Telecommunications Network Equilibrium with Price and Quality-of-Service Characteristics

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Abstract

We present a competitive model that describes the interaction between several competing telecommunications service providers (SPs), their subscribers, and a network owner. Competition between the service providers is assumed to take place in their pricing decisions as well as in terms of the quality of service (QoS) they offer. In turn, the subscribers' demand for the service of an SP depends not only on the price and QoS of that SP but also upon those proposed by all of its competitors. We consider two types of games to describe the competitive interactions and analyze the resulting equilibria. As quality of service measures, we consider delay, packet losses and call rejections. We establish conditions for existence and uniqueness of the equilibria, compute them explicitly and characterize their properties.

I. Introduction

Competitive routing and equilibrium models for telecommunications networks generally assume a single characteristic through which an equilibrium is computed, such as delay [17], prices [15], [19], or even loss probabilities [2]. However, in order to take into account Quality of Service (QoS) on a network, it is generally necessary to incorporate into the model more than one parameter. A straightforward example is to include both prices and some measure of QoS. Other multi-criteria models may incorporate, for example, delay and reliability, the latter representing the QoS, price or delay and jitter, etc.

In this paper, we present a general model for computing a bi-criteria Nash equilibrium of several telecom providers. We shall then examine particular functional forms, and the resulting equilibrium properties, when the two parameters of each providers are prices and a measure of QoS.

Often, in research on competitive network routing and equilibrium models, the demands are assumed to be either given, as constants, or elastic but given by a function of a single parameter. In this paper we assume that the demand for the services of a given service provider can be described by a function that depends on the vectors of prices and qualities of service (QoS) of *all service providers*.

Our methodology is strongly inspired by [4] who studied a dynamic competitive inventory control model that includes both pricing as well as a quality parameter. The authors reduced the problem to a static game which is a special case of the general form of games that we shall study in our paper.

In the next section, we present a brief review of the related literature. Section III provides the general model setting and some general theorems for existence and uniqueness of equilibria. Section III-B then presents the linear demand model and establishes some preliminary properties of the equilibrium for fairly general relation between the cost functions and QoS of the service providers, in the case where the QoS are fixed and

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competition occurs only through prices. The following two sections obtain stronger characterizations of the equilibria and analyze competition both in prices and in QoS. Two specific QoS measures are considered: delay (Section IV) and loss probabilities (Section V). Section V includes several different possible ways of modeling loss probabilities including packet loss versus session rejection probability, and the influence of incorporating large-deviation scaling. We conclude with a number of promising avenues for extending these results.

II. Literature review

Whereas equilibria in transportation networks have been investigated for more than fifty years, this issue appeared much later in telecommunication networks, with the initial work on problems of competitive routing [7], [17], decentralized flow control [5], [11] and resource allocation [24]. One reason for the growing interest in competitive approaches to network management is the deregulation of the telecommunication monopolies. For surveys on competitive games in telecommunications, see [1], [3].

In our model, we do not take into account network topology, as in the above references, but rather model the total service proposed by each **SP** as a single entity. In other words, the price and QoS proposed by an **SP** will not depend on the physical source or destination, distance, etc. that underlies the request of each user. On the other hand, our model does take into account the quantities requested by users.

Telecommunications equilibria as a function of the price charged have been considered in great detail, as pricing has become a subject of intense research debate in the community. Numerous models exist, from utility maximizing approaches [6], [19] to auction-type approaches [21]. In these contributions, price and quantity requested have been the only parameters.

Another body of work has considered price as an endogeneous variable, which is determined as a function of the degree of saturation on the network. Typically in these approaches, the price is a *shadow price*, or Lagrange multiplier on inequality constraints, such as capacity restrictions. The reader is referred to [12], [13], [14], [23] for further details on those approaches. Finally, we mention work on defining equilibria on telecom networks as a function of other characteristics, such as loss probabilities, such as in [2].

In all of these approaches, however, a common point is that only one exogeneous characteristic gives rise to a competitive equilibrium. However, it is clear that, if one wishes to model QoS effects, more than one exogenous variable will need to be used. In this vein, J. Van Mieghem and P. Van Mieghem [22] consider the problem of pricing and scheduling of telecom services, in which the services are differentiated according to throughput, delay, and loss. While the authors consider fairness in scheduling, they do not compute equilibria or model competitive behavior. Park, Sitharam, and Chen [18] on the other hand model a "QoS provision game" in which QoS is defined by a vector of attributes. In their model, however, they do not provide existence or uniqueness results on the equilibria, as the particular user utility functions they employ do not satisfy sufficient concavity conditions. Rather, they provide results on identifying Nash equilibria, and Pareto, and system-optimal solutions.

III. Model and general properties

A. Model setting

Let us consider a game in which there are N service providers, $\mathbf{SP}$, the set of which is denoted by $\mathcal{I} = \{1, 2, .., N\}$. Each provider has two parameters to set as regards the service it offers: $(\mathbf{p}, \mathbf{f}) \in \mathbb{R}_+^{N+N}$. (We use bold face to denote a vector, e.g. $\mathbf{p}$ and denote its ith component by p_i.) Although the two parameters

can be quite general, we shall regard the first as prices (p_i is the price that **SP** i charges his subscribers per unit demand) and the second as some QoS measure (delay, loss or rejection probability, etc). Then, **SP** i experiences a demand for its service, $D_i : \mathbb{R}_+^{N+N} \mapsto \mathbb{R}_+$ which depends not only on its own parameters, p_i and f_i, but also on the prices and QoS offered by its competitors; that is, each demand function D_i depends upon the entire price vector $\mathbf{p} = (p_1, p_2, .., p_N)$ and the entire QoS vector $\mathbf{f} = (f_1, ..., f_N)$. The demand functions themselves can take on a number of functional forms, each with its own consequences upon the resulting equilibrium. In this work, however, we shall consider only a linear demand model, which will be defined further in the following section.

The utility functions of the service providers are given by $U : \mathbb{R}_+^{N+N} \mapsto \mathbb{R}_+$. Finally, we must define the strategy space of the N service providers. We shall suppose that all service providers' strategies are defined by the following orthogonal constraints; the strategy space, $\mathcal{R}_i$, of provider i is given by the subset of $\mathbb{R}^2$:

$$\mathcal{R}_i = \{(p_i, f_i) : \ 0 \leq \hat{g}_i(f_i) \leq p_i \leq p^{\max}; \quad 0 \leq f_i^{\min} \leq f_i \leq f_i^{\max}\}, \tag{1}$$

where $p_i^{\min} = \hat{g}_i(f_i)$. (The dependence of $p_i^{\min}$ on f_i through the function $\hat{g}_i$ will allow us to eliminate explicitly policies that will result in negative profits for a service provider). The fact that there are maximum prices reflects the fact that beyond some reasonable price, the demand will be zero (what ever the prices of competitors and quality of services are). In addition, $\hat{g}_i$ is assumed convex so that the strategy space of player i is convex, for all $i \in \mathcal{I}$. (This will be needed in Theorem III.1). We further define:

$$\mathcal{R}_{pi}(f_i) = \{p_i : \ 0 \leq \hat{g}_i(f_i) \leq p_i \leq p^{\max}\}, \ \ \mathcal{R}_{fi} = \{f_i : \ 0 \leq f_i^{\min} \leq f_i \leq f_i^{\max}\}, \tag{2}$$

$$\mathcal{R} = \mathcal{R}_1 \times \cdots \mathcal{R}_N, \ \ \mathcal{R}_p(\mathbf{f}) = \mathcal{R}_{p1}(\mathbf{f}) \times \cdots \mathcal{R}_{pN}(\mathbf{f}), \ \ \mathcal{R}_f = \mathcal{R}_{f1} \times \cdots \mathcal{R}_{fN}. \tag{3}$$

Along with the model of the service providers, we include a model of the (single) network owner, who may charge each service provider, i, a cost per unit of bandwidth requested as a function of the two criteria it requires, p_i and f_i. Then, the total price charged to the service provider will be a function of that per-unit price and of the demand that the provider i experiences. That is, given the aggregate demand $d_i = D_i(\mathbf{p}, \mathbf{f})$ of **SP** i, we associate with **SP** i a cost function given by $R_i(p_i, f_i, d_i)$, representing the fee that i pays to the network owner.

We next define the Nash equilibria that we are interested in computing. The first involves each service provider setting a single parameter, as a function of those of all other providers, and the second involves simultaneous setting of both parameters by all providers, as a function of those of all other providers. As already mentioned, our approach is inspired by [4] who studied a dynamic competitive inventory control model that include both pricing as well as a quality parameter.

Definition III.1: [Single-parameter Nash equilibrium] Let $U_i(\mathbf{p}, \mathbf{f})$ be the net revenue[1] of service provider i, when the vector of prices set by all the providers is given by $\mathbf{p}$, and the vector of Quality of Service parameters, $\mathbf{f}$, of all the providers is fixed at some predetermined point, $\hat{\mathbf{f}}$. Then, a single-parameter Nash equilibrium in $\mathbf{p}$ at $\hat{\mathbf{f}}$ is the vector $\mathbf{p}^*$ that solves the following system for all $i \in \mathcal{I}$.

$$U_i(\mathbf{p}^*, \hat{\mathbf{f}}) = \max_{p_i \in \mathcal{R}_{pi}(\hat{\mathbf{f}})} U_i(p_1^*, ..., p_{i-1}^*, p_i, p_{i+1}^*, .., p_N^*, \hat{\mathbf{f}}). \tag{4}$$

[1]The net revenue can be expressed as the income minus the cost R_i. Examples will be given in later sections.

This equilibrium corresponds to a price equilibrium for a fixed QoS vector $\hat{\mathbf{f}}$.

Definition III.2: [Two-parameter Nash equilibrium] Let $U_i(\mathbf{p}, \mathbf{f})$ be the net revenue of service provider i, when the pair of vectors of parameters set simultaneously by all the providers is given by $(\mathbf{p}, \mathbf{f})$. Then, a two-parameter Nash equilibrium in $(\mathbf{p}, \mathbf{f})$ is the vector couple $(\mathbf{p}^*, \mathbf{f}^*)$ that solves the following system for all $i \in \mathcal{I}$.

$$U_i(\mathbf{p}^*, \mathbf{f}^*) = \max_{(p_i, f_i) \in \mathcal{R}_i} U_i(p_1^*, ..., p_{i-1}^*, p_i, p_{i+1}^*, .., p_N^*, f_1^*, ..., f_{i-1}^*, f_i, f_{i+1}^*, .., f_N^*). \tag{5}$$

B. General demand model

We shall suppose that the average demand, $D_i(\mathbf{p}, \mathbf{f})$, is linear in all prices and *QoS* levels. For a particular **SP** i, the demand for its service, D_i should be decreasing in the price it charges, p_i, but increasing in the prices charged by its competitors, p_j, $j \neq i$. The analogous relationship holds in service quality, but in that case, D_i is increasing in f_i and decreasing in f_j, for $j \neq i$. Then, we may write

$$D_i(\mathbf{p}, \mathbf{f}) = a_i - b_i p_i + \sum_{j \in \mathcal{I}, j \neq i} c_{ij} p_j + \beta_i f_i - \sum_{j \in \mathcal{I}, j \neq i} \gamma_{ij} f_j, \tag{6}$$

with b_i, c_{ij}, β_i and γ_{ij} positive constants. We shall make the following blanket assumptions, which will be in effect throughout this study:

Assumption 1: Without loss of generality, we shall assume that the demand function of each service provider satisfies, for all $i \in \mathcal{I}$,

$$D_i(p^{\max} - p_i^{\min}, \mathbf{0}_{-i}, \mathbf{f}) \leq 0, ; \ \ \forall \mathbf{f} \in \mathcal{R}_f, \tag{7}$$

where, as before, $p_i^{\min} = \hat{g}_i(f_i)$, and $\mathbf{0_{-i}}$ is the $N-1$-dimensional null vector.

The above assumption should be understood simply as imposing some relation between the parameters that define the demand functions. It could seem an realistic assumption, since it requires the demand to possibly be negative. However, we note that the nonpossitivity of the demand function is required to hold for values of the prices that are not necessarily within the set of feasible strategies $\mathcal{R}_i$'s. Therefore the assumption could be understood as requiring that if we extrapolated linearly the demand out of the feasible region and allowed the prices of all other service providers to be zero, then we would get a nonpositive demand.

We would expect, in practice, that an even stronger assumption would likely be satisfied, namely that $D_i(p_i, \mathbf{0}_{-i}, \mathbf{f}) \leq 0, \ \forall \mathbf{f} \in \mathcal{R}_f; \ \ \forall \mathbf{p}_i \in \mathcal{R}_{pi}(f_i)$. We require, for technical purposes, however, only this weaker assumption, that the demand for each provider's services be non-positive when competitor's prices are null, and his own prices are given by $p^{\max} - p_i^{\min} = p^{\max} - \hat{g}_i(f_i)$.

Assumption 2: The constants b and c satisfy: $b_i > \sum_{j \neq i} c_{ij}, \ i \in \mathcal{I}$.Moreover, assume that D_i is non-negative over the strategy space.

a) Sufficient condition for Assumption 1: : Let Assumption 2 hold. In view of (6), if we choose $p^{\max}$ large enough then Assumption 1 will hold.

Assumption 2 will be needed to ensure the uniqueness of the resulting equilibrium. It is furthermore a reasonable condition, in that Assumption 2 implies that the influence of an **SP**'s price is significantly greater on its observed demand than the prices of its competitors. This condition could then take into account the presence of customer loyalties and/or imperfect knowledge of competitors' prices.

C. General QoS model

Along with the model of the service providers, the (single) network owner charges each service provider, i, a cost per unit of bandwidth requested. We assume that the amount of bandwidth μ_i requested by **SP** i depends on the demand it experiences and on the QoS it wishes to offer (the higher the demand and the better the QoS, the higher μ_i will be). For example, if the QoS is linear in the standard Kleinrock delay function (obtained from an M/M/1 queuing model), i.e. $f_i = c - 1/(\mu_i - d_i)$, then $\mu_i(f_i, d_i) = d_i + 1/(c - f_i)$.

Thus, given the aggregate demand $d_i = D_i(p_i, f_i)$, the fee paid by **SP** i can be written as: $R_i = v_i\mu_i(f_i, d_i)$, where $\mu_i(f_i, d_i) \in [0, B]$ is again the bandwidth required in order to guarantee the QoS, f_i, at a demand level of d_i for **SP** i, and v_i is the price to be paid per unit of bandwidth.

D. Utility model

The revenue of **SP** i is given by $p_iD_i(\mathbf{p}, \mathbf{f})$, whereas the profit (or net revenue) of **SP** i is the difference between the revenue and the fee it pays to the network owner,

$$U_i(\mathbf{p}, \mathbf{f}) = p_iD_i(\mathbf{p}, \mathbf{f}) - R_i(f_i, d_i). \tag{8}$$

In the sequel, we shall show the uniqueness of Nash equilibrium for the price game (4) in which the capacity μ_i required by user i has the following form:

$$\mu_i(f_i, d_i) = d_ig_i(f_i) + h_i(f_i), \tag{9}$$

where g_i and h_i are positive functions. Hence, the profit function becomes

$$U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_ig_i(f_i)) - v_ih(f_i).$$

In the single-parameter Nash game, the strategy space of each **SP** i is given by (1) and in the two-parameter game, the strategy space is given by (2) where $p^{\min} = \hat{g}_i(f_i) = v_ig_i(f_i)$. Note that the sets $\mathcal{R}_i$, and $\mathcal{R}_{pi}$, $i \in \mathcal{I}$ are convex when g_i is a convex function.

E. Properties of the general model

The question is then, under general assumptions on the model functions and data, when can we ensure the existence and uniqueness of the resulting equilibrium across service providers. To this end, we make the following assumptions that concern the properties of the utility functions.

Assumption 3: Without loss of generality, express the net revenue (or utility) of service provider i as $U_i = \hat{U}_i(\mathbf{p}, \mathbf{f}, d_i) = U_i(\mathbf{p}, \mathbf{f})$, $i \in \mathcal{I}$ where, as before, $d_i = D_i(\mathbf{p}, \mathbf{f})$. Assume that U_i is upper semi-continuous in $(\mathbf{p}, \mathbf{f}) \in \mathcal{R}$, for all $i \in \mathcal{I}$.

Under Assumption 3, one has the following general result on the existence of a Nash equilibrium among the N service providers.

Theorem III.1: [Existence] Under the Assumption 3 and the definition of the strategy spaces $\mathcal{R} = \mathcal{R}_i \times \cdots \mathcal{R}_N$, there exists at least one single-parameter (resp., two-parameter) Nash equilibrium of the service providers' game.

Proof: The strategy space for each $i \in \mathcal{I}$ is convex and compact. The utility functions are assumed u.s.c in both arguments. The existence of a Nash point follows (see e.g. [20]). □

We first consider the single-parameter price game (4) which arises when the vector of quality of service levels of all providers is fixed. Then, we shall consider the two-parameter price and QoS game.

The following theorem is a generalization of [4, Thrm. 5] that established uniqueness of an equilibrium for a similar setting but when $h_i(.)$ are all zero and for some specific functions $g_i(.)$. By allowing for general g_i and h_i we shall be able to handle in the next sections a large number of quality of service functions that are typical in networks.

Theorem III.2: [Uniqueness of Price-based Nash equilibrium] Consider the price game (4) which arises when the vector of quality of service levels $\mathbf{f}$ is fixed, and assume that $\mathcal{R}_i(\mathbf{f})$ are nonempty, $i \in \mathcal{I}$. Then

1) The price game has a unique equilibrium $\mathbf{p}^*(\mathbf{f})$, with

$$\mathbf{p}^*(\mathbf{f}) = (I - T)^{-1}\Gamma^{-1}q \tag{10}$$

where $\Gamma = diag(2b_1, ..., 2b_N)$, $T_{ii} = 0$, $T_{ij} = \frac{c_{ij}}{2b_j}$ for $i \neq j$ and $q_i = b_i v_i g_i(f_i) + a_i + \beta_i f_i - \sum_{j\neq i}\gamma_{ij}f_j$.

2) The equilibrium prices p_i^* are increasing in each of the unit prices v_j.

3) The equilibrium prices p_i^* are decreasing in f_j, $j \in \mathcal{I}$ when $g_j'(f_j) < \frac{\sum_{l\in\mathcal{I}}(L^{-1})_{il}\gamma_{jl} - (L^{-1})_{ij}\beta_j}{(L^{-1})_{ij}b_j v_j}$ and increasing otherwise.

Proof: The existence of a Nash equilibrium of the price game (4) was demonstrated in Theorem III.1. Therefore, we show the uniqueness of the equilibrium point.

Since $\mathbf{p}^*(\mathbf{f})$ is a Nash equilibrium, it follows that p_i^* maximizes the function $U_i(\mathbf{p}_i, \mathbf{p}_{-i}^*, \mathbf{f})$, where $\mathbf{p}_{-i} = (p_1, .., p_{i-1}, p_{i+1}, .., p_N)$. Moreover , we have

$$\lim_{p_i \to v_i g_i(f_i)} \frac{\partial U_i(p_i, p_{-i}^*, \mathbf{f})}{\partial p_i} = d_i(v_i g_i(f_i), p_{-i}^*, \mathbf{f}) \geq 0 \tag{11}$$

$$\begin{aligned}
\lim_{p_i \to p^{max}} \frac{\partial U_i(p_i, p_{-i}^*, \mathbf{f})}{\partial p_i} &= -b_i p^{max} + b_i v_i g_i(f_i) + D_i(p^{\max}, \mathbf{p}_{-i}^*, \mathbf{f}) \\
&= a_i - 2b_i p^{\max} + \sum_{j\neq i} c_{ij} p_j^* + b_i v_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i}\gamma_{ij} f_j \\
&\leq a_i - 2b_i p^{\max} + p^{\max}\sum_{j\neq i} c_{ij} + b_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i}\gamma_{ij} f_j \qquad (12) \\
&< a_i - b_i p^{\max} + b_i v_i g_i(f_i) + \beta_i f_i - \sum_{j\neq i}\gamma_{ij} f_j \qquad (13) \\
&= d_i(p^{\max} - v_i g_i(f_i), \mathbf{0}_{-i}, \mathbf{f}) \leq 0 \qquad (14)
\end{aligned}$$

where the transition (12)-(13) is due to the Assumption 2 and the last transition is due to Assumption (1). Thus, p_i^* is unique solution of the optimality conditions:

$$0 = \frac{\partial U_i(\mathbf{p}, \mathbf{f})}{\partial p_i} = -b_i(p_i - v_i g_i(f_i)) + d_i(\mathbf{p}, \mathbf{f}). \tag{15}$$

Then, $\mathbf{p}^*$ is a solution to the following linear system

$$Lp = q \tag{16}$$

where $L_{ii} = 2b_i$, $L_{ij} = -c_{ij}$ for $i \neq j$ and $q_i = b_i v_i g_i(f_i) + a_i + \beta_i f_i - \sum_{j \neq i} \gamma_{ij} f_j$.

To show the uniqueness of Nash equilibrium, it suffices to show that the linear system (16) admits a unique solution. We observe that the matrix L can be written as: $L = \Gamma(I - T)$ with $\Gamma = diag(2b_1, ..., 2b_N)$, and

$$T = \begin{pmatrix} 0 & \vdots & \frac{c_{1N}}{2b_1} \\ & \ddots & \\ \frac{c_{N1}}{2b_N} & \vdots & 0 \end{pmatrix}.$$

The matrix T is substochastic since $\frac{\sum_{j \neq i} c_{ij}}{2b_i} \leq \frac{b_i}{2b_i} < 1$. Thus L^{-1} exists and is equal to $(I - T)^{-1}\Gamma^{-1}$ and $p^*(\mathbf{f}) = L^{-1}q = (I - T)^{-1}\Gamma^{-1}q$. Writing explicitly one term from the equilibrium price vector, $p_i^*(\mathbf{f})$, and rearranging terms to identify dependencies on f_i and f_j independently, $j \in \mathcal{I}$, $j \neq i$, we have that

$$\begin{aligned} p_i^*(\mathbf{f}) &= (L^{-1})_{ii}(a_i + (b_i v_i g_i(f_i)) + f_i((L^{-1})_{ii}\beta_i - \sum_{j \neq i}(L^{-1})_{ij}\gamma_{ji}) + \\ & \sum_{j \neq i}(L^{-1})_{ij}(a_j + b_j v_j g_j(f_j) + \beta_j f_j) - \sum_{j \in \mathcal{I}} \sum_{l \neq i}(L^{-1})_{ij}\gamma_{jl} f_l \end{aligned} \quad (17)$$

Then, taking partial derivatives, we obtain: $\frac{\partial p_i^*(\mathbf{f})}{\partial f_j} = (L^{-1})_{ij} b_j v_j g_j'(f_j) + (L^{-1})_{ij}\beta_j - \sum_{l \in \mathcal{I}}(L^{-1})_{il}\gamma_{jl}$,, which is negative when $g_j'(f_j) < \frac{\sum_{l \in \mathcal{I}}(L^{-1})_{il}\gamma_{jl} - (L^{-1})_{ij}\beta_j}{(L^{-1})_{ij} b_j v_j}$. □

Next we make the following assumptions on the functions g_i and h_i appearing in Definition (9) of μ_i. This will guarantee the uniqueness of the equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$ in the price-QoS game (5) which we shall compute. Several examples of this structure will be given in Sections IV and V.

Assumption 4: Let $h_i(.) = 0$. Moreover, suppose that $g(.) \in C^2$ is convex increasing, and its derivative satisfies for all $i \in \mathcal{I}$: $\lim_{f_i \to f_i^{\max}} g_i'(f_i) = +\infty$.

This assumption will be satisfied in most of the queuing systems we study. The following result establishes the uniqueness of Nash equilibrium in which prices and QoS levels are set by all players simultaneously.

Theorem III.3: [Uniqueness of Price and QoS-based Nash equilibrium] Ket Assumption 4 hold. The price-QoS problem (5) has a unique two-parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the following equations, for every $i \in \mathcal{I}$,

$$\begin{aligned} g_i'(f_i^*) &= \frac{\beta_i}{v_i b_i} \text{ if } \frac{\beta_i}{v_i b_i} \geq g_i'(f_i^{\min}) \\ f_i^* &= 0 \text{ if } \frac{\beta_i}{v_i b_i} < g_i'(f_i^{\min}) \end{aligned}$$

Proof: Let

$$\tilde{U}_i(\mathbf{p}, \mathbf{f}) = \log[p_i - v_i g_i(f_i)] + \log[D_i(\mathbf{p}, \mathbf{f})].$$

Note that a vector $(\mathbf{p}^*, \mathbf{f}^*)$ is an equilibrium in the original utilities if and only if it is an equilibrium in terms of their logarithms, i.e. of the functions $\tilde{U}_i$.

Since g_i is convex by assumption, U_i is concave in $(\mathbf{p}, \mathbf{f})$. Under Assumption 4, we have that

$$\lim_{f_i \to f_i^{\max}} \frac{\partial \tilde{U}_i(\mathbf{p}, \mathbf{f})}{\partial f_i} = \lim_{f_i \to f_i^{\max}} \frac{-v_i g'(f_i)}{p_i - v_i g_i(f_i)} + \frac{\beta_i}{D_i(\mathbf{p}^*, \mathbf{f})} = -\infty. \tag{18}$$

This implies there will not be a Nash equilibrium at the constraint boundary where $f_i = f_i^{\max}$, $i \in \mathcal{I}$. To show the uniqueness of the Nash solution, we must thus examine the interior of the feasible region with respect to the upper bound on the f_i and show that there is precisely one point, $(\mathbf{p}^*, \mathbf{f}^*)$, which solves simultaneously the utility maximization problems of each provider, $i \in \mathcal{I}$. Note from (18) that the lower limit, $\lim_{f_i \to f_i^{\min}} \partial \tilde{U}_i(\mathbf{p}, \mathbf{f})/\partial f_i$ may be either positive or negative, and as such, the Nash equilibrium may arise at a point for which $f_i^* = f_i^{\min}$ for some $i \in \mathcal{I}$.

On the other hand, we do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (15) is zero, possibly at the lower boundary, (11) and (14).

That is, it is necessary to consider the following Kuhn-Tucker conditions in which a Lagrange multiplier, α_i represents the lower bound constraint on f_i for each i. In other words, $(\mathbf{p}^*, \mathbf{f}^*)$ is a Nash equilibrium if for every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i such that

$$0 = \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial p_i} = \frac{1}{(p_i - v_i g_i(f_i))} + \frac{-b_i}{D_i(p^*, \mathbf{f})} \tag{19}$$

$$-\alpha^i = \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial f_i} = \frac{-v_i g_i'(f_i)}{p_i - v_i g_i(f_i)} + \frac{\beta_i}{D_i(p^*, \mathbf{f})} \tag{20}$$

$$\alpha^i (f_i - f_i^{\min}) = 0, \quad \alpha^i \geq 0 \tag{21}$$

By multiplying (19) by $1/b_i$ and (20) by $1/\beta_i$ and substituting the resulting equations, we obtain

$$\frac{\alpha^i}{\beta_i} = \frac{v_i g_i'(f_i)/\beta_i - 1/b_i}{p_i - v_i g_i(f_i)}, \quad \text{and } \alpha_i (f_i - f_i^{\min}) = 0 \tag{22}$$

It follows that $g_i'(f_i^*) = \frac{\beta_i}{v_i b_i}$ if $\frac{\beta_i}{v_i b_i} \geq g_i'(f_i^{\min})$ and $f_i^* = f_i^{\min}$ if $\frac{\beta_i}{v_i b_i} < g_i'(f_i^{\min})$. Fixing $f = f^*$, we obtain p^* through (10). Thus $(\mathbf{p}^*, \mathbf{f}^*)$ is the unique Nash equilibrium. □

Next, we consider two applications of this framework, making use of two different QoS measures.

IV. Case study I: expected delay as QoS

We shall suppose in this section that the measure defining the QoS, f, corresponds to some function of the expected delay. We shall use the Kleinrock delay function[2] which is a common delay function used in networking games [17]. Instead of Minimizing delay, we consider the maximization of the reciprocal of its square-root (in that way the QoS f_i indeed increases as the delays decrease (this is similar to [19]):

$$f_i = \frac{1}{\sqrt{Delay}} = \sqrt{\mu_i - d_i} \tag{23}$$

[2]This function corresponds to a queuing delay in an $M/M/1$ queue with first-in-first-out discipline or to the more general $M/G/1$ queue under processor sharing delay.

Thus $\mu_i(f_i, d_i) = d_i g(f_i) + h_i(f_i)$ where $g_i(f_i) = 1$ is indeed convex and where $h_i(f_i) = f_i^2$. The cost that **SP** i pays the network owner is $v_i[(f_i^2) + d_i]$. Then, the net revenue of **SP** i that it wishes to maximize is then

$$U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_i) - v_i f_i^2. \tag{24}$$

We observe that this profit function is special case of the one defined in (9), by taking $g_i(f_i) = 1$ and $h_i(f_i) = f_i^2$. We shall make the following assumption:

Assumption 5: The constants b, v, and β satisfy: $2b_i v_i > \beta_i^2, \quad i \in \mathcal{I}$

Remark IV.1: Note that, for price-sensitive demands, i, this assumption is likely to hold, since in that case, $b_i >> \beta_i$. For relatively price-*in*-sensitive demands, who value above all QoS, Assumption 5 may still hold if the unit price for capacity charged by the network owner to **SP** i**,** v_i **is high.**

We now proceed with the analysis of the price-QoS game (5) (two-parameter) in which prices and services levels are set by all **SP**s simultaneously.

Theorem IV.1: Let Assumption 5 hold. The price-QoS game (5) with QoS, $\mathbf{f}$ given by (23) has a unique Nash equilibrium $\mathbf{p}^*, \mathbf{f}^*)$, with p_i^* increasing in each v_j, $j \in \mathcal{I}$.

Proof: Assumption 3 is satisfied by (5) with (23) and (24). Hence, by Theorem III.1, there exists a Nash equilibrium.

We next show that the Nash equilibrium is unique. We have seen, in the previous section, that the partial derivatives of the profit functions with respect to price variables p_i of each providers i will be zero. In this section, the upper and lower bounds on the quality of service variables, f_i may or may not be active, and so we take a Lagrangian relaxation of those constraints and examine the Kuhn Tucker conditions of the resulting Lagrangian.

To show the uniqueness of the Nash equilibrium solution, we show first that the profit functions U_i are jointly strictly concave in (p_i, f_i) for all providers, $i \in \mathcal{I}$, so that the solution to system defined by the Kuhn Tucker conditions is uniquely defined.

To this end, note first that

$$\begin{aligned}
\frac{\partial U_i(p, \mathbf{f})}{\partial p_i} &= -b_i(p_i - v_i) + D_i(p^*, \mathbf{f}) \text{ and } \frac{\partial U_i(p, \mathbf{f})}{\partial f_i} = \beta_i(p_i - v_i) - 2v_i f_i \\
\frac{\partial^2 U_i(p, \mathbf{f})}{\partial p_i^2} &= -2b_i < 0, \ \frac{\partial^2 U_i(p, \mathbf{f})}{\partial p_i \partial f_i} = \beta_i, \text{ and } \frac{\partial^2 U_i(p, \mathbf{f})}{\partial f_i^2} = -2v_i < 0.
\end{aligned}$$

Since $\frac{\partial^2 U_i}{\partial f_i^2} = -2v_i < 0$ and $\frac{\partial^2 U_i}{\partial p_i^2} = -2b_i < 0$, it suffices to show that the determinant of the Hessian matrix H is positive, where

$$H = \begin{pmatrix} \frac{\partial^2 U_i}{\partial p_i^2} & \frac{\partial^2 U_i}{\partial p_i \partial f_i} \\ \\ \frac{\partial^2 U_i}{\partial f_i \partial p_i} & \frac{\partial^2 U_i}{\partial f_i^2} \end{pmatrix}$$

We have $\det H[U_i(\mathbf{p}, \mathbf{f})] = \frac{\partial^2 U_i}{\partial f_i^2}\frac{\partial^2 U_i}{\partial p_i^2} - \left(\frac{\partial^2 U_i}{\partial p_i \partial f_i}\right)^2 = 4b_i v_i - \beta_i^2 > 0$. The latter inequality follows from Assumption 5.

We note that any solution to the following Kuhn-Tucker conditions is a Nash equilibrium: For every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i and λ^i such that

$$0 = \frac{\partial U_i(\mathbf{p},\mathbf{f})}{\partial p_i} = -b_i(p_i - v_i) + D_i(\mathbf{p}^*,\mathbf{f}) \tag{25}$$

$$\alpha^i - \lambda^i = \frac{\partial U_i(\mathbf{p},\mathbf{f})}{\partial f_i} = \beta_i(p_i - v_i) - 2v_i f_i \tag{26}$$

$$\alpha^i(f_i - f_i^{\min}) = 0, \quad \lambda^i(f_i^{\max} - f_i) = 0 \tag{27}$$

where α^i and λ^i are the Lagrange multipliers corresponding to the min and max constraints on the values of f_i. As before, we do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (25) is zero, possibly at the lower boundary, (11) and (14).

It follows from (26) that

$$\text{if } \lambda^i = 0 \text{ and } \alpha^i = 0,\ f_i^* = \frac{\beta_i}{2v_i}(p_i - v_i) \tag{28}$$

$$\text{if } \alpha^i > 0,\ f_i^* = f_i^{\min} \tag{29}$$

$$\text{if } \lambda^i > 0,\ f_i^* = f_i^{\max} \tag{30}$$

Hence, it follows that $\mathbf{p}^*$ is solution of the following linear system

$$Ap = r \tag{31}$$

where

$$A_{ii} = 2b_i - \delta_{\{\lambda^i - \alpha^i = 0\}}\frac{\beta_i^2}{2v_i}, \quad A_{ij} = -c_{ij} + \delta_{\{\lambda^j - \alpha^i = 0\}}\gamma_{ij}\frac{\beta_j}{2v_j},$$

$$R_i = b_i v_i + a_i + \delta_{\{\lambda^i - \alpha^i = 0\}}\frac{\beta_i^2}{2} - \sum_{j \neq i}\delta_{\{\lambda^j - \alpha^j = 0\}}\gamma_{ij}\frac{\beta_j}{2} + \beta_i(\delta_{\{\lambda^i > 0\}}f_i^{\max} + \delta_{\{\alpha^i > 0\}}f_i^{\min})$$

$$- \sum_{j \neq i}\gamma_{ij}(\delta_{\{\lambda^i > 0\}}f_j^{\max} + \delta_{\{\alpha^j > 0\}}f_i^{\min}).$$

To show the uniqueness of Nash equilibrium, it suffices to show that the linear system (31) admits a unique solution. We observe that the matrix A can be written as: $A = \Gamma_1(I - T_1)$ with $\Gamma_1 = diag(2b_1 - \delta_{\{\lambda^i - \alpha^i = 0\}}\frac{b_1^2}{2v_1}, ..., 2b_N - \delta_{\{\lambda^i - \mu^i = 0\}}\frac{b_N^2}{2v_N})$, and

$$T_1 = \begin{pmatrix} 0 & \vdots & \frac{c_{1N} - \delta_{\{\lambda^i - \mu^i = 0\}}\gamma_{1N}\frac{\beta_N}{2v_N}}{2b_1 - \delta_{\{\lambda^i - \mu^i = 0\}}\frac{\beta_1^2}{2v_1}} \\ & \ddots & \\ \frac{c_{N1} - \delta_{\{\lambda^i - \mu^i = 0\}}\gamma_{N1}\frac{\beta_1}{2v_1}}{2b_N - \delta_{\{\lambda^i - \mu^i = 0\}}\frac{\beta_N^2}{2v_N}} & \vdots & 0 \end{pmatrix}$$

We observe that the matrix T is substochastic since

$$\frac{\sum_{j\neq i} c_{ij} - \delta_{\{\lambda^i-\mu^i=0\}}\gamma_{ij}\frac{\beta_j}{2v_j}}{2b_i - \delta_{\{\lambda^i-\mu^i=0\}}\frac{\beta_i^2}{2v_i}} \leq \frac{\sum_{j\neq i} c_{ij}}{2b_i - \frac{\beta_i^2}{2v_i}} \leq \frac{b_i}{2b_i - \frac{\beta_i^2}{2v_i}} < \frac{b_i}{b_i} = 1.$$

The third inequality follows from Assumption 5. Thus $A^{-1} = (I - T_1)^{-1}\Gamma_1^{-1}$ and

$$\begin{aligned} p_i^* &= \sum_{j\in\mathcal{I}}(A^{-1})_{ij}(b_j v_j + a_j + \delta_{\{\lambda^j-\alpha^j=0\}}\frac{\beta_j^2}{2} - \sum_{k\neq j}\delta_{\{\lambda^k-\alpha^k=0\}}\gamma_{jk}\frac{\beta_k}{2} \\ &\quad +\beta_j(\delta_{\{\lambda^j>0\}}f_j^{\max} + \delta_{\{\alpha^j>0\}}f_j^{\min}) \\ &\quad -\sum_{k\neq i}\gamma_{jk}(\delta_{\{\lambda^j>0\}}f_k^{\max} + \delta_{\{\alpha^k>0\}}f_j^{\min})). \end{aligned}$$

We conclude that $(\mathbf{p}^*, \mathbf{f}^*)$ is the unique solution of Kuhn-Tucker conditions (25)-(26) and therefore the unique Nash equilibrium to the price-QoS game (5). □

V. Case study II: Loss or rejection probability as QoS

In this section, the quality of service corresponds to the loss probability, where one may model packet loss probabilities (e.g. at some common input queue) or rejection probability in a loss network (such as a circuit switched network).

$$f_i = (1 - P_{loss}^i)^{1/s}, \tag{32}$$

where P_{loss}^i is the loss probability. i.e,. $P_{loss}^i := G(\rho_i)$, with, as usual, $\rho_i = d_i/\mu_i$ is the traffic intensity, and $s \geq 1$ is a scaling coefficient that adjusts the relative importance of the QoS parameter with respect to price. Note that when $s = 1$, QoS enters linearly in the demand function, D, whereas for $s > 1$, the QoS increases as a concave function of its parameters; in other words, for $s > 1$, we allow for *decreasing rates of return* on the quality of service provided.[3]

We will use the notation $G'(\rho)$ to denote the derivative with respect to ρ and we assume that $G'(\rho)$ is positive function over the interval $(0, +\infty)$. Hence, the loss probability is strictly increasing in ρ, and the function G^{-1} exists. Consequently, for all $i \in \mathcal{I}$,

$$g_i(f_i) = \frac{1}{G^{-1}(1 - f_i^s)} \tag{33}$$

Indeed, since G^{-1} exists, then $d_i/\mu_i = \rho_i = G^{-1}(1 - f_i)$, it follows that $\mu_i = \frac{1}{G^{-1}(1-f_i)}d_i$. Hence from (9), the function g_i is given by (33). Note that $f_i^{\min} = 0$ and $f_i^{\max} = 1$ for every $i \in \mathcal{I}$.

We are going to present some important special cases for which the assumption A4 is verified. In V-A and V-C we consider only the linear relationship between QoS and loss probability, that is $s = 1$, whereas in V-B we give results for strictly concave f as well.

[3] *Decreasing rates of return* means that the marginal benefit of QoS decreases with increasing QoS levels This phenomenon is observed often in practice; the amount that a user is willing to pay, per unit of QoS, decreases as the level increases.

A. Packet loss probability: a basic M/M/1/K queuing model

In this paragraph, the quality of service corresponds to the loss probability, or probability of a full buffer, in the M/M/1/K queue. We note that this could be a good approximation for the loss probabilities if the service provider offers access to a network, and the bottleneck in terms of losses is in some common input buffer to that network. Thus,

$$G(\rho_i) := \begin{cases} (1-\rho_i)\rho_i^{K_i}/(1-\rho_i^{K_i+1}) & \text{if} \quad 0 \leq \rho_i < 1,\ K_i = 1,2,3,.., \\ 1/(K_i+1) & \text{if} \qquad \rho_i = 1,\ K_i = 1,2,3,.., \end{cases} \tag{34}$$

where K_i is the buffer size. Clearly, the derivative $G'(\rho)$ is positive, hence we can use the above framework. Let the concavity coefficient satisfy $s = 1$, implying that QoS measure f and loss probabilities are linearly related [see (32)].

The next lemma characterizes some properties of the function g_i with respect to service levels f_i, $i \in \mathcal{I}$.

Lemma V.1: The function $g_i(.) \in C^2$ is convex, increasing, and its derivative satisfies

$$\lim_{f_i \to 1} g_i'(f_i) = +\infty, \text{ and } \lim_{f_i \to 0} g_i'(f_i) = 1.$$

Proof: See Appendix in the full version of the paper [8] . □

The above result implies that the function g_i verifies Assumption 4. Hence, from Theorem III.3, we have the following result.

Corollary V.1: The price-QoS game (5) has a unique two-parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the equations

$$\begin{aligned} g_i'(f_i^*) &= \frac{\beta_i}{v_i b_i} \text{ if } \frac{\beta_i}{v_i b_i} \geq 1 \\ f_i^* &= 0 \text{ if } \frac{\beta_i}{v_i b_i} < 1 \end{aligned}$$

and $\mathbf{p}^*$ is then obtained through (10).

B. Packet loss probability: incorporating large-deviation scaling

Let P_{loss} be, as above, the loss probability for a M/M/1/K queue with K the buffer size at the queue. Given the probability of loss from (34), where $\rho_i := \frac{d_i}{\mu_i}$, and letting the arrival rate, capacity and buffer size be scaled by a factor n, as in many-source large-deviation scaling, we obtain, by letting $n \to \infty$,

$$\lim_{n\to\infty} \frac{(1-\rho_i)\rho_i^{nK}}{1-\rho_i^{nK+1}} = \frac{(d_i-\mu_i)^+}{d_i}, \tag{35}$$

with $(\cdot)^+ = \max\{0, \cdot\}$. Note that even though the buffer size goes to ∞, the delay remains constant as the capacity, μ_i, also goes to ∞. In a deterministic fluid model, this has the simple interpretation of fraction of fluid lost when the arrival rate exceeds capacity. Using (34), the quality of service is then given by:

$$f_i = \left[1 - \frac{(d_i-\mu_i)^+}{d_i}\right]^{1/s}. \tag{36}$$

Note that (36) is not a differentiable function due to the max operation. However, since the QoS measure remains constant ($f_i = 1$) when the capacity exceeds the demand, it is sufficient to consider that the largest capacity implemented would be $\mu_i = d_i$. Hence, we shall neglect the $\max$ operation in (36), and then the bandwidth required in order to guarantee the QoS f_i is given by: $\mu_i = g_i(f_i)d_i$ with $g_i(f_i) = f_i^s$

The profit of **SP** i is in turn given by: $U_i(\mathbf{p}, \mathbf{f}) = D_i(\mathbf{p}, \mathbf{f})(p_i - v_i f_i^s)$, where the strategy space is given by the subset $\mathcal{R}_i$ where $\mathcal{R}_i = \{(p_i, f_i) : \ p_i^{\min} \leq p_i \leq p_i^{\max};\ 0 \leq f_i \leq 1\}$.

We shall require the following assumption, which is likely to hold, since none of the constants β, v or b should be zero in this setting.

Assumption 6: The constants b, v, and β satisfy: $\beta_i - b_i v_i \neq 0$

Theorem V.1: The price-QoS problem (5) with Quality of Service $\mathbf{f}$ as in (36) has a unique two parameter Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* is given by:

1) For concave QoS coefficient, $s > 1$, that is, under decreasing rate of return on the QoS,

$$f_i^* = \begin{cases} \left(\frac{\beta_i}{s v_i b_i}\right)^{\frac{1}{s-1}} & \text{if} \quad \frac{\beta_i}{s v_i b_i} < 1, \\ 0 & \text{if} \quad \frac{\beta_i}{s v_i b_i} \geq 1. \end{cases}$$

2) For $s = 1$, that is, under constant rate of return on the QoS level, $\beta_i \neq b_i v_i$, $i \in \mathcal{I}$, so that

$$f_i^* = \begin{cases} 1 & \text{if} \quad \beta_i < v_i b_i, \\ 0 & \text{if} \quad \beta_i > v_i b_i. \end{cases}$$

Proof: For all $s > 0$, the function g_i is convex, this implies that the function U_i is concave. Hence, we need only to show that that the profit function of each provider i is jointly log-concave in (p_i, f_i) for $s = 1$. Recall that $\tilde{U}_i$ denotes $\log U_i$. Note that

$$\begin{aligned} \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial p_i} &= \frac{1}{(p_i - v_i f_i)} - \frac{b_i}{D_i(p^*, \mathbf{f})} \text{ and } \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial f_i} = -\frac{v_i}{(p_i - v_i f_i)} + \frac{\beta_i}{D_i(p^*, \mathbf{f})} \\ \frac{\partial^2 \tilde{U}_i(p, \mathbf{f})}{\partial p_i^2} &= \frac{-1}{(p_i - v_i f_i)^2} - \frac{b_i^2}{D_i^2(p^*, \mathbf{f})} < 0, \ \frac{\partial^2 \tilde{U}_i(p, \mathbf{f})}{\partial p_i \partial f_i} = \frac{v_i}{(p_i - v_i f_i)^2} + \frac{b_i \beta_i}{D_i^2(p^*, \mathbf{f})} \\ \frac{\partial^2 \tilde{U}_i(p, \mathbf{f})}{\partial f_i^2} &= \frac{-v_i^2}{(p_i - v_i f_i)^2} - \frac{\beta_i^2}{D_i^2(p^*, \mathbf{f})} < 0. \end{aligned}$$

Since $\frac{\partial^2 \tilde{U}_i}{\partial f_i^2} < 0$ and $\frac{\partial^2 \tilde{U}_i}{\partial p_i^2} < 0$, it suffices to show that the determinant of the Hessian matrix H is positive, where

$$H = \begin{pmatrix} \frac{\partial^2 \tilde{U}_i}{\partial p_i^2} & \frac{\partial^2 \tilde{U}_i}{\partial p_i \partial f_i} \\ \\ \frac{\partial^2 \tilde{U}_i}{\partial f_i \partial p_i} & \frac{\partial^2 \tilde{U}_i}{\partial f_i^2} \end{pmatrix}$$

We have

$$\begin{aligned} \det H[\tilde{U}_i(\mathbf{p}, \mathbf{f})] &= \frac{\partial^2 \tilde{U}_i}{\partial f_i^2} \frac{\partial^2 \tilde{U}_i}{\partial p_i^2} - \left(\frac{\partial^2 \tilde{U}_i}{\partial p_i \partial f_i}\right)^2 = \Big(D_i(p_i - v_i f_i)\Big)^{-4} \\ & \Big([D_i^2 + b_i^2 (p_i - v_i f_i)^2][v_i^2 D_i^2 + \beta_i^2 (p_i - v_i f_i)^2] - [v_i D_i^2 + b_i \beta_i (p_i - v_i f_i)^2]^2\Big) \end{aligned}$$

$$= \frac{(\beta_i - b_i v_i)^2}{D_i^2 (p_i - v_i f_i)^2} > 0$$

The latter inequality follows from Assumption 6.

Next, we shall establish the uniqueness of the equilibrium. Note first that any solution to the following Kuhn-Tucker conditions is a Nash equilibrium: For every $i \in \mathcal{I}$, there exist non-negative Lagrange multipliers α^i and λ^i such that

$$0 = \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial p_i} = \frac{1}{(p_i - v_i f_i^s)} - \frac{b_i}{D_i(p^*, \mathbf{f})} \tag{37}$$

$$\lambda^i - \alpha^i = \frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial f_i} = -\frac{s v_i f_i^{s-1}}{(p_i - v_i f_i^s)} + \frac{\beta_i}{D_i(p^*, \mathbf{f})} \tag{38}$$

$$\alpha^i f_i = 0, \quad \lambda^i (1 - f_i) = 0, \tag{39}$$

where, as before, α^i and λ^i are the Lagrange multipliers corresponding to the min and max constraints on the values of f_i; note, also as before, that we do not need Lagrange multipliers for the constraints on p_i since p_i^* is obtained at a point where the partial derivative (25) is zero, possibly at the lower boundary, (11) and (14). By multiplying (37) by β_i and (38) by b_i, and summing the resulting equations, we obtain

$$b_i(\lambda^i - \alpha^i) = \frac{\beta_i - s v_i b_i f_i^{s-1}}{(p_i - v_i f_i^s)} \tag{40}$$

Now, we are going to detemine the *QoS* for $s > 1$. We have use the fact that the partial derivative $\frac{\partial \tilde{U}_i(p, \mathbf{f})}{\partial f_i}$ is positive. Hence we do not need the Langrage mulitiplier α^i for the constraint $f_i - f_i^{min} \geq 0$. According to this, (40) becomes $b_i \lambda^i = \frac{\beta_i - s v_i b_i f_i^{s-1}}{(p_i - v_i f_i^s)}$. We deduce that $f_i^* = \begin{cases} \left(\frac{\beta_i}{s v_i b_i}\right)^{\frac{1}{s-1}} & \text{if} \quad \frac{\beta_i}{s v_i b_i} \leq 1, \\ 1 & \text{if} \quad \frac{\beta_i}{s v_i b_i} > 1. \end{cases}$.

Letting now $s = 1$. It follows, then, from (40) with $s = 0$ that $f_i^* = \begin{cases} 0 & \text{if} \quad \beta_i < v_i b_i \\ 1 & \text{if} \quad \beta_i \geq v_i b_i \end{cases}$.

Fixing $\mathbf{f} = \mathbf{f}^*$, we obtain that $\mathbf{p}^*$ is the *unique* solution of the linear system given by (16). Thus $(\mathbf{p}^*, \mathbf{f}^*)$ is consequently is the unique Nash equilibrium. □

C. Session rejection probability: the Erlang loss formula

Consider now a model handling blocked calls in which there are K servers available, and each newly arriving customer is given his own server; if a customer arrives when all servers are occupied, that customer's request is lost. This system is of particular interest in telephone networks, and is known as Erlang's loss formula. In terms of our notation, we have:

$$P_{loss}^i = G_i(\rho_i) = \frac{(\rho_i)^{K_i}/K_i!}{\sum_{j=0}^{K_i} (\rho_i)^j / j!} \tag{41}$$

We begin by taking the first derivatives of $G_i(.)$ with respect to ρ. After some simplification, we obtain

$$G_i'(\rho_i) = \frac{\sum_{j=0}^{K_i-1}(K_i-j)\rho_i^{K_i+j-1}/K_i!j!}{[\sum_{j=0}^{K_i}(\rho_i)^j/j!]^2} > 0 \tag{42}$$

Hence the function G_i^{-1} exists.

Let the concavity coefficient, $s = 1$ implying that QoS measure f and loss probabilities are linearly related (see (32). The following lemma characterizes some properties of the function g_i with respect to f_i.

Lemma V.2: The function $g_i(.) \in C^2$ is convex, increasing, and its derivative satisfies

$$\lim_{f_i \to 1} g_i'(f_i) = +\infty, \text{ and } \lim_{f_i \to 0} g_i'(f_i) = \frac{1}{K_i}$$

Proof: See Appendix in the full version of the paper citeEAW. □

Combining the last lemma with Theorem III.3 we obtain the following result.

Corollary V.2: The two-parameter price-QoS game (5), with QoS defined as in (34) with $s = 1$ and session loss probability given by (41), has a unique Nash equilibrium $(\mathbf{p}^*, \mathbf{f}^*)$, with f_i^* the unique solution to the equations $\begin{cases} g_i'(f_i^*) = \frac{\beta_i}{v_i b_i} \text{ if } K_i \geq \frac{v_i b_i}{\beta_i} \\ f_i^* = 0 \text{ if } K_i < \frac{v_i b_i}{\beta_i} \end{cases}$ and $\mathbf{p}^*$ is obtained through (10).

VI. Conclusions

We have presented a framework for modeling the complex interactions among telecom and Internet service providers through a class of one- and two-parameter Nash equilibrium models. The novel properties of the models are the use of demand functions which describe each service providers' customer set, and take into account not only the characteristics of that service provider, (**SP**), i, but also of all other **SP**, j, the presence of two parameters describing each **SP**'s service price and Quality of Service (QoS) level, and the generality of the utility functions, which permit incorporating complex functions such as those describing loss probabilities, decreasing rate of return, and other general QoS forms.

The following potentially interesting avenues for further research have arisen during this study.

- It could be valuable to consider nonlinear expressions for the demand experienced by each **SP**. One could envisage some general concave functions, or the logit or probit models, for example. The logit model has the positive feature of allowing closed form expressions for the choice probabilities across **SP**, but suffers the drawback of requiring an i.i.d. assumption on those probabilities. The probit model allows for covariances across **SP**s, but requires numerical simulation, in general, to obtain solutions.
- It could be of interest to incorporate explicitly the network of each **SP** into the model. Indeed, in this work, the characteristics of each provider's network have been aggregated into a single pair of functions, representing some sort of *average* service one would experience using that service provider. However, it is well known that QoS, and in some cases, price, vary considerably across the paths of a single **SP**. In addition, the demand experienced by an **SP** depends naturally on the source and destination of the demand.

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