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The Lewowicz number of linear diffeomorphisms on the torus

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Abstract. We prove that 2 is a Lewowicz number of every linear Anosov diffeomorphism on the torus. This result is independent of any linear metric and provides an explicit Lyapounov function for the diffeomorfisms.

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1. Introduction

For an Anosov diffeomorphism f defined on a compact Riemannian manifold, J. Lewowicz [1] has proved the existence of an integer m which provides information about properties of f like expansivity and persistence. Such number allows us to obtain a quadratic form a with the properties given below. That quadratic form has been used to define, via suspension of f, two-variable Lyapunov functions which can be used to establish structural stability of Anosov diffeomorphims (see for instance [2]). In [3], Rueda observed such a number as associated to an invariant compact hyperbolic set, proving therefrom a kind of stability of those sets; which had also been noticed by Shubb [4] in 1968. In Theorem 1 below we compute explicitly the Lewowicz's Number for the class of linear diffeomorphisms on the torus relative to any quadratic form obtained, on the tangent bundle, from a constant quadratic form on $\mathbb{R}^2$. In terms of applications, the size of m provides an idea of the difficulty to obtain a. However, for the cases considered here the second iteration is enough, as established by Theorem 1.

2. The Lewowicz number concept

Let q be a quadratic form on a compact manifold M, and suppose f is a diffeomorphism of M. Then we can define

$$(f^*q)(\vec{u},\vec{v}) = q(Df(\vec{u}), Df(\vec{v})).$$

We shall by mainly concerned with the particular case when q is a quadratic positive definite form. We will write $m \in L(q,f)$ if the following conditions are fulfilled:

1. $a = \sum_{i=0}^{m-1} f^{*i}q$ is positive definite.
2. $b = f * a - a$ is non-degenerate everywhere on M.
3. $c = f^*b - b$ is positive definite.

J. Lewowicz [1] has proved that

Lemma 1. *(Lewowicz). If $f : M \to M$ is an Anosov diffeomorphism then for any positive definite quadratic form q in M there exists an integer m such that $m \in L(q,f)$.*

3. Linear diffeomorphisms on the torus

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are integers. Then A induces a function $\varphi_A : T_2 \to T_2$ such that, when $\det A \neq 0, \varphi_A$ is a covering, and when $\det A = \pm 1, \varphi_A$ is a diffeomorphism. Furthermore,

det A	$\lvert$trace $A\rvert$	φ_A
1	> 2	Anosov
-1	> 0	Anosov

In order to verify that $m \in L(q, \varphi_A)$ for a q with constant coefficients,

$$q(\vec{u}) = (u_1)^2 + 2su_1u_2 + r^2(u_2)^2,$$

we will resort to the non-negativity of certain polynomials. The use of a symbolic processor may be helpful.

Theorem 1. *Let A be such that φ_A is Anosov. Then*

(1) $1 \in L(q, \varphi_A)$ *if q is positive definite and* det $A = 1$.
(2) $1 \in L(q, \varphi_A)$ *if q is positive definite,* det $A = -1$ *and* $|$trace $A| = 2$.
(3) $2 \in L(q, \varphi_A)$, *if q is positive definite.*

Proof. Assume first det $A = 1$. Then $1 \in L(q, \varphi_A)$ if and only if

$$F = q\left(A^2\begin{pmatrix} x \\ y \end{pmatrix}\right) - 2q\left(A\begin{pmatrix} x \\ y \end{pmatrix}\right) + q\begin{pmatrix} x \\ y \end{pmatrix} > 0$$

for $(x, y) \neq 0$, and since F is a polynomial of degree 2 in x, y, say $F = F_0x^2 - 2F_1xy + F_1y^2$, the proof simply reduces to verifying that $D = F_2F_0 - \left(\frac{F_1}{2}\right)^2 > 0$. We can write $b = \frac{-1+ad}{c}$, because if A is a hyperbolic and integer matrix then $c \neq 0$. Then we consider $D_1 = c^2D$, where b has been replaced by its expression in terms of a, d and c. This is a polynomial of degree 4 in c. Its coefficients are polynomials in the variables a, d, s, r. However, D_1 is a multiple of $4 - (\text{trace } A)^2$, and since φ_A is Anosov, we must have $|\text{trace } A| > 2$, and the problem reduces to establishing the positivity of $Q = \frac{D1}{(\text{trace } A)^2 - 4}$. We note that since Q is a polynomial of second degree in a, $Q = Q_2a^2 - Q_2^a + Q_0$, its positivity can be deduced from the fact that

$$L = Q_2Q_0 - \left(\frac{Q_1}{2}\right)^2 = -c^2(s-r)(s+r)(-2csd - 1 + d^2 + r^2c^2)^2$$

is positive. Because q being positive definite is equivalent to $|s| < r$, it is clear that $L \geq 0$. But $L = 0$ implies that $a = \frac{Q1}{2Q^2}$, which leads to trace $A = 0$. Thus, F, as a quadratic form whose coefficients are continuous in the variables a, c, d, s, r, has negative discriminant on the set

$$U = \{(a, d, c, s, r)/|a + d| \geq 3, |c| \geq 1, |s| < r\}.$$

But in each connected component of U there exists a point where the coefficient of x^2 is positive. Then F is positive definite for every point in U (this holds for any integer matrix with determinant equal to 1) and the proof of (1) is finished. Now, if A is an integer hyperbolic matrix with determinant -1, A^2 is hyperbolic with determinant 1. Then, according to the definition of m, $2 \in L(q, \varphi_A)$ is equivalent to $1 \in L(q, \varphi_{A^2})$, and (3). in Theorem 1 is proved.

To prove (2), we note that D_1 can be written in the case of det $A = 1$, but D_1 is not divisible by $(\text{trace } A)^2 - 4$. However, considered as a polynomial in the variable c, D_1 is of degree 4, $D_1 = P_0 + P_1c + P_2c^2 + P_3c^3 + P_4c^4$, and among P_0, P_1, P_2, P_3 and P_4, the only one which is not divisible by $(\text{trace } A)^2 - 4$ is P_2, which is

$$P_2 = ((\text{trace } A)^2 - 4)R(a, d, s, r) + 16(r^2 - s^2).$$

Then, if $|\text{trace } A| = 2$, D_1 is positive for every positive definite q. On the other hand, L, the coefficient of x^2 in F, takes, after replacing $b = \frac{1 \pm ad}{c}$ and having taken into account that trace $A = \pm 2$, the form

$$L = 18 \mp 12d + 2c^2r^2 - 4csd + 2d^2 \pm 12cs = L_0 + L_1 d + L_2 d^2$$

with

$$L_0 = 18 + 2cR^2 \pm 12cs, \; L_1 = \mp 12 - 4cs, \; L_2 = 2.$$

Here the discriminant of L is $D_L = L_2 L_0 - (L_1/2)^2 = c^2(r^2 - s^2)$, which is positive for all positive definite q. Since L_2 is positive then L is positive. So, in all cases where q is positive definite, F has a positive coefficients in x^2, and since the discriminant of F has the same sign as D_1, then F is positive definite provided trace $A = \pm 2$ and $\det A = -1$. The theorem is proved. ☑

Following a different line of argument we can improve Theorem 1 in the case of $\det A = -1$.

Theorem 2. *If* $\det A = -1$ *and* $|\text{trace } A| \geq 3$, *then* $1 \in L(q, \varphi_A)$ *for every positive definite* q.

Proof. Let $q(\vec{u}) = x^2 + 2sxy + r^2y^2$, where $\vec{u} = (x, y)$. Then

$$q(\vec{u}) = (x + sy)^2 + (r^2 - s^2)y^2,$$

and taking as new coordinates for $\vec{u}$, $x_1 = x + sy$ and $y_1 = y\sqrt{r^2 - s^2}$, this becomes

$$q(\vec{u}) = x_1{}^2 + y_1{}^2.$$

In order to emphasize the geometrical character of the argument we write $||\vec{u}||_q = \sqrt{q(\vec{u})}$. Now observe that if $|\lambda|$ is the eigenvalue with absolute value bigger than 1 in the matrix A, then $|\text{trace } A| = |\lambda| - \frac{1}{|\lambda|}$ and

$$|\lambda| \geq \frac{3 + \sqrt{13}}{2}.$$

It is clear that every vector $\vec{u}$ can be written in the form $\vec{u} = \vec{u}_1 + \vec{u}$ where $\vec{u}_1$ and $\vec{u}_2$ are eigenvectors corresponding to λ, λ^{-1}, and that

$$(||\vec{u}||_q)^2 = (||\vec{u}_1||_q)^2 + (||\vec{u}_2||_q)^2 + 2||\vec{u}_1||_q|\vec{u}_2|_q \cos\varphi,$$

where φ is the angle between $\vec{u}_1$ and $\vec{u}_2$ in the scalar product corresponding to the norm. From

$$A(\vec{u}) = \lambda\vec{u}_1 - \frac{1}{\lambda}\vec{u}_2,$$

it follows that

$$(||(A\vec{u})||_q)^2 = \lambda^2(||\vec{u}_1||_q)^2 + \frac{1}{\lambda^2}(||\vec{u}_2||_q)^2 - 2||\vec{u}_1||_q||\vec{u}_2||_q)^2 \cos\varphi.$$

Hence, if we assume $\|\vec{u}_1\|_q \geq \|\vec{u}_2\|_q$, we obtain that

$$\begin{aligned}(\|A\vec{u}\|_q)^2 &= (\lambda-8)(\|\vec{u}_1\|_q)^2 + 2(\|\vec{u}_1\|_q)^2 + 4\|\vec{u}_1\|_q\|\vec{u}_1\|_q + \frac{1}{\lambda^2}(\|\vec{u}_2\|_q)^2 \\ &\quad + (2\|\vec{u}_1\|_q\|\vec{u}_1\|_q - 2\|\vec{u}_1\|_q\|\vec{u}_2\|_q\cos\varphi) \\ &\geq (\lambda^2-8)(\|\vec{u}_1\|_q)^2 + 2(\|\vec{u}_2\|_q)^2 + 4\|\vec{u}_1\|_q\|\vec{u}_2\|_q\cos\varphi \\ &= 2\big((\|\vec{u}_1\|_q)^2 + (\|\vec{u}_2\|_q)^2 + 2\|\vec{u}_1\|_q\|\vec{u}_2\|_q\cos\varphi\big) + (\lambda^2-10)(\|\vec{u}_1\|_q)^2.\end{aligned}$$

Thus $q(A(\vec{u})) < 2q(\vec{u})$ if $\lambda^2 \geq 10$, and a similar analysis shows in the case $q(\vec{u}_1) < q(\vec{u}_2)$ that $q(A^{-1}(\vec{u})) < 2q(\vec{u})$.

So, since $\left(\frac{3+\sqrt{13}}{2}\right)^2 > 10$, we are led to the alternative $q(A\vec{u}) > 2q(\vec{u})$ or $q(A^{-1}(\vec{u})) > q(\vec{u})$. But

$$\begin{aligned}q(A^2(\vec{u})) - 2q(A\vec{u})) + q(\vec{u}) &= q(A(A(\vec{u}))) - q(A(\vec{u})) + q(\vec{u}) \\ &= q(A^{-1}(A(\vec{u}))) - q(A(\vec{u})) + q(A^2(\vec{u})).\end{aligned}$$

Then, by the preceding alternative applied to vector $A(\vec{u})$, we get

$$q(A^2(\vec{u})) - 2q(A(\vec{u})) + q(\vec{u}) > 0$$

for every $\vec{u} \neq 0$, which completes the proof. ☑

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References

1. J. Lewowicz, *Persistence in expansive systems*, Erg. Th. and dyman. Sys. **3** (1983), 567–578.
2. J. Lewowicz, *Lyapunov function and topological stability*, Journal of Differential Equations **38** (1980), 192–209, (MR 82d:58032).
3. A. Rueda, *Structural stability of hyperbolic sets*, Rev. Tec. Fac. Ingr. Univ. Zulia **12** (1989), 101–103.
4. M. Shub, *Global Stability of Dynamical System*, Springer Verlag, 1987.
5. V. I. Arnold, *Geometric Methods in the Theory of Ordinary Differential Equations*, Springer Verlag, 1977.

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