Axioms and Theorems

This Chapter has the purpose to provide a simple set of behavioral axioms under which the decision maker's preference is represented by the Choquet expected utility with the contamination of confidence. It turns out that a natural extension of the Anscombe and Aumann (1963)'s theory leads to the contamination representation of preferences (which also extends Nishimura and Ozaki (2006)'s theory). The parameter ψ here is taken as a primitive rather than deriving it from preference, as in Kopylov (2009). In this Chapter it will be restrict restrict attention to the axiomatizations in a lottery-acts framework à la Anscombe-Aumann: the Schmeidler (1989)' Choquet expected utility (CEU) model and the Gilboa and Schmeidler (1989)'s maximin expected utility (MEU) model.

The organization of this Chapter is as follows. In Section 4.1 we present the set up. Section 4.2 presents subject expected utility (SEU) in a lottery-acts framework à la Anscombe-Aumann, and Section 4.3 presents CEU à la Schmeidler. In Section 4.4 it is presented new axioms that extends Anscombe and Aumann (1963) and Nishimura and Ozaki (2006)'s axiomatizations and then in Section 4.5 is presented a new Theorem and a Corollary. We then turn to classic models and in Section 4.6 presents CEU with convex capacities, Section 4.7 presents MEU à la Gilboa-Schmeidler and, finally, in Section 4.8 is discussed under which conditions there exists equivalence between CEU and MEU.

4.1 Set up

The definitions and notations in this Chapter are standard in the literature. The basic reference is Schmeidler (1989). Let (S, Σ) be a measurable space, where S denote the set of *states of nature (world)*, which we assume to be nonempty and endowed with an algebra Σ of subsets called *events*. We use a version of the Anscombe and Aumann (1963) model as re-started by Fishburn (1970). Let X be a nonempty and finite set and Y be the von Neumann and Morgenstern (1944) set of distributions over X with finite

supports:

$$Y = \{y : X \to [0,1] / \#\{x : y(x) > 0\} < \infty \text{ and } \sum_{x \in X} y(x) = 1\}$$

where the elements of X are (deterministic) outcomes and the elements of Y are random outcomes or *(roulette) lotteries.* [1]

For notational simplicity the set X is identified with the subset $\{y \in Y : y(x) = 1$ for some $x \in X\}$ of Y. Note that Y is a mixture space, which means that it is endowed with a mixing operation, i.e., for every $y, y' \in Y$ and $\lambda \in [0,1]$ a *compound lottery* $\lambda y + (1-\lambda)y'$ is defined such that for all $x \in X$:

$$(\lambda y + (1-\lambda)y')(x) = \lambda y(x) + (1-\lambda)y'(x).$$

A *lottery act*, or more simply, an *act*, is a Y-valued Σ-measurable function on S and is also known as a *horse lottery*. The set of *acts* is denoted by L.

$$L = \{f : S \to Y : \ f \text{ is } Y\text{-valued } \Sigma\text{-measurable function on } S\}.$$

A *simple lottery act*, or more simple, an *act* is a Y-valued Σ-measurable function on S whose range is a finite subset of Y. The set of *simple acts* is denoted by L_0.

$$L_0 = \{f : S \to Y : \ f \text{ is } Y\text{-valued } \Sigma\text{-measurable function on } S \text{ whose range is a finite subset of } Y\}.$$

It is endowed with a mixture operator as well, performed pointwise. That is, for every $f, g \in L_0$ and every $\lambda \in (0,1)$, the *compound lottery act*, or more simple, a *compound act* $\lambda f + (1-\lambda)g$ or $f_\lambda g$ is defined such that for all $s \in S$:

$$f_\lambda g(s) \equiv (\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s).$$

A *simple lottery act* whose range is a singleton is referred to as a *constant lottery act*, or more simple, a *constant act*. The set of *constant acts* is denoted by L_c.

$$L_c = \{f : S \to Y : \ f \in L_0 \text{ and } f(s) = y \text{ for all } s \in S\}.$$

The decision-maker's preference is characterized by a binary relation $\succeq$ on L_0. Two binary relations, $\succ$ and $\sim$ (respectively, the asymmetric and symmetric parts of $\succeq$), are defined from $\succeq$ by: $\succ \Leftrightarrow [\succeq \text{ and } \not\preceq]$ and $\sim \Leftrightarrow [\succeq \text{ and } \preceq]$.

A binary relation is defined over Y by restricting $\succeq$ on L_c and is denoted by the same symbol $\succeq$, that is, for all $y, y' \in Y$:

$$y \succeq y' \Leftrightarrow \exists\ f, g \in L_c \text{ such that for all } s \in S : f(s) = y,\ g(s) = y' \text{ and } f \succeq g.$$

[1] In fact, $Y \subseteq \Delta$, i.e., Y is a subset of probability measures on (S, Σ) endowed with the eventwise convergence topology (which means that $\{p_k\}$ converges to p if and only if $\{p_k(E)\} \to p(E)$ for all $E \in \Sigma$).

Two acts f and g, are said to be *co-monotonic* if for all $s, t \in S$:[2]

$$f(s) \succ f(t) \;\Rightarrow\; g(t) \not\succ g(s).$$

A *utility function* is a real-valued function on Y, $u : Y \to \mathbb{R}$ and an *expected utility function* is defined on the class of *simple acts* L_0, and for all $y \in Y$ it will be used the following notation:

$$E_y U = \sum_{x \in X} y(x)u(x).$$

One may replace Y by any convex subsets of a vector set, or even any mixture space, and $E_y U$ with evaluation at y of an affine function V on Y. All results remain valid. Thus, if the decision maker chooses $f \in L_0$ and nature chooses $s \in S$, the decision gets a lottery $f(s)$, which has the *expected utility value* of:

$$E_{f(s)} U = \sum_{x \in X} f(s)(x)u(x).$$

4.2 SEU à la Anscombe and Aumann (1963)

It will be considered the following set of axioms which may be imposed on an arbitrary binary relation $\succeq$ defined on L_0. [3]

Axiom 1 (Weak order)**.** $\succeq$ *is a* weak order *on* L_0 *if* $\succeq$ *is* complete *and* transitive*:*[4]

i) $\succeq$ *is* complete *if for all* $f, g \in L_0$*:* $f \succeq g$ *or* $g \succeq f$*; and*

ii) $\succeq$ *is* transitive *if for all* $f, g, h \in L_0$*: if* $f \succeq g$ *and* $g \succeq h$*, then* $f \succeq h$*.*

Axiom 2 (Continuity)**.** *For all* simple acts $f, g, h \in L_0$*, if* $f \succ g$ *and* $g \succ h$ *then* $\exists$ $\lambda, \lambda' \in (0, 1)$ *such that:*

$$\lambda f + (1 - \lambda)h \succ g \text{ and } g \succ \lambda' f + (1 - \lambda')h.$$

[2]Alternatively, two acts f and g are said to be *co-monotonic* if and only if for all $s, t \in S$: $(f(s) - f(t)) \cdot (g(t) - g(s)) \geq 0$.

[3]In the axioms, f and g denotes arbitrary elements in L_0 and λ denotes an arbitrary real number such that $\lambda \in [0, 1]$.

[4]Alternatively, a binary relation $\succeq$ is a *weak order* on L_0 by definition if $\succ$ is *asymmetric* and *negatively transitive*: (i) $\succ$ is *asymmetric* if for all $f, g \in L_0$: $f \succ g \Rightarrow g \not\succ f$; and (ii) $\succ$ is *negatively transitive* if for all $f, g, h \in L_0$: $f \not\succ g$ and $g \not\succ h \Rightarrow f \not\succ h$ (Kreps (1988)).

Axiom 3 (Independence)**.** *For all* simple acts $f, g, h \in L_0$ *and for all* $\lambda \in (0,1)$:

$$f \succ g \Rightarrow \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h.$$

An interesting property that should be satisfied is that preferences should not depend on the state that occurred. The next two axioms present two different versions of this fact.[5]

Axiom 4 (Monotonicity)**.** *For all* simple acts $f, g \in L_0$ *if* $f(s) \succeq g(s)$*, for all* $s \in S$*, then* $f \succeq g$*.*

Axiom 5 (Strict Monotonicity)**.** *For all* simple acts $f, g \in L_0$*,* $y, z \in Y$ *and* $E \in \Sigma$ *if* $f \succ g$*,* $f(s) = y$ *on* E*,* $g(s) = z$ *on* E *and* $f(s) = g(s)$ *on* E^c*, for all* $s \in S$*, then* $y \succ z$*.*

Axiom 6 (Non-degeneracy)**.** *Not for all* simple acts $f, g \in L_0$*,* $f \succeq g$*.*

Anscombe and Aumann (1963)'s main result (Fishburn (1970)'s version) asserts that a binary relation $\succeq$ defined on L_0 satisfies Axioms 1, 2, 3, 5 and 6 if and only if there exists a unique finitely probability measure p on (S, Σ) and an *affine* (utility) function $u : Y \to \mathbb{R}$ such that for all f and g in L_0:[6]

$$f \succeq g \Leftrightarrow \int_S u(f(\cdot))dp \geq \int_S u(g(\cdot))dp.$$

Moreover, the (utility) function is unique up to a positive affine transformation.

4.3 CEU à la Schmeidler (1989)

The Schmeidler (1989) model uses a less restrictive Axiom than the Axiom 3 (independence, also called the "sure thing principle") in the Anscombe and Aumann (1963) model, called co-monotonic independence axiom:

Axiom 7 (Co-monotonic Independence)**.** *For all pairwise* co-monotonic simple acts $f, g, h \in L_0$ *and for all* $\lambda \in (0,1]$*:*

$$f \succ g \Rightarrow \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h.$$

[5]It is easy to prove that if Axiom 1 holds then Axiom 4 and Axiom 5 are equivalent.

[6]Because Axiom 1 holds, the use of Axiom 4 here instead of Axiom 5 (as in Anscombe and Aumann (1963) and Fishburn (1970)) does not make a difference, as already mentioned in footnote 5 of this Chapter.

Schmeidler (1989)'s axiomatic model of decision making under uncertainty is based on capacities (non-probability measures) and on the notion of integration due to Choquet (1953), also known as Choquet expected utility (CEU) over a set of priors decision rule.

Schmeidler (1989)'s main result asserts that a binary relation $\succeq$ defined on L_0 satisfies Axioms 1, 2, 4, 6 and 7 if and only if there exists a unique capacity v on (S, Σ) and an *affine* (utility) function $u : Y \to \mathbb{R}$ such that for all f and g in L_0:[7]

$$f \succeq g \Leftrightarrow \int_S u(f(\cdot))dv \geq \int_S u(g(\cdot))dv$$

where the *Choquet expected utility* w. r. t. a *capacity* v and to an *affine* function $u : Y \to \mathbb{R}$ for a given *simple act* $a \in L_0$ is defined by: [8]

$$\int_S u(a(\cdot))dv \equiv \int_0^\infty v(L_\alpha(u(a(\cdot))))d\alpha + \int_{-\infty}^0 [v(L_\alpha(u(a(\cdot)))) - 1]d\alpha$$

where $L_\alpha(u(a(\cdot)) = \{s \in S : u(a(s)) \geq \alpha\}$ and the integral on the right hand side is a well defined integral in the usual sense (Riemann or Lebesgue), because a is bounded and v is monotone.

Moreover, the (utility) function is unique up to a positive affine transformation.

4.4 New axioms

In the following discussion, the ordering underlying *constant acts* plays a crucial role. Given $f \in L_0$, let $Y_{f(s_1)}$ be the subset of Y representing f's worst-limit constant act, defined by

$$Y_{f(s_1)} = \{y \in Y : \text{ for all } s \in S,\ y \preceq f(s) \text{ and } \exists\ s \in S \text{ such that } y = f(s)\}.$$

Since that f is a simple act, $Y_{f(s_1)}$ is non empty when $\succeq$ is a *weak order*. Denote by $y_{f(s_1)}$ an arbitrary element of $Y_{f(s_1)}$.

Similarly, let $Y_{f(s_2)}$ be the subset of Y representing f's second worst constant act, defined by

$$Y_{f(s_2)} = \{y \in Y \backslash Y_{f(s_1)} : \text{ for all } s \in S,\ f(s_1) \prec y \preceq f(s) \text{ and } \exists\ s \in S \text{ such that } y = f(s)\}.$$

Denote by $y_{f(s_2)}$ an arbitrary element of $Y_{f(s_2)}$.

[7] Relative to Anscombe-Aumann, Schmeidler replaced strict monotonicity with monotonicity, independence with co-monotonic independence and the finitely probability measure with a capacity.

[8] If the capacity is in fact a finitely probability measure then the expression on the right-hand side collapses to the standard subject expected utility formulation.

An analogous procedure can be done for each $j \in \{3, \cdots, k\}$. Thus, let $Y_{f(s_j)}$ be the subset of Y representing the f's $j-$worst constant act, defined by

$$Y_{f(s_j)} = \{y \in Y \backslash Y_{f(s_{j-1})} : \text{ for all } s \in S,\ f(s_{j-1}) \prec y \preceq f(s) \text{ and } \exists\ s \in S \text{ such that } y = f(s)\}.$$

Denote by $y_{f(s_j)}$ an arbitrary element of $Y_{f(s_j)}$, for all $j \in \{3, \cdots, k\}$.

In addition to Schmeidler (1989)'s well-known axioms: 1) weak order, 2) continuity, 4) monotonicity, 6) nondegeneracy and 7) comonotonic independence, we consider the following axioms which may be imposed on an binary relation $\succeq$ defined on L_0.

The first axiom requires that any *simple act* f is dominated by some compound lottery act which consider the ordering underlying constant acts ($y_{f(s_k)} \succeq y_{f(s_{k-1})} \succeq \cdots \succeq y_{f(s_2)} \succeq y_{f(s_1)}$) and the best limit constant act ($y_{f(s_k)}$). In the axiom, ψ is a Knightian uncertainty level associated to the (unique) support.

Axiom 8 (ψ-Dominance). [9]

$$\psi(A_1)y_{f(s_1)} + (1 - \psi(A_1))y_{f(s_k)} + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \succeq f.$$

Under Axioms 1, 2, 4, 6, 7 and 8, it can be shown (see *Lemma* 4 of Appendix B) that all $f \in L_0$ has the following contamination equivalence.

For all $f \in L_0$, $\exists\ y_f \in L_c$ such that:

$$f \sim \psi(A_1)y_{f(s_1)} + (1 - \psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

where ψ is the Knightian uncertainty level associated to the (unique) support with which Axiom 8 holds. This property shows that all *simple acts* have their own equivalent *compound act*. Clearly, y_f is one way of representing f. We hereafter call it f's *equivalent constant act* in *contamination equivalence.*

The next axiom concerns ordering among these equivalent constant acts in contamination equivalence. In the axiom, each ψ is a Knightian uncertainty level associated to the (unique) support.

Axiom 9 (Irrelevance of the ordering of constant acts). [10] *Both of the following hold:*

1) *(Affine irrelevance) If there exists* y_f, y_g, $y_{f\lambda g} \in L_c$ *such that*

$$f \sim \psi(A_1)y_{f(s_1)} + (1 - \psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

[9] This Axiom extends the Axiom of ε-Dominance in Nishimura and Ozaki (2006).

[10] This Axiom extends the Axiom of Worst-limit irrelevance in Nishimura and Ozaki (2006)

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$f\lambda g \sim \psi(A_1)y_{f\lambda g(s_1)} + (1-\psi(A_1))y_{f\lambda g} + \sum_{i=3}^{k} \left(y_{f\lambda g(s_i)} - y_{f\lambda g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

then

$$y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$$

and

2) *(Monotone irrelevance) If, for all* $s \in S$, $f(s) \succeq g(s)$ *and there exist* y_f, $y_g \in L_c$ *such that*

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

then

$$y_f \succeq y_g.$$

Axiom 9−1 means that if y_f, y_g and $y_{f\lambda g}$ are *equivalent acts* of f, g and $f\lambda g$ in contamination equivalence, respectively, then $y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$, regardless of characteristics of the ordering underlying *constant acts*:

$$y_{f(s_k)} \succ y_{f(s_{k-1})} \succ \cdots \succ y_{f(s_2)} \succ y_{f(s_1)}$$

$$y_{g(s_k)} \succ y_{g(s_{k-1})} \succ \cdots \succ y_{g(s_2)} \succ y_{g(s_1)}$$

and

$$y_{f\lambda g(s_k)} \succ y_{f\lambda g(s_{k-1})} \succ \cdots \succ y_{f\lambda g(s_2)} \succ y_{f\lambda g(s_1)}$$

Similarly, Axiom 9−2 implies that if $f(s) \succeq g(s)$ for all $s \in S$, then $y_f \succeq y_g$, regardless of characteristics of the ordering underlying *constant acts:*

$$y_{f(s_k)} \succ y_{f(s_{k-1})} \succ \cdots \succ y_{f(s_2)} \succ y_{f(s_1)}$$

and

$$y_{g(s_k)} \succ y_{g(s_{k-1})} \succ \cdots \succ y_{g(s_2)} \succ y_{g(s_1)}$$

These two axioms imply that the ordering underlying constant acts are irrelevant in ordering among equivalent constant acts in contamination equivalence.

Axioms 8 and 9, which generalizes Nishimura and Ozaki (2006)'s axioms are closely related to the axioms of Anscombe and Aumann (1963), especially their independence axiom (or equivalently, Schmeidler (1989)'s (iii) independence axiom). In fact they can be considered as a natural extension of the Anscombe-Aumann theory to the case in which the decision-maker consider the ordering underlying constant acts associated all the time.

As in Schmeidler (1989)'s main result, the Axioms 1, 2, 4, 6 and 7 as a whole characterize the preference which is represented by the Choquet expected utility with respect to some capacity.[11]

4.5 New theorem and corollary

The main result of this Chapter is the following theorem and corollary. The proof is relegated to Appendix B.

Theorem 2. *Given any Knightian uncertainty level associated to the (unique) support (i.e., $\psi \in \Psi|_D$), a binary relation $\succeq$ defined on L_0 satisfies the Axioms* 1, 2, 4, 6, 7, 8 *and* 9 *if and only if there exist a unique finitely probability measure p on (S, Σ), an affine function $u : Y \to \mathbb{R}$, such that for all f and g in L_0:*

$$f \succeq g \Leftrightarrow \int_S u(f(\cdot))dv \geq \int_S u(g(\cdot))dv$$

where v is a generalized simple capacity with an unique support and

[11] For a related axiomatization, see Gilboa and Schmeidler (1989).

$\int_S u(a(\cdot))dv$ *can be defined as:*

$$\int_S u(a(\cdot))dv \equiv \psi(A_1)u(a(s_1)) + (1-\psi(A_1))\int_S u(a(\cdot))dp + \\ + \sum_{i=3}^{k}\left(u(a(s_i)) - u(a(s_{i-1}))\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)$$

where $p : \Sigma \to [0,1]$ *is a finitely probability measure on* (S, Σ)*; and* $\int_S u(a(s))dp$ *is its subject expected utility.*

Moreover, the (utility) function is unique up to a positive affine transformation.

Proof. See Appendix B. □

Let p be a probability measure (i.e., $p \in \Delta$) with $supp(v) = \{D\}$ and let ψ be a Knightian uncertainty level associated to the (unique) support (i.e., $\psi \in \Psi|_D$). The *contamination of confidence* of p with support D, $cont(p,\psi)|_D \equiv \{p\}^{\psi}|_D$, is a subset of Δ (i.e., $\{p\}^{\psi}|_D \subsetneq \Delta$), defined in the following way: [12]

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:

$$\{p\}^{\psi}|_D = \{(1-\varepsilon)\cdot p(E) + \varepsilon \cdot q(E) : q \in \Delta\}.$$

and

ii) If $\exists E \supseteq D$, with $\psi(E) < \varepsilon$, then:

$$\{p\}^{\psi}|_D = \{(1-\psi(E))\cdot p(E) + \psi(E)\cdot q(E) : q \in \Delta \text{ such that } supp(q) \subseteq D\}.$$

Then, it follows that for all *simple acts* $f \in L_0$:

$$\int_S u(a(\cdot))d\{p\}^{\psi}|_D \equiv min\left\{\int_S u(f(\cdot))dp : p \in \{p\}^{\psi}|_D\right\} = \\ = \psi(A_1)u(s_1) + (1-\psi(A_1))\int_S u(f(\cdot))dp + \\ + \sum_{i=3}^{k}\left(u(f(s_i)) - u(f(s_{i-1}))\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right).$$

Therefore, the following corollary is immediate.

[12] The convex capacity v corresponding to a contamination of confidence of p is given by:

$$v \equiv (1-\psi)\cdot p.$$

Corollary 3. *Given any Knightian uncertainty level associated to the (unique) support* ψ *(i.e.,* $\psi \in \Psi|_D$*), a binary relation* $\succeq$ *defined on* L_0 *satisfies Axioms* 1, 7, 2, 4, 6, 8 *and* 9 *if and only if there exists a unique finitely probability measure* p *on* (S, Σ)*, an affine function* $u : Y \to \mathbb{R}$ *such that for all* f *and* g *in* L_0*:*

$$f \succeq g \Leftrightarrow \int_S u(f(\cdot))\ d\ \{p\}^{\psi}|_D \geq \int_S u(g(\cdot))\ d\ \{p\}^{\psi}|_D.$$

Moreover, the (utility) function is unique up to a positive affine transformation.

4.6 CEU with convex capacities

The following axiom should be considered if we are interested in restrict attention over convex capacities. It captures the phenomenon of hedging, especially when the preference is strict.

Axiom 10 (Uncertainty aversion)**.** *For all* simple acts $f, g \in L_0$*, and* $\lambda \in (0, 1)$*:* $f \sim g \Rightarrow \lambda f + (1 - \lambda)g \succeq f$.

A Corollary of Schmeidler (1989)'s main result asserts that a binary relation $\succeq$ defined on L_0 satisfies Axioms 1, 2, 3, 4, 6 and 10 if and only if there exists a unique convex capacity v on (S, Σ) and an affine utility function $u : Y \to \mathbb{R}$ such that for all f and g in L_0:

$$f \succeq g \Leftrightarrow \int_S u(f(\cdot))dv \geq \int_S u(g(\cdot))dv.$$

where the *Choquet expected utility* w. r. t. a *convex capacity* v and to an *affine* function $u : Y \to \mathbb{R}$ for a given *simple act* $a \in L_0$ is defined by: [13]

$$\int_S u(a(\cdot))dv \equiv \int_0^{\infty} v(L_\alpha(u(a(\cdot))))d\alpha + \int_{-\infty}^{0} [v(L_\alpha(u(a(\cdot)))) - 1]d\alpha$$

where $L_\alpha(u(a(\cdot)) = \{s \in S : u(a(s)) \geq \alpha\}$ and the integral on the right hand side is a well defined integral in the usual sense (Riemann or Lebesgue), because a is bounded and v is monotone.

Moreover, the (utility) function is unique up to a positive affine transformation.

[13] Note that if the capacity is in fact a finitely probability measure then the expression on the right-hand side collapses to the standard subject expected utility formulation.

4.7 MEU à la Gilboa and Schmeidler (1989)

In Gilboa-Schmeidler model a weaker version of the co-monotonic independence axiom is used, the certainty independence axiom. The following axiom is less restrictive than the co-monotonic independence axiom. [14]

Axiom 11 (Certainty Independence (C-independence for short)). *For all* $f, g \in L_0$ *and* $h \in L_c$ *and for all for all* $\lambda \in (0, 1]$*:*

$$f \succ g \Rightarrow \lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h.$$

Gilboa and Schmeidler (1989)'s axiomatic model of decision making under uncertainty is based on a set of prior probabilities, rather than by a single probability coupled with a decision rule that chooses an act whose minimal expected utility (over all prior probabilities in the set) is the highest, also known as maximin expected utility (MEU) over a set of priors decision rule.

Gilboa and Schmeidler (1989)'s main result asserts that a binary relation $\succeq$ defined on L_0 satisfies Axioms 1, 2, 4, 6, 10 and 11 if and only if there exists a unique convex and compact subset of the set of finitely probability measures on (S, Σ), say C ($C \subsetneq \Delta$) and an *affine* (utility) function $u : Y \to \mathbb{R}$ such that for all f and g in L_0:[15]

$$f \succeq g \Leftrightarrow min\{\int_S u(f(\cdot))dp : p \in C\} \geq min\{\int_S u(g(\cdot))dp : p \in C\}.$$

Moreover, the (utility) function is unique up to a positive affine transformation.

4.8 Equivalence between CEU and MEU

Schmeidler (1986) shows that the Choquet integral of a simple function a with respect to a convex capacity v is equal to the minimum of $\{\int adp : p \in core(v)\}$. Gilboa and Schmeidler (1989)'s main result is equivalent to the Schmeidler (1989)'s main result added by the uncertainty aversion measure if the subset C is the core of the convex capacity v. That is equivalent to say that CEU à la Schmeidler with a convex capacity is equivalent to MEU à la Gilboa-Schmeidler when the set of possible priors is the core of the convex capacity v (i.e., $C = core(v)$).

[14] Note that any *simple act* and a *constant act* are *co-monotonic*. So *C-independence* is a less restrictive condition than the *co-monotonic independence* condition.

[15] Relative to Schmeidler added by the uncertainty aversion axiom, Gilboa-Schmeidler replaced co-monotonic independence with C-independence and the capacity for a convex and compact subset C of finitely probability measures on (S, Σ).

When an arbitrary (closed and convex) set of priors C is given, and one defines $v(A) = min\{p(A) : p \in C\}$ v need not to be convex.[16] Furthermore, even if v happens to be convex (does note have to be its core) it is not hard to construct an example in which C is a proper subset of the core of v.

Dow and Werlang (1992) is refereed as the first application of this theory (they justify the possibility of inertia in a portfolio choice). Mukerji and Tallon (2004) survey economic applications for models that uses CEU and MEU models.

[16] In this case it is exact as shown in Schmeidler (1972) and Huber and Strassen (1973).

Part II

Economic applications

Chapter 5

Nash Equilibrium under Knightian uncertainty

We will present a definition of Nash equilibrium for two-person normal-form games in the presence of Knightian uncertainty that extends the notion which is due to Dow and Werlang (1994). With our framework it is possible to explain explicitly the paradox on which players of a finitely repeated prisoners´dilemma breaks down backward induction. Considering the use of contaminations (instead of ε-contaminations) it is possible to find all pure-strategies Nash equilibrium under uncertainty of any two-person finite normal form game and much more equilibria than in Dow and Werlang (1994)'s article. Our result is also related to Marinacci (2000)'s ambiguous games.

The organization of this Chapter is as follows. In Section 5.1 we introduce a brief discussion on the theme. In Section 5.2 we discuss about Nash Equilibrium under uncertainty. Finally, Section 5.3 provides the example that shows that players in finitely repeated prisoners' dilemma breaks down backward induction.

5.1 Introduction

In the literature of finitely repeated games of complete information, it is a very known fact that if the stage game has a unique Nash equilibrium, then according to the procedure of backward induction the unique subgame perfect Nash equilibrium of the finitely game is to play the stage game in every stage, after every story. A classical example of a game on which this occurs is the prisoners' dilemma where defect is a strictly dominant strategy for both players. This implies that the prediction of the finitely prisoners' dilemma, using the procedure of backward induction, is for both players to defect in every period. However, despite this theoretical prediction, a lot of experimental evidence shows that

players in the prisoners' dilemma do in fact tend to cooperate in many periods.[1] Dow and Werlang (1994) extended the notion of Nash equilibrium to incorporate the fact that players act in a game considering the presence of uncertainty, and show an example that justify cooperative behavior of the players in the finitely repeated prisoners´ dilemma, depending on how averse to uncertainty players are. [2]

According to Dow and Werlang (1994)'s model, uncertainty is introduced through the use of simple capacities with parameter ε (also known as ε-contamination of confidence). They suggest that depending on the uncertainty aversion it is possible to justify that players do not backward induct in the finitely repeated prisoners´ dilemma. In fact, with the simple capacities used in their proof of the existence theorem of Nash equilibrium under uncertainty it is impossible to explain explicitly the fact that players breaks down backward induction in the prisoners' dilemma. But they suggest, through an example, that there exists convex capacities that can solve the paradox on which players breaks down backward induction in the prisoners' dilemma although these convex capacities can not be equivalently represented as simple capacities (as in their existence theorem). In this Chapter we will extend the Dow and Werlang (1994) notion of Nash equilibrium under uncertainty using the less restrictive class of convex capacities studied in the first part of this thesis, the generalized simple capacities (or contamination of confidence) in place of simple capacities (or ε-contamination) as in Dow and Werlang (1994)'s paper. An interesting fact is that the convex capacity described in Dow and Werlang (1994)'s example can be represented as a generalized simple capacity. This occurs because the support has only one element (strategy). So, with our framework it is possible to explain precisely the paradox on which players breaks down backward induction in a finitely repeated prisoners' dilemma. Besides this, through the use of generalized simple capacities it is possible to find all pure-strategy Nash equilibria under uncertainty of any finite two-person game and among other advantages is the ease of doing comparative statics exercises. The uncertainty is an exogenous parameter in our framework and we also relate our work to Marinacci (2000).

5.2 A generalization of Dow and Werlang's existence Theorem

We will follow Dow and Werlang (1994) set up. We restrict attention to two-person finite normal form games. Let $\Gamma = (A_1, A_2; u_1, u_2)$ be a bi-matrix game where for any $i \in \{1, 2\}$ the A_i's are pure strategy sets and u_i's are utilities (*payoffs*) depending on the

[1] Experiments conducted, among others, by Neelin and Spiegel (1988), and the well know prisoners´ dilemma tournament of Axelrod (1981) show that players of a finitely repeated prisoners´ dilemma do not act according to the logic of backward induction.

[2] Kreps and Wilson (1982) proposed a solution through a reputation model.

pure strategy combination $(a_1, a_2) \in A_1 \times A_2$. This will be called the primitive game, or game without uncertainty.

Let us now turn to the definition of Nash equilibrium under uncertainty. The point of departure will be a well known definition of mixed strategy in standard theory: a probability on the set of pure strategies of the player. A mixed strategy Nash equilibrium can be defined as follows. Let (p_1, p_2) be a pair of probability measures and let $supp(p_i)$ denote the support of p_i. In Nash equilibrium, every $a_1 \in supp(p_1)$ is a best response to p_2 (i.e. a_1 maximizes the expected utility of player 1 given that player 2 is playing the mixed strategy p_2; conversely, every $a_2 \in supp(p_2)$ is a best response to p_1).

Now, under uncertainty, what happens is that each player no longer views the strategy of the other player as a probability measure, but as a convex capacity on the other player's strategy set.

Let (v_1, v_2) be a pair of generalized simple capacities, that represents the beliefs of players 1 and 2; v_i represents the belief player $j \neq i$ holds about pure-strategy play of player i. A subjective interpretation can be given to the Nash equilibrium, as firstly suggested by Aumann (1987): the mixed strategy of player 1, v_1, may be viewed as the belief that player 2 has about the pure strategy play of player 1. Conversely, the mixed strategy of player 2, v_2, may be viewed as the beliefs player 1 has about the pure strategy play of player 2. In the absence of uncertainty these subjective beliefs reduces to the pair of additive probability measures as in a game without uncertainty.

The definition that we will present for Nash equilibrium under uncertainty extends Dow and Werlang (1994)' definition. We use a less restrictive class of convex capacities, the generalized simple capacities, which will enable us to find all pure-strategy Nash equilibria and much more equilibria than in Dow and Werlang (1994) paper. If there is no uncertainty our definition reduces to the traditional definition, due to Nash (1951).

Definition 5. Nash Equilibrium under Uncertainty For each player $i \in \{1, 2\}$ let p_i be a probability measure (i.e., $p_i \in \Delta$) with $supp(p_i) = \{D_i\}$ that represents player's $-i$'s beliefs without uncertainty about the strategies of player i and let $\psi^i \in \Psi^i(A_{-i}, 2^{A_{-i}})|_{D_{-i}}$, where $\psi^i : 2^{A_{-i}} \to [0, 1)$ is the Knightian uncertainty level of $v_{-i} \in cont(p_{-i}, \psi^i)|_{D_{-i}}$.[3]

A pair (v_1, v_2) of convex capacities that can be represented as a generalized simple capacity $(v_i \in cont(p_i, \psi^{-i})|_{D_i}, i \in \{1, 2\})$, v_1 over A_1 and v_2 over A_2 is a Nash equilibrium under uncertainty if given the support of v_1 and the support of v_2 it it is true that:

i) for all a_1 in the support of v_1, a_1 maximizes the Choquet expected utility of player 1, given that v_2 represents player 1's beliefs about the strategies of player 2; and conversely,

[3]The reason for the interchange in the subscripts is that v_{-i} is what player i thinks player $-i$ is going to do, so that the uncertainty aversion of v_{-i} is a characteristic of player i, for $i \in \{1, 2\}$.

ii) for all a_2 in the support of v_2, a_2 maximizes the Choquet expected utility of player 2, given that v_1 represents player 2 beliefs about the strategies of player 1.[4]

The following theorem that extends Dow and Werlang (1994)'s result with the use of the Knightian uncertainty level of each player $i \in \{1,2\}$ as a parameter in the game (instead of a constant Knightian uncertainty level as they used).

Theorem 3. ***Existence of Nash Equilibrium under Knightian uncertainty*** *Let* $\Gamma = (A_1, A_2; u_1, u_2)$ *be a two-person finite normal form game.*

For each player $i \in \{1,2\}$ *let* p_i *be a probability measure (i.e.,* $p_i \in \Delta$*) with* $supp(p_i) = \{D_i\}$ *that represents player's* $-i$*'s beliefs without uncertainty about the strategies of player* i *and let* $\psi^i \in \Psi^i(A_{-i}, 2^{A_{-i}})|_{D_i}$*, where* $\psi^i : 2^{A_{-i}} \to [0,1)$ *is the Knightian uncertainty level associated to* D_i *of* $v_{-i} \in cont(p_{-i}, \psi^{-i})|_{D_i}$*.*

For all $(\psi^1, \psi^2) \in [0,1) \times [0,1)$*, there exists a Nash equilibrium under uncertainty* (v_1, v_2)*, such that* ψ^1 *is the Knightian uncertainty level associated to* D_2 *of* v_2*, and* ψ^2 *is the Knightian uncertainty level associated to* D_1 *of* v_1*.*

Proof. See Section $C.1$ of Appendix C. □

The following proposition shows that this theorem enables to find all pure strategy Nash equilibrium under uncertainty of any two-person finite normal form game:

In particular, for a constant Knightian uncertainty level (as in Dow and Werlang (1994)'s article):

Proposition 8. Every Dow-Werlang's Nash equilibrium under uncertainty is Nash equilibrium under uncertainty as our definition.

Proof. See Section $C.2$ of Appendix C. □

[4]Note that this definition reduces to the standard definition of Nash equilibrium whenever there is no uncertainty (which means that the v_i's are additive). In other words if there is no uncertainty then the pair (v_1, v_2) reduces to a pair of probability measures and the Choquet expected utility reduces to the subjective expected utility. Clearly, a standard mixed strategy Nash equilibrium is also a Nash equilibrium under uncertainty.

5.3 Breaking down backward induction

Dow and Werlang (1994, p.316)'s paper illustrates in Example 3 that cooperation may arise in the twice repeated prisoners' dilemma, which demonstrates that (Knightian) rational players may not backward induct as predicted by the theory. We will consider the same version of the prisoner's dilemma of Dow and Werlang (1994), as shown in figure below. The strategies are F (for "fink") and C (for "cooperate").

	F	C
F	$(0,0)$	$(1.25,-0.5)$
C	$(-0.5,1.25)$	$(1,1)$

The same remarks in Dow and Werlang (1994) applies here:

> *"In the twice repeated version of this game, there are eight strategies for each of the players. Four are unconditional, or history independent strategies, F^2, FC, CF, C^2, which stand for, respectively, F in both rounds, F in the first round and then C in the second round, C and then F, and C in both rounds. There are four history dependent strategies, which we name W, X, Y and Z, for convenience. W is: start with F. If the other player played C in the first round, then play C in the second round. Otherwise play F in the second round. The letter X stands for: start with F, and if the other player played C then play F, otherwise play C. The letter Y stands for the tit-for-tat: start with C, and play C in the second round if the other player played C in the first round. Otherwise play F. Finally, Z stands for: start with C, and play F in the second round if C was also played by the other player. Otherwise, play C."*
>
> *Dow and Werlang (1994, p. 317)*

Consider the game without discounting (i.e., the payoffs are sums of payoffs of each one-shot game). It may be verified that if each player $i \in \{1,2\}$ has beliefs that the other player will play $\{CF\}$ (and so this strategy corresponds to the only support) and if each player $i \in \{1,2\}$ has the following Knightian uncertainty level associated to $\{CF\}$, $\psi^i : 2^{A_{-i}} \to [0,1]$ defined by:

(1) $\psi(\{F^2, CF, C^2, W\}) = 0$;
(2) $\psi(\{F^2\}) = \psi(\{C^2, CF, W\}) = 0.2$;
(3) $\psi(\{CF\}) = \psi(\{C^2\}) = \psi(\{W\}) = \psi(\{CF, C^2\}) = \psi(\{CF, W\}) = \psi(\{C^2, W\}) = \psi(\{F^2, C^2\}) = \psi(\{F^2, CF\}) = \psi(\{F^2, CF, C^2\}) = \psi(\{F^2, CF, W\}) = \psi(\{F^2, CF, C^2\}) = 0.6$;
(4) For all events B, $\psi(\{B\}) = \psi(\{B \cap \{F^2, CF, FC, C^2, W\}\})$.

Then there exists a Nash equilibrium under uncertainty in which both players has beliefs given by the following convex capacity restricted in $\{CF\}$, $v : 2^{A_{-i}} \to [0,1]$:

(1) $v(\{F^2, CF, C^2, W\}) = 1$;
(2) $v(\{C^2, CF, W\}) = 0.8$;
(3) $v(\{F^2, CF, W\}) = v(\{F^2, CF, C^2\}) = 0.4$;
(4) $v(\{F^2\}) = v(\{CF, W\}) = v(\{CF, C^2\}) = 0.4$;
(5) $v(\{W\}) = v(\{C^2\}) = v(\{F^2\}) = v(\{F^2, C^2\}) = v(\{C^2, W\}) = v(\{F^2, C^2, W\}) = 0$;
(6) $v(\{CF\}) = 0.4$;
(7) For all events B, $v(\{B\}) = v(\{B\} \cap \{F^2, CF, FC, C^2, W\})$.

To check that $\{CF\}$ is in fact the prediction of the game according with these beliefs note that the Choquet expected *payoffs* for each players, given the belief v about the other player's action is, for each strategy: $U(F^2) = 0,5$, $U(FC) = 0$, $U(CF) = 0,6$, $U(C^2) = 0,2$, $U(W) = 0,3$, $U(X) = 0,2$, $U(Y) = 0,3$, $U(Z) = 0,4$. This means that cooperation in the first round may occur, again with Knightian rationality.

Comparative statics exercises

One may ask what happen if the uncertainty perceived by some player increases (which implies a increasing in the Knightian uncertainty level). Suppose, for instance, that $\psi(F^2) = 0,2$ increases to $\psi'(F^2) = 0,25$ every else remaining the same. It is easy to note that CF is still the prediction of the repeated prisoners' dilemma game. However if $\psi'(F^2) = 0,3$ then CF will no longer be the best response (because in this case it will be strict dominated by F^2).

Chapter 6

Portfolio inertia and contamination

According to Arrow (1965) there is no sense in some stocks not to be traded or not to be quoted at any time. However everyday is observed portfolio inertia in the stock markets. Using Gilboa and Schmeidler (1989)'s maximin expected utility in an Anscombe and Aumann (1963) framework, Dow and Werlang (1992), in a seminal article, show that portfolio inertia may appear, due to some uncertainty.

Asano (2008) analyses investor's portfolio selection problem in a two-period dynamic model of Knightian uncertainty that extends Dow and Werlang (1992)'s static framework to a two-period dynamic framework in order to incorporate decision makers updating behavior and shows that obtaining new information affects portfolio inertia, using ε-contamination. In this chapter we will extend Asano (2008)'s framework in a Knightian uncertainty model à la Gilboa and Schmeidler (1989) in an Anscombe and Aumann (1963) framework on which the set of probability measures is based on the contamination of confidence of some probability measure.

Our results extends Asano (2008)'s results and as in his analysis, we also show that observing new information in the first period will expand portfolio inertia in the second period compared with the case in which observing new information has not been gained in the first period if the uncertainty is sufficiently large. It is important to note that through the use of contamination it is possible to do much more comparative statics exercises, compared to the ε-contamination case.

The organization of this Chapter is as follows. In Section 6.1 we provide the stochastic environment of a two-period model à la Asano (2008). Section 6.2 presents that the Fagin - Halpern and the Dempster - Shafer update rules. Section 6.3 discuss about contamination as the set of priors, which is a restriction of the set of decision makers' beliefs that extends ε-contamination. Section 6.4 analyze a portfolio selection problem à la Arrow (1965) under the non-expected utility framework which extends Asano (2008). Section 6.5 provides the main theorem.

6.1 Stochastic environment

In this section, we provide the formal description of our two period dynamic model, which is similar to Asano (2008).

Let S be a state set. Let (s_1, s_2) denote a generic element of $S \times S$. Let $\Sigma = \{\Sigma_t\}_{t=0,1,2}$, where $\Sigma_0 \equiv \{\emptyset, S \times S\}$, and Σ_1 is defined to be the algebra generated by the set of finite partitions of $S \times S$ of the form: $\{\Pi_i \times S\}_i$ for some finite partition $\{\Pi_i\}_{i=1}^m$ of S, and Σ_2 is defined to be the algebra generated by the set of finite partitions of $S \times S$ of the form: $\{\Pi_i \times \Pi_j\}_{i,j}$ for some finite partitions $\{\Pi_j\}_{j=1}^n$ of S.

For notational abuse, we denote the *first-period measurable space* by (S, Σ_1), where Σ_1 is the algebra generated by the partition $\{\Pi_i\}$. We denote the set of all the probability measures on it by $\Delta(S, \Sigma_1)$. Similar notational abuse applies to the *second-period measurable space* denoted by $(S \times S, \Sigma_2)$. Furthermore, we denote the set of all the probability measures on $(S \times S, \Sigma_2)$ by $\Delta(S \times S, \Sigma_2)$, and denote decision maker's set of probability measures by $P \subseteq \Delta(S \times S, \Sigma_2)$. In this chapter, we consider that decision maker's Knightian uncertainty is represented by the set P.

Let $p \in \Delta(S \times S, \Sigma_2)$. Then we define $p|_1(\cdot) \equiv p(\cdot \times S)$. Formally speaking, $p|_1$ is not a measure on (S, Σ_1), but it can be considered as the first-period marginal probability measure of p on $(S \times S, \Sigma_2)$. In a similar way, we define the second-period marginal probability measure, denoted by $p|_2$, and we consider it as a measure on $(S \times S, \Sigma_2)$.

Let $P \subseteq \Delta(S \times S, \Sigma_2)$. Then we define the first and second period marginal Knightian uncertainty by

$$P|_1 = \{p|_1 : p \in P\} \text{ and } P|_2 = \{p|_2 : p \in P\},$$

respectively.

In order to describe updating rules after observing an event E in the first period, we provide the definition of the Bayesian updating procedure. Let $p \in P$ and let $E \in \Sigma_1$ such that $p(E \times S) > 0$. Then for all $E \in \Sigma_1$ and for all $F \in \Sigma_2$ the probability measure on $(S \times S, \Sigma_2)$ conditional on the event $E \times S$ is defined by

$$p|_2(E \times F|E) \equiv \frac{p(E \times F)}{p(E \times S)}.$$

We define the Bayesian procedure by a function $(p, E) \to p|_2(\cdot|E)$, where $p|_2(\cdot|E) \equiv p|_2(E \times \cdot|E)$. That is, it maps a pair of a probability measure p on $(S \times S, \Sigma_2)$ and an event E in Σ_1 in the first period, to $p|_2(\cdot|E)$, which can be considered as the probability measure on $(S \times S, \Sigma_2)$.

Let P be the decision maker's set of beliefs, and let E be an Σ_1-measurable space such that $p(E \times S) > 0$ for all $p \in P$. Then we define an updating rule by a function that maps a pair (P, E) to a set of probability measures on $(S \times S, \Sigma_2)$. In this chapter, an updating rule is denoted by ϕ.

6.2 Updating rules: the Fagin Halpern and Dempster Shafer rules

In this section, we provide the definitions of two updating rules: the Fagin-Halpern and the Dempster-Shafer update rules.

The Fagin-Halpern updating rule is a Bayesian procedure. The Fagin-Halpern updating rule (we abbreviate it as the FH rule), denoted by ϕ_{FH}, is such that for all $P \subseteq \Delta(S \times S, \Sigma_2)$ and for all $E \in \Sigma_1$ is defined by:[1]

$$\phi_{FH}(P, E) = \{p|_2(\cdot|E) : p \in P\}$$

Now, let´s turn to the Dempster-Shafer updating rule.

Firstly, we need to define the set of first-period probability measures, which is such that for all $E \in \Sigma_1$ is given by:

$$P^*(E) = \arg\max\{p|_1(E) : p \in P\}.$$

The Dempster-Shafer updating rule (we abbreviate it as the DS rule) is such that, for all $P \subseteq \Delta(S \times S, \Sigma_2))$ and for all $E \in \Sigma_1$ is given by:

$$\phi_{DS}(P, E) = \{p|_2(\cdot|E) : p \in P^*(E)\}.$$

Note that in the first period, a decision maker who follows the Dempster-Shafer updating rule survives probability measures $p \in P$ that assign the maximum value to an event E, and in the second period, she applies the Bayesian procedure to $P^*(E)$.

From the definitions of FH and DS rules, it follows that for all $P \subseteq \Delta(S\text{x}S, \Sigma_2)$ and for all $E \in \Sigma_1$,

$$\phi_{DS}(P, E) \subseteq \phi_{FH}(P, E).$$

Based in the concept of dilation of Knightian uncertainty [2] Asano (2008) shows that the converse set inclusion holds by restricting the set of probability measures to the ε-contamination (which is a particular case of the contamination that we are studying here). The results are still true if we consider contamination, and in the sequel we will restrict the set of priors P to be the contamination of confidence of some probability measure.

[1] If a investor adopts the Fagin Halpern updating rule then she updates all probability measures $p \in P$ following the Bayesian procedure. Also note that when $E = S$, $\phi_{FH}(P, S) = P|_2$ for any P. That is, if $E = S$, then $\phi_{FH}(P, S)$ is equal to the second period marginal Knightian uncertainty $P|_2$.

[2] The dilation of Knightian uncertainty states that if a decision maker has observed an event $E \in \Sigma_1$ in the first period and updates her set of beliefs P by an updating rule ϕ, then the updated set of probability measures $\phi(P, E)$ expands compared with the case in which she observes no event in the first period, that is, $E = S$.

6.3 Contamination as the set of priors

In this section, we will restrict the set of priors P to be the contamination of confidence. Let p^0 be a probability measure on $(S \times S, \Sigma_2)$, with $supp(p^0) = \{D\}$, such that $p^0(E \times S) > 0$ and let ψ be a Knightian uncertainty level associated to D. So, the set of priors P is assumed to be represented by the contamination of confidence of p^0, denoted by $cont(p^0, \psi)|_D$, is a set of probability measures defined in the following way:

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:

$$P = cont(p^0, \psi)|_D = \{(1 - \varepsilon) \cdot p^0(E) + \varepsilon \cdot q(E) : q \in \Delta'\}.$$

and

ii) If $\exists E \supseteq D$, with $\psi(E) < \varepsilon$, then:

$$P = cont(p^0, \psi)|_D = \{(1-\psi(E))\cdot p^0(E)+\psi(E)\cdot q(E) : q \in \Delta' \text{ such that } supp(q) \subseteq D\}.$$

where with notational abuse $\Delta' \equiv \Delta(S \times S, \Sigma_2)$.

Remark 1. If the Knightian uncertainty level $\psi \in \Psi|_D$ is defined by: $\psi(E) = \varepsilon$, for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $\{\emptyset, S\}$ then it reduces to the case of ε-contamination of confidence of p^0 (i.e., $P = cont(p^0, \psi)|_D$ reduces to $P = cont(p^0, \varepsilon)$).

Remark 2. If $\psi(E) = 0$, for all $E \in \Sigma$, which means that there is no uncertainty, then $P = cont(p^0, \psi)|_D$ reduces to $P \equiv \{p^0\}$. This corresponds to the case in which investor's belief is captured by the single probability measure p^0.

For later use, we define the contamination of $p^0|_2(\cdot, E)$ analogously.

Nishimura and Ozaki (2002) shows that the converse set inclusion

$$\phi_{DS}(P, E) \supseteq \phi_{FH}(P, E)$$

is true for the ε-contamination.[3] The following theorem naturally extends Nishimura and Ozaki (2002)'s Theorem 1 to the case of contamination.

Theorem 4. *Let p^0 be a probability measure (i.e., $p^0 \in \Delta$) with $supp(p^0) = \{D\}$, let ψ be a Knightian uncertainty level associated to D, (i.e., $\psi \in \Psi|_D$) and let $E \in \Sigma_1$. Then*

$$\phi_{FH}(cont(p^0, \psi)|_D, E) = \phi_{DS}(cont(p^0, \psi)|_D, E) = cont(p^0|_2(\cdot|E), \psi)|_D$$

[3]That is, sets of probability measures updated by FH and DS rules are identical under ε-contamination.

Proof. See Section $D.1$ of Appendix D. □

As in Chapter 3, Section 3.4, it can be proved that the next equality holds:

$$core(v_1)|_D = cont(p^0|_2(\cdot|E), \psi)|_D$$

where given $\psi \in \Psi|_D$, $v : \Sigma_2 \to [0,1]$ is defined by:

$$v_1 \equiv (1-\psi) \cdot p^0|_2(\cdot|E).$$

It can be proved that v_1 is a convex capacity. Thus it follows that:

$$\int_S I(s_1,s_2)v_1(ds_2) = min\{\int_S I(s_1,s_2)p(ds_2) : p \in core(v_1)|_D\} =$$
$$= min\{\int_S I(s_1,s_2)p(ds_2) : p \in cont(p^0|_2(\cdot|E), \psi)|_D\},$$

where the integral of the left-hand side is in the sense of Choquet integral.

Next, we investigate the relation between the set of the first-period marginals of the contamination, $P|_1$, and the contamination of the first-period marginal probability measures, $cont(p^0|_1, \psi)|_D$.

Lemma 1. *Let p^0 be a probability measure ($p^0 \in \Delta$) with $supp(p^0) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi_D$). Let $P \equiv cont(p^0, \psi)|_D$ and let $P|_1 \equiv \{p|_1 : p \in cont(p^0, \psi)|_D\}$. Then $P|_1 \equiv cont(p^0|_1, \psi)|_D$, is a set of probability measures defined in the following way:*

i) *If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:*

$$P|_1 = cont(p^0|_1, \psi)|_D = \{(1-\varepsilon) \cdot p^0|_1(E) + \varepsilon \cdot q(E) : q \in \Delta'\}.$$

and

ii) *If $\exists E \supseteq D$, with $\psi(E) < \varepsilon$, then:*

$$P|_1 = cont(p^0|_1, \psi)|_D = \{(1-\psi(E)) \cdot p^0|_1(E) + \psi(E) \cdot q(E) : q \in \Delta' \text{ such that } supp(q) \subseteq D\}.$$

where with notational abuse $\Delta' \equiv \Delta(S \times S, \Sigma_2)$.

Proof. See Section $D.2$ of Appendix D. □

By this lemma, we can show that

$$P|_1 = cont(p^0|_1, \psi)|_D = core(v_0)|_D$$

where $v_0 : \Sigma_1 \to [0,1]$ is defined by:

$$v_0 \equiv (1-\psi)p^0|_1$$

as in Chapter 3.

It can be proved that v_0 is a convex capacity. Then it follows that:

$$\int_S I(s_1,s_2)v_0(ds_1) = min\{\int_S I(s_1,s_2)p(ds_1) : p \in core(v_0)|_D\} =$$
$$= min\{\int_S I(s_1,s_2)p(ds_1) : p \in cont(p^0|_1, \psi)|_D\}$$

where the integral of the left-hand side is in the sense of Choquet integral, and I is Σ_2-measurable.

Similarly, we can characterize the set of second-period marginals of the contamination, $P|_2$, by the contamination of the second-period marginal probability measures, $cont(p^0|_2, \psi)|_D$.

Lemma 2. *Let p^0 be a probability measure ($p^0 \in \Delta$) with $supp(p^0) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi_D$). Let $P|_2 \equiv \{p|_2 : p \in cont(p^0|_2, \psi)|_D\}$. Then, $P|_2 \equiv p^0|_2$, denoted by $cont(p^0|_2, \psi)|_D$, is a set of probability measures defined in the following way:*

i) *If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then:*

$$P|_2 = cont(p^0|_2, \psi)|_D = \{(1-\varepsilon)\cdot p^0|_2(E) + \varepsilon \cdot q(E) : q \in \Delta'\}.$$

and

ii) *If $\exists E \supseteq D$, with $\psi(E) < \varepsilon$, then:*

$$P|_2 = cont(p^0|_2, \psi)|_D = \{(1-\psi(E))\cdot p^0|_2(E) + \psi(E)\cdot q(E) : q \in \Delta' \text{ such that } supp(q) \subseteq D\}.$$

where with notational abuse $\Delta' \equiv \Delta(S \times S, \Sigma_2)$.

Proof. See Section $D.3$ of Appendix D. □

By this lemma, it follows that

$$P|_2 = cont(p^0|_2, \psi)|_D = core(v_2),$$

where $v_2 : \Sigma_2 \to [0,1]$ is defined such that for all $F \in \Sigma_2$:

$$v_2 \equiv (1-\psi)p^0|_2$$

as in Chapter 3. It can be proved that v_2 is a convex capacity.

6.4 Portfolio selection problem

In this section, we consider a portfolio selection problem à la Arrow (1965) in a two period framework, using the same procedure as Asano (2008). At first, the setup is in order. After that, we analyze the portfolio selection problem in two different setting. The first setting is the one in which an investor observes an event E in the first period and updates her set of probability measures $P = cont(p^0)|_D$ by the Fagin Halpern updating rule. The second is the one in which she observes no event in the first period and updates $P = cont(p^0)|_D$ by the Fagin Halpern updating rule.

Let $W \in \mathbb{R}_+$ be the wealth at $t = 0$, $N \in \mathbb{R}_+$ be the amount of money invested at $t = 0$, $q > 0$ be the price of a risky asset at $t = 0$, and X be the random payoff of the asset at $t = 2$, where X is Σ_2-measurable. Moreover, let $u : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic increasing [4] concave function. We assume that an investor obtains new information at $t = 1$, but she does not gain any payment at $t = 1$.

So, the objective function to be maximized by the investor is given by:

$$V(X(s_1, s_2)) = min\{E^p\left[min\{E^p[X|\Sigma_2]|\Sigma_1\}\right] : p \in P\} =$$

$$= min\{\int min\{E^p[X|\Sigma_2]|p|_1(ds_1) : p \in P\} : p \in P\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in \phi_{FH}(P, E)\}\right] p|_1(ds_1) : p \in P\} =$$

[4] We say that a function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing if $u(x) > u(y)$ whenever $x \succ y$, and a function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing if $u(x) \geq u(y)$ whenever $x \succ y$.

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in \phi_{FH}(P, E)\} \right] p(ds_1) : p \in P|_1\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in cont(p^0|_2(\cdot|E), \psi)|_D\} \right] p(ds_1) : p \in cont(p^0|$$

$$= \int_S \left[\int_S u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v_1(ds_2) \right] v_0(ds_1)$$

where $v_0 : \Sigma_1 \to [0, 1]$ is defined by $v_0 \equiv (1 - \psi) \cdot p^0|_1$ and $v_1 : \Sigma_2 \to [0, 1]$ is defined by $v_1 \equiv (1 - \psi) \cdot p^0|_2$.

Note that the investor's set of beliefs P is captured by the contamination of p^0, and her updating rule $\phi(P, E)$ is characterized by the Fagin Halpern updating rule $\phi_{FH}(P, E)$.

Suppose that the investor may invest in two assets, a riskless asset and a risky asset. The problem of the investor is to choose the amount of money N so as to maximize

$$\int_S \left[\int_S u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v_1(ds_2) \right] v_0(ds_1)$$

This means that the investor is supposed to choose the amount of money N to invest in the risky asset so as to maximize her non-expected utility of the terminal payoff.

The next theorem states that there exists portfolio inertia in a two-period setting in which the investor's set of probability measures $P = cont(p^0, \psi)|_D$ is updated by the Fagin Halpern updating rule.

Theorem 5. *The investor will neither buy nor sell the risky asset, if q satisfies:*

$$\int_S \int_S X(s_1, s_2)v_1(ds_2)v_0(ds_1) < q < \int_S \int_S X(s_1, s_2)v_1'(ds_2)v_0'(ds_1),$$

where v_0' is the conjugate of v_0 defined by $v_0 \equiv (1 - \psi) \cdot p^0|_1$ and v_1' is the conjugate of v_1 defined by $v_1 \equiv (1 - \psi) \cdot p^0|_2$.

Proof. See Section $D.4$ of Appendix D. □

Finally, we consider the case in which the investor does not observe any event. Then, she observes the whole set S in the first period and she updates her set of probability measures P by the Fagin Halpern updating rule. Recall that her set of probability measures P is characterized by the contamination. As we have already pointed out, for the case in which $E = S$, the Fagin Halpern updating rule $\phi_{FH}(P, S)$ is equal to the second-period marginal of P, $P|_2$. In this case, her objective function turns out to be[5]

$$V(X(s_1, s_2)) = min\{E^p\left[min\{E^p[X|\Sigma_2]|\Sigma_1 : p \in P\}\right] : p \in P\} =$$

$$= min\{\int min\{E^p[X|\Sigma_2]|p|_1(ds_1) : p \in P\} : p \in P\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in \phi_{FH}(P, S)\}\right] p|_1(ds_1) : p \in P\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in \phi_{FH}(P, S)\}\right] p(ds_1) : p \in P|_1\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in P|_2\}\right] p(ds_1) : p \in P|_1\} =$$

$$= min\{\int \left[min\{\int u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v(ds_2) : v \in \phi_{FH}(P, S)\}\right] p(ds_1) : p \in cont(p^0|_2, \psi)|_D\} =$$

$$= \int_S \left[\int_S u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v_1(ds_2)\right] v_0(ds_1)$$

where $v_0 : \Sigma_1 \to [0, 1]$ is defined by $v_0 \equiv (1 - \psi) \cdot p^0|_1$ and $v_1 : \Sigma_2 \to [0, 1]$ is defined by $v_1 \equiv (1 - \psi) \cdot p^0|_2$.

So, the problem of the investor is to choose the amount of money N so as to maximize

$$\int_S \left[\int_S u(W - N + (\frac{N}{q}) \cdot X(s_1, s_2))v_1(ds_2)\right] v_0(ds_1).$$

Similarly, we can prove the following theorem.

[5]We do not assume, even implicitly, a stochastic independence such as $p(E \times F) = p|_1(E) \times p|_2(F)$ for any $p \in P$ and for any $E \in \Sigma_1$ and any $F \in \Sigma_2$.

Theorem 6. *The investor will neither buy nor sell the risky asset, if q satisfies:*

$$\int_S \int_S X(s_1, s_2) v_2(ds_2) v_0(ds_1) < q < \int_S \int_S X(s_1, s_2) v_2'(ds_2) v_0'(ds_1),$$

where v_0' is the conjugate of v_0 defined by $v_0 \equiv (1-\psi) \cdot p^0|_1$ and v_2' is the conjugate of v_2 defined by $v_2 \equiv (1-\psi) \cdot p^0|_2$.

Proof. See Section $D.5$ of Appendix D. □

This theorem states that there also exists portfolio inertia in a two-period model in which the investor does not obtain any information at $t = 1$, and her set of probability measures is updated by the Fagin Halpern updating rule for $E = S$.

6.5 Expansion of portfolio inertia

In this section, we are interested to know if obtaining new information in the first period compared with the case in which she has not gain any information in the first period will shrink or expand portfolio inertia. Our answer extends Asano (2008)'s result.

In a two-period setting, we consider the following two cases. The first case is the one in which a investor observes an event $E \in \Sigma_1$ in the first period, and she evaluates a second-period event by the probability measures updated by the Fagin Halpern updating rule. The second case is the one in which she observes no event in the first period, that is, $E = S$, and she evaluates the second-period event by the probability measures updated by the Fagin Halpern updating rule.

In order to analyze this problem, we define the "informational value" according to Nishimura and Ozaki (2002). Let $E \in \Sigma_1$ and let $\delta(E)$ be defined by

$$\delta(E) = max\{j = 1, \cdots n : |p^0|_2(F_j|E) - p^0|_2(F_j)\}$$

where $\delta(E) \in [0, 1)$. This number $\delta(E)$ is one of the measures to capture the informational value of knowing that an event E has occurred in the first period. Note that $\delta(E) = 0$ implies that observing an event E in the first period will not affect investor's evaluation of an event F in the second period.

In the following theorem, we assume two additional conditions. The first is the condition that $m \geq 2$, which implies that the first-period state set is partitioned with more than two events. This condition together with the assumption that $p^0(E \times S) > 0$ implies that $0 < p^0|_1(E) < 1$. The second is the condition that $F \in \Sigma_2$ is such that $p^0|_2(F) > 0$.

So, because ε is associated to the events that strictly belongs to the support of the contamination of confidence of some probability measure p^0, and which can be considered as sufficiently large with respect to the informational value of the observation E, (i.e., $\varepsilon\delta(E)$), then obtaining new information expands portfolio inertia compared with the case in which an investor does not obtain new information at $t = 1$.

Theorem 7. *An investor will neither buy nor sell the risky asset, if q satisfies:*

$$\int_S \int_S X(s_1, s_2) v_2(ds_2) v_0(ds_1) < q < \int_S \int_S X(s_1, s_2) v_2'(ds_2) v_0'(ds_1),$$

where v_0' is the conjugate of v_0 defined by $v_0 \equiv (1-\psi) \cdot p^0|_1$ and v_2' is the conjugate of v_2 defined by $v_2 \equiv (1-\psi) \cdot p^0|_2$.

Furthermore, let $F \in \Sigma_2$ such that $p^0|_2(F) > 0$. Suppose that the following inequality holds:

$$\varepsilon > \frac{p^0|_1(E)}{(1 - p^0|_1(E)) min\{p^0|_2(F_j) : j\}} \delta(E)$$

then, for such an asset price, the following inequalities hold:

$$\begin{aligned} \int_S \int_S X(s_1, s_2) v_1(ds_2) v_0(ds_1) < \int_S \int_S X(s_1, s_2) v_2(ds_2) v_0(ds_1) < q < \\ < \int_S \int_S X(s_1, s_2) v_2'(ds_2) v_0'(ds_1) < \int_S \int_S X(s_1, s_2) v_1'(ds_2) v_0'(ds_1), \end{aligned}$$

where v_0' is the conjugate of v_0 defined by $v_0 \equiv (1-\psi) \cdot p^0|_1$, v_1' is the conjugate of v_1 defined by $v_1 \equiv (1-\psi) \cdot p^0|_1$ and v_2' is the conjugate of v_2 defined by $v_2 \equiv (1-\psi) \cdot p^0|_2$.

Proof. See Section $D.6$ of Appendix D. □

Chapter 7

Conclusion

The first part of this thesis starts with a review of the basic statements on capacity integration. In the sequel we investigate a special class of convex capacity, which we call generalized simple capacity. Then we provide a simple set of behavioral axioms under which the decision maker's preference is represented by the Choquet expected utility with the contamination of confidence. The new axioms and theorem presented extends not only the Anscombe and Aumann (1963)'s axiomatization but also the Nishimura and Ozaki (2006)'s axiomatization.

In the second part we present two applications of the use of the generalized simple capacities. In game theory, we show that it is possible to explain explicitly the paradox on which players of a finitely repeated prisoners´dilemma breaks down backward induction. The other application extended Asano (2008)'s results and as in his analysis, we also show that observing new information in the first period will expand portfolio inertia in the second period compared with the case in which observing new information has not been gained in the first period if the uncertainty is sufficiently large.

It is important to note that, through the use of contamination of confidence, it is possible to do much more comparative statics exercises, compared to the ε-contamination case.

Bibliography

Alon, S. and D. Schmeidler (2008), "Purely subjective maxmin expected utility." *mimeo (first draft at 06/19/2008). Tel Aviv university.*

Anscombe, F. J. and R. J. Aumann (1963), "A definition of subjective probability." *Annals of Mathematical Statistics*, 34, 199–205.

Arrow, K. J. (1965), *Aspects of the Theory of Risk-Bearing*. Chicago: Markham Publishing Co.

Asano, T. (2008), "Portfolio inertia and epsilon-contamination." *Theory and Decision (on line).*

Aumann, R. J. (1987), "Correlated equilibrium as an expression of bayesian rationality." *Econometrica*, 55, 1–18.

Axelrod, R. (1981), "The emergence of cooperation among egoists." *American Political Science Review*, 75, 306–318.

Bacon, F. (1620), *Novum Organum.*

Barrios, R. E. A. (1996), *Equilíbrio, impaciência e neutralidade à incerteza em uma economia do tipo Bewley*. Ph.D. thesis, Instituto de Matemática Pura e Aplicada - IMPA.

Berger, J. O. (1985), *Statistical Decision Theory and Bayesian Analysis*. Second edition, Springer-Verlang.

Berger, J. O. and L. M. Berliner (1986), "Robust bayes and empirical bayes analisys with $\varepsilon-$contaminated priors." *The Annals of Statistics*, 14, 461–486.

Boff, H. and S. R. C. Werlang (1996), "Cournot competition under knightian uncertainty." *Brazilian Review of Econometrics*, 18, 265–308.

Casadesus-Masanell, R.; P. Klibanoff and E. Ozdenoren (2000), "Maxmin expected utility over savage acts with a set of priors." *Journal of Economic Theory*, 92, 35–65.

Chen, Z. and L. Epstein (2002), "Ambiguity, risk and asset returns in continuous time." *Econometrica*, 70, 1403–1443.

Choquet, G. (1953), "Theory of capacities." *Annales de l´Instute Fourier (Grenoble)*, 5, 131–295.

Coimbra, P. C. (2003), *The theory of individual choice under Knightian uncertainty: a survey (in portuguese)*. Master's thesis, Graduate School of Economics at Getulio Vargas Foundation.

de Finetti, B. (1937), "La prevision, ses lois logiques, ser sources subjectives." *Annales de l´Instute Henri Poincare*, 7, 1–68.

Dellacherie, C. (1970), *Quelques Commentaries sur les Prolongements de Capacites*. Springer Verlag.

Dempster, A. P. (1967), "Upper and lower probabilities induced by a multivalued map." *Annals of Mathematical Statistics*, 38, 325–339.

Denneberger, D. (1994), *Non-additive Measure and Integral*. Kluwer, Dordrecht.

Dow, J. and S. R. C. Werlang (1992), "Uncertainty aversion, risk aversion and the optimal choice of portfolio." *Econometrica*, 60, 197–204.

Dow, J. and S. R. C. Werlang (1994), "Nash equilibrium under knightian uncertainty: breaking down backward induction." *Journal of Economic Theory*, 64, 305–324.

Dunford, N. and J. T. Schwartz (1958), *Linear operators. Part I: general theory*. London: Wiley - Interscience.

Eichberger, J. and D. Kelsey (2000), "Non-additive beliefs and strategia equilibria." *Games and Economic Behavior*, 30, 183–215.

Ellsberg, D. (1961), "Risk, ambiguity and the savage axioms." *Quarterly Journal of Economics*, 75, 643–669.

Fishburn, P. C. (1970), *Utility Theory for Decision Making*. John Wiley and Sons.

Gilboa, I. (1987), "Expected utility with purely subjective non-additive prior." *Journal of Mathematical Economics*, 16, 65–88.

Gilboa, I. and D. Schmeidler (1989), "Maxmin expected utility with non-unique prior." *Journal of Mathematical Economics*, 18, 141–153.

Gilboa, I. and D. Schmeidler (1993), "Updanting ambiguous beliefs." *Journal of Economic Theory*, 59, 33–49.

Huber, P. J. (1973), "The use of choquet capacities in statistics." *Bulletin of the Institute of International Statistics*, 45, 181–191.

Huber, P. J. and Strassen (1973), "Minimax tests and the tests of the neymann-pearson lemma for capacities." *Annals of Statistics*, 1, 252–263.

Kajii, A. and T. Ui (2005), "Equivalence of the dempster-shafer rule and the maximum likelihood rule implies convexity." *Economics Bulletin*, 4, 1–6.

Keynes, J. M. (1921), *A treatise on probability*. London: MacMillan and Co.

Knight, F. H. (1921), *Risk, uncertainty and profit*. Boston, New York: Houghton Mifflin.

Kopylov, I. (2009), "Choice deferral and ambiguity aversion." *Theoretical Economics*, 4, 199–225.

Kreps, D. (1988), *Notes on the theory of choice*. Boulder, CO: Westview Press.

Kreps, P. Milgrom J. Roberts, D. and R. Wilson (1982), "Rational cooperation in the finitely repeated prisoners' dilemma." *Journal of Economic Theory*, 27, 245–252.

Marinacci, M. (2000), "Ambiguous games." *Games and Economic Behavior*, 31, 191–219.

Marinacci, M. and L. Montrucchio (2004), "Introduction to the mathematics of ambiguity." In Uncertainty in Economic Theory*: Essays in Honor of David Schmeidler's 65th Birthday* (I. Gilboa, ed.), chapter 4, 46–107, New York: Routledge Frontiers of Political Economy.

Montaigne, M. (1595), *Essais*.

Mukerji, S. and J.-M. Tallon (2004), "An overview of economic applications of david schmeidler's models of decision making under uncertainty." In Uncertainty in Economic Theory*: Essays in Honor of David Schmeidler's 65th Birthday* (I. Gilboa, ed.), chapter 13, 283–302, New York: Routledge Frontiers of Political Economy.

Nash, J. (1951), "Non-cooperative games." *The Annals of Mathematics*, 54, 286–295.

Neelin, H. Sonnenschein, J. and M. Spiegel (1988), "A further test of noncooperative bargaining theory: comment." *American Economic Review*, 78, 824–836.

Nishimura, K. G. and H. Ozaki (2002), "Economics of self-feeding fear." *mimeo, University of Tokio*.

Nishimura, K. G. and H. Ozaki (2004), "Search and knightian uncertainty." *Journal of Economic Theory*, 119, 299–333.

Nishimura, K. G. and H. Ozaki (2006), "An axiomatic approach to $\varepsilon-$contamination." *Economic Theory*, 27, 333–340.

Nishimura, K. G. and H. Ozaki (2007), "Irreversible investiment and knightian uncertainty." *Journal of Economic Theory*, 136, 668–694.

Ramsey, F. P. (1931), "Truth and probability." In *The Foundations of Mathematics and other Logical Essays, Ch. VII, p.156-198* (R. B. Braithwaite, ed.), 156–198, London: Kegan, Paul, Trench, Trubner Co., New York: Harcourt, Brace and Company.

Rosenmüller, J. (1971), "On core and value." *Methods of Operations Research*, 09, 84–104.

Rosenmüller, J. (1972), "Some properties of convex set functions, part ii." *Methods of Operations Research*, 17, 287–307.

Savage, L. J. (1954), *The Foundations of Statistics*. New York: John Willey and Sons. ((Second Edition) 1972, New York: Dover).

Schmeidler, D. (1972), "Core of exact games, i." *Journal of Math. Annal. and Aplications*, 40, 214–225.

Schmeidler, D. (1982), "Subjective probability without additivy.(temporary title)." *working paper, The Foereder Institute for Economic Research, Tel Aviv University.*

Schmeidler, D. (1984), "Non-additive probabilities and convex games." *mimeo, University of Pennsylvania.*

Schmeidler, D. (1986), "Integral representation without additivity." *Proceedings of the American Mathematical Society*, 97, 255–261.

Schmeidler, D. (1989), "Subjective probability and expected utility without additivity." *Econometrica*, 57, 571–587.

Shafer, G. (1976), *A Mathematical Theory of Evidence*. Princeton University Press.

Shapley, L. (1965), "Notes on n-person games vii: cores of convex games." *The RAND Corporation R. M.*

Shapley, L. (1972), "Cores of convex games (reprinted)." *International Journal of Game Theory*, 1, 11–26.

Simonsen, M. H. and S. R. C. Werlang (1991), "Subadditive probabilities and portfolio inertia." *Revista Brasileira de Econometria*, 10, 1–14.

von Neumann, J. and O. Morgenstern (1944), *Theory of Games and Economic Behavior*. Princeton University Press.

Wasserman, L. A. and J. B. Kadane (1990), "Bayes' theorem for choquet capacities." *The Annals of Statistics*, 18, 1328–1339.

Appendix A

Proofs of Chapter 3

This appendix provide the proofs of Chapter 3.

A.1 Proof of Proposition 1

In order to prove this proposition, we will start with the following lemma:

Lemma 3. $\emptyset \notin supp(v)$.

Proof. Suppose not and so $\emptyset \in supp(v)$. But this implies that $v(\emptyset^c) = v(S) = 0$.

A contradiction to $C.1$. □

According to the definition of the support of a capacity, if $D \in supp(v)$ then:

i) $v(D^c) = 0$; and

ii) for all $\emptyset \neq C \subsetneq D$: $v(C^c) > 0$.

Suppose that $D \notin supp(v)$.

One possibility is that $v(D^c) > 0$.

$\emptyset \neq C \subsetneq D$ implies:

$$\begin{aligned} v(C \cup D^c) + v(C \cap D^c) = v(C \cup D^c) + v(\emptyset) = \\ = v(C \cup D^c) \geq \\ \geq v(C) + v(D^c) \end{aligned}$$

A contradiction by $C.8$, because $v(C \cup D^c) = v(C)$ which implies that $v(D^c) = 0$.

Now, another possibility is that $v(D^c) = 0$ and $\exists\ \emptyset \neq C \subsetneq D$ such that $V(C^c) = 0$.

$$v(D) = v(C \cup (C^c \cap D)) \geq v(C) + v(C^c \cap D)$$

Note that according to $C.8$:

$$v(C^c) = v(D^c \cup (C^c \cap D)) = v(C^c \cap D)$$

Substituting in the above equation and by $C.9$:

$$v(D) \geq v(C) + v(C^c) > 0$$

The above inequality is valid for every $\emptyset \neq C \subsetneq D$.

Note also that:

$$v(C^c) = v(C' \cup (C \cup C')^c) \geq v(C') + v((C \cup C')^c)$$

Substituting in the above equation:

$$v(D) \geq v(C) + v(C') + v((C \cup C')^c) > 0$$

According to $C.7$:

$$v(D) \geq v(C \cup C') + v((C \cup C')^c) > 0$$

So, this implies that $v(C^c) > 0$ it is true for every $\emptyset \subseteq C \subsetneq D$, which contradicts the fact that $\exists\ \emptyset \subseteq C \subsetneq D$ such that $V(C^c) = 0$.

If, in fact, $v(C^c) = 0$ then $D \notin supp(v)$ and C is the natural candidate to $supp(v)$.

Using analogously arguments, $\exists\ \emptyset \neq B \subsetneq C$ such that $V(B^c) = 0$ and so $C \notin supp(v)$. And so on.

This reasoning conduce to the conclusion that $supp(v) = \emptyset$. A contradiction to $Lemma3$, and so $v(C^c) > 0$ and $D \in supp(v)$.

Fix $D = C \cup B$. Using $C.8$:

$$\begin{aligned} v(B^c) = v(C \cup D^c) = \\ = v(C) > 0 \end{aligned}$$

so, it follows that for all $\emptyset \neq C \subsetneq D$:

$$v(C) > 0.$$

If $\emptyset \neq C \subsetneq D$ and $E \subseteq D^c$ then $C \cap E = \emptyset$. From $C.3$, it follows that:

$$\begin{aligned} v(C \cup E) + v(C \cap E) &= v(C \cup E) + v(\emptyset) = \\ &= v(C \cup E) \geq v(C) + v(E). \end{aligned}$$

Combining this result with $C.8$ we get that for all $E \subseteq D^c$:

$$v(E) = 0.$$

So, using this result and $C.3$ it is easy to verify that for all $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = D^c$:

$$0 < v(D) \leq v(D \cup E_1) \leq v(D \cup E_2) \leq \cdots \leq v(D \cup E_n) = v(D \cup D^c) = v(S) = 1$$

Finally, it is easy to verify that $\#supp(v) = 1$.

Let $s \in S$ be such that $\{s\} \subsetneq D^c$ and consider $\emptyset \neq E \subseteq D^c$. According to the previous results we know that:

$$v(\{s\}) = v(E) = 0$$

And so, for all $\{s\} \subsetneq D^c$ and all $\emptyset \neq E \subseteq D^c$:

$$v(E \cup \{s\}) = v(E) + v(s)$$

which implies that according to Marinacci (2000)'s proposition that $\#supp(v) = 1$ and so $supp(v) = \{D\}$.

Q.E.D.

A.2 Proof of Proposition 2

Verifying property i:

$$c(v, \emptyset) = 1 - v(\emptyset) - v(S) = 1 - 0 - 1 = 0 = c(v, S).$$

Verifying property ii:

$$c(v, E) = 1 - v(E) - v(E^c) = c(v, E^c) < 1.$$

Because of property $C.9$ (not fully uncertainty).

Verifying property *iii (concavity)*:

$$\begin{aligned} c(v, E \cup F) + c(v, E \cap F) &= [1 - v(E \cup F) - v((E \cup F)^c)] + [1 - v(E \cap F) - v((E \cap F)^c)] = \\ &= [1 - v(E \cup F) - v(E \cap F)] + [1 - v((E \cup F)^c) - v((E \cap F)^c)] = \\ &(\text{Morgan´s law}) = [1 - v(E \cup F) - v(E \cap F)] + [1 - v(E^c \cap F^c) - v(E^c \cup F^c)] \leq \\ &\leq [1 - v(E) - v(F)] + [1 - v(E^c) - v(F^c)] = \\ &= [1 - v(E) - v(E^c)] + [1 - v(F) - v(F^c)] = \\ &= c(v, E) + c(v, F). \end{aligned}$$

Verifying property *iv (uniformity in proper subsets of D)*:

If $supp(v) \equiv D$ ($\emptyset \neq D \subseteq S$) and C and C' satisfy $\emptyset \neq C, C' \subsetneq D$ with $C \cup C' \neq D$ and $C \cap C' = \emptyset$ then there exists $\emptyset \neq B \subsetneq S$ such that $D = C \cup C' \cup B$.

According to property $C.7$ *(additivity in proper subsets of D)*:

$$v(C \cup C') = v(C) + v(C');\ v(C \cup B) = v(C) + v(B); \text{ and } v(C' \cup B) = v(C') + v(B)$$

and according to property $C.8$ *(nullity in the complement of D)*:

$$v((C \cup C')^c) = v(B \cup (D^c)) = v(B);\ v(C^c) = v((C' \cup B) \cup (D^c)) = v(C' \cup B); \text{ and}$$

$$v((C')^c) = v((C \cup B) \cup (D^c)) = v(C \cup B).$$

So,

$$\begin{aligned} c(v, C) &= 1 - v(C) - v(C^c) = \\ &= 1 - v(C) - v(C' \cup B) = \\ &= 1 - v(C) - v(C') - v(B) = \\ &= 1 - v(C') - v(C \cup B) = \\ &= 1 - v(C') - v((C')^c) = \\ &= c(v, C') = \\ &= 1 - v(C \cup C') - v(B) = \\ &= 1 - v(C \cup C') - v((C \cup C')^c) = \\ &= c(v, C \cup C') \geq \\ &\geq 1 - v(D) = \\ &= 1 - v(D) - v(D^c) = \\ &= c(v, D) \end{aligned}$$

Verifying property *v (nullity in the complement of D)*:

If $supp(v) \equiv D$ ($\emptyset \neq D \subseteq S$) and $\emptyset \neq C \subsetneq D$ then there exists $\emptyset \neq B' \subsetneq S$ such that $D = C \cup B'$.

According to property $C.8$ *(nullity in the complement of D)*:

$$v(C \cup E) = v(C);\ v((C \cup E)^c) = v(B'); \text{ and } v(C^c) = v(B').$$

So,

$$\begin{aligned} c(v, C \cup E) = 1 - v(C \cup E) - v((C \cup E)^c) = \\ = 1 - v(C) - v(B') = \\ = 1 - v(C) - v(C^c) = \\ = c(v, C) \end{aligned}$$

Verifying property *vi (monotonic decreasing in D)*:

It follows trivially from the property i and iii and from the fact that for each E_j, $v((D \cup E_j)^c) = 0$.

$$\begin{aligned} c(v, D) = 1 - v(D) - v(D^c) = 1 - v(D) \geq \\ \geq 1 - v(D \cup E_1) = c(v, D \cup E_1) \geq \\ \geq 1 - v(D \cup E_2) = c(v, D \cup E_2) \geq \\ \geq \cdots \geq \\ \geq 1 - v(D \cup E_n) = c(v, D \cup E_n) = \\ = 1 - v(D \cup D^c) = c(v, S) = 0. \end{aligned}$$

Q.E.D.

A.3 Proof of Theorem 1

In order to verify that the real-valued set-function $w : \Sigma \to \mathbb{R}_+$ defined by:

$$w \equiv (1 - \psi) \cdot p$$

is a generalized simple capacity with support D associated to the Knightian uncertainty level ψ in $\Psi|_D$ (i.e., $w \in \Lambda|_D(\psi)$, $\psi \in \Psi|_D$), note firstly, that $w(\emptyset) = (1 - \psi(\emptyset)) \cdot p(\emptyset) = 0$.

Verifying property *C.* 1 **(normalized)**:

$$w(S) = (1 - \psi(S)) \cdot p(S) = 1.$$

Verifying property *C.* 3 **(convexity)**:

We must show that $w(E \cup F) + w(E \cap F) \geq w(E) + w(F)$, for all $E, F \in \Sigma$.

There exist three trivial cases and three non-trivial cases.

Let start with the three trivial cases:

1st case: $\emptyset \subseteq E \subsetneq F = S$ (note that $E \cup F = F = S$ and $E \cap F = E$).

$$w(E \cup F) + w(E \cap F) = w(F) + w(E).$$

2nd case: $\emptyset \subseteq E \subsetneq F \neq S$ (note that $E \cup F = F$ and $E \cap F = E$).

$$w(E \cup F) + w(E \cap F) = w(F) + w(E).$$

3rd case: $E \cup F = S$ and $E \cap F = \emptyset$ (note that $E^c = F$).

$$\begin{aligned} w(E \cup F) + w(E \cap F) = w(S) + w(\emptyset) = \\ = 1 = p(E) + p(E^c) = \\ = p(E) + p(F) \geq \\ \geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

Now the three non-trivial cases: [1]

1st case: $E \cup F \neq S$ and $E \cap F = \emptyset$.

$$w(E \cup F) + w(E \cap F) = w(E \cup F) + w(\emptyset) = (1 - \psi(E \cup F)) \cdot p(E \cup F).$$

i) If $D \subseteq (E \cup F)^c$.

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot 0 = \\ = (1 - \psi(E)) \cdot 0 + (1 - \psi(F)) \cdot 0 = \\ = w(E) + w(F). \end{aligned}$$

ii) If [2] $D \subseteq E$ then according to property *U.6 (monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(E)) \geq (1 - \psi(D)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) = \\ = (1 - \psi(E \cup F)) \cdot p(E) \geq \\ \geq (1 - \psi(E)) \cdot p(E) = \\ = w(E) + w(F). \end{aligned}$$

[1] In what follows, let $\emptyset \neq X \subseteq (E \cap F^c)$, $\emptyset \neq Y \subseteq (E \cap F)$ and $\emptyset \neq Z \subseteq (F \cap E^c)$.

[2] The subcase on which $D \subseteq F$ is analogous.

iii) If [3] $D \subseteq F^c$ then according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(E \cup F)) = (1 - \psi(E)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) = \\ = (1 - \psi(E)) \cdot p(E) = \\ = w(E) + w(F). \end{aligned}$$

iv) If [4] $D \subseteq (E \cup F)$ then according to property $U.6$ *(monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(D)) \geq (1 - \psi(E)).$$

and

$$(1 - \psi(E \cup F)) \geq (1 - \psi(D)) \geq (1 - \psi(F)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) = \\ = (1 - \psi(E \cup F)) \cdot (p(E) + p(F)) = \\ = (1 - \psi(E \cup F)) \cdot p(E) + (1 - \psi(E \cup F)) \cdot p(F)) \geq \\ \geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F)) = \\ = w(E) + w(F). \end{aligned}$$

v) If [5] $D \subseteq S$ then according to property $U.4$ *(uniformity in proper subsets of D)*:

$$(1 - \psi(E \cup F)) = (1 - \psi(E)) \geq (1 - \psi(F)).$$

and

$$(1 - \psi(E \cup F)) \geq (1 - \psi(D)) \geq (1 - \psi(F)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) = \\ = (1 - \psi(E \cup F)) \cdot (p(E) + p(F)) = \\ = (1 - \psi(E \cup F)) \cdot p(E) + (1 - \psi(E \cup F)) \cdot p(F)) = \\ = (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F)) = \\ = w(E) + w(F). \end{aligned}$$

[3] The subcase on which $D \subseteq E^c$ is analogous.

[4] The subcase on which $D \subseteq E^c$ is analogous.

[5] The subcase on which $D \subseteq E^c$ is analogous.

2nd case: $E \cup F = S$ and $E \cap F \neq \emptyset$.

$w(E \cup F) + w(E \cap F) = w(S) + w(E \cap F) = 1 + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F)$.

i) If $D \subseteq E \cap F$.

$$\begin{aligned} w(E \cup F) + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = 1 + (1 - \psi(E \cap F)) \geq \\ \geq (1 - \psi(E)) + (1 - \psi(F)) = \\ = w(E) + w(F). \end{aligned}$$

ii) If [6] $D \subseteq E \cap F^c$.

$$\begin{aligned} w(E \cup F) + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = 1 = p(E \cap F^c) \geq \\ (1 - \psi(E \cap F^c)) \cdot p(E \cap F^c) \geq \\ (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

iii) If [7] $D \subseteq E$, with $D \cap F^c = X$ and $D \cap (E \cap F) = Y$ then according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(Y)) = (1 - \psi(E \cap F)) = (1 - \psi(F)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = 1 + (1 - \psi(F)) \cdot p(F) \geq \\ \geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

iv) If $D \subseteq (E \cap F)^c$, with $D \cap F^c = X$ and $D \cap E^c = Z$ then according to property $U.5$ *(nullity in the complement of D)*:

$(1 - \psi(X)) = (1 - \psi(F^c)) = (1 - \psi(E))$ and $(1 - \psi(Z)) = (1 - \psi(E^c)) = (1 - \psi(F))$.

[6] The subcase on which $D \subseteq E^c \cap F$ is analogous.

[7] The subcase on which $D \subseteq F$ is analogous.

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = 1 = p(E) + p(F) \geq \\ \geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) \\ = w(E) + w(F). \end{aligned}$$

v) If $D \subseteq S$, with $D \cap F^c = X$, $D \cap (E \cap F) = Y$ and $D \cap E^c = Z$ then according to property $U.4$ *(uniformity in proper subsets of D)*:

$$\begin{aligned} (1 - \psi(X)) = (1 - \psi(Y)) = (1 - \psi(Z)) = \\ = (1 - \psi(X \cup Y)) = (1 - \psi(X \cup Z)) = (1 - \psi(Y \cup Z)); \end{aligned}$$

and according to property $U.5$ *(nullity in the complement of D)*:

$$\begin{aligned} (1 - \psi(X \cup Y)) = (1 - \psi(E)); \; (1 - \psi(Y)) = (1 - \psi(E \cap F)); \text{ and} \\ (1 - \psi(Y \cup Z)) = (1 - \psi(F)). \end{aligned}$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = 1 + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = 1 + (1 - \psi(E \cap F)) \cdot (p(E) + p(F) - 1) = \\ = \psi(E \cap F) + (1 - \psi(E \cap F)) \cdot (p(E) + p(F)) = \\ = \psi(E \cap F) + (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F)) \geq \\ \geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F)) = \\ = w(E) + w(F). \end{aligned}$$

3rd case: $E \cup F \neq S$ and $E \cap F \neq \emptyset$.

$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F)$.

i) If $D \subseteq (E \cap F)$.

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) + (1 - \psi(E \cap F)) \geq \\ \geq (1 - \psi(E)) + (1 - \psi(F)) = \\ = w(E) + w(F). \end{aligned}$$

ii) If [8] $D \subseteq (E \cap F^c)$, with $D \cap (E \cap F^c) = X$ then according to property $U.6$ *(monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(E)).$$

[8] The subcase on which $D \subseteq E^c \cap F$ is analogous.

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = (1 - \psi(E \cup F)) \cdot p(E) \geq \\ \geq (1 - \psi(E)) \cdot p(E) = \\ (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

iii) If $D \subseteq ((E \cap F) \cup (E \cup F)^c)$.

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) + (1 - \psi(E \cap F)) \geq \\ \geq (1 - \psi(E)) + (1 - \psi(F)) = \\ = w(E) + w(F). \end{aligned}$$

iv) If [9] $D \subseteq ((E \cap F^c) \cup (E \cup F)^c)$, with $D \cap (E \cap F^c) = X$ then according to property *U*.5 *(nullity in the complement of D)*:

$$(1 - \psi(X)) = (1 - \psi(E)) = (1 - \psi(E \cup F)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = (1 - \psi(E)) \cdot p(E) = \\ = (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

v) If [10] $D \subseteq E$, with $D \cap (E \cap F^c) = X$ and $D \cap (E \cap F) = Y$ then according to property *U*.5 *(nullity in the complement of D)*:

$$(1 - \psi(Y)) = (1 - \psi(E \cap F)) = (1 - \psi(F)).$$

and according to property *U*.6 *(monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(E)).$$

[9] The subcase on which $D \subseteq ((E^c \cap F) \cup (E \cup F)^c)$ is analogous.

[10] The subcase on which $D \subseteq F$ is analogous.

So,

$$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) \geq$$
$$\geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) =$$
$$= w(E) + w(F).$$

vi) If $D \subseteq ((E \cap F^c) \cup (E^c \cap F))$, with $D \cap (E \cap F^c) = X$ and $D \cap (F \cap E^c) = Z$ then according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(X)) = (1 - \psi(E)) \text{ and } (1 - \psi(Y)) = (1 - \psi(F));$$

and according to the property $U.6$ *(monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(E)) \text{ and } (1 - \psi(E \cup F)) \geq (1 - \psi(F)).$$

So,

$$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) =$$
$$= (1 - \psi(E \cup F)) \cdot (p(E) + p(F)) \geq$$
$$\geq (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) =$$
$$= w(E) + w(F).$$

vii) If [11] $D \subseteq (E \cup (E \cup F)^c)$, with $D \cap (E \cap F^c) = X$ and $D \cap (E \cap F) = Y$ then according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(X \cup Y)) = (1 - \psi((E \cap F^c) \cup (E \cap F)) = (1 - \psi(E)) = (1 - \psi(E \cup F)) \text{ and}$$
$$(1 - \psi(Y)) = (1 - \psi(E \cap F)) = (1 - \psi(F)).$$

So,

$$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) =$$
$$= (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) =$$
$$= w(E) + w(F).$$

viii) If $D \subseteq (E \cap F)^c$, with $D \cap (E \cap F^c) = X$ and $D \cap (F \cap E^c) = Z$ then according to property $U.4$ *(uniformity in proper subsets of D)*:

$$(1 - \psi(X)) = (1 - \psi(Z)) = (1 - \psi(X \cup Z));$$

and according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(X)) = (1 - \psi(E)); (1 - \psi(Z)) = (1 - \psi(F)); \text{ and } (1 - \psi(X \cup Z)) = (1 - \psi(E \cup F)).$$

[11] The subcase on which $D \subseteq (F \cup (E \cup F)^c)$ is analogous.

So,

$$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = (1 - \psi(E \cup F)) \cdot (p(E) + p(F)) = \\ = (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F).$$

ix) If $D \subseteq (E \cup F)$, $D \cap (E \cap F^c) = X$; $D \cap (E \cap F) = Y$ and $D \cap (E^c \cap F) = Z$ then according to property $U.4$ *(uniformity in proper subsets of D)*:

$$(1 - \psi(X)) = (1 - \psi(Y)) = (1 - \psi(Z)) = (1 - \psi(X \cup Y)) = (1 - \psi(X \cup Z)) = (1 - \psi(Y \cup Z));$$

according to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(X \cup Y)) = (1 - \psi(E)); (1 - \psi(Y)) = (1 - \psi(E \cap F)); \text{ and} \\ (1 - \psi(Y \cup Z)) = (1 - \psi(F));$$

and according to the property $U.6$ *(monotone decreasing in D)*:

$$(1 - \psi(E \cup F)) \geq (1 - \psi(E \cap F)).$$

So,

$$w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) \geq \\ \geq (1 - \psi(E \cap F)) \cdot (p(E \cup F) + p(E \cap F)) = \\ = (1 - \psi(E \cap F)) \cdot (p(E) + p(F)) = \\ = (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F).$$

x) If $D \subseteq S$, $D \cap (E \cap F^c) = X$; $D \cap (E \cap F) = Y$ and $D \cap (E^c \cap F) = Z$ then according to property $U.5$ *(uniformity in proper subsets of D)*:

$$(1 - \psi(X)) = (1 - \psi(Y)) = (1 - \psi(Z)) = \\ (1 - \psi(X \cup Y)) = (1 - \psi(X \cup Z)) = (1 - \psi(Y \cup Z)) = (1 - \psi(X \cup Y \cup Z));$$

and according to property $U.6$ *(nullity in the complement of D)*:

$$(1 - \psi(X \cup Y)) = (1 - \psi(E)); (1 - \psi(Y) = (1 - \psi(E \cap F)); \\ (1 - \psi(Y \cup Z)) = (1 - \psi(F)) \text{ and } (1 - \psi(X \cup Y \cup Z)) = (1 - \psi(E \cup F)).$$

So,

$$\begin{aligned} w(E \cup F) + w(E \cap F) = (1 - \psi(E \cup F)) \cdot p(E \cup F) + (1 - \psi(E \cap F)) \cdot p(E \cap F) = \\ = (1 - \psi(E \cup F)) \cdot (p(E \cup F) + p(E \cap F)) = \\ = (1 - \psi(E \cup F)) \cdot (p(E) + p(F)) = \\ = (1 - \psi(E)) \cdot p(E) + (1 - \psi(F)) \cdot p(F) = \\ = w(E) + w(F). \end{aligned}$$

Verifying property $C.\ 7$ **(additivity in proper subsets of the support)**:
We must show that for all $\emptyset \neq C, C' \subsetneq D$, with $C \cup C' \neq D$ and $C \cap C' = \emptyset$:

$$w(C \cup C') = w(C) + w(C').$$

According to property $U.4$ *(uniformity in proper subsets of D)*:

$$(1 - \psi(C \cup C')) = (1 - \psi(C)) = (1 - \psi(C'))$$

So,

$$\begin{aligned} w(C \cup C') = (1 - \psi(C \cup C')) \cdot p(C \cup C') = \\ = ((1 - \psi(C \cup C') \cdot (p(C) + p(C')) = \\ = (1 - \psi(C)) \cdot p(C) + (1 - \psi(C')) \cdot p(C') = \\ = w(C) + w(C'). \end{aligned}$$

Verifying property $C.\ 8$ **(nullity in the complement of the support)**:
We must show that for all $\emptyset \neq C \subsetneq D$ and all $E \subseteq D^c$:

$$w(C \cup E) = w(C).$$

According to property $U.5$ *(nullity in the complement of D)*:

$$(1 - \psi(C \cup E)) = (1 - \psi(C))$$

So,

$$\begin{aligned} w(C \cup E) = (1 - \psi(C \cup E)) \cdot p(C \cup E) = \\ = (1 - \psi(C)) \cdot p(C) = \\ = w(C). \end{aligned}$$

Verifying property $C.\ 9$ ($w(E) + w(E^c) > 0$, for all $E \in \Sigma$):

$$\begin{aligned} w(E) + w(E^c) = (1 - \psi(E)) \cdot p(E) + (1 - \psi(E^c)) \cdot p(E^c) = \\ = (1 - \psi(E)) \cdot (p(E) + p(E^c)) = \\ = (1 - \psi(E)) > 0. \end{aligned}$$

(because $\psi(E) < 1$ for all $E \in \Sigma$).

In order to verify that the real-valued set-function $q : \Sigma \to \mathbb{R}_+$ defined by:

$$q \equiv \frac{1}{(1 - c(v, \cdot))} \cdot v$$

is a probability measure (i.e., $q \in \Delta$), note firstly, that:

$$q(\emptyset) = \frac{1}{(1 - c(v, \emptyset))} \cdot v(\emptyset) = \frac{1}{v(\emptyset) + v(S)} \cdot v(\emptyset) = 0.$$

Verifying property $C.\ 1$ **(normalized)**:

$$q(S) = \frac{1}{(1 - c(v, S))} \cdot v(S) = \frac{1}{v(S) + v(\emptyset)} \cdot v(S) = 1.$$

Verifying property $C.\ 4$ **(additivity)**:

We must show that for all $E, F \in \Sigma$ such that $E \cap F = \emptyset$:

$$q(E \cup F) = q(E) + q(F).$$

Note, firstly, that if $S = E \cup F$ then the condition is always satisfied.

Otherwise, when $S \neq E \cup F$, there exist one trivial subcase and three non trivial subcases.

Let start with the trivial subcase:

i) If [12] $D \subseteq E$:

So,

$$\begin{aligned} q(E \cup F) = \frac{1}{(1 - c(v, E \cup F))} \cdot v(E \cup F) = \\ = \frac{1}{v(E \cup F) + v((E \cup F)^c)} \cdot v(E \cup F) = \\ = 1 = q(E) = \\ = q(E) + q(F). \end{aligned}$$

Now the three non trivial subcases:[13]

[12] The subcase on which $D \subseteq F$ is analogous.

[13] In what follows, let $\emptyset \neq X \subseteq E$, $\emptyset \neq Y \subseteq F$ and $\emptyset \neq Z \subseteq (E \cup F)^c$.

i) If [14] $D \subseteq F^c$, with $D \cap E = X$ then $v(F) = 0$ and according to property $C.8$ *(nullity in the complement of D)*:

$$v(X) = v(E) = v(E \cup F) \text{ and } v((E \cup F)^c) = v(E^c).$$

So,

$$\begin{aligned} q(E \cup F) = \frac{1}{(1 - c(v, E \cup F))} \cdot v(E \cup F) = \\ = \frac{1}{v(E \cup F) + v((E \cup F)^c)} \cdot v(E \cup F) = \\ = \frac{1}{v(E) + v(E^c)} \cdot v(E) = \\ = \frac{1}{(1 - c(v, E))} \cdot v(E) = \\ = q(E) = \\ = q(E) + q(F). \end{aligned}$$

ii) If $D \subseteq (E \cup F)$, with $D \cap E = X$ and $D \cap F = Y$ then $v((E \cup F)^c) = 0$ and according to property $C.8$ *(nullity in the complement of D)*:

$$v(X) = v(E) = v(F^c) \text{ and } v(Y) = v(F) = v(E^c).$$

So,

$$\begin{aligned} q(E \cup F) = \frac{1}{(1 - c(v, E \cup F))} \cdot v(E \cup F) = \\ = \frac{1}{v(E \cup F) + v((E \cup F)^c)} \cdot v(E \cup F) = \\ = \frac{1}{v(E \cup F)} \cdot v(E \cup F) = \\ = 1 = q(E) + q(E^c) = \\ = q(E) + q(F). \end{aligned}$$

iii) If $D \subseteq S$, with $D \cap E = X$; $D \cap F = Y$ and $D \cap (E \cup F)^c = Z$ then, according to property $C.7$ *(additivity in proper subsets of D)*:

$v(X \cup Y) = v(X) + v(Y)$; $v(X \cup Z) = v(X) + v(Z)$; and $v(Y \cup Z) = v(Y) + v(Z)$;

and according to property $C.8$ *(nullity in the complement of D)*:

$$\begin{aligned} v(X) = v(E);\ v(Y) = v(F);\ v(Z) = v((E \cup F)^c);\ v(X \cup Y) = v(E \cup F); \\ v(X \cup Z) = v(E \cup (E \cup F)^c) = v(F^c); \text{ and } v(Y \cup Z) = v(F \cup (E \cup F)^c) = v(E^c). \end{aligned}$$

[14] The subcase on which $D \subseteq E^c$ is analogous.

So,

$$\begin{aligned}
q(E \cup F) &= \frac{1}{(1 - c(v, E \cup F))} \cdot v(E \cup F) = \\
&= \frac{1}{v(E \cup F) + v((E \cup F)^c)} \cdot v(E \cup F) = \\
&= \frac{1}{(v(E) + v(F)) + (v(E^c) - v(F))} \cdot (v(E) + v(F)) = \\
&= \frac{1}{v(E) + v(E^c)} \cdot (v(E) + v(F)) = \\
&= \frac{1}{v(E) + v(E^c)} \cdot v(E) + \frac{1}{v(E) + v(E^c)} \cdot v(F) = \\
&= \frac{1}{v(E) + v(E^c)} \cdot v(E) + \frac{1}{v(E) + v((E \cup F)^c) + v(F)} \cdot v(F) = \\
&= \frac{1}{v(E) + v(E^c)} \cdot v(E) + \frac{1}{v(F^c) + v(F)} \cdot v(F) = \\
&= \frac{1}{(1 - c(v, E))} \cdot v(E) + \frac{1}{(1 - c(v, F))} \cdot v(F) = \\
&= q(E) + q(F).
\end{aligned}$$

Q.E.D.

A.4 Proof of Proposition 3

Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

Let, w. l. g., the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by

$$v \equiv (1 - \psi) \cdot p$$

be represented by:

$$v(E) \equiv \begin{cases} (1 - \varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then $q \in cont(p, \psi)|_D$ implies that $\exists q' \in \Delta$ such that:

$$q(E) = (1 - \varepsilon) \cdot p(E) + \varepsilon \cdot q'(E) \geq (1 - \varepsilon) \cdot p(E) = v(E);$$

and

ii) If $\exists E \supseteq D$, $\psi(E) < \varepsilon$, then there exits two subcases:

ii.1)If $(E \cap D) \subsetneq D$ then $q \in cont(p, \psi)|_D$ implies that $\exists q' \in \Delta$ with $supp(q') \subseteq D$ such that:

$$q(E) = (1 - \varepsilon) \cdot p(E) + \varepsilon \cdot q'(E) \geq (1 - \varepsilon) \cdot p(E) = v(E);$$

ii.2)If $D \subseteq E \neq S$ then $q \in cont(p, \psi)|_D$ implies that $\exists q' \in \Delta$ with $supp(q') \subseteq D$ (which implies that $q'(E) = 1$ if $p(E) = 1$) such that:

$$q(E) = (1 - \psi(E)) \cdot 1 + \psi(E) \cdot 1 \geq (1 - \psi(E)) = v(E).$$

So, $q \in cont(p, \psi)|_D$ implies $q \in core(v)$.

Q.E.D.

A.5 Proof of Proposition 4

Let p be a probability measure (i.e., $p \in \Delta$) with $supp(p) = \{D\}$ and let ψ be a Knightian uncertainty level associated to D (i.e., $\psi \in \Psi|_D$).

Let, w. l. g., the real-valued set-function $v : \Sigma \to \mathbb{R}_+$ defined by

$$v \equiv (1 - \psi) \cdot p$$

be represented by:

$$v(E) \equiv \begin{cases} (1 - \varepsilon) \cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1 - \psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

where $\psi(E) \leq \varepsilon$, for all $E \in \Sigma$ and $\varepsilon \in [0, 1)$.

Because of Proposition 3, it is enough to prove that $core(v)|_D \subseteq cont(p, \psi)|_D$.

i) If $\psi(E) = \varepsilon$ for $E \neq \{\emptyset, S\}$ and $\psi(E) = 0$ for $E = \{\emptyset, S\}$, then $q \in core(v)|_D$ implies that $q(E) \geq (1 - \varepsilon) \cdot p(E)$ and $\exists q' \in \Delta$ such that:

$$q(E) = (1 - \varepsilon) \cdot p(E) + \varepsilon \cdot q'(E)); and$$

ii) If $\exists E \supseteq D$, $\psi(E) < \varepsilon$, then there exits two subcases:

ii.1)If $(E \cap D) \subsetneq D$ then $q \in core(v)|_D$ implies that $q(E) \geq (1 - \varepsilon) \cdot p(E)$ and $\exists q' \in \Delta$ with $supp(q') \subseteq D$ such that:

$$q(E) = (1 - \varepsilon) \cdot p(E) + \varepsilon \cdot q'(E);$$

ii.2)If $D \subseteq E \neq S$ then $q \in core(v)|_D$ implies that $q(E) \geq (1 - \psi(E)$ and $\exists q' \in \Delta$ with $supp(q') \subseteq D$ (which implies that $q'(E) = 1$ if $p(E) = 1$) such that:

$$q(E) = (1 - \psi(E)) \cdot 1 + \psi(E) \cdot 1.$$

So, $q \in core(v)|_D$ implies $q \in cont(p,\psi)|_D$.

$Q.E.D.$

A.6 Proof of Proposition 5

The Dempster-Shafer update rule is given by:

$$v_F(E) = \frac{v(E \cup F^c) - v(F^c)}{1 - v(F^c)}$$

substituting v:

$$v_F(E) = \frac{(1-\psi(E \cup F^c)) \cdot p(E \cup F^c) - (1-\psi(F^c)) \cdot p(F^c)}{1 - (1-\psi(F^c)) \cdot p(F^c)} =$$

$$= \frac{(1-\psi(E \cup F^c)) \cdot [p(E) + p(F^c) - p(E \cap F^c)] - (1-\psi(F^c)) \cdot p(F^c)}{1 - (1-\psi(F)) \cdot [1 - p(F)]} =$$

$$= \frac{[(1-\psi(E \cup F^c)) - (1-\psi(F^c))] \cdot p(F^c) + (1-\psi(E \cup F^c)) \cdot [p(E) - p(E \cap F^c)]}{\psi(F) + (1-\psi(F)) \cdot p(F)} =$$

$$= \frac{(\psi(F^c) - \psi(E \cup F^c)) \cdot (1 - p(E \cap F) - p(E^c \cap F)) + (1-\psi(E \cup F^c)) \cdot p(E \cap F)}{\psi(F) + (1-\psi(F)) \cdot p(F)} =$$

$$= \frac{[-(\psi(F^c) - \psi(E \cup F^c)) + (1-\psi(E \cup F^c)] \cdot p(E \cap F) + (\psi(F^c) - \psi(E \cup F^c)) \cdot (1 - p(E^c \cap F))}{\psi(F) + (1-\psi(F)) \cdot p(F)} =$$

$$= \frac{(1-\psi(F^c)) \cdot p(E \cap F) + (\psi(F^c) - \psi(E \cup F^c)) \cdot p(E \cup F^c)}{\psi(F) + (1-\psi(F)) \cdot p(F)} =$$

$$= \frac{(1-\psi(F^c))\cdot p(F)\cdot p_F(E)}{\psi(F)+(1-\psi(F))\cdot p(F)} + \frac{(\psi(F^c)-\psi(E\cup F^c))\cdot p(E\cup F^c)}{\psi(F)+(1-\psi(F))\cdot p(F)} =$$

$$= \frac{[(1-\psi(F))p(F)+\psi(F)]\cdot p_F(E)-\psi(F)\cdot p_F(E)}{\psi(F)+(1-\psi(F))\cdot p(F)} + \frac{(\psi(F^c)-\psi(E\cup F^c))\cdot p(E\cup F^c)}{\psi(F)+(1-\psi(F))\cdot p(F)} =$$

$$= \left(1-\frac{\psi(F)}{\psi(F)+(1-\psi(F))\cdot p(F)}\right)\cdot p_F(E) + \frac{(\psi(F^c)-\psi(E\cup F^c))\cdot p(E\cup F^c)}{\psi(F)+(1-\psi(F))\cdot p(F)}.$$

Q.E.D.

A.7 Proof of Proposition 6

Firstly note that:

$$p \in \arg\max\{p'(F) : p' \in core(v)\}$$

is equivalent to:

$$p(F) = 1-(1-\psi(F^c))\cdot p(F^c)$$

and

$$p(F^c) = (1-\psi(F^c))\cdot p(F^c).$$

So, $p \in core(v)$.

This implies that:

$$P_F = \{p_F \in \Delta : p \in \arg\max\{p'(F) : p' \in core(v)\}\}$$

can be written as:

$$P_F = \{p_F \in \Delta : p \in core(v)_{F^c}\}$$

So,

$$\begin{aligned} min\{p_F(E) : p_F \in P_F\} = min\{\frac{p(E \cap F^c)}{p(F)} : p \in core(v)_{F^c}\} = \\ = \frac{min\{p(E \cup F^c) : p \in core(v)_{F^c}\} - (1 - \psi(F^c)) \cdot p(F^c)}{1 - (1 - \psi(F^c)) \cdot p(F^c)} = \\ = \frac{(1 - \psi(E \cup F^c)) \cdot p(E \cup F^c) - (1 - \psi(F^c)) \cdot p(F^c)}{1 - (1 - \psi(F^c)) \cdot p(F^c)} = \\ = \frac{v(E \cup F^c) - v(F^c)}{1 - v(F^c)} \end{aligned}$$

and so, $v \equiv (1 - \psi) \cdot p$ is a convex capacity.

Q.E.D.

A.8 Proof of Corollary 1

Let a be a positive simple function (i.e., $a \in \mathbb{R}^S_+$) so that $\alpha_1 > 0$. If $\lambda > \alpha_k$ then $\{s \in S : a(s) \geq \lambda\} = \emptyset$. If $\lambda \in (\alpha_{i-1}, \alpha_i]$, then (recall that $\alpha_0 = 0$) $\{s \in S : a(s) \geq \lambda\} = \bigcup_{j=i}^{k} A_j$. Hence, for each $\lambda \in \mathbb{R}_+$

$$v(L_\lambda(a)) = \sum_{i=1}^{k} v\left(\bigcup_{j=i}^{k} A_j\right) 1_{(\alpha_{i-1}, \alpha_i]}(\lambda)$$

Using the definition of $v(\cdot)$, for each $\lambda \in \mathbb{R}_+$

$$v(L_\lambda(a)) = \sum_{i=1}^{k} \left(1 - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(\bigcup_{j=i}^{k} A_j\right) 1_{(\alpha_{i-1}, \alpha_i]}(\lambda)$$

So that:

$$\int a dv := \int_0^\infty v(L_\lambda(a)) d\lambda$$

where $L_\lambda(a) = \{s \in S : a(s) \geq \lambda\}$

$$\begin{aligned}\int adv &= \int_0^{\infty} \sum_{i=1}^{k} \left(1-\psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(\bigcup_{j=i}^{k} A_j\right) 1_{(\alpha_{i-1},\alpha_i]}(\lambda)d\lambda \\ &= \sum_{i=1}^{k} \left(1-\psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(\bigcup_{j=i}^{k} A_j\right) \int_0^{\infty} 1_{(\alpha_{i-1},\alpha_i]}(\lambda)d\lambda \\ &= \sum_{i=1}^{k} (\alpha_i - \alpha_{i-1}) \left(1-\psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(\bigcup_{j=i}^{k} A_j\right) \\ &= \alpha_1\psi(A_1) + (1-\psi(A_1)) \int adp + \sum_{i=3}^{n} (\alpha_i - \alpha_{i-1}) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)\end{aligned}$$

as desired.

This proof for a positive simple function a. The case of a general function a is easily obtained using translation invariance (the interest reader should see Marinacci and Montrucchio (2004)).

Q.E.D.

A.9 Proof of Proposition 7

For each $p \in \Delta$ if $\psi' \geq \psi$ then $v \geq v'$. So, it follows from the property of monotonicity of the convex capacities (see Simonsen and Werlang (1991)) that:

$$\int_S adv' \leq \int_S adv.$$

Q.E.D.

A.10 Proof of Corollary 2

In order to verify that the real-valued set-function $e : \Sigma \to \mathbb{R}_+$ defined by:

$$w \equiv (1-\psi) \cdot p$$

is convex, we will use Schmeidler's Proposition #3 (see Section $A.5$).

Verifying property i:

Let $q : \Sigma \to \mathbb{R}_+$ be a real-valued set-function defined by:

$$q(s_i) = \begin{cases} 1 - (1 - \psi(A_1))\, p\left(\bigcup_{j=2}^{k} A_j\right) & \text{if} \quad i = 1; \\ \left(1 - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right) p\left(A_i\right) + \left(\psi\left(\bigcup_{j=1}^{i} A_j\right) - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right) p\left(\bigcup_{j=i+1}^{k} A_j\right) & \text{if} \quad i = 2, \cdots, k-1; \\ \left(1 - \psi\left(\bigcup_{j=1}^{k-1} A_j\right)\right) p\left(A_k\right) & \text{if} \quad i = k \end{cases}$$

It is easy to verify that q is a probability measure.
In fact, $q \in core(w)$. To see this, note that:
If $i = 1$:

$$\begin{aligned} 1 - (1 - \psi(A_1))\, p\left(\bigcup_{j=2}^{k} A_j\right) &= 1 - (1 - \psi(A_1))\left(1 - p\left(A_1\right)\right) = \\ &= 1 - (1 - \psi(A_1)) + (1 - \psi(A_1)) \cdot p(A_1) = \\ &= \psi(A_1) + (1 - \psi(A_1))p(A_1) \geq \\ &\geq (1 - \psi(A_1))p(A_1) \end{aligned}$$

If $i = 2, \cdots, k-1$:[15]

$$\begin{aligned} &\left(1 - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right) p\left(A_i\right) + \left(\psi\left(\bigcup_{j=1}^{i} A_j\right) - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right) p\left(\bigcup_{j=i+1}^{k} A_j\right) = \\ &= \left(1 - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(A_i\right) + \left(\psi\left(\bigcup_{j=i+1}^{k} A_j\right) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right) p\left(\bigcup_{j=i+1}^{k} A_j\right) \geq \\ &\geq (1 - \psi(A_i)) \cdot p(A_i). \end{aligned}$$

And if: $i = k$:

$$\begin{aligned} \left(1 - \psi\left(\bigcup_{j=1}^{k-1} A_j\right)\right) p\left(A_k\right) &= \\ &= \left(1 - \psi\left(A_k\right)\right) p\left(A_k\right) \end{aligned}$$

[15]It follows from the property $C.8$:

$$(1 - \psi\left(\bigcup_{j=i}^{k} A_j\right)) \geq (1 - \psi(A_i))$$

and

$$\psi\left(\bigcup_{j=1+1}^{k} A_j\right) - \psi\left(\bigcup_{j=i}^{k} A_j\right) \geq 0$$

Verifying property *ii*:

$$\int_S adq = \min\{q' \in core(w) : \int_S adq'\}.$$

Then, it holds that:

$$\int_S adq = \alpha_{_1} \left[1 - (1 - \psi(A_1))\, p\left(\bigcup_{j=2}^{k} A_j\right)\right] +$$

$$+ \sum_{i=2}^{k-1} \alpha_i \left[\left(1 - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right)\ p(A_{_i}) + \left(\psi\left(\bigcup_{j=1}^{i} A_j\right) - \psi\left(\bigcup_{j=1}^{i-1} A_j\right)\right)\ p\left(\bigcup_{j=i+1}^{k} A_j\right)\right] +$$

$$+ \alpha_k \left(1 - \psi\left(\bigcup_{j=1}^{k-1} A_j\right)\right)\ p(A_{_k}) =$$

$$= \alpha_1 \psi(A_{_1}) + (1 - \psi(A_{_1})) \int adp + \sum_{i=3}^{n} (\alpha_i - \alpha_{i-1}) \left[\psi(A_{_1}) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)$$

Q.E.D.

Appendix B

Proof of the main Theorem of Chapter 4

This Appendix is an extension of Section 4 of Nishimura and Ozaki (2006). The necessity of the axioms in the Theorem can be easily verified. In what follows it will be proved the sufficiency of them.

B.1 A lemma and the Anscombe-Aumann theory

This section proves a lemma and then shows that our set of axioms can be considered as an extension of the Nishimura-Ozaki model, which in turn can be considered as an extension of the Anscombe-Aumann model to the case where the decision-maker considers the ordering underlying constant acts and a Knightian uncertainty level associated to the set D (i.e., $\psi \in \Psi|_D$).

Lemma 4. *Let ψ be a Knightian uncertainty level associated to the support D (i.e., $\psi \in \Psi|_D$) and assume that Axioms 1, 2, 4, 6, 7 and 8 hold. Then, for all $f \in L_0$, $\exists$ $y_f \in L_c$ such that:*

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

Proof. Let $f \in L_0$. Then,

$$y^* \equiv \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_{f(s_k)} + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \succeq f \succeq$$

$$\succeq \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_{f(s_1)} + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \equiv y_*$$

where the first and second ordering hold true by Axiom 12 and by Axiom 4, respectively.

In the rest of the proof, it will be assumed that $y^* \succ f \succ y_*$ since the lemma would follow immediately otherwise. Then, since any pair of constant acts is comonotonic, a similar argument to Kreps (1988, p. 47) shows that $f \sim a^* y^* + (1-a^*) y_*$, where a^* is defined by $a^* = sup\{a \in [0,1] : f \succeq a y^* + (1-a) y_*\}$. On the other hand,

$$a^* y^* + (1-a^*) y_* = a^* \left[\psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_{f(s_k)} + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]\right] +$$

$$+ (1-a^*) \left[\psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_{f(s_1)} + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]\right]$$

$$= \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) \left[a^* y_{f(s_k)} + (1-a^*) y_{f(s_1)}\right] + \sum_{i=3}^{n} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

Therefore, defining $y_f = a^* y_{f(\alpha_k)} + (1-a^*) y_{f(s_1)}$ completes the proof.

□

Anscombe and Aumann (1963)'s main result (Fishburn (1970)'s version) asserts that a binary relation $\succ$ defined on L_0 satisfies Axioms 1, 2, 3, 4 and 6 if and only if there exists a unique finitely probability measure p on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that: [1]

$$f \succ g \Leftrightarrow \int_S u(f(\cdot)) dp > \int_S u(g(\cdot)) dp$$

Moreover, the (utility) function is unique up to a positive affine transformation.

[1] See footnote 5 in Chapter 4.

The following proposition, proved in Nishimura and Ozaki (2006), shows that Axioms 1, 2, 4, 6, 7 combined with the special cases on which Axioms 8 and 9 are such that $\psi(A) = 0$ for all $A \subseteq S$) are necessary and sufficient for the Anscombe-Aumann's Axioms 1, 2, 3, 4 and 6.

Proposition 9. Axioms $1, 2, 4, 6, 7$, 8 and 9 hold with $\psi(A) = 0$ for all $A \subseteq S \Leftrightarrow$

$\Leftrightarrow$ Axioms $1, 2, 3, 4$ and 6 hold.

Proof. See the proof of Proposition 1 in Nishimura and Ozaki (2006, p. 337). □

B.2 A definition of $\succ^*$ and some preliminary lemmas

Let $\succ^*$ define a binary relation on L_0 induced by $\succ$ as follows:

$$f \succ^* g \Leftrightarrow \left[\begin{array}{c} f \sim \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \\ \text{and} \\ g \sim \psi(A_1) y_{g(s_1)} + (1-\psi(A_1)) y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \end{array} \right] \Rightarrow y_f \succ y_g$$

where y_f and y_g are arbitrary elements of L_c. By definition, $f \succ^* g$ holds true whenever there does not exist such a y_f and/or y_g. Clearly, y_f (y_g) is, when it exists, $f(g)$'s equivalent constant act in contamination equivalence. Thus, the binary relation $\succ^*$ is induced by the original preferences over these equivalent constant acts.

The binary relations $\succeq^*$ and $\sim^*$ are defined from $\succ^*$ by: $\succeq^* \Leftrightarrow \not\prec^*$ and $\sim^* \Leftrightarrow [\not\succ^*$ and $\not\prec^*]$. A binary relation on Y is naturally induced from $\succeq^*$ as its restriction on L_c and it is denoted by the same symbol, $\succeq^*$. Then, the following lemmas hold.

Lemma 5. *Assume that Axiom* 1 *holds, let* $f \in L_o$ *and let* $y_f \in L_c$.

If

$$f \sim \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right],$$

then $f \sim^* y_f$

Proof. Suppose that

$$f \sim \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right].$$

It always holds that

$$y_f \sim \psi(A_1)y_f + (1-\psi(A_1))y_f + \sum_{i=3}^{k}(y_f - y_f)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right].$$

Furthermore, $y_f \not\succ y_f$ since $\succ$ is asymmetric by Axiom 1.

Therefore, by the definition of $\succ^*$, it follows that $f \not\succ^* y_f$. Similarly, $y_f \not\succ^* f$. Therefore, $f \sim^* y_f$. □

Lemma 6. *Assume that Axioms* 1 *and* 7 *hold. Then,* $\succ$ *and* $\succ^*$ *coincide on* L_c.

Proof. Let $y, y' \in Y$.

First, assume that $y \succ^* y'$. Note that:

$$y \sim \psi(A_1)y + (1-\psi(A_1))y + \sum_{i=3}^{k}(y - y)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$y' \sim \psi(A_1)y' + (1-\psi(A_1))y' + \sum_{i=3}^{k}(y' - y')\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

hold for all $y, y' \in Y$.

Hence, it follows from the definition of $\succ^*$ that $y \succ y'$.

Second, assume that $y \succ y'$. Let $\overline{y}$ and $\overline{y}'$ be arbitrary *constant acts* such that:

$$\text{(a) } y \sim \psi(A_1)y + (1-\psi(A_1))\overline{y} + \sum_{i=3}^{k}(y - \overline{y})\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$\text{(b) } y' \sim \psi(A_1)y' + (1-\psi(A_1))\overline{y}' + \sum_{i=3}^{k}(y' - \overline{y}')\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

Therefore, Axioms 1 and 7 imply that $y \sim \overline{y}$ (recall that any pair of constant acts is comonotonic). Similarly, it holds from (b) that $y' \sim \overline{y}'$.

Finally, Axiom 1 and the assumption that $y \succ y'$ show that $\overline{y} \succ \overline{y}'$, which in turn shows that $y \succ^* y'$ by the definition of $\succ^*$. □

B.3 $\succ^*$ satisfies the Anscombe-Aumann axioms

In this subsection, we show that the binary relation $\succeq^*$ satisfies the axioms postulated in Anscombe and Aumann Anscombe and Aumann (1963). Let ψ be the Knightian uncertainty level associated to the set D (i.e., $\psi \in \Psi|_D$). Throughout this and the next subsections, it will always be assumed that $\succeq$ satisfies Axioms 1, 2, 4, 6, 7, 8 and 9.

Lemma 7. *The binary relation $\succeq^*$ satisfies Axioms 1, 2, 3, 4 and 6.*

Proof. **1 (Weak order)**: We prove this by showing that $\succ^*$ is asymmetric and negatively transitive (see footnote 3 of Chapter 4).

Asymmetry: Assume that $f \succ^* g$. Also suppose that

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

The existence of constant acts, y_f and y_g, is guaranteed by Lemma 4. Then, it follows from the definition of $\succ^*$ that $y_f \succ y_g$ and the asymmetry of $\succ$ implies that $y_g \not\succ y_f$. Hence, the definition of $\succ^*$ implies that $g \not\succ^* f$.

Negative Transitivity: Assume that $f \not\succ^* g$ and $g \not\succ^* h$. Then, there exist constant acts y_f and y_g such that:

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and $y_f \not\succ y_g$, and there exist constant acts y'_g and y_h such that:

$$g \sim \psi(A_1)y'_{g(s_1)} + (1-\psi(A_1))y'_g + \sum_{i=3}^{k} \left(y'_{g(s_i)} - y'_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$h \sim \psi(A_1)y_{h(s_1)} + (1-\psi(A_1))y_h + \sum_{i=3}^{k} \left(y_{h(s_i)} - y_{h(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and $y'_g \not\succ y_h$.

It then holds that

$$\psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \sim g \sim$$

$$\sim \psi(A_1)y'_{g(s_1)} + (1-\psi(A_1))y'_g + \sum_{i=3}^{k} \left(y'_{g(s_i)} - y'_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] \sim$$

$$\sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y'_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

where the last indifference holds since $y_{g(s_i)} \sim y'_{g(s_i)}$ for all $i \in \{1, 2, \cdots, k\}$ and since for all constant acts $y_f, y_g, y_h \in L_c$ holds that

$$y_f \sim y_g \Rightarrow \lambda y_f + (1-\lambda)y_h \sim \lambda y_g + (1-\lambda)y_h$$

because $\succeq$ satisfies Axioms 1, 2 and 3 when restricted on L_c (see proof of *Proposition 1* in Nishimura and Ozaki (2006, p. 337)). Therefore, Axioms 1 and 7 imply that $y_g \sim y'_g$ (recall that any pair of constant acts is comonotonic). Hence Axiom 1 implies that $y_f \not\succ y_h$, which shows that $f \not\succ^* h$.

2 (Continuity): Assume that $f \succ^* g$ and $g \succ^* h$ and let y_f, y_g and y_h be any constant acts such that

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$h \sim \psi(A_1)y_{h(s_1)} + (1-\psi(A_1))y_h + \sum_{i=3}^{k} \left(y_{h(s_i)} - y_{h(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

Such y_f, y_g and y_h exist by Lemma 5. By the assumption that $f \succ^* g$ and $g \succ^* h$ and the definition of $\succ^*$, it follows that $y_f \succ y_g$ and $y_g \succ y_h$. Then, Axiom 2 implies that there exits $\alpha \in (0,1)$ such that $\alpha y_f + (1-\alpha) y_h \succ y_g$. Let $y_{f\alpha h}$ be any constant act such that

$$\alpha f + (1-\alpha)h \sim \psi(A_1) y_{f\alpha h(s_1)} + (1-\psi(A_1)) y_{f\alpha h} + \sum_{i=3}^{k} (y_{f\alpha h(s_i)} - y_{f\alpha h(s_{i-1})}) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

then, $(13-1)$ implies that

$$y_{f\alpha h} \sim \alpha y_f + (1-\alpha) y_h$$

Therefore, Axiom 1 shows that $y_{f\alpha h} \succ y_g$, which in turn shows that $\alpha f + (1-\alpha)h \succ^* g$ by the definition of $\succ^*$. A similar proof applies for the existence of $\beta \in (0,1)$ such that $g \succ^* \beta f + (1-\beta)h$.

3 (Independence): Assume that $f \succ^* g$ and let $y_{f\lambda h}$ and $y_{g\lambda h}$ be any constant acts such that

$$\lambda f + (1-\lambda)h \sim \psi(A_1) y_{f\lambda h(s_1)} + (1-\psi(A_1)) y_{f\lambda h} + \sum_{i=3}^{k} (y_{f(s_i)} - y_{f\lambda h(s_{i-1})}) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

and

$$\lambda g + (1-\lambda)h \sim \psi(A_1) y_{g\lambda h(s_1)} + (1-\psi(A_1)) y_{g\lambda h} + \sum_{i=3}^{k} (y_{g\lambda h(s_i)} - y_{g\lambda h(s_{i-1})}) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

It will be showed that $y_{f\lambda h} \succ y_{g\lambda h}$, which completes the proof by the definition of $\succ^*$. By Lemma 4 and the assumption that $f \succ^* g$, there exist constant acts y_f, y_g and y_h such that

$$f \sim \psi(A_1) y_{f(s_1)} + (1-\psi(A_1)) y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})} \right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

$$g \sim \psi(A_1) y_{g(s_1)} + (1-\psi(A_1)) y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})} \right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

and

$$h \sim \psi(A_1) y_{h(s_1)} + (1-\psi(A_1)) y_h + \sum_{i=3}^{k} \left(y_{h(s_i)} - y_{h(s_{i-1})} \right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j \right) \right]$$

and $y_f \succ y_g$.

Since any pair of constant acts is co-monotonic, Axiom 7 implies that

$$\lambda y_f + (1-\lambda)y_h \succ \lambda y_g + (1-\lambda)y_h$$

On the other hand, $(ix-1)$ implies that

$$\lambda y_f + (1-\lambda)y_h \sim \lambda y_{f\lambda h}$$

and

$$\lambda y_g + (1-\lambda)y_h \sim \lambda y_{g\lambda h}$$

Therefore, Axiom 1 shows that $y_{f\lambda g} \succ y_{g\lambda h}$.

4 (Monotonicity): Suppose that for all $s \in S$, $f(s) \succeq^* g(s)$. Since $\succ^*$ and $\succ$ coincide on L_c (lemma 7), it follows that for all $s \in S$, $f(s) \succeq g(s)$. Let y_f and y_g be constant acts such that

$$f \sim \psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k} \left(y_{f(s_i)} - y_{f(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

and

$$g \sim \psi(A_1)y_{g(s_1)} + (1-\psi(A_1))y_g + \sum_{i=3}^{k} \left(y_{g(s_i)} - y_{g(s_{i-1})}\right) \left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]$$

Such y_f and y_g exist by Lemma 5. Then, $9-2$ implies that $y_f \succeq y_g$, or equivalently, $y_g \not\succ y_f$. Therefore, it follows from the definition of $\succ^*$ that $g \not\succ^* f$, implying that $f \succeq^* g$.

6 (Nondegeneracy): From Axioms 4 and 6, it follows that $\exists\ y, y' \in Y$ such that $y \succ y'$. Since $\succ^*$ and $\succ$ coincide on L_c (Lemma 6), $y \succ^* y'$. □

B.4 End of the proof

Anscombe and Aumann (1963)'s theorem and Lemma 7 show that there exist a unique finitely probability measure p on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that

$$f \succ^* g \Leftrightarrow \int_S u(f(\cdot))dp > \int_S u(g(\cdot))dp$$

Define $J^* : L_0 \to \mathbb{R}$ such that for all $f \in L_0$

$$J^*(f) = \int_S u(f(\cdot))dp$$

and define $J : L_0 \to \mathbb{R}$ such that for all $f \in L_0$

$$J(f) = u\left(\psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k}\left(y_{f(s_i)} - y_{f(s_{i-1})}\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]\right)$$

where $y_f \in L_c$ is f's equivalent constant act in generalized $\psi-$contamination equivalence whose existence is guaranteed by Lemma 5.

Since u represents $\succ^*$ on L_c by $f \succ^* g$ and $\succ^*$ and $\succ$ coincide on L_c by Lemma 7, u represents $\succ$ on L_c. Therefore, J is well-defined and represents $\succ$ on L_0.

Finally, we have

$$J(f) = u\left(\psi(A_1)y_{f(s_1)} + (1-\psi(A_1))y_f + \sum_{i=3}^{k}\left(y_{f(s_i)} - y_{f(s_{i-1})}\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right]\right)$$

$$= \psi(A_1)u(y_{f(s_1)}) + (1-\psi(A_1))u(y_f) + \sum_{i=3}^{k}\left(u(y_{f(s_i)}) - u(y_{f(s_{i-1})})\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)$$

$$= \psi(A_1)u(y_{f(s_1)}) + (1-\psi(A_1))J^*(y_f) + \sum_{i=3}^{k}\left(u(y_{f(s_i)}) - u(y_{f(s_{i-1})})\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)$$

$$= \psi(A_1)u(y_{f(s_1)}) + (1-\psi(A_1))J^*(f) + \sum_{i=3}^{k}\left(u(y_{f(s_i)}) - u(y_{f(s_{i-1})})\right)\left[\psi(A_1) - \psi\left(\bigcup_{j=i}^{k} A_j\right)\right] p\left(\bigcup_{j=i}^{k} A_j\right)$$

$$= \psi(A_1)u(y_{f(s_1)}) + (1-\psi(A_1))\int_S u(f(\cdot))dp +$$

$$+\sum_{i=3}^{k}\left(u(y_{f(s_i)})-u(y_{f(s_{i-1})})\right)\left[\psi(A_1)-\psi\left(\bigcup_{j=i}^{k}A_j\right)\right]p\left(\bigcup_{j=i}^{k}A_j\right)$$

where the second equality holds by u's affinity; the third equality holds by the definition of J^* and the fact that u represents $\succ$ on Y; the fourth equality holds by *Lemma 6* and the fact that J^* represents $\succ^*$; and the last equality holds by $J^*(f)$. Since J represents $\succ$ on L_0, the proof is complete.

Q.E.D.

Appendix C

Proofs of Chapter 5

C.1 Proof of Theorem 3

We now that if $v_i \in cont(p_i, \psi^{-i})|_{D_i}$ then for some probability measure p (i.e.,$p \in \Delta$) with $supp(p) = \{D\}$ it is true that:

$$v_i(E) \equiv \begin{cases} (1-\varepsilon)\cdot p(E) & \text{if} \quad (E \cap D) \subsetneq D \\ (1-\psi(E)) & \text{if} \quad D \subseteq E \subsetneq S \\ 1 & \text{if} \quad E = S \end{cases}$$

Thus, Choquet integral has the form:

$$\int adv \equiv \psi_1 a(s_1) + (1-\psi_1)\int adp + \sum_{j=3}^{n}(\psi_1 - \psi_{j,\ldots,n})\left(\sum_{i=j}^{n} p(\{s_i\})\right)(a(s_j) - a(s_{j-1}))$$

Suppose, without loss of generality, that player $-i$ has n pure strategies. So we can order the payoffs of player $i \in \{1,2\}$ to each pure strategy, $a_i \in A_i$, as:

$$u_i^1(a_i) \leq u_i^2(a_i) \leq \ldots \leq u_i^n(a_i)$$

where

$u_i^j(a_i) = u_i(a_i, a_j)$ is the j-position ($j = 1, \ldots, n$).

We modify the original game Γ to $\Gamma_{(\psi^1,\psi^2)} = (A_1, A_2; u_1', u_2')$, where if $j = 1, 2$:

$$u_i'(a_i, a_j) = \psi_1^i u_i^1(a_i) + (1-\psi_1^i)u_i^j(a_i)$$

and, if $j = 3, \ldots, n$:

$$u_i\prime(a_i, a_j) = \psi_1^i u_i^1(a_i) + (1-\psi_1^i)u_i^j(a_i) + \sum_{k=3}^{j}(\psi_1^i - \psi_{k,\ldots,n}^i)(u_i^k(a_i) - u_i^{k-1}(a_i))$$

Note that ψ_1^i is the Knightian uncertainty level associate to the strategy of player $-i$ who gives the worst payoff to player $i(\in \{1,2\})$ when his choice is the pure strategy a_i.

Let (p_1, p_2) be a standard mixed strategy Nash equilibrium of the modified game $\Gamma_{(\psi^1,\psi^2)}$. We will show that the pair (v_1, v_2), where:

$$v_1(A) \equiv \begin{cases} (1-\varepsilon)\cdot p(A) & \text{if} \quad (A \cap D) \subsetneq D \\ (1-\psi(A)) & \text{if} \quad D \subseteq E \subsetneq A_1 \\ 1 & \text{if} \quad A = A_1 \end{cases}$$

and

$$v_2(A) \equiv \begin{cases} (1-\varepsilon)\cdot p(A) & \text{if} \quad (A \cap D) \subsetneq D \\ (1-\psi(A)) & \text{if} \quad D \subseteq E \subsetneq A_2 \\ 1 & \text{if} \quad A = A_2 \end{cases}$$

is a Nash equilibrium under uncertainty for the original game Γ, with the specified Knightian uncertainty levels associated. It is immediate to verify that the uncertainty aversion measure of v_2 is given by ψ^1 and that the uncertainty aversion measure of v_1 is given by ψ^2.

To check that this is a Nash equilibrium under uncertainty, note that (except in case where, for each $i \in \{1,2\}$ and for all $A \in 2^{A_i}$, $\emptyset \neq A \neq A_i$: $\psi^i(A) = 1$) the support of v_i is unique and coincides with the support of p_i for each player $i \in \{1,2\}$. Since (p_1, p_2) is a standard mixed strategy Nash equilibrium for the modified game, it follows that any $a_i \in supp(p_i)$ is a best response to p_{-i} (for the modified utility u_i'). In other words, a_i maximizes the following expression over $a \in A_i$:

$$\begin{aligned}
\int u_i'(a_i,\cdot)dp_{-i} &= p_{-i}^1\left(\psi_1^i u_i^1(a_i) + (1-\psi_1^i)u_i^1(a_i)\right) + ...+ \\
&+ p_{-i}^k\left[\left(\psi_1^i u_i^1(a_i) + (1-\psi_1^i)u_i^k(a_i)\right) + \sum_{l=3}^{k}(\psi_1^i - \psi_{l,...n}^i)\left(u_i^l(a_i) - u_i^{l-1}(a_i)\right)\right] + ...+ \\
&+ p_{-i}^n\left[\left(\psi_1^i u_i^1(a_i) + (1-\psi_1^i)u_i^n(a_i)\right) + \sum_{l=3}^{n}(\psi_1^i - \psi_{l,...n}^i)\left(u_i^l(a_i) - u_i^{l-1}(a_i)\right)\right] \\
&= \psi_1^i u_i^1(a_i) + (1-\psi_1^i)\int u_i(a_i,\cdot)dp_{-i} + \sum_{l=3}^{n}(\psi_1^i - \psi_{l,...n}^i)\left(\sum_{r=1}^{n} p_{-i}^r\right)\left(u_i^l(a_i) - u_i^{l-1}(a_i)\right) \\
&= \int u_i(a_i,\cdot)dv_{-i}
\end{aligned}$$

Thus, a_i is also a best response in the original game Γ. In the case where, for each player $i \in \{1,2\}$ and for all $A \in 2^{A_i}$, $\emptyset \neq A \neq A_i$: $\psi^i(A) = 1$ any singleton a_i is a support of v_i. Therefore any best response for player i is in a support.

Thus (v_1, v_2) is a Nash equilibrium under uncertainty for the original game.

Q.E.D.

C.2 Proof of Proposition 8

Consider, for each player $i \in \{1,2\}$: $\psi^i \in \Psi^i(A_{-i}, 2^{A_{-i}})|_{D_{-i}}$, where $\psi^i : 2^{A_{-i}} \to [0,1)$ is the Knightian uncertainty level defined such that, for all $A \in 2^{A_{-i}}$,$\emptyset \neq A \neq A_{-i}$:

$$\psi^i(A) = \varepsilon_i,\ \varepsilon_i \in [0,1)$$

Thus defined, ψ^i exhibits constant Knightian uncertainty level. So, for each player $i \in \{1,2\}$, v_i is a uniform squeeze of p_i. These beliefs (v_1, v_2) form Nash equilibrium under uncertainty as in Definition 5.

Q.E.D.

Appendix D

Proofs of Chapter 6

D.1 Proof of Theorem 4

Omitted. This is a natural extension of Theorem 1 in Nishimura and Ozaki (2002). The interested reader should read their proof.

Q.E.D.

D.2 Proof of Lemma 1

At first, we show that $cont(p^0, \psi)|_{1,D} \subseteq cont(p^0|_1, \psi)|_D$.

Let $p_1 \in cont(p^0, \psi)|_{1,D}$. Then there exists $p \in cont(p^0, \psi)|_D$ such that $p|_1 = p(\cdot \times S)$. Since $p \in cont(p^0, \psi)|_D$, there exists $q \in \Delta(S \times S, \Sigma_2)$ such that $p = (1-\psi(E))p^0 + \psi(E)q$. Thus, $p_1 = p(\cdot \times S) = (1 - \psi(E))p^0|_1(\cdot)$ which implies that $p_1 \in cont(p^0|_1, \psi)|_D$.

Now, we show that $cont(p^0|_1, \psi)|_D \subseteq cont(p^0, \psi)|_{1,D}$.

Let $p_1 \in cont(p^0|_1, \psi)|_D$. Then there exists $q_1 \in \Delta(S, \Sigma_1)$ such that $p_1 = (1 - \psi(E))p^0|_1 + \psi(E)q_1$. Let $q_2 \in \Delta(S, \Sigma_2)$ and let $p = (1 - \psi(E))p^0 + \psi(E)(q_1 \times q_2)$.

Then, $p \in cont(p^0, \psi)|_D$ and $p|_1 = (1 - \psi(E))p^0|_1 + \psi(E)q_1 = p_1$, which implies that $p_1 \in cont(p^0, \psi)|_{1,D}$.

Q.E.D.

D.3 Proof of Lemma 2

Omitted. Similar to the proof of Lemma 1.

Q.E.D.

D.4 Proof of Theorem 5

We just consider the purchasing problem. Similar argument provides the corresponding results for short sales. Assume first that

$$q > \int_S \int_S X(s_1, s_2) v_1(ds_2) v_0(ds_1).$$

Then, for all $N \geq 0$,

$$\begin{aligned}
&\int S \left[\int_S u \left(W - N + (\frac{N}{q}) X(s_1, s_2) \right) v_1(ds_2) \right] v_0(ds_1) \leq \\
&\leq \int_S u \left(\int_S \left(W - N + (\frac{N}{q}) X(s_1, s_2) \right) v_1(ds_2) \right) v_0(ds_1) = \\
&= \int_S u \left(W - N + (\frac{N}{q}) \left[\int_S X(s_1, s_2) v_1(ds_2) \right] \right) v_0(ds_1) \leq \\
&\leq u \left(\int SW - N + (\frac{N}{q}) \int_S X(s_1, s_2) v_1(ds_2) \right) v_0(ds_1) = \\
&= u \left(W - N + (\frac{N}{q}) \int_S \int_S X(s_1, s_2) v_1(ds_2) v_0(ds_1) \right) < \\
&< u \left(W - N + (\frac{N}{q}) \cdot q \right) = \\
&= u(w)
\end{aligned}$$

Suppose that:

$$q < \int_S \int SX(s_1, s_2) v_1'(ds_2) v_0'(ds_1)$$

then, for all $N \geq 0$,

$$\begin{aligned}
&\int_S \left[\int_S u\left(W - N + (\frac{N}{q})X(s_1,s_2)\right) v_1(ds_2)\right] v_0(ds_1) \leq \\
&\leq \int_S u\left(\int_S \left(W - N + (\frac{N}{q})X(s_1,s_2)\right) v_1(ds_2)\right) v_0(ds_1) \leq \\
&\leq \int_S u\left(W - N - (\frac{N}{q})\int_S X(s_1,s_2)v_1(ds_2)\right) v_0(ds_1) = \\
&= \int_S u\left(W - N - (\frac{N}{q})\left(-\int_S X(s_1,s_2)v_1'(ds_2)\right)\right) v_0(ds_1) \leq \\
&\leq u\left(\int_S \left(W - N + (\frac{N}{q})\int_S X(s_1,s_2)v_1'(ds_2)\right) v_0(ds_1)\right) = \\
&= u\left(W - N - (\frac{N}{q})\int_S - \left(\int_S X(s_1,s_2)v_1'(ds_2)\right) v_0(ds_1)\right) = \\
&= u\left(W - N - (\frac{N}{q})\left(-\int_S\int_S X(s_1,s_2)v_1'(ds_2)v_0'(ds_1)\right)\right) < \\
&< u\left(W - N - (\frac{N}{q})(-q)\right) = \\
&= u(W)
\end{aligned}$$

Hence no holding is strictly better than investing in the asset. Similar argument provides the corresponding results for short sales.

Q.E.D.

D.5 Proof of Theorem 6

Omitted. Similar to the proof of Theorem 4.

Q.E.D.

D.6 Proof of Theorem 7

Note that it suffices to show that for events $F \in \Sigma_2$ such that $p^0|_2(F) > 0$:

$$v_1(F) < v_2(F)$$

Because the Knightian uncertainty levels associated to the events in the support are constant, the remaining of the proof is similar to the proof of Theorem 4 in Asano (2008).

When $\delta(E) = 0$, it follows that

$$\begin{aligned}(1-\varepsilon)p^0|_2(F) - (1-\varepsilon')p^0|_2(F|E) = \\ = (1-\varepsilon)\left(p^0|_2(F) - p^0|_2(F|E)\right) - (\varepsilon - \varepsilon')p^0|_2(F|E) = \\ = (\varepsilon - \varepsilon')p^0|_2(F|E) > 0\end{aligned}$$

where $\varepsilon' = \frac{\varepsilon}{(1-\varepsilon)p|^0|_1(E)+\varepsilon}$.

When $\delta(E) > 0$, it follows that:

$$\begin{aligned}(1-\varepsilon)p^0|_2(F) - (1-\varepsilon')p^0|_2(F|E) = \varepsilon'\left(\frac{1-\varepsilon}{\varepsilon'}p^0|_2(F) - \frac{1-\varepsilon'}{\varepsilon'}p^0|_2(F|E)\right) = \\ = (1-\varepsilon)\varepsilon'\left[\left(\frac{1-\varepsilon}{\varepsilon'}p^0|_1(E)+1\right)p^0|_2(F) - \frac{1}{\varepsilon}p^0|_1(E)p^0|_2(F|E)\right] \geq \\ \geq (1-\varepsilon)\varepsilon'\left[\left(\frac{1-\varepsilon}{\varepsilon'}p^0|_1(E)+1\right)p^0|_2(F) - \frac{1}{\varepsilon}p^0|_1(E)\left(p^0|_2(F)+\delta(E)\right)\right] = \\ = (1-\varepsilon)\varepsilon'\left[\left(1-p^0|_1(E)\right) - \frac{\delta(E)}{\varepsilon}p^0|_1(E)\right] \geq \\ \geq (1-\varepsilon)\varepsilon'\left[\left(1-p^0|_1(E)\right)min\{p^0|_2(F_j) - \frac{\delta(E):j}{\varepsilon}p^0|_1(E)\}\right] > \\ > (1-\varepsilon)\varepsilon'\left[\left(1-p^0|_1(E)\right)min\{p^0|_2(F_j) - \delta(E)p^0|_1(E)(\frac{p^0|_1(E)}{(1-p^0|_1(E))min\{p^0|_2(F_j)\delta(E):j\}})^{-1}:j\}\right] = \\ = 0.\end{aligned}$$

$Q.E.D.$