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ON THE HELICITY IN 3D-PERIODIC NAVIER–STOKES EQUATIONS I: THE NON-STATISTICAL CASE

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Abstract

We consider the three-dimensional Navier–Stokes equations with potential forces and study the helicity of the regular solutions which are periodic in the space variables. We will give a detailed description of the behavior of the helicity for large times. In particular, the following asymptotic dichotomy of the helicity will be established: the helicity either is identically zero or is eventually non-zero and converges to zero as $t^d e^{-2h_0 t}$ for time $t \to \infty$. The relation between the helicity and the energy is also investigated in connection with that between the energy and enstrophy. Our study relies on the theory of the asymptotic expansion of the regular solutions of the Navier–Stokes equations and its associated normalization map as well as a Phragmen–Linderlöf principle. The application of this principle is possible due to our proof that the domain of analyticity (in complexified time) of the regular solutions contains (up to a logarithmic correction) a right half plane.

1. *Introduction*

In this paper we study the incompressible viscous flows which are periodic in the space variables and are driven by potential body forces. Then the velocity field

$$\mathbf{u}(\mathbf{x},t), \quad \text{where } \mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3,$$

satisfies the periodicity condition

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}(\mathbf{x}+L\mathbf{e}_j,t), \quad \text{for } \mathbf{x}\in\mathbb{R}^3 \text{ and } j=1,2,3,$$

(where $L>0$ is the spatial period and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$) as well as the Navier–Stokes equations

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{x},t)-\nu\Delta\mathbf{u}(\mathbf{x},t)=-\nabla p(\mathbf{x},t)-\nabla\varphi(\mathbf{x},t),$$
$$\nabla\cdot\mathbf{u}(\mathbf{x},t)=0,$$

where ν is the viscosity of the fluid, p is the pressure, and φ is the potential of the body force; here we assume that the mass density is equal to 1. Using the well-known remarkable Galileian invariance of the Navier–Stokes equations we can also (by a change of the reference system) consider only the flows satisfying the following zero-space average condition:

$$\int_\Omega \mathbf{u}(\mathbf{x},t)\,dx=0, \quad \Omega=(-L/2,L/2)^3,$$

where $dx=dx_1dx_2dx_3$ is the usual volume element in $\mathbb{R}^3$.

Recall that the curl of the velocity, that is, $\nabla\times\mathbf{u}$, is usually called the vorticity of the flow and is denoted by $\boldsymbol{\omega}$. The kinetic energy/mass, the rate of energy dissipation/mass, and the helicity/mass are respectively defined by

$$\mathcal{E}(t)=\tfrac{1}{2}\int_\Omega|\mathbf{u}(\mathbf{x},t)|^2\,dx, \quad \mathcal{F}(t)=\int_\Omega|\boldsymbol{\omega}(\mathbf{x},t)|^2\,dx \quad\text{and}\quad \mathcal{H}(t)=\int_\Omega\mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},t)\,dx.$$

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These entities satisfy the following (balance) equations:

$$\frac{d}{dt}\mathcal{E}(t) + \nu\mathcal{F}(t) = 0, \quad \frac{d}{dt}\mathcal{H}(t) + 2\nu\mathcal{J}(t) = 0,$$

where

$$\mathcal{J}(t) = \int_\Omega \boldsymbol{\omega}(\mathbf{x},t)\cdot(\nabla\times\boldsymbol{\omega}(\mathbf{x},t))\,dx.$$

Above, $\mathbf{u}\cdot\boldsymbol{\omega}$ is the helicity density of the flow. The invariance of the helicity for the inviscid flows (case $\nu = 0$) seems to have first been noticed by Moreau [**20**]. However the first thorough study of the helicity and its density for inviscid incompressible flows was done by Moffatt [**17**]. He gave a connection of the helicity to the topological invariants and dynamics of the vortex tubes as well as the first examples of physically relevant fluid flows with non-zero helicity. (In connection with the latter contribution, let us mention that for the flows considered in our paper, the non-zero helicity case represents not the exception, but the generic situation (see Theorem 3.2 below).) The work [**17**] was followed by many studies devoted to the role of the helicity in dissipative turbulent flows (for example, [**16**–**19**] and the references therein). There is a general agreement that helicity plays an important role in magneto-hydro dynamics, but not in the dynamics of 'neutral' flows (that is, solutions of the Euler or the Navier–Stokes equations) [**1**]. However, theoretical, empirical and numerical evidence indicate that the helicity can provide insights into the nature of the fluid flows, at least in the case when the viscosity is small [**19**]. Moreover, the behavior of the helicity in turbulent flows and its influence in the statistical dynamical behavior of those flows at different length scales were extensively studied [**1**, **12**, **19**, **16**]. This also led to a large array of empirical facts which are not yet subsumed to a generally accepted theory. Therefore the study of the helicity in decaying turbulence based on the mathematical analysis of the Navier–Stokes equations may be of interest. This paper is a contribution in this direction. In this paper, we focus on the study of the asymptotic behavior (for $t\to\infty$) of the helicity of the solutions of the Navier–Stokes equations defined above. The choice of those solutions is motivated by the fact that they are more amenable to analysis, and can carry turbulent statistics, as exploited in numerical simulations (in, for example, [**2**]). The statistical corollaries of our study will be presented in a forthcoming paper (part II of the current work).

To give a brief presentation of our results, let us start by recalling that when $\mathbf{u}(\mathbf{x},t)$ is not identically zero we have

$$\lim_{t\to\infty}\frac{\ln\left(\frac{1}{\nu^2 L}\int_\Omega|\mathbf{u}(\mathbf{x},t)|^2\,dx\right)}{\nu t} = \lim_{t\to\infty}\frac{\ln\left(\frac{L}{\nu^2}\int_\Omega|\boldsymbol{\omega}(\mathbf{x},t)|^2\,dx\right)}{\nu t} = -2\lambda_1 n_0, \tag{1.1}$$

where $\lambda_1 = 4\pi^2/L^2$ is the first eigenvalue of the Stokes operator (that is, $\nabla\times(\nabla\times\cdot) = -\Delta$ restricted to divergence-free vector fields which are L-periodic in the spatial variables and with zero-space averages) and $n_0\lambda_1$ is an eigenvalue of the Stokes operator [**5**]. Moreover, the asymptotic behavior of the regular solution $\mathbf{u}(\mathbf{x},t)$ is

$$e^{\nu\lambda_1 n_0 t}\mathbf{u}(\mathbf{x},t) \to \mathbf{w}_{n_0}(\mathbf{x}), \quad \text{as } t\to\infty, \tag{1.2}$$

uniformly for $\mathbf{x}\in\mathbb{R}^3$, where $\mathbf{w}_{n_0}(\mathbf{x})$ is an eigenfunction of the Stokes operator corresponding to the eigenvalue $\lambda_1 n_0$.

As a consequence of (1.1) and the Schwarz inequality, the helicity will converge to zero for $t\to\infty$ at least with exponential rate $2\nu\lambda_1 n_0$. However, (1.1) and (1.2) do not imply that the helicity will decay exponentially at exactly the same rate $2\nu\lambda_1 n_0$ as that of the energy or even that its decay is exponential. This is due to the possibilities of the cancellations in the scalar product $\mathbf{u}\cdot\boldsymbol{\omega}$ and in its sign while integrating the helicity density. In particular, it was not known whether the helicity can switch sign or vanish infinitely many times when $t\to\infty$. In this paper we show that the helicity has only two possibilities: it is either identically zero (that is, the flow is helicity-free) or eventually non-zero. We also show that these two cases turn out to be quite substantive so that neither can be neglected in the study of fluid flows. The helicity-free flows are 'unstable', in the sense that small perturbations can always produce non-helicity-free flows (see Theorem 3.2 and its proofs in § 6). Also inspired by the discussion in [**19**] on the generation of non-zero helicity for initially helicity-free flows, we prove that for our simpler type of flows, the initially helicity-free flows which have non-zero helicity are generic (see Proposition 3.3). For these latter flows, the helicity eventually has one sign (positive or negative) and

$$\lim_{t\to\infty} \frac{\ln\left(\frac{1}{\nu^2}\left|\int_\Omega \boldsymbol{\omega}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t)\,dx\right|\right)}{\nu t} = -2\lambda_1 h_0, \tag{1.3}$$

where h_0 is a positive integer which is always greater than or equal to the above n_0. In fact, we show that the asymptotic behavior of the helicity is always of the form $ct^d e^{-2\nu\lambda_1 h_0 t}$, where d is a non-negative integer and c is a constant with the appropriate physical dimension. If $d > 0$ then necessarily $h_0 > n_0$ and there are flows for which this is the case. The set of these integers h_0 in (1.3) is strictly larger than the set of the integers n_0 occurring in (1.1) and there are flows for which the ratio h_0/n_0 can be as large as desired. However, unlike the case of (1.1) the problem of identifying all possible values of h_0 is still open (see Remark 7.6).

In the study of the relation between the helicity and the energy, we will also show that their quotient $\mathcal{H}(t)/\mathcal{E}(t)$ converges to $2\theta\sqrt{n_0\lambda_1}$ for $t\to\infty$, where the number θ, belonging to $[-1,1]$, measures the asymptotic alignment between velocity and vorticity. In particular, when θ attains its maximum or minimum value, that is, $\theta=\pm 1$, the above eigenfunction $\mathbf{w}_{n_0}(\mathbf{x})$ of the Stokes operator is an eigenfunction of the curl operator $\nabla\times$, that is, a Beltrami flow; hence, in this case, $\boldsymbol{\omega}(\mathbf{x},t)\approx \pm\sqrt{n_0\lambda_1}\,\mathbf{u}(\mathbf{x},t)$ when time t is large. Moreover for any eigenvalue $n_0\lambda_1$ of the Stokes operator and θ in $[-1,1]$, there exists a flow for which $\mathcal{H}(t)/\mathcal{E}(t)$ converges exactly to $2\theta\sqrt{n_0\lambda_1}$ when $t\to\infty$. Clearly the case $h_0 > n_0$ can occur only when $\theta = 0$.

Let us now briefly discuss our method of studying the helicity in this paper. Using the asymptotic expansion of the regular solutions of the Navier–Stokes equations we derive and study the asymptotic expansion of the helicity. It follows that each coefficient in the asymptotic expansion of the helicity is a polynomial in time and in finitely many components of the normalization map, each of which is an eigenvector of the Stokes operator. The explicit computation of the polynomials necessitates the superposition of algebraic relations involving the non-linear term in the Navier–Stokes equations and the curl operator. When the asymptotic expansion of the helicity is not identically zero we can concentrate the study on the first non-zero polynomial under the constraints imposed by the vanishing of its preceding polynomials. This part of our study has a rather algebraic character and it exploits a surjective property of the normalization map (see the statement (4.5)). However, to prove that the helicity has zero asymptotic expansion only if the helicity itself is identically zero, we need techniques from complex analysis. For this, we will use the analyticity of the solutions in the complexified time. In particular, we will show that the domain of time-analyticity in this case is sufficiently large to permit us to apply the Polya–Szegö improvement of the Phragmen–Linderlöf theorem to conclude that the helicity is identically zero for all times.

The structure of this paper is the following. In § 2, we present the functional settings of the Navier–Stokes equations. Our main results are stated in § 3. Section 4 is a review of the asymptotic expansions of regular solutions and the normalization map as well as of some algebraic properties of the non-linear term in the Navier–Stokes equations which we need in the proofs. In § 5 we derive and study the asymptotic expansion of the helicity, particularly when this asymptotic expansion is not identically zero. In § 6 we study the set $\mathcal{R}_0$ of the regular initial data such that the solutions are regular and have zero helicity for all time $t \geqslant 0$. Though 'small' compared to the set of regular initial data for which the helicity is eventually non-zero, $\mathcal{R}_0$ contains infinitely many invariant closed linear manifolds of infinite dimension. In § 7, we present examples of solutions for which the helicity and energy display similar and dissimilar asymptotic behavior. There we also show that the helicity is, in fact, asymptotically decaying at least as fast as the energy. Moreover, we give examples showing that the exponential decay of the helicity may have an exponent which is much larger than that of the energy (that is, $h_0 \gg n_0$). In § 8, we show that the helicity can be extended by analyticity in time to the whole image of a right half plane by a conformal map which is a 'small' perturbation of the identity map. This allows us to prove that the helicity is identically zero when its asymptotic expansion is identically zero. In Appendix A, the abstract asymptotic expansions in normed spaces are presented for the completeness of the paper. In Appendices B and C, we prove the technical results in complex analysis which are used in § 8.

In fairness to the reader, we must discuss the abbreviated functional notation we will use throughout the proofs. Most of those proofs are based on calculations with long formulas and estimates. Using the classical notation would substantially increase the space necessary for those proofs. Therefore we introduce the functional notation in §§ 2 and 3 and, in order to help the reader become familiar with the notation, we use it simultaneously with the classical one in those two sections.

2. *Preliminaries*

We consider the Navier–Stokes equations in the whole three-dimensional (3D) space $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - \nabla\varphi, \\ \operatorname{div}\,\mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{u}(\mathbf{x},t) = (u_1,u_2,u_3)(\mathbf{x},t)$ is the unknown velocity field, $p \in \mathbb{R}$ is the unknown pressure, the body force is $(-\nabla\varphi)$ with a given function $\varphi \in \mathbb{R}$, and the initial velocity $\mathbf{u}_0$ is also known. Recall that for each $t \geqslant 0$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x},t)$ satisfies

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j) = \mathbf{u}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \text{ and } j = 1,2,3, \tag{2.2}$$

and

$$\int_\Omega \mathbf{u}(\mathbf{x})\,dx = 0, \tag{2.3}$$

where $\{\mathbf{e}_j : j = 1,2,3\}$ is the canonical basis in $\mathbb{R}^3$, $L > 0$ and $\Omega = (-L/2, L/2)^3$. We say that those functions satisfying (2.2) are L-periodic functions. Throughout $L = 2\pi$ and $\nu = 1$, the general case being easily recovered by a change of scale.

For $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3)$ in $\mathbb{C}^3$, we define $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}^*}$, where $\mathbf{a}^* = (\bar{a}_1,\bar{a}_2,\bar{a}_3)$. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the usual scalar product and norm, respectively, in $L^2(\Omega)^3$, that is,

$$\langle u,v\rangle = \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,dx, \quad |u| = \langle u,u\rangle^{1/2} = \left(\int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,dx\right)^{1/2}, \tag{2.4}$$

for $u = \mathbf{u}(\cdot)$ and $v = \mathbf{v}(\cdot)$ in $L^2(\Omega)^3$. Here and in the sequel a vector field $\mathbf{u}(\cdot)$, with values $\mathbf{u}(\mathbf{x})$ at $\mathbf{x} \in \mathbb{R}^3$, is denoted simply by u when viewed as an element of a specified functional space. Also note that we use $|\cdot|$ for the length of vectors in $\mathbb{C}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$, but in each case the context clarifies the precise meaning of this notation.

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials on Ω with values in $\mathbb{R}^3$ which are divergence-free as well as satisfy the condition (2.3). We define

$$\begin{cases} H \text{ to be the closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V \text{ to be the closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \end{cases} \tag{2.5}$$

where $H^l(\Omega)$ (with $l = 0, 1, 2, \ldots$) denotes the Sobolev space of functions $\varphi \in L^2(\Omega)$ such that for every multi-index α with $|\alpha| \leqslant l$ the distributional derivative $D^\alpha\varphi$ belongs to $L^2(\Omega)$. On V we consider the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the norm $\|\cdot\|$ defined by

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{j,k=1}^{3} \int_\Omega \frac{\partial u_j(\mathbf{x})}{\partial x_k} \frac{\partial v_j(\mathbf{x})}{\partial x_k}\, dx, \quad \|u\|^2 = \langle\!\langle u, u \rangle\!\rangle = \sum_{j,k=1}^{3} \int_\Omega \left| \frac{\partial u_j(\mathbf{x})}{\partial x_k} \right|^2 dx, \tag{2.6}$$

for $u = \mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))$ and $v = \mathbf{v}(\cdot) = (v_1(\cdot), v_2(\cdot), v_3(\cdot))$ in V. In order to study the dissipation term $(-\nu\Delta\mathbf{u})$ which involves the second derivatives, let $\mathcal{D}_A = V \cap H^2(\Omega)^3$ and define the Stokes operator by

$$Au = -\Delta\mathbf{u}(\cdot) = \left(-\sum_{j=1}^{3} \frac{\partial^2 u_k}{\partial x_j^2} \right)_{k=1,2,3} \quad \text{for all } u = \mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot)) \in \mathcal{D}_A. \tag{2.7}$$

The inner product of $u, v \in \mathcal{D}_A$ and the norm of $w \in \mathcal{D}_A$ are defined by $\langle Au, Av \rangle$ and $|Aw|$, respectively. In fact, this norm $|Aw|$, with $w \in \mathcal{D}_A$, is equivalent to the usual Sobolev norm of $H^2(\Omega)^3$.

Recall that the curl of a vector field $\mathbf{u}(\mathbf{x})$ is $\nabla \times \mathbf{u}(\mathbf{x})$ and that in the case $\mathbf{u}(\mathbf{x})$ is a velocity field of a fluid, $\boldsymbol{\omega}(\mathbf{x}) = \nabla \times \mathbf{u}(\mathbf{x})$ is called the vorticity field of the fluid. In our study the restriction of this curl operator to the space V will be denoted by $\mathfrak{C}$, that is, $\mathfrak{C}w$ is the field $\nabla \times \mathbf{w}(\mathbf{x})$ (with $\mathbf{x} \in \mathbb{R}^3$) in H (where the derivatives are taken in the weak sense) for $w = \mathbf{w}(\cdot) \in V$. With this notation, we have, due to the fact that $\mathbf{w}(\cdot)$ is divergence-free,

$$|\mathfrak{C}w|^2 = \int_\Omega |\nabla \times \mathbf{w}(\mathbf{x})|^2\, dx = \int_\Omega \sum_{k,j=1}^{3} \left| \frac{\partial w_k(\mathbf{x})}{\partial x_j} \right|^2 dx = \|w\|^2, \tag{2.8}$$

for $w = \mathbf{w}(\cdot) = (w_1(\cdot), w_2(\cdot), w_3(\cdot)) \in V$. Thus $\mathfrak{C}$ is an isometric operator from V into H, in fact it is *surjective*. Again, because $\mathbf{u}(\cdot)$ is divergence-free, we have for $u = \mathbf{u}(\cdot) \in \mathcal{D}_A$,

$$\mathfrak{C}^2 u = \nabla \times (\nabla \times \mathbf{u}(\cdot)) = -\Delta\mathbf{u}(\cdot) = Au. \tag{2.9}$$

Moreover, this curl operator $\mathfrak{C}$ is symmetric, that is,

$$\langle \mathfrak{C}u, v \rangle = \int_\Omega [\nabla \times \mathbf{u}(\mathbf{x})] \cdot \mathbf{v}(\mathbf{x})\, dx = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot [\nabla \times \mathbf{u}(\mathbf{x})]\, dx = \langle u, \mathfrak{C}v \rangle, \tag{2.10}$$

for $u, v \in V$.

Another operator playing a major role in this paper is associated with the non-linear term in the equations (2.1). This operator is defined on $\mathcal{D}_A \times \mathcal{D}_A$ (and valued in H) by the formula

$$B(u, v) = P[(\mathbf{u}(\cdot) \cdot \nabla)\mathbf{v}(\cdot)] \quad \text{for all } u = \mathbf{u}(\cdot) \text{ and } v = \mathbf{v}(\cdot) \text{ in } \mathcal{D}_A, \tag{2.11}$$

where P is the orthogonal projection of $L^2(\Omega)^3$ onto the space H. Recall that for $v = \mathbf{v}(\cdot)$ in $L^2(\Omega)^3$ the projection $w = Pv \in H$ is defined by the following decomposition:

$$v = w + \nabla q, \tag{2.12}$$

where $w \in H$ and q is a 2π-periodic function in $H^1(\Omega)$. Note that w is divergence-free and belongs to H. Thus (2.12) is an adequate generalized variant of the classical Helmholtz decomposition of a vector field $\mathbf{v}(\mathbf{x})$ into its divergence-free $\mathbf{w}(\mathbf{x})$ and gradient ∇q components. Thus for $u, v \in \mathcal{D}_A$ and $w \in H$,

$$\langle B(u, v), w \rangle = \int_\Omega [(\mathbf{u}(\mathbf{x}) \cdot \nabla)\mathbf{v}(\mathbf{x})] \cdot \mathbf{w}(\mathbf{x})\, dx.$$

The following relations are well known:

$$B(u, u) = -P(u \times \mathfrak{C}u) = -P[\mathbf{u}(\cdot) \times (\nabla \times \mathbf{u}(\cdot))] \quad \text{for all } u \in \mathcal{D}_A \tag{2.13}$$

and

$$\langle B(u,v),v\rangle = \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{v}\,dx = 0 \quad \text{for } u,v\in\mathcal{D}_A. \tag{2.14}$$

Also for $u,v\in\mathcal{D}_A$, we have

$$\begin{aligned}\langle B(\mathfrak{C}v,v),u\rangle &= \int_\Omega [((\nabla\times\mathbf{v})\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\,dx\\ &= \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot(\nabla\times\mathbf{v})\,dx = \langle B(u,v),\mathfrak{C}v\rangle,\end{aligned} \tag{2.15}$$

whence

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot(\nabla\times\mathbf{u})\,dx = \langle B(u,u),\mathfrak{C}u\rangle = \langle B(\mathfrak{C}u,u),u\rangle = 0 \quad \text{for } u\in\mathcal{D}_A, \tag{2.16}$$

by virtue of (2.14).

A time-dependent velocity field of a fluid with finite kinetic space average energy/mass, that is, $\int_\Omega |\mathbf{u}(\mathbf{x},t)|^2\,dx<\infty$, can be viewed as an H-valued function $u(t)=\mathbf{u}(\cdot,t)$. If this $u(t)\in V$, then the vorticity $\boldsymbol{\omega}(\cdot,t)=\nabla\times\mathbf{u}(\cdot,t)$ can be also denoted by $\omega(t)=\mathfrak{C}u(t)$ when considered as an element of H. Thus $\mathfrak{C}^2u(t)$ is the field $\nabla\times\boldsymbol{\omega}(\cdot,t)=\nabla\times(\nabla\times\mathbf{u}(\cdot,t))$ provided $u(t)\in V$ and $\mathfrak{C}u(t)\in V$.

A classical result (tracking back to Leray's pioneering works in the 1930s, for example, [**13**, **14**, **15**]) in the theory of the Navier–Stokes equations is that for any initial data $\mathbf{u}_0(\mathbf{x})$ in H there exists a weak solution $\mathbf{u}(\mathbf{x},t)$ defined for all $t>0$ which *eventually* becomes regular and $|u(t)|^2=\int_\Omega|\mathbf{u}(\mathbf{x},t)|^2\,dx$ converges exponentially to zero as $t\to\infty$. We recall that a weak solution $u(t)$ is called *regular* in a time interval $[t_0,t_1)$ (where $t_0<t_1\leqslant\infty$) if it is continuous from $[t_0,t_1)$ to V. In this case, its values $u(t)$ (for $t_0<t<t_1$) are uniquely determined by the value $u(t_0)$. Recall that in this paper we consider only the case when the driving force is potential (that is, of the form $-\nabla\varphi$) and the velocity field is periodic in the space variables. In this case if $u(t)$ is regular on $[t_0,t_1)$ the velocity field $\mathbf{u}(\mathbf{x},t)$ is analytic in $(\mathbf{x},t)$ for $\mathbf{x}\in\mathbb{R}^3$ and $t\in(t_0,t_1)$ (see, for example, [**3**, **9**, **4**]). It easily follows that the sum $p+\varphi$ of the pressure p and the potential φ of the force is also analytic in $(\mathbf{x},t)$ for $\mathbf{x}\in\mathbb{R}^3$ and $t\in(t_0,t_1)$. We stress that these analyticity results do not automatically hold when the driving force is not potential or when the fluid is bounded by walls. In our present situation the regularity and analyticity results recalled above imply that for any weak solution $\mathbf{u}(\cdot,t)$ of the Navier–Stokes equations there exists a time $t_0\geqslant 0$ (depending on the solution) such that for $t>t_0$ both the vector field $\mathbf{u}(\mathbf{x},t)$ and the sum $p+\varphi$ are analytic in $(\mathbf{x},t)$ and satisfy the Navier–Stokes equations. Moreover, such a solution $u(t)=\mathbf{u}(\cdot,t)$ satisfies the following functional form of the Navier–Stokes equations which is derived by applying the operator P to the momentum equations in (2.1) which eliminates the gradient terms:

$$\frac{du(t)}{dt}+Au(t)+B(u(t),u(t))=0, \quad \text{for } t>t_0;$$

here the equation holds in H and $u(t)$ is continuous from $[t_0,\infty)$ into V. Since the interest of this paper is to study the asymptotic behavior of a solution for time $t\to\infty$, we will restrict all our considerations to *the set $\mathcal{R}$ of all the initial data u_0 in V such that the solution $u(t)$ satisfying $u(0)=u_0$ is regular* (*and hence also unique*) *in* $[0,\infty)$; whenever not otherwise specified, the topology of $\mathcal{R}$ is that of V. In other words, we will study the following functional form of Navier–Stokes equations:

$$\begin{cases}\dfrac{du(t)}{dt}+Au(t)+B(u(t),u(t))=0, & \text{for } t>0,\\ u(0)=u_0\in\mathcal{R},\end{cases} \tag{2.17}$$

where the equation holds in H for all $t > 0$, and $u(t)$ is continuous from $[0, \infty)$ into V. In this way both the pressure and the driving force have no effect on the asymptotic behavior of $u(t)$.

A subset $\mathcal{S}$ of $\mathcal{R}$ is called *invariant* if for any $u_0 \in \mathcal{S}$, the regular solution $u(t)$ of (2.17) stays in $\mathcal{S}$ for all time $t \geqslant 0$.

In particular, the analyticity of a regular solution $\mathbf{u}(\mathbf{x}, t)$ in its variables for $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$ easily implies that the energy balance

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |\mathfrak{C}u(t)|^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2\, dx + \int_\Omega |\boldsymbol{\omega}(\mathbf{x}, t)|^2\, dx = 0, \tag{2.18}$$

and helicity balance

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\langle u(t), \mathfrak{C}u(t)\rangle + \langle \mathfrak{C}u(t), \mathfrak{C}^2 u(t)\rangle \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbf{u}(\mathbf{x}, t)\cdot\boldsymbol{\omega}(\mathbf{x}, t)\, dx + \int_\Omega \boldsymbol{\omega}(\mathbf{x}, t)\cdot(\nabla\times\boldsymbol{\omega}(\mathbf{x}, t))\, dx = 0\end{aligned} \tag{2.19}$$

hold for all $t > 0$. Above $\frac{1}{2}|u|^2 = \frac{1}{2}\int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2\, dx$ and $\langle u, \mathfrak{C}u\rangle = \int_\Omega \mathbf{u}(\mathbf{x}, t)\cdot\boldsymbol{\omega}(\mathbf{x}, t)\, dx$ are the space average (kinetic) energy/mass and helicity/mass of $\mathbf{u}(\cdot, t)$.

Finally, since the eigenvalues and eigenspaces of the Stokes operator play an important role in our analysis, we recall their explicit forms here. The Stokes operator A has a sequence of eigenvalues $\{\lambda_j : j = 1, 2, 3, \ldots\} = \sigma(A)$ of the form $\lambda_j = |\mathbf{k}|^2$ for some $\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}$. Note that $\lambda_1 = 1 = |\mathbf{e}_1|^2$ and hence the additive semigroup generated by these eigenvalues coincides with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of all natural numbers. For $n \in \mathbb{N}$ we denote by R_n the orthogonal projection of H onto the eigenspace of A associated to n, namely,

$$R_n H = \{u \in H : Au = nu\}. \tag{2.20}$$

If n is an eigenvalue of A, then $R_n H$ is generated by functions of the forms

$$(\mathbf{a}^1_{\mathbf{k}} + i\mathbf{a}^2_{\mathbf{k}})e^{i(\mathbf{k}\cdot\mathbf{x})} + (\mathbf{a}^1_{\mathbf{k}} - i\mathbf{a}^2_{\mathbf{k}})e^{-i(\mathbf{k}\cdot\mathbf{x})}, \quad \text{with } \mathbf{k} \in \mathbb{Z}^3 \text{ and } |\mathbf{k}|^2 = n,$$

where

$$\mathbf{a}^1_{\mathbf{k}}, \mathbf{a}^2_{\mathbf{k}} \in \mathbb{R}^3 \quad \text{and} \quad \mathbf{a}^1_{\mathbf{k}}\cdot\mathbf{k} = \mathbf{a}^2_{\mathbf{k}}\cdot\mathbf{k} = 0.$$

Otherwise, $R_n = 0$, for example, $R_7 = 0$, $R_{15} = 0$, $R_{23} = 0$, etc.

3. *Main results*

To describe the asymptotic behavior of the helicity, we will need some asymptotic properties of the solutions to (2.17). It is known from [**5**] that for any solution $u(t)$ of the Navier–Stokes equations (2.17) with initial data $u_0 \in \mathcal{R}$, there is an eigenvalue n_0 of the Stokes operator A such that

$$\lim_{t\to\infty}\frac{|\mathfrak{C}u(t)|^2}{|u(t)|^2} = \lim_{t\to\infty}\frac{\int_\Omega |\nabla\times\mathbf{u}(\mathbf{x}, t)|^2\, dx}{\int_\Omega |\mathbf{u}(\mathbf{x}, t)|^2\, dx} = n_0 \tag{3.1}$$

and

$$\lim_{t\to\infty} u(t)e^{n_0 t} = w_{n_0}(u_0) \in R_{n_0}H \setminus \{0\}, \tag{3.2}$$

where the limit is taken either in H or in V (see § 4). Recall that $\boldsymbol{\omega}(\cdot, t) = \nabla\times\mathbf{u}(\cdot, t)$ is the vorticity field and $\mathbf{u}(\mathbf{x}, t)\cdot\boldsymbol{\omega}(\mathbf{x}, t)$ is the helicity density.

In this paper, we study the behavior of the helicity

$$\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle = \langle \omega(t), u(t)\rangle = \int_\Omega \boldsymbol{\omega}(\mathbf{x}, t)\cdot\mathbf{u}(\mathbf{x}, t)\, dx$$

and its related quantity

$$\mathcal{J}(t) = \langle \mathfrak{C}^2 u(t), \mathfrak{C}u(t)\rangle = \int_\Omega [\nabla \times (\nabla \times \mathbf{u}(\mathbf{x},t))] \cdot [\nabla \times \mathbf{u}(\mathbf{x},t)]\, dx$$
$$= \langle \mathfrak{C}\omega(t), \omega(t)\rangle = \int_\Omega [\nabla \times \boldsymbol{\omega}(\mathbf{x},t)] \cdot \boldsymbol{\omega}(\mathbf{x},t)\, dx$$

for time $t \to \infty$. These two satisfy the equation (see (2.19))

$$\frac{1}{2}\frac{d\mathcal{H}(t)}{dt} + \mathcal{J}(t) = 0. \tag{3.3}$$

This resembles the energy relation (2.18) connecting $|\mathfrak{C}u(t)|^2 = |\omega(t)|^2$ and $|u(t)|^2$. We will find the analogues of the relations (3.1) and (3.2) for $\mathcal{J}(t)/\mathcal{H}(t)$ and $\mathcal{H}(t)$. The asymptotic behavior of $\mathcal{H}(t)$ and $\mathcal{J}(t)$ as $t \to \infty$ will also be described.

The main results of this paper are summed up in the following theorems.

Theorem 3.1. *The helicity $\mathcal{H}(t)$ of a regular solution $u(t)$ of (2.17) is either*

(i) *eventually non-zero and decaying when t goes to infinity as $t^d e^{-2h_0 t}$, where $d \geqslant 0$ and $h_0 > 0$ are integers depending on the solution and*

$$\lim_{t\to\infty} \frac{\mathcal{J}(t)}{\mathcal{H}(t)} = \lim_{t\to\infty} \left[\int_\Omega \boldsymbol{\omega}(\mathbf{x},t) \cdot (\nabla \times \boldsymbol{\omega}(\mathbf{x},t))\, dx \Big/ \int_\Omega \mathbf{u}(\mathbf{x},t) \cdot \boldsymbol{\omega}(\mathbf{x},t)\, dx \right] = h_0, \tag{3.4}$$

or

(ii) *identically zero.*

The next theorem shows that both cases (i) and (ii) above are substantive.

Theorem 3.2. *Let $\mathcal{R}_1$ and $\mathcal{R}_0$ denote the subsets formed by the initial data $u_0 \in \mathcal{R}$ for which the solutions are in case* (i), *respectively* (ii), *of Theorem* 3.1. *Then $\mathcal{R}_1$ is open and dense in $\mathcal{R}$, while $\mathcal{R}_0$ is closed and contains an infinite union of invariant closed linear manifolds of infinite dimension.*

Moreover, we also have the following similar relation between the initially helicity-free flows and the identically helicity-free flows (see the proof after Remark 6.7).

Proposition 3.3. *The set $\mathcal{R}_0$ is a closed nowhere dense subset of $\{u_0 \in \mathcal{R} : \langle \mathfrak{C}u_0, u_0\rangle = 0\}$.*

The connections between the exponential decay in (3.1) and Theorem 3.1 can be explicated as follows.

Theorem 3.4. *Let $u_0 \in \mathcal{R} \setminus \{0\}$ and let $u(t)$ be the regular solution of* (2.17) *and let n_0 be defined by* (3.1). *Then*

$$\lim_{t\to\infty} \frac{\mathcal{H}(t)}{|u(t)|^2} = \lim_{t\to\infty} \left[\int_\Omega \boldsymbol{\omega}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t)\, dx \Big/ \int_\Omega |\mathbf{u}(\mathbf{x},t)|^2\, dx \right] = \alpha_0, \tag{3.5}$$

and $\alpha_0 \in [-\sqrt{n_0}, \sqrt{n_0}\,]$. Moreover, for any $n \in \sigma(A)$ and $\alpha \in [-\sqrt{n}, \sqrt{n}\,]$, there is $u_0 \in \mathcal{R}$ such that the solution $u(t)$ of (2.17) *satisfies*

$$\lim_{t\to\infty} \frac{|\mathfrak{C}u(t)|^2}{|u(t)|^2} = \lim_{t\to\infty} \frac{\int_\Omega |\nabla \times \mathbf{u}(\mathbf{x},t)|^2\, dx}{\int_\Omega |\mathbf{u}(\mathbf{x},t)|^2\, dx} = n, \tag{3.6}$$

and

$$\lim_{t\to\infty} \frac{\mathcal{H}(t)}{|u(t)|^2} = \lim_{t\to\infty} \left[\int_\Omega \boldsymbol{\omega}(\mathbf{x},t) \cdot \mathbf{u}(\mathbf{x},t)\, dx \Big/ \int_\Omega |\mathbf{u}(\mathbf{x},t)|^2\, dx \right] = \alpha. \tag{3.7}$$

It is clear that in the preceding theorem, if $\alpha_0 \neq 0$ then $u(t)$ is in case (i) of Theorem 3.1 with $d = 0$ and $h_0 = n_0 \in \sigma(A)$. In fact, $h_0 = n_0$ if and only if $\alpha_0 \neq 0$ (see Remark 7.5). The other situation when $\alpha_0 = 0$ will be analyzed further in the next theorem.

THEOREM 3.5. *For any $n \in \sigma(A)$ and $M > 0$, there exists an initial data $u_0 \in \mathcal{R}$ such that the corresponding regular solution $u(t)$ of* (2.17) *satisfies* (3.1), *is in case* (i) *of Theorem* 3.1, *with $n_0 = n$ and $h_0 \geqslant n_0 + M$, and is such that*

$$\frac{\mathcal{H}(t)}{|u(t)|^2} = \int_\Omega \boldsymbol{\omega}(\mathbf{x},t)\cdot \mathbf{u}(\mathbf{x},t)\,dx \Big/ \int_\Omega |\mathbf{u}(\mathbf{x},t)|^2\,dx = O(e^{-2Mt}) \quad \text{as} \quad t\to\infty. \tag{3.8}$$

Also, there are solutions with the helicity satisfying the condition

$$\lim_{t\to\infty} \mathcal{H}(t) t^{-d} e^{2h_0 t} = \lim_{t\to\infty} t^{-d} e^{2h_0 t} \int_\Omega \boldsymbol{\omega}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t)\,dx \quad \text{exists and is not zero,}$$

where $d > 0$ or h_0 is not an eigenvalue of A.

Theorems 3.4 and 3.5 provide supplementary information only about the solutions in the set $\mathcal{R}_1$ in Theorem 3.2. We also note that if $u_0 \in \mathcal{R}_1$ (respectively, $\mathcal{R}_0$) and $u(t)$ ($t \geqslant 0$) is the regular solution of (2.17), then $u(t) \in \mathcal{R}_1$ (respectively, $\mathcal{R}_0$) for all $t \geqslant 0$. In other words, $\mathcal{R}_1$ and $\mathcal{R}_0$ are invariant.

The remainder of this article is devoted to the proofs of the results stated above.

4. *Mathematical prerequisites*

In this section, first we review some asymptotic properties of the solutions of (2.17) which are outlined below (see [**5**, **6**, **7**, **10**] for more details).

Let $u(t)$ be a regular solution of the Navier–Stokes equations with initial data $u_0 \in \mathcal{R}$. Then it has the following asymptotic expansion:

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \dots, \tag{4.1}$$

where $q_j(t)$ is a polynomial in t of degree at most $(j-1)$ and with values trigonometric polynomials in H. This means that for any $N \in \mathbb{N}$ the correction term

$$v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}$$

satisfies

$$|v_N(t)| = O(e^{-(N+\varepsilon)t}) \quad \text{for } t\to\infty \text{ with some } \varepsilon = \varepsilon_N > 0. \tag{4.2}$$

In fact, $v_N(t)$ belongs to $C^1([0,\infty); V) \cap C^\infty((0,\infty); C^\infty(\mathbb{R}^3))$ and for each $m \in \mathbb{N}$,

$$\|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}) \quad \text{as } t\to\infty \text{ for some } \varepsilon = \varepsilon_{N,m} > 0. \tag{4.3}$$

Define the normalization map W by

$$W(u_0) = W_1(u_0) \oplus W_2(u_0) \oplus \dots,$$

where $W_j(u_0) = R_j q_j(0)$, for $j = 1, 2, 3, \dots$. Then W is a one-to-one analytic mapping from $\mathcal{R}$ to the Fréchet space $S_A = R_1 H \oplus R_2 H \oplus \dots$ equipped with the component-wise topology. Also, $W'(0) = \mathrm{Id}$ meaning

$$W'(0)u_0 = R_1 u_0 \oplus R_2 u_0 \oplus R_3 u_0 \oplus \dots. \tag{4.4}$$

Therefore, if Π_N denotes the canonical projection of S_A onto $R_1 H \oplus \dots \oplus R_N H$ then

$$\{\Pi_N W(u_0) : \|u_0\| < \rho\} \text{ is a neighborhood of 0 in } \Pi_N S_A, \tag{4.5}$$

for all $\rho > 0$ and $N = 1, 2, 3, \ldots$.

The case (3.1) holds if and only if

$$W_1(u_0) = W_2(u_0) = \ldots = W_{n_0-1}(u_0) = 0 \quad \text{and} \quad W_{n_0}(u_0) \neq 0;$$

in this case

$$q_1 \equiv q_2 \equiv \ldots \equiv q_{n_0-1} \equiv 0 \quad \text{and} \quad q_{n_0} \equiv w_{n_0}(u_0) = W_{n_0}(u_0). \tag{4.6}$$

Moreover, if $u_0 \in \mathcal{R}$ and $W(u_0) = (\xi_1, \xi_2, \ldots)$, then the polynomials q_j are the unique polynomial solutions of the following equations:

$$q_j' + (A - j)q_j + \beta_j = 0, \tag{4.7}$$

with $R_j q_j(0) = \xi_j$, where the terms β_j are defined by

$$\beta_1 = 0 \quad \text{and} \quad \text{for } j > 1,\ \beta_j = \sum_{k+l=j} B(q_k, q_l). \tag{4.8}$$

Explicitly, the polynomials $q_j(t)$ are recurrently given by the formulas

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau)\, d\tau + \sum_{n \geqslant 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} \left(\frac{d}{dt}\right)^n (I - R_j)\beta_j, \tag{4.9}$$

for $j = 1, 2, 3, \ldots$, where $[(A - j)(I - R_j)]^{-n-1}$ is defined by

$$[(A - j)(I - R_j)]^{-n-1} u = \sum_{|\mathbf{k}|^2 \neq j} \frac{\mathbf{a_k}}{(|\mathbf{k}|^2 - j)^{n+1}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.10}$$

for $u = \sum_{|\mathbf{k}|^2 \neq j} \mathbf{a_k} e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\mathbf{x} \in \mathbb{R}^3$, in $\mathcal{V}$. Here I denotes the identity map on H.

We will also need some algebraic properties of the bilinear map $B(\cdot,\cdot)$ introduced in (2.11). Replacing v by $v + w$ in (2.14) gives

$$\langle B(u, v), w\rangle = -\langle B(u, w), v\rangle, \quad \text{for } u, v, w \in \mathcal{D}_A. \tag{4.11}$$

By writing $u = v + \lambda w$ in (2.15), where λ is a real parameter, and collecting the coefficients of λ one obtains

$$\langle B(u, u), \mathfrak{C}v\rangle + \langle B(u, v), \mathfrak{C}u\rangle + \langle B(v, u), \mathfrak{C}u\rangle = 0, \quad \text{for } u, v \in \mathcal{D}_A. \tag{4.12}$$

For $u, \mathfrak{C}u \in \mathcal{D}_A$, it follows from (4.12), (4.11) and (2.16) that

$$\langle B(u, u), \mathfrak{C}v\rangle = \langle B(u, \mathfrak{C}u), v\rangle - \langle B(\mathfrak{C}u, u), v\rangle$$

is valid for all $v \in \mathcal{D}_A$ and by continuity also for all $v \in V$; hence $B(u, u)$ belongs to V and

$$\mathfrak{C}B(u, u) = B(u, \mathfrak{C}u) - B(\mathfrak{C}u, u). \tag{4.13}$$

Replacing u by $u + v$ in (4.13), we obtain

$$\mathfrak{C}B(u, v) + \mathfrak{C}B(v, u) = B(u, \mathfrak{C}v) + B(v, \mathfrak{C}u) - B(\mathfrak{C}u, v) - B(\mathfrak{C}v, u), \tag{4.14}$$

for $\mathfrak{C}u, \mathfrak{C}v \in \mathcal{D}_A$.

5. *Exponential decay of the helicity*

We will derive the *asymptotic expansion* of the helicity $\mathcal{H}(t)$. This can be easily seen as a formal expansion of $\langle \mathfrak{C}u(t), u(t)\rangle$ based on the expansion (4.1) of $u(t)$:

$$\langle \mathfrak{C}u(t), u(t)\rangle \sim \sum_{j \geqslant 1} e^{-jt} \sum_{k+l=j} \langle \mathfrak{C}q_k, q_l\rangle.$$

In fact, this formal computation will be proved to be valid in the rigorous sense of Definition A.1 in Appendix A. In the spirit of Definition A.1, we can say from (4.2) and (4.3) that the regular solution $u(t)$ of (2.17) has a unique asymptotic expansion

$$u(t) \sim \sum_{j\geqslant 1} q_j(t)e^{-jt} \quad \text{in } H^m(\Omega) \text{ for } m = 0, 1, 2, \ldots. \tag{5.1}$$

The asymptotic expansions of $\mathfrak{C}u$ and $\mathfrak{C}^2u$ are formally calculated by applying $\mathfrak{C}$ and $\mathfrak{C}^2$, respectively, to (5.1) term by term. This is justified by Lemma A.3 and we have the following expansions.

LEMMA 5.1. *Let $u(t)$ be a regular solution of* (2.17) *with the asymptotic expansion* (5.1). *Then the vorticity and its curl have the asymptotic expansions*

$$\boldsymbol{\omega}(\cdot, t) = \mathfrak{C}u(t) \sim \sum_{j=1}^{\infty} \mathfrak{C}q_j(t)e^{-jt} \tag{5.2}$$

and, respectively,

$$\nabla \times \boldsymbol{\omega}(\cdot, t) = \mathfrak{C}^2u(t) \sim \sum_{j=1}^{\infty} \mathfrak{C}^2q_j(t)e^{-jt}, \tag{5.3}$$

in all $H^m(\Omega)$ for $m = 0, 1, 2, \ldots$.

By Lemmas 5.1 and A.4, we easily and rigorously obtain the asymptotic expansions of $\mathcal{H}(t)$ and $\mathcal{J}(t)$.

LEMMA 5.2. *Let $u(t)$ be a regular solution of* (2.17) *with the asymptotic expansion* (5.1). *Then the helicity $\mathcal{H}(t)$ and $\mathcal{J}(t)$ have the following asymptotic expansions:*

$$\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle \sim \sum \phi_n(t)e^{-nt}, \tag{5.4}$$

where

$$\phi_n(t) = \sum_{j+l=n} \langle \mathfrak{C}q_j(t), q_l(t)\rangle, \tag{5.5}$$

and, respectively,

$$\mathcal{J}(t) = \langle \mathfrak{C}^2u(t), \mathfrak{C}u(t)\rangle \sim \sum \psi_n(t)e^{-nt}, \tag{5.6}$$

where

$$\psi_n(t) = \sum_{j+l=n} \langle \mathfrak{C}^2q_j(t), \mathfrak{C}q_l(t)\rangle. \tag{5.7}$$

Proof. Apply Lemma A.4 with $X = Y = L^2(\Omega)$, and $Z = \mathbb{R}$ and the use of (5.1) as well as Lemma 5.1 for individual expansions of $u(t)$, $\mathfrak{C}u(t)$ and $\mathfrak{C}^2u(t)$. □

So far we have obtained the asymptotic expansions (5.4) and (5.6) of $\mathcal{H}(t)$ and $\mathcal{J}(t)$, respectively. The functions $\mathcal{H}(t)$ and $\mathcal{J}(t)$ are connected by equation (3.3). The corresponding connection between ϕ_n and ψ_n in those asymptotic expansions is given in the following lemma.

LEMMA 5.3. *For all $n \in \mathbb{N}$ and $t > 0$,*

$$\phi_n'(t) - n\phi_n(t) + 2\psi_n(t) = 0. \tag{5.8}$$

Proof. Taking the derivative of $\phi_n(t)$, we obtain

$$\begin{aligned}\phi_n'(t) &= \sum_{m+k=n} \langle \mathfrak{C} q_m, q_k' \rangle + \sum_{m+k=n} \langle \mathfrak{C} q_m', q_k \rangle \\ &= \sum_{m+k=n} \langle \mathfrak{C} q_m, q_k' \rangle + \sum_{m+k=n} \langle q_m', \mathfrak{C} q_k \rangle \\ &= 2 \sum_{m+k=n} \langle \mathfrak{C} q_m, q_k' \rangle = -2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \mathfrak{C}^2 q_k - k q_k + \beta_k \rangle,\end{aligned}$$

by virtue of (4.7); hence

$$\phi_n'(t) = -2\psi_n + 2 \sum_{m+k=n} \langle \mathfrak{C} q_m, k q_k \rangle - 2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \beta_k \rangle. \tag{5.9}$$

The second term on the right-hand side of (5.9) is

$$\sum_{m+k=n} \langle \mathfrak{C} q_m, k q_k \rangle + \langle \mathfrak{C} q_k, m q_m \rangle = \sum_{m+k=n} \langle \mathfrak{C} q_m, (m+k) q_k \rangle = n \sum_{m+k=n} \langle \mathfrak{C} q_m, q_k \rangle = n\phi_n.$$

Using (4.8), (2.10) and (4.14) we can write the last term of (5.9) as

$$\begin{aligned}-2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \beta_k \rangle &= -2 \sum_{m+j+l=n} \langle \mathfrak{C} q_m, B(q_j, q_l) \rangle \\ &= - \sum_{m+j+l=n} (\langle \mathfrak{C} q_m, B(q_j, q_l) \rangle + \langle \mathfrak{C} q_m, B(q_l, q_j) \rangle) \\ &= - \sum_{m+j+l=n} \langle q_m, \mathfrak{C} B(q_j, q_l) + \mathfrak{C} B(q_l, q_j) \rangle \\ &= \sum_{m+j+l=n} \langle q_m, B(\mathfrak{C} q_j, q_l) + B(\mathfrak{C} q_l, q_j) - B(q_j, \mathfrak{C} q_l) - B(q_l, \mathfrak{C} q_j) \rangle \\ &= 2 \sum_{m+j+l=n} \langle q_m, B(\mathfrak{C} q_j, q_l) \rangle - 2 \sum_{m+j+l=n} \langle q_m, B(q_j, \mathfrak{C} q_l) \rangle.\end{aligned}$$

We have, from (4.11),

$$\begin{aligned}-2 \sum_{m+j+l=n} \langle q_m, B(q_j, \mathfrak{C} q_l) \rangle = 2 \sum_{m+j+l=n} \langle \mathfrak{C} q_l, B(q_j, q_m) \rangle &= 2 \sum_{m+j+l=n} \langle \mathfrak{C} q_m, B(q_j, q_l) \rangle \\ &= 2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \beta_k \rangle,\end{aligned}$$

and

$$2 \sum_{m+j+l=n} \langle q_m, B(\mathfrak{C} q_j, q_l) \rangle = \sum_{m+j+l=n} \langle q_m, B(\mathfrak{C} q_j, q_l) \rangle + \sum_{m+j+l=n} \langle q_l, B(\mathfrak{C} q_j, q_m) \rangle = 0.$$

Therefore,

$$-2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \beta_k \rangle = 2 \sum_{m+k=n} \langle \mathfrak{C} q_m, \beta_k \rangle,$$

and hence it is zero. Thus, $\phi_n' = -2\psi_n + n\phi_n$ and this proves (5.8). □

The behavior of $\phi_n(t)$ and $\psi_n(t)$ for large times t is constrained by Lemma 5.3 and the fact that these functions are polynomials. Indeed we also have the following.

LEMMA 5.4. *For each $n \in \mathbb{N}$, either*
(i) *ϕ_n and ψ_n are identically zero, or*
(ii) *ϕ_n and ψ_n are eventually non-zero.*

In the second case,

$$\lim_{t\to\infty}\frac{\psi_n(t)}{\phi_n(t)}=\frac{n}{2}. \tag{5.10}$$

Proof. In the case where ϕ_n is identically zero, so is ψ_n by virtue of (5.8), and hence (i) holds. If ϕ_n is not identically zero, since it is a polynomial, we have deg $\phi_n' <$ deg ϕ_n. Hence $\phi_n' - n\phi_n$ is not identically zero and due to (5.8), neither is ψ_n. Since both ϕ_n and ψ_n are polynomials, it follows that for all large enough t, say, $t > t_0$, $\psi_n(t) \neq 0$ and $\phi_n(t) \neq 0$; this proves (ii). Moreover, for $t > t_0$, we have

$$\frac{\phi_n'}{\phi_n} - n + 2\frac{\psi_n}{\phi_n} = 0, \tag{5.11}$$

where the first term goes to zero as $t \to \infty$. Clearly, (5.11) now implies (5.10). ☐

Continuing our study of the asymptotic expansions (5.4) and (5.6) we first consider the case when the asymptotic expansion (5.4) of $\mathcal{H}(t)$ is not identically zero, that is when at least one of the polynomials $\phi_n(t)$ $(n = 1, 2, 3, \dots)$ is not identically zero. The other case when all $\phi_n(t) \equiv 0$ will be relegated to § 8. In the former case, let N be the smallest $n \in \mathbb{N}$ such that the polynomial $\phi_n(t)$ is not identically zero. By Lemma 5.4, this number N is also the smallest $n \in \mathbb{N}$ such that the polynomial $\psi_n(t)$ in the asymptotic expansion (5.6) of $\mathcal{J}(t)$ is not identically zero. Hence, we have

$$\phi_n(t) \equiv \psi_n(t) \equiv 0 \quad \text{for } 1 \leqslant n < N, \quad \phi_N(t) \not\equiv 0 \quad \text{and} \quad \psi_N(t) \not\equiv 0. \tag{5.12}$$

The asymptotic expansions (5.4) and (5.6) of $\mathcal{H}(t)$ and, respectively, $\mathcal{J}(t)$ now become

$$\mathcal{H}(t) = \phi_N(t)e^{-Nt} + O(e^{-(N+\varepsilon)t}) \quad \text{and} \quad \mathcal{J}(t) = \psi_N(t)e^{-Nt} + O(e^{-(N+\varepsilon)t}), \tag{5.13}$$

for some $\varepsilon > 0$. Therefore

$$\lim_{t\to\infty}\frac{\mathcal{J}(t)}{\mathcal{H}(t)} = \lim_{t\to\infty}\frac{\psi_N(t)}{\phi_N(t)} = \frac{N}{2}, \tag{5.14}$$

by (5.10). Thus, if we denote $d =$ deg $\phi_N(t)$, then $d =$ deg $\psi_N(t)$ and from (5.13), we see that

$$\lim_{t\to\infty}\frac{\mathcal{H}(t)}{t^d e^{-Nt}} \quad \text{and} \quad \lim_{t\to\infty}\frac{\mathcal{J}(t)}{t^d e^{-Nt}} \quad \text{exist and are non-zero.} \tag{5.15}$$

A priori, it may happen that the limit number $N/2$ is not an integer and hence is not an eigenvalue of A. However, a more careful analysis will restrict the value $N/2$ to integers only. In order to prove this, it is convenient to introduce the following definition.

Definition 5.5. For $r \in \mathbb{N} \cup \{0\}$, we denote by F_r the set of functions $\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{f}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}$ belonging to $\mathcal{V}$ such that $\mathbf{f}_\mathbf{k} = 0$ whenever $|\mathbf{k}|^2 \not\equiv r(\text{mod } 2)$.

It is clear that
(a) for each $r = 0, 1, 2, \dots, F_r$ is a linear space and $R_n H \subset F_n$ for all $n \in \mathbb{N}$;
(b) $F_r = F_s$ if $r \equiv s(\text{mod } 2)$;
(c) $\mathfrak{C}F_r \subset F_r$, $R_n F_r \subset F_r$, $(I - R_n)F_r \subset F_r$, and $[(A - n)(I - R_n)]^{-1}F_r \subset F_r$ (see formula (4.10));
(d) if $u(t), v(t) \in F_r$ for all $t > 0$ then the integral $\int_0^t u(\tau)\,d\tau$ and the derivatives $d^n v(t)/dt^n$, for $n = 0, 1, 2, 3, \dots$ and $t > 0$ are also in F_r whenever these quantities are defined;
(e) if $u \in F_r$, $v \in F_s$, and $r \not\equiv s(\text{mod } 2)$ then $\langle u, v\rangle = 0$.

The action of the non-linear term $B(\cdot,\cdot)$ on the classes F_r is given by the following lemma.

Lemma 5.6. *For any $r, s \in \mathbb{N} \cup \{0\}$, $B(F_r, F_s) \subset F_{r+s}$.*

Proof. Let $u = \sum_{\mathbf{k}\in\mathbb{Z}^3} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in F_r$, $v = \sum_{\mathbf{l}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{l}\cdot\mathbf{x}} \in F_s$ and

$$B(u,v) = \sum_{\mathbf{m}\in\mathbb{Z}^3} B_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}}.$$

Then an elementary computation shows that

$$B_0 = 0 \quad \text{and} \quad \text{for } \mathbf{m}\neq 0, \quad B_{\mathbf{m}} = Q_{\mathbf{m}} - \frac{\mathbf{m}\cdot Q_{\mathbf{m}}}{|\mathbf{m}|^2}\mathbf{m}, \quad \text{where } Q_{\mathbf{m}} = i\sum_{\mathbf{k}+\mathbf{l}=\mathbf{m}} (u_{\mathbf{k}}\cdot\mathbf{l})v_{\mathbf{l}}.$$

Therefore, if $B_{\mathbf{m}}\neq 0$, there must exist $\mathbf{k},\mathbf{l}\in\mathbb{Z}^3$ such that $u_{\mathbf{k}}\neq 0$, $v_{\mathbf{l}}\neq 0$ and $\mathbf{m}=\mathbf{k}+\mathbf{l}$. According to Definition 5.5, these $\mathbf{k}$ and $\mathbf{l}$ satisfy $|\mathbf{k}|^2\equiv r(\text{mod } 2)$ and $|\mathbf{l}|^2\equiv s(\text{mod } 2)$. Hence

$$|\mathbf{m}|^2 = |\mathbf{k}+\mathbf{l}|^2 = |\mathbf{k}|^2+|\mathbf{l}|^2+2\mathbf{k}\cdot\mathbf{l} \equiv (|\mathbf{k}|^2+|\mathbf{l}|^2)(\text{mod } 2) \equiv (r+s)(\text{mod } 2).$$

This means that $B_{\mathbf{m}}\neq 0$ implies $|\mathbf{m}|^2\equiv(r+s)(\text{mod } 2)$, and thus $B(u,v)\in F_{r+s}$. □

We can now establish a relation between the polynomials $q_n(t)$ in (5.1) and the classes F_r.

PROPOSITION 5.7. *For all $n\in\mathbb{N}$ and $t>0$, $q_n(t)\in F_n$.*

Proof. We will prove this by induction. Since $q_1=\xi_1\in R_1H$, we have $q_1\in F_1$ by (a). Let $n>1$ and suppose that the statement holds for all q_j with $j<n$. Consider $q_n(t)$ given by (4.9):

$$q_n(t) = \xi_n - \int_0^t R_n\beta_n(\tau)\,d\tau + \sum_{h\geqslant 0}(-1)^{h+1}[(A-n)(I-R_n)]^{-h-1}\left(\frac{d}{dt}\right)^h (I-R_n)\beta_n.$$

Note that $\xi_n, R_n\beta_n\in R_nH$; hence they belong to F_n. Furthermore, if $r+s=n$, then $q_r\in F_r$ and $q_s\in F_s$. Hence $B(q_r,q_s)\in F_{r+s}=F_n$, by Lemma 5.6. Thus $\beta_n=\sum_{r+s=n}B(q_r,q_s)$ is in F_n. Thanks to (c) and (d),

$$\int_0^t R_n\beta_n(\tau)\,d\tau \quad\text{and}\quad [(A-n)(I-R_n)]^{-h-1}\left(\frac{d}{dt}\right)^h(I-R_n)\beta_n, \quad \text{for } h=0,1,2,\dots,$$

are in F_n. In conclusion, $q_n(t)$ is a sum of F_n-functions. Therefore $q_n(t)\in F_n$, due to (a). □

We are ready now to prove the following supplementary property of the expansions (5.4) and (5.6).

PROPOSITION 5.8. *For odd numbers n, the polynomials ϕ_n and ψ_n are identically zero.*

Proof. Suppose that n is odd, and consider $\phi_n=\sum_{j+l=n}\langle\mathfrak{C}q_j,q_l\rangle$. For each term of the sum, Lemma 5.7 implies that $q_l\in F_l$ and $q_j\in F_j$ and hence $\mathfrak{C}q_j\in F_j$ by virtue of (c). Since $j+l=n$ is odd or equivalently $j\not\equiv l(\text{mod } 2)$, property (e) yields $\langle\mathfrak{C}q_j,q_l\rangle=0$. It follows that $\phi_n=0$. A similar argument applies to the polynomials ψ_n. □

The preceding proposition has the following direct consequence.

COROLLARY 5.9. *The integer N in (5.12) is even.*

The above discussion establishes Theorem 3.1 in a particular case. Indeed, we can sum up that discussion as follows.

Proposition 5.10. *Let $u(t)$ be a regular solution of* (2.17) *such that its helicity $\mathcal{H}(t)$ has a non-zero asymptotic expansion. Then there exist integers $d \geqslant 0$ and $h_0 > 0$ such that*

$$\lim_{t\to\infty} \frac{\mathcal{J}(t)}{\mathcal{H}(t)} = h_0 \tag{5.16}$$

and

$$\lim_{t\to\infty} \frac{\mathcal{H}(t)e^{2h_0 t}}{t^d} \quad \text{exists and is non-zero.} \tag{5.17}$$

Proof. Set $N = 2h_0$ in (5.12). Then (5.16) coincides with (5.14), while (5.17) is already contained in (5.15). Finally, the fact that h_0 is an integer was established in Corollary 5.9. □

Clearly, to conclude the proof of Theorem 3.1, it suffices to establish the following.

Proposition 5.11. *Let $u(t)$, with $t \in [0,\infty)$, be a regular solution of* (2.17). *Then the asymptotic expansion* (5.4) *of its helicity $\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle$ is identically zero if and only if $\mathcal{H}(t)$ is identically zero.*

We postpone the proof of Proposition 5.11 to § 8 since it involves some finely tuned properties of the regular solutions which are of independent interest. However, Theorem 3.2 is a consequence of Proposition 5.11 (and hence of Theorem 3.1) as will be shown in the next section.

6. *On the flows with zero helicity*

In this section, we will prove that the set $\mathcal{R}_0$ of all initial data in $\mathcal{R}$ for which the solutions have identically zero helicity is a closed, rich but nowhere dense subset of $\mathcal{R}$ endowed with the topology of V.

First it will be useful to give formulas for the helicity $\langle \mathfrak{C}u, u\rangle$ and the non-linear term $B(u,u)$ (see equation (2.11)) in terms of Fourier coefficients of u. Recall that for $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{C}^3$, we define $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, $\mathbf{a}^* = (\bar{a_1}, \bar{a_2}, \bar{a_3})$ and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}^*}$. Again, we use $|\cdot|$ for the length of vectors in $\mathbb{C}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$, but in each case the context clarifies the precise meaning of this notation. The same holds true for the following abuse of notation. The exponential $e^{i\mathbf{k}\cdot\mathbf{x}}$ will denote both the function of $\mathbf{x}$ viewed as an element of $L^2(\Omega)$ as well as its value at point $\mathbf{x}$. With this notation, every $u \in \mathcal{V}$ has a Fourier series of the form

$$u = \sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.1}$$

where $a_{\mathbf{k}} = a^1_{\mathbf{k}} + ia^2_{\mathbf{k}} \in \mathbb{C}^3$, $a_{\mathbf{k}}\cdot\mathbf{k} = 0$, $a_{-\mathbf{k}} = a^*_{\mathbf{k}}$, $a_0 = 0$ and $a_{\mathbf{k}} \neq 0$ for only finitely many vectors $\mathbf{k}$ of $\mathbb{Z}^3$. Also if $v = \sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ is an $\mathbb{R}^3$-valued trigonometric polynomial (that is, if $v_{-\mathbf{k}} = v^*_{\mathbf{k}}$, $v_0 = 0$ and $v_{\mathbf{k}} \neq 0$ for only finitely many $\mathbf{k}$) then the Leray projection of v is simply given by

$$P\left(\sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\right) = \sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{where } a_0 = 0 \text{ and } a_{\mathbf{k}} = v_{\mathbf{k}} - \frac{v_{\mathbf{k}}\cdot\mathbf{k}}{|\mathbf{k}|^2}\mathbf{k} \text{ for } \mathbf{k} \neq 0. \tag{6.2}$$

We apply the curl and Stokes operators to (6.1) term by term and obtain

$$\mathfrak{C}u = \sum_{\mathbf{k}\in\mathbb{Z}^3} i\mathbf{k}\times a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.3}$$

$$Au = \sum_{\mathbf{k}\in\mathbb{Z}^3} |\mathbf{k}|^2 a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.4}$$

Furthermore, for $u=\sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$ and $v=\sum_{\mathbf{k}\in\mathbb{Z}^3} b_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$ in $\mathcal{V}$, the Fourier series of the bilinear mapping $B(u,v)$ is

$$B(u,v)=P\left(\sum_{\mathbf{n}\in\mathbb{Z}^3} Q_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}\right)=\sum_{\mathbf{n}\in\mathbb{Z}^3} B_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}, \tag{6.5}$$

where $\sum_{\mathbf{n}\in\mathbb{Z}^3} Q_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}$ is an $\mathbb{R}^3$-valued trigonometric polynomial since

$$Q_{\mathbf{n}}=i\sum_{\mathbf{k}+\mathbf{j}=\mathbf{n}}(a_{\mathbf{k}}\cdot\mathbf{j})b_{\mathbf{j}} \quad \text{for } \mathbf{n}\in\mathbb{Z}^3. \tag{6.6}$$

Therefore

$$B_0=0 \quad \text{and} \quad B_{\mathbf{n}}=Q_{\mathbf{n}}-\frac{Q_{\mathbf{n}}\cdot\mathbf{n}}{|\mathbf{n}|^2}n \quad \text{for } \mathbf{n}\neq 0. \tag{6.7}$$

We also have

$$\langle\mathfrak{C}u,v\rangle=iL^3\sum_{\mathbf{k}\in\mathbb{Z}^3}(\mathbf{k}\times a_{\mathbf{k}})\cdot b_{\mathbf{k}}^*=iL^3\sum_{\mathbf{k}\in\mathbb{Z}^3}\mathbf{k}\cdot(a_{\mathbf{k}}\times b_{\mathbf{k}}^*). \tag{6.8}$$

In particular, when $u=v$ this becomes

$$\langle\mathfrak{C}u,u\rangle=iL^3\sum_{\mathbf{k}\in\mathbb{Z}^3}\mathbf{k}\cdot[(a_{\mathbf{k}}^1+ia_{\mathbf{k}}^2)\times(a_{\mathbf{k}}^1-ia_{\mathbf{k}}^2)]=2L^3\sum_{\mathbf{k}\in\mathbb{Z}^3}\mathbf{k}\cdot(a_{\mathbf{k}}^1\times a_{\mathbf{k}}^2). \tag{6.9}$$

Remark 6.1. The general term of the last sum in (6.9) vanishes if $a_{\mathbf{k}}^1\times a_{\mathbf{k}}^2=0$. This happens when $a_{\mathbf{k}}^2=0$ or $a_{\mathbf{k}}^1=0$ or $a_{\mathbf{k}}^1$ is parallel with $a_{\mathbf{k}}^2$. In particular, if $a_{\mathbf{k}}\in z_{\mathbf{k}}\mathbb{R}^3$ for all $\mathbf{k}$ with some $z_{\mathbf{k}}\in\mathbb{C}$, then $\langle\mathfrak{C}u,u\rangle=0$. We often use the case $a_{\mathbf{k}}\in\mathbb{R}^3$, for all $\mathbf{k}\in\mathbb{Z}^3$, or $a_{\mathbf{k}}\in(i\mathbb{R}^3)$, for all $\mathbf{k}\in\mathbb{Z}^3$, in our examples in §§6 and 7 below.

Remark 6.2. Let u and v in $\mathcal{V}$ be such that whenever $a_{\mathbf{k}}\neq 0$ and $b_{\mathbf{j}}\neq 0$, the vectors $\mathbf{k}$ and $\mathbf{j}$ are parallel or $a_{\mathbf{k}}$ and $b_{\mathbf{j}}$ are (complex) collinear. For such $\mathbf{k}$ and $\mathbf{j}$, $a_{\mathbf{k}}\cdot\mathbf{j}=0$; hence $Q_{\mathbf{n}}$ given in (6.6) is zero and so is $B_{\mathbf{n}}$. Therefore, $B(u,v)=0$. Also, if $\mathfrak{C}u=\mu u$ for some $\mu\in\mathbb{R}$, then by the relation (2.13), the non-linear term $B(u,u)=0$.

We give here a simple description of eigenspaces of the curl operator $\mathfrak{C}$. Suppose that $u=\mathbf{u}(\cdot)$ is an eigenfunction of $\mathfrak{C}$ corresponding to some eigenvalue μ. Then

$$\mathfrak{C}u=\mu u \quad \text{and} \quad Au=\mathfrak{C}^2u=\mu^2u.$$

Thus μ^2 is an eigenvalue of A and u is also an eigenfunction of A corresponding to μ^2. Therefore $u\in R_{\mu^2}H$ and has the Fourier expansion of the form

$$u=\sum_{|\mathbf{k}|=|\mu|}a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad a_{\mathbf{k}}=a_{\mathbf{k}}^1+ia_{\mathbf{k}}^2\in\mathbb{C}^3, \quad a_{\mathbf{k}}\cdot\mathbf{k}=0, \quad a_{-\mathbf{k}}=a_{\mathbf{k}}^*. \tag{6.10}$$

Moreover, from (6.3) it follows that

$$i\mathbf{k}\times a_{\mathbf{k}}=\mu a_{\mathbf{k}} \quad \text{or} \quad i\mathbf{k}\times(a_{\mathbf{k}}^1+ia_{\mathbf{k}}^2)=\mu(a_{\mathbf{k}}^1+ia_{\mathbf{k}}^2).$$

This is equivalent to

$$\mathbf{k}\times a_{\mathbf{k}}^1=\mu a_{\mathbf{k}}^2, \quad \mathbf{k}\times a_{\mathbf{k}}^2=-\mu a_{\mathbf{k}}^1$$

or

$$a_{\mathbf{k}}^2=\frac{\mathbf{k}}{\mu}\times a_{\mathbf{k}}^1, \quad a_{\mathbf{k}}^1=-\frac{\mathbf{k}}{\mu}\times a_{\mathbf{k}}^2. \tag{6.11}$$

Obviously, (6.11) holds if and only if either $a_{\mathbf{k}}^1=a_{\mathbf{k}}^2=0$, or $|a_{\mathbf{k}}^1|=|a_{\mathbf{k}}^2|\neq 0$ and

$$\{\mathbf{k}/\mu, a_{\mathbf{k}}^1/|a_{\mathbf{k}}^1|, a_{\mathbf{k}}^2/|a_{\mathbf{k}}^2|\} \text{ is a positively oriented orthonormal basis of } \mathbb{R}^3.$$

The orientation in $\mathbb{R}^3$ considered here is that of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We sum up this discussion with the following result.

LEMMA 6.3. *The spectrum of the curl operator $\mathfrak{C}$ is*

$$\sigma(\mathfrak{C}) = \{\pm\sqrt{n} : n \in \sigma(A)\} = \{\pm|\mathbf{k}| : \mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}\}. \tag{6.12}$$

Moreover, the eigenfunctions of $\mathfrak{C}$ corresponding to an eigenvalue μ are the functions of the form (6.10) *such that non-zero coefficients $a_\mathbf{k} = a^1_\mathbf{k} + ia^2_\mathbf{k}$ in* (6.10) *satisfy*

(i) $|a^1_\mathbf{k}| = |a^2_\mathbf{k}| \neq 0$ *and*
(ii) $\{\mathbf{k}/\mu,\ a^1_\mathbf{k}/|a^1_\mathbf{k}|,\ a^2_\mathbf{k}/|a^2_\mathbf{k}|\}$ *is a positively oriented orthonormal basis of* $\mathbb{R}^3$.

We now establish some properties of the sets $\mathcal{R}_0$ and $\mathcal{R} \setminus \mathcal{R}_0$ in the following proposition which is essentially the same as Theorem 3.2 with $\mathcal{R}_1$ being replaced by $\mathcal{R} \setminus \mathcal{R}_0$ here.

PROPOSITION 6.4. *The set $\mathcal{R} \setminus \mathcal{R}_0$ is open in V and dense in $\mathcal{R}$, while $\mathcal{R}_0$ is closed and consists of an infinite union of invariant closed linear manifolds of infinite dimension.*

Proof. First, let us recall that

$$\mathcal{R} \text{ is open in } V. \tag{6.13}$$

The reason is that for any $u_0 \in \mathcal{R}$, we know that $\|u(t)\| = O(e^{-n_0 t})$ as $t \to \infty$ for some $n_0 > 0$. Therefore $\int_0^\infty \|u(t)\|^4\, dt < \infty$, and hence, there is a neighborhood of u_0 in V which is also a subset of $\mathcal{R}$ (see, for example, [**22**]).

We now show that $\mathcal{R}_0$ is closed in $\mathcal{R}$. Suppose that $u_0 \in \mathcal{R}$ and there is a sequence $\{u^n_0\}_{n\in\mathbb{N}}$ in $\mathcal{R}_0$ such that $\lim_{n\to\infty} \|u^n_0 - u_0\| = 0$. Let $u(t)$ and $u^n(t)$ be the regular solutions of (2.17) with initial data u_0 and, respectively, u^n_0. Again, in our case of potential forces,

$$\int_0^\infty \|u(t)\|^4\, dt < \infty. \tag{6.14}$$

For large n, the norm difference $\|u^n_0 - u_0\|$ is so small that we have (cf. [**22**])

$$\|u^n(t) - u(t)\| \leqslant \|u^n_0 - u_0\| e^{-C_1 t/2} e^{M(u_0)C_2},$$

for some constants $C_1, C_2 > 0$ independent of n. Thus for each $t > 0$,

$$\lim_{n\to\infty} \|u^n(t) - u(t)\| = 0.$$

Since $u^n_0 \in \mathcal{R}_0$, $\langle \mathfrak{C}u^n(t), u^n(t)\rangle = 0$ for all $t \geqslant 0$. For each $t > 0$,

$$\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle = \lim_{n\to\infty} \langle \mathfrak{C}u^n(t), u^n(t)\rangle = 0.$$

This implies that $u_0 \in \mathcal{R}_0$, and hence $\mathcal{R}_0$ is closed in $\mathcal{R}$. Since $\mathcal{R}_0$ is closed in $\mathcal{R}$, its complement $\mathcal{R} \setminus \mathcal{R}_0$ is open in $\mathcal{R}$ and hence (due to (6.13)) in V.

We next prove that $\mathcal{R} \setminus \mathcal{R}_0$ is dense in $\mathcal{R}$. Given $u_0 \in \mathcal{R}_0$, the corresponding solution $u(t)$ of (2.17) satisfies $\langle \mathfrak{C}u(t), u(t)\rangle = 0$ for all $t \geqslant 0$, and in particular, $\langle \mathfrak{C}u_0, u_0\rangle = 0$. If $u_0 = 0$, we can take $u^\varepsilon_0 = \varepsilon v_0$ where $\varepsilon > 0$ and $\mathfrak{C}v_0 = \mu v_0 \neq 0$ with some $\mu \in \sigma(\mathfrak{C})$ (see Lemma 6.3). The solution of (2.17) with initial condition u^ε_0 is $u^\varepsilon(t) = \varepsilon e^{-\mu^2 t} v_0$. The helicity is

$$\mathcal{H}_\varepsilon(t) = \langle \mathfrak{C}u^\varepsilon(t), u^\varepsilon(t)\rangle = \varepsilon\mu|v_0|^2 e^{-2\mu^2 t} \neq 0.$$

Hence $u^\varepsilon_0 \in \mathcal{R} \setminus \mathcal{R}_0$ and $\|u^\varepsilon_0 - u_0\| = \|u^\varepsilon_0\| = \varepsilon\|v_0\|$ which goes to zero as ε goes to zero.

In the case when $u_0 \neq 0$, then $u_0 = \mathbf{u}_0(\cdot)$ has the Fourier series

$$u_0 = \sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in V, \quad \text{where } a_{\mathbf{k}} = \frac{1}{L^3}\int_Q \mathbf{u}_0(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\,dx$$

and there is $\mathbf{k}^0 \in \mathbb{Z}^3 \setminus \{0\}$ such that $a_{\mathbf{k}^0} \neq 0$. Given $\varepsilon > 0$, consider the perturbed initial condition

$$u_0^\varepsilon = u_0 + (\mathbf{b}e^{i\mathbf{k}^0\cdot\mathbf{x}} + \mathbf{b}^* e^{-i\mathbf{k}^0\cdot\mathbf{x}}),$$

where $\mathbf{b} = \mathbf{b}(\varepsilon) = \mathbf{b}^1 + i\mathbf{b}^2$ with $\mathbf{b}^1, \mathbf{b}^2 \in \mathbb{R}^3$ specified later satisfying $\mathbf{k}^0 \cdot \mathbf{b} = 0$. By denoting $a^1_{\mathbf{k}} = (a_{\mathbf{k}} + a^*_{\mathbf{k}})/2$ and $a^2_{\mathbf{k}} = (a_{\mathbf{k}} - a^*_{\mathbf{k}})/2i$ for all $\mathbf{k} \in \mathbb{Z}^3$, we infer from (6.9) that

$$\begin{aligned}\mathcal{H}_\varepsilon(0) = \langle \mathfrak{C}u_0^\varepsilon, u_0^\varepsilon\rangle &= 2L^3 \sum_{\mathbf{k}\neq\mathbf{k}^0,-\mathbf{k}^0} \mathbf{k}\cdot(a^1_{\mathbf{k}} \times a^2_{\mathbf{k}}) + 4L^3\mathbf{k}^0 \cdot (a^1_{\mathbf{k}^0} + \mathbf{b}^1) \times (a^2_{\mathbf{k}^0} + \mathbf{b}^2) \\ &= \langle \mathfrak{C}u_0, u_0\rangle + 4L^3\mathbf{k}^0 \cdot (a^1_{\mathbf{k}^0} \times \mathbf{b}^2 + \mathbf{b}^1 \times a^2_{\mathbf{k}^0} + \mathbf{b}^1 \times \mathbf{b}^2) \\ &= 4L^3\mathbf{k}^0 \cdot (a^1_{\mathbf{k}^0} \times \mathbf{b}^2 + \mathbf{b}^1 \times a^2_{\mathbf{k}^0} + \mathbf{b}^1 \times \mathbf{b}^2).\end{aligned}$$

If $a^1_{\mathbf{k}^0} \neq 0$, we choose $\mathbf{b}^1 = 0$ and $\mathbf{b}^2 = \varepsilon\mathbf{k}^0 \times a^1_{\mathbf{k}^0}$. Then $|\mathbf{b}| = \varepsilon|\mathbf{k}^0||a^1_{\mathbf{k}^0}| \leqslant \varepsilon|\mathbf{k}^0||a_{\mathbf{k}^0}|$,

$$a^1_{\mathbf{k}^0} \times \mathbf{b}^2 + \mathbf{b}^1 \times a^2_{\mathbf{k}^0} + \mathbf{b}^1 \times \mathbf{b}^2 = a^1_{\mathbf{k}^0} \times \mathbf{b}^2 = \varepsilon\mathbf{k}^0|a^1_{\mathbf{k}^0}|^2 - a^1_{\mathbf{k}^0}(\mathbf{k}^0 \cdot a^1_{\mathbf{k}^0}) = \varepsilon\mathbf{k}^0|a^1_{\mathbf{k}^0}|^2$$

and

$$\mathcal{H}_\varepsilon(0) = 4L^3\varepsilon|\mathbf{k}^0|^2|a^1_{\mathbf{k}^0}|^2 \neq 0.$$

If $a^1_{\mathbf{k}^0} = 0$, we must have $a^2_{\mathbf{k}^0} \neq 0$, and consequently we can choose $\mathbf{b}^2 = 0$ and $\mathbf{b}^1 = \varepsilon\mathbf{k}^0 \times a^2_{\mathbf{k}^0}$. Then $|\mathbf{b}| = \varepsilon|\mathbf{k}^0||a^2_{\mathbf{k}^0}| \leqslant \varepsilon|\mathbf{k}^0||a_{\mathbf{k}^0}|$,

$$a^1_{\mathbf{k}^0} \times \mathbf{b}^2 + \mathbf{b}^1 \times a^2_{\mathbf{k}^0} + \mathbf{b}^1 \times \mathbf{b}^2 = \mathbf{b}^1 \times a^2_{\mathbf{k}^0} = -\varepsilon\mathbf{k}^0|a^2_{\mathbf{k}^0}|^2$$

and

$$\mathcal{H}_\varepsilon(0) = -4L^3\varepsilon|\mathbf{k}^0|^2|a^2_{\mathbf{k}^0}|^2 \neq 0.$$

Therefore, in both cases (that is, regardless of whether $a^1_{\mathbf{k}^0}$ is zero or not), we have

$$\mathcal{H}_\varepsilon(0) \neq 0 \quad \text{and} \quad |\mathbf{b}| \leqslant \varepsilon|\mathbf{k}^0||a_{\mathbf{k}^0}| \quad \text{for all } \varepsilon > 0. \tag{6.15}$$

Moreover, for the difference between the two initial data we have

$$\|u_0^\varepsilon - u_0\|^2 \leqslant 2L^3|\mathbf{k}^0|^2|\mathbf{b}|^2 \leqslant 2\varepsilon^2|\mathbf{k}^0|^4|a_{\mathbf{k}^0}|^2 \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.16}$$

From (6.13) and (6.16), we have $u_0^\varepsilon \in \mathcal{R}$ for ε small enough. Hence the corresponding perturbed solution $u^\varepsilon(t)$ exists for all time $t \geqslant 0$ and the first property in (6.15) shows that $u_0^\varepsilon \in \mathcal{R}\setminus\mathcal{R}_0$. We have proved that in any neighborhood (in V) of $u_0 \in \mathcal{R}_0$, there exists some $u_0^\varepsilon \in \mathcal{R}\setminus\mathcal{R}_0$. Therefore $\mathcal{R}\setminus\mathcal{R}_0$ is a dense subset of $\mathcal{R}$.

To complete our proof we must show that $\mathcal{R}_0$ contains many infinite-dimensional invariant linear manifolds. We will present three different types of such manifolds. Our first example is the class of 2D solutions to the 3D Navier–Stokes equations. We define

$$\mathcal{M}^{2D}_{1,2} = \left\{ u \in V : u \sim \sum_{\mathbf{k}\in\mathbb{Z}\times\mathbb{Z}\times\{0\}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ a_{\mathbf{k}} \in \mathbb{C}\times\mathbb{C}\times\{0\} \right\}, \tag{6.17}$$

where $\mathbb{Z}\times\mathbb{Z}\times\{0\} = \{(k_1, k_2, 0) : k_1, k_2 \in \mathbb{Z}\}$ and $\mathbb{C}\times\mathbb{C}\times\{0\} = \{(z_1, z_2, 0) : z_1, z_2 \in \mathbb{C}\}$. From the theory of 2D Navier–Stokes equations, for $u_0 \in \mathcal{M}^{2D}_{1,2}$ the corresponding regular solution $u(t) = \mathbf{u}(\cdot, t)$ exists for all time $t \geqslant 0$ and belongs to $\mathcal{M}^{2D}_{1,2}$; hence $\mathcal{M}^{2D}_{1,2}$ is invariant. Note that in this case $\nabla \times \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}\mathbf{e}_3$ which is orthogonal to $\mathbf{u}(\mathbf{x}, t) \in (\mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_2)$ as vectors in $\mathbb{R}^3$. Therefore,

$$\mathcal{H}(t) = \int_\Omega (\nabla \times \mathbf{u}(\mathbf{x}, t)) \cdot \mathbf{u}(\mathbf{x}, t)\,dx = 0, \quad \text{for } t \geqslant 0,$$

and hence $\mathcal{M}^{2D}_{1,2} \subset \mathcal{R}_0$. We can also equally define the other two similar manifolds, namely,

$$\mathcal{M}^{2D}_{1,3} = \left\{ u \in V : u \sim \sum_{\mathbf{k} \in \mathbb{Z}\times\{0\}\times\mathbb{Z}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\, a_{\mathbf{k}} \in \mathbb{C}\times\{0\}\times\mathbb{C} \right\} \tag{6.18}$$

and

$$\mathcal{M}^{2D}_{2,3} = \left\{ u \in V : u \sim \sum_{\mathbf{k} \in \{0\}\times\mathbb{Z}\times\mathbb{Z}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\, a_{\mathbf{k}} \in \{0\}\times\mathbb{C}\times\mathbb{C} \right\}. \tag{6.19}$$

Then $\mathcal{M}^{2D}_{1,3}$ and $\mathcal{M}^{2D}_{2,3}$ are invariant submanifolds of $\mathcal{R}_0$.

The second example is an extension of the family of manifolds given in Remark 7 of [**5**]. Let $\mathbf{a}$ be a vector in $\mathbb{R}^3$ such that its orthogonal plane has non-trivial intersection with $\mathbb{Z}^3$; this means that

$$\mathbf{a}^{\perp} := \{\mathbf{k} \in \mathbb{Z}^3 : \mathbf{k}\cdot\mathbf{a} = 0\} \neq \{0\}.$$

Define the linear manifold $\mathcal{M}_{\mathbf{a}^\perp}$ in V by

$$\mathcal{M}_{\mathbf{a}^\perp} = \left\{ u \in V : u = \sum_{\mathbf{k}\in\mathbf{a}^\perp} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ a_{\mathbf{k}} \text{ is (complex) collinear to } \mathbf{a}, \text{ for all } \mathbf{k}\in\mathbf{a}^\perp \right\}. \tag{6.20}$$

For $u_0 \in \mathcal{M}_{\mathbf{a}^\perp}$, $u(t) = e^{-tA}u_0$ belongs to $\mathcal{M}_{\mathbf{a}^\perp}$ for all $t \in [0,\infty)$ and solves the linearized Navier–Stokes equations

$$\begin{cases} \dfrac{du}{dt} + Au = 0, & \text{for } t > 0, \\ u(0) = u_0 \in V, \end{cases} \tag{6.21}$$

as well as the Navier–Stokes equations (2.17) (the non-linear term $B(u,u)$ vanishes according to Remark 6.2). Also, in view of Remark 6.1, $\langle \mathfrak{C}u, u\rangle = 0$ for all $u \in \mathcal{M}_{\mathbf{a}^\perp}$. Hence

$$\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle = 0 \quad \text{for all } t \geqslant 0.$$

Therefore, $u_0 \in \mathcal{R}_0$ and $\mathcal{M}_{\mathbf{a}^\perp}$ is an invariant linear manifold in $\mathcal{R}_0$.

Before moving on to the next manifold, let us state a simple lemma on closed linear subspaces of H and invariant sets. □

Lemma 6.5. *Let X be a closed linear subspace of H such that $P_n u \in X$ for all $u \in X$ and $n \in \mathbb{N}$, where $P_n = R_1 + R_2 + \ldots + R_n$ and $Au, B(u,u) \in X$ for all $u \in X \cap \mathcal{V}$. Then $X \cap \mathcal{R}$ is invariant.*

Proof. Let u_0 be the initial condition in $X \cap \mathcal{R}$ and $u(t)$ be the corresponding regular solution of (2.17). For each $n \in \mathbb{N}$, we consider the Galerkin approximation of the Navier–Stokes equations on the finite-dimensional subspace $P_nX \subset \mathcal{V}$ with initial condition $P_n u_0 \in P_nX$ (see, for example, [**3**]). Since $Av, P_nB(v,v) \in P_nX$ for $v \in P_nX$, this approximation problem has a unique solution $u_n(t) \in P_nX$, with $t \in [0,T]$ for some $T > 0$. From the theory of Navier–Stokes equations, there is a subsequence $\{u_{n'}\}$ such that

$$\lim_{n'\to\infty} u_{n'}(t) = u(t) \quad \text{in } H \text{ for almost every } t \in (0,T). \tag{6.22}$$

Since X is closed in H, we have $u(t) \in X$ for almost every $t \in (0,T)$. By the strong continuity (in H) of the regular solution, $u(t) \in X$ for all $t \in [0,T)$, and because $u(t)$ is a global solution, we can take $T = \infty$. □

Proof of Proposition 6.4 (*continued*). The last manifold we present in this proof is the one formed by anti-symmetric (odd) functions $u \in V$. Namely, define

$$\begin{aligned} H_{\mathrm{odd}} &= \{u = \mathbf{u}(\cdot) \in H : \mathbf{u}(-\mathbf{x}) = -\mathbf{u}(\mathbf{x}), \text{ a.e. in } \mathbb{R}^3\} \\ &= \left\{ u \in H : u = \sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ a_{\mathbf{k}} \in i\mathbb{R}^3 \text{ for all } \mathbf{k} \in \mathbb{Z}^3 \right\}. \end{aligned} \tag{6.23}$$

For each $u \in H_{\mathrm{odd}} \cap \mathcal{V}$, using (6.4)–(6.7) one can verify that $Au, B(u,u) \in H_{\mathrm{odd}}$. It is also clear from (6.23) that H_{odd} is a closed linear subspace of H and $P_n u \in H_{\mathrm{odd}}$ for all $n \in \mathbb{N}$ and $u \in H_{\mathrm{odd}}$. By virtue of Lemma 6.5,

$$H_{\mathrm{odd}} \cap \mathcal{R} \text{ is invariant.} \tag{6.24}$$

Moreover, if $u \in H_{\mathrm{odd}} \cap V$, then it follows from (6.23) and Remark 6.1 that $\langle \mathfrak{C}u, u\rangle = 0$. Therefore, due to (6.24),

$$H_{\mathrm{odd}} \cap \mathcal{R} \subset \mathcal{R}_0. \tag{6.25}$$

Also, it is well known that for $\delta > 0$ small enough and $\|u_0\| < \delta$, the solution $u(t)$ exists for all times $t \geqslant 0$ and $d\|u(t)\|^2/dt \leqslant 0$ for $t \geqslant 0$. Therefore, $\|u(t)\|^2 \leqslant \|u_0\|^2 < \delta^2$, for $t \geqslant 0$ and hence $\mathcal{M}^\delta = \{u \in V : \|u\| < \delta\}$ is invariant. Together with (6.24) and (6.25) we infer that $\mathcal{M}^\delta_{\mathrm{odd}} = H_{\mathrm{odd}} \cap \mathcal{M}^\delta$ is a subset of $\mathcal{R}_0$ and invariant. $\square$

Remark 6.6. The manifolds $\mathcal{M}^{2D}_{1,2}$, $\mathcal{M}^{2D}_{1,3}$, $\mathcal{M}^{2D}_{2,3}$ and $\mathcal{M}_{\mathbf{a}^\perp}$ above are closed linear subspaces of V, while $\mathcal{M}^\delta_{\mathrm{odd}}$ is only a portion of the closed linear subspace $V \cap H_{\mathrm{odd}}$. Clearly, these manifolds are of infinite dimension.

Remark 6.7. Theorem 3.2 is a direct consequence of Propositions 6.4 and 5.11 (to be proved in § 8 as already indicated). Indeed, it is obvious that Proposition 5.11 implies that $\mathcal{R}_1 = \mathcal{R} \setminus \mathcal{R}_0$.

Taking into account Theorems 3.1 and 3.2 we now prove Proposition 3.3.

Proof of Proposition 3.3. Let $\mathcal{M} = \{u \in \mathcal{R} : \langle \mathfrak{C}u, u\rangle = 0\}$. Since $\mathcal{R}_0$ is clearly closed in $\mathcal{M}$, we only have to prove that $\mathcal{R}_0$ is nowhere dense in $\mathcal{M}$. In the contrary case, let $u_0 \in \mathcal{R}_0$ be such that

$$\mathcal{M} \cap \{v \in V : \|v - u_0\| < \varepsilon\} \subset \mathcal{R}_0, \tag{6.26}$$

for some $\varepsilon > 0$. Since $\mathcal{R}$ is open in V, we can also assume that $\{v \in V : \|v - u_0\| < \varepsilon\} \subset \mathcal{R}$. Let $u(t)$ be the regular solution with initial data u_0 and take $t_1 > 0$ such that $u_1 = u(t_1)$ satisfies $\|u_0 - u_1\| < \varepsilon/2$. Then $u_1 \in \mathcal{D}_A$ and $\langle \mathfrak{C}u_1, u_1\rangle = \langle \mathfrak{C}^2 u_1, \mathfrak{C}u_1\rangle = 0$. Note that $u_1 \in \mathcal{N}_0 \cap \mathcal{N}_1$, where

$$\mathcal{N}_0 = \{v \in \mathcal{D}_A : \langle \mathfrak{C}v, v\rangle = 0\} \quad \text{and} \quad \mathcal{N}_1 = \{v \in \mathcal{D}_A : \langle \mathfrak{C}^2 v, \mathfrak{C}v\rangle = 0\}.$$

Let $\mathcal{N} = \{v \in \mathcal{D}_A : \|v - u_1\| < \varepsilon/2\} \subset \mathcal{R}$. Then (6.26) implies that

$$\mathcal{N}_0 \cap \mathcal{N} \subset \mathcal{M} \cap \mathcal{N} \subset \mathcal{D}_A \cap \mathcal{R}_0 \subset \mathcal{N}_1. \tag{6.27}$$

Since the identity map from $\mathcal{D}_A$ to V is continuous, $\mathcal{N}$ is an open neighborhood of u_1 in $\mathcal{D}_A$. Therefore the tangent space $T_0 = \{w \in \mathcal{D}_A : \langle \mathfrak{C}w, u_1\rangle = 0\}$ of $\mathcal{N}_0$ at u_1 is a subset of the tangent space $T_1 = \{w \in \mathcal{D}_A : \langle \mathfrak{C}w, Au_1\rangle = 0\}$ of $\mathcal{N}_1$ at u_1. Hence $Au_1 = \lambda u_1$, for some eigenvalue λ (depending on u_1) of A. Let $u_2 = u_1 + v + w$ where $\mathfrak{C}v = \sqrt{k}\,v$ and $\mathfrak{C}w = -\sqrt{h}\,w$ with k and

h any fixed eigenvalues of A satisfying $\lambda < k < h$. We have

$$\langle \mathfrak{C}u_2, u_2\rangle = \langle \mathfrak{C}v, v\rangle + \langle \mathfrak{C}w, w\rangle = \sqrt{k}\,|v|^2 - \sqrt{h}\,|w|^2,$$

and

$$\langle \mathfrak{C}^2 u_2, \mathfrak{C}u_2\rangle = \langle \mathfrak{C}^2 v, \mathfrak{C}v\rangle + \langle \mathfrak{C}^2 w, \mathfrak{C}w\rangle = k^{3/2}|v|^2 - h^{3/2}|w|^2.$$

Since $h \neq k$, we can take small $|v|$ and $|w|$ so that $u_2 \in \mathcal{N}$ and

$$\sqrt{k}\,|v|^2 - \sqrt{h}\,|w|^2 = 0 \quad \text{and} \quad k^{3/2}|v|^2 - h^{3/2}|w|^2 \neq 0.$$

Thus $u_2 \in (\mathcal{N}_0 \cap \mathcal{N}) \setminus \mathcal{N}_1$ which contradicts (6.27). Therefore $\mathcal{R}_0$ is nowhere dense in $\mathcal{M}$. □

Remark 6.8. For $u_0 \in \mathcal{M} \setminus \mathcal{R}_0$, the analyticity in the time $t \in (0,\infty)$ of the helicity implies that the non-zero helicity is generated instantly right after the initial time.

7. *Rates of exponential decay of the helicity*

In this section we will discuss in more detail the possible values that N, h_0 and d (see Proposition 5.10) can take. This will give a clearer picture of the difference between the asymptotic behavior of the helicity and that of the energy of the flow. Recall that the helicity $\mathcal{H}(t) = \langle \mathfrak{C}u(t), u(t)\rangle$ has the asymptotic expansion (5.4). We consider the case when one of the polynomials $\phi_n(t)$ in this expansion is not identically zero. The previously obtained asymptotic properties of $\mathcal{H}(t)$ and $\mathcal{J}(t) = \langle \mathfrak{C}^2 u(t), \mathfrak{C}u(t)\rangle$ in this case are summarized in Proposition 5.10. We start by comparing the limits

$$h_0 = \lim_{t\to\infty} \frac{\mathcal{J}(t)}{\mathcal{H}(t)} \quad \text{in (5.16)}, \qquad \lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \in \sigma(A) \quad \text{in (3.1)},$$

and

$$\alpha_0 = \lim_{t\to\infty} \frac{\mathcal{H}(t)}{|u(t)|^2} \quad \text{in (3.5)}.$$

The last limit exists due to the asymptotic expansions of the helicity and energy (see the proof of Theorem 3.4 below). Note that

$$\frac{|\mathcal{H}(t)|}{|u(t)|^2} \leqslant \frac{|\mathfrak{C}u(t)||u(t)|}{|u(t)|^2} = \frac{\|u(t)\|}{|u(t)|} \to \sqrt{n_0} \quad \text{as } t \to \infty.$$

Hence $|\alpha_0| \leqslant \sqrt{n_0} \in \sigma(\mathfrak{C})$. In the sequel, we will present some samples of solutions with different relations between the limits h_0, n_0 and α_0. Due to these examples, Theorems 3.4 and 3.5 will have straightforward proofs. The first example will show that n_0 and h_0 can be any eigenvalue of A while α_0 can attain the eigenvalues $\pm\sqrt{n_0}$ of $\mathfrak{C}$.

Example 7.1. Let $n \in \sigma(A)$. Using Lemma 6.3, we can find $u_0^\pm \in R_n H \setminus \{0\}$ such that $\mathfrak{C}u_0^\pm = \pm\sqrt{n}\,u_0^\pm$. By (2.13) one has $B(u_0^\pm, u_0^\pm) = 0$ and the solution of (2.17) with the initial data $u_0^\pm$ is

$$u^\pm(t) = e^{-nt}u_0^\pm, \quad \text{for } t \geqslant 0.$$

Therefore

$$|u^\pm(t)|^2 = e^{-2nt}|u_0^\pm|^2, \qquad \|u^\pm(t)\|^2 = ne^{-2nt}|u_0^\pm|^2,$$
$$H^\pm(t) = \langle \mathfrak{C}u^\pm(t), u^\pm(t)\rangle = \pm\sqrt{n}\,e^{-2nt}\langle u_0^\pm, u_0^\pm\rangle = \pm\sqrt{n}\,|u^\pm(t)|^2$$

and

$$J^\pm(t) = \langle \mathfrak{C}^2 u^\pm(t), \mathfrak{C}u^\pm(t)\rangle = \pm n\sqrt{n}\,e^{-2nt}\langle u_0^\pm, u_0^\pm\rangle = nH^\pm(t).$$

Hence the corresponding limits $n_0^{\pm}$, $h_0^{\pm}$ and $\alpha_0{}^{\pm}$ above are given by $n_0{}^{\pm} = h_0{}^{\pm} = n \in \sigma(A)$ and $\alpha_0{}^{\pm} = \pm\sqrt{n} = \pm\sqrt{n_0{}^{\pm}} \in \sigma(\mathfrak{C})$.

The next example shows that there are initial data u_0 for which the limits h_0 are no longer in the spectrum of A although we always have $n_0 \in \sigma(A)$.

EXAMPLE 7.2. There is $u_0 \in \mathcal{R}$ such that the corresponding limit h_0 in (5.16) is 7 which does not belong to the spectrum of A.

Proof. *Part* I. We choose $\xi_4 \in R_4H$ and $\xi_5 \in R_5H$ such that $|\xi_4|$ and $|\xi_5|$ are very small satisfying

$$\langle \mathfrak{C}\xi_4, \xi_4\rangle = 0 = \langle \mathfrak{C}\xi_5, \xi_5\rangle \quad \text{and} \quad \langle \mathfrak{C}\xi_4, B(\xi_5, \xi_5)\rangle \neq 0. \tag{7.1}$$

A concrete choice of such ξ_4 and ξ_5 will be given in Part II of this proof. Let

$$\xi = (0, 0, 0, \xi_4, \xi_5, 0, 0, \dots, 0) \in \Pi_{14}S_A.$$

Since ξ is small, (4.5) implies that $\xi = \Pi_{14}W(u_0)$ for some $u_0 \in \mathcal{R}$. Trivial calculations using (4.8) and (4.9) yield

$$q_1 = 0, \quad \beta_1 = \beta_2 = 0, \quad q_2 = 0, \quad \beta_3 = 0, \quad q_3 = 0, \quad \beta_4 = 0, \tag{7.2}$$

$$q_4 = \xi_4, \quad \beta_5 = 0, \quad q_5 = \xi_5, \quad \beta_6 = 0, \tag{7.3}$$

$$q_6 = 0, \quad \beta_7 = 0, \quad q_7 = 0. \tag{7.4}$$

By Proposition 5.8, for odd integers n, the polynomials ϕ_n in the asymptotic expansion (5.4) of $\mathcal{H}(t)$ are identically zero. Concerning the polynomials ϕ_n for even n, from (7.2)–(7.4) we easily obtain

$$\phi_2 = \phi_4 = \phi_6 = 0,$$

as well as

$$\phi_8 = \langle \mathfrak{C}q_4, q_4\rangle + 2\sum_{k<4, k+l=8} \langle \mathfrak{C}q_k, q_l\rangle = \langle \mathfrak{C}q_4, q_4\rangle = \langle \mathfrak{C}\xi_4, \xi_4\rangle = 0, \tag{7.5}$$

and

$$\phi_{10} = \langle \mathfrak{C}q_5, q_5\rangle + 2\langle \mathfrak{C}q_4, q_6\rangle + 2\sum_{k<4, k+l=10} \langle \mathfrak{C}q_k, q_l\rangle = \langle \mathfrak{C}q_5, q_5\rangle = \langle \mathfrak{C}\xi_5, \xi_5\rangle = 0, \tag{7.6}$$

where we also used the condition (7.1). Since $\xi_4 \in R_4H$, the Fourier expansion (6.1) of $u = \xi_4$ is $\xi_4 = \sum_{|\mathbf{k}|=2} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$. Therefore, by (6.5), we have

$$\beta_8 = B(q_4, q_4) = B(\xi_4, \xi_4) = \sum_{\mathbf{m}} B_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}},$$

where the summation is over all $\mathbf{m} = \mathbf{k} + \mathbf{k}'$ for $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^3$ with $|\mathbf{k}| = |\mathbf{k}'| = 2$ and $\mathbf{k} \neq -\mathbf{k}'$. Moreover, for each $\mathbf{k}$ as above, Remark 6.2 implies that $Q_{2\mathbf{k}} = B_{2\mathbf{k}} = 0$. Therefore, $B_{\mathbf{m}} \neq 0$ only if $\mathbf{m}$ belongs to the set

$$\begin{aligned} E &= \{\mathbf{k} \pm \mathbf{k}' : \mathbf{k}, \mathbf{k}' \in \mathbb{Z}^3,\ |\mathbf{k}| = |\mathbf{k}'| = 2,\ \mathbf{k} \neq \pm\mathbf{k}'\} \\ &= \{(\epsilon 2, \epsilon' 2, 0), (0, \epsilon 2, \epsilon' 2), (\epsilon 2, 0, \epsilon' 2) : \epsilon, \epsilon' = 1, -1\}. \end{aligned}$$

Thus $\beta_8 = \sum_{\mathbf{m}\in E} B_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}}$ and $|\mathbf{m}|^2 = 8$ for each $\mathbf{m} \in E$; hence $\beta_8 \in R_8H$ and $q_8(t) = -t\beta_8$, by (4.9). Since $q_4 \in R_4H$ and $q_8 \in R_8H$, we have $\phi_{12} = 2\langle \mathfrak{C}q_4, q_8\rangle = 0$. From the relations (4.8), (4.9) and (4.10) one can readily infer that

$$\beta_9 = B(q_5, q_4) + B(q_4, q_5), \tag{7.7}$$

$$q_9(t) = -tR_9\beta_9 - [(A-9)(I-R_9)]^{-1}(I-R_9)\beta_9, \tag{7.8}$$

$$\beta_{10} = B(q_5, q_5) = B(\xi_5, \xi_5), \tag{7.9}$$

$$q_{10}(t) = -tR_{10}\beta_{10} - [(A-10)(I-R_{10})]^{-1}(I-R_{10})\beta_{10}, \tag{7.10}$$

and consequently

$$\phi_{14} = 2(\mathfrak{C}q_5, q_9) + 2(\mathfrak{C}q_4, q_{10}), \tag{7.11}$$

by (5.4). Now, using (7.8), (7.7), (4.10) and (4.12), we obtain

$$\begin{aligned}\langle \mathfrak{C}q_5, q_9\rangle &= \langle \mathfrak{C}\xi_5, -[(A-9)(I-R_9)]^{-1}(I-R_9)\beta_9\rangle \\ &= \langle \mathfrak{C}\beta_9, -[(A-9)(I-R_9)]^{-1}(I-R_9)\xi_5\rangle = \tfrac{1}{4}\langle \mathfrak{C}\xi_5, \beta_9\rangle \\ &= \tfrac{1}{4}\langle \mathfrak{C}\xi_5, B(\xi_4,\xi_5) + B(\xi_5,\xi_4)\rangle = -\tfrac{1}{4}\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle,\end{aligned}$$

and similarly,

$$\begin{aligned}\langle \mathfrak{C}q_4, q_{10}\rangle &= \langle \mathfrak{C}\xi_4, -[(A-10)(I-R_{10})]^{-1}(I-R_{10})\beta_{10}\rangle \\ &= \tfrac{1}{6}\langle \mathfrak{C}\xi_4, \beta_{10}\rangle = \tfrac{1}{6}\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle.\end{aligned}$$

Thus (7.11) finally gives us

$$\phi_{14} = 2\left(-\tfrac{1}{4} + \tfrac{1}{6}\right)\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle = -\tfrac{1}{6}\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle,$$

which is not zero by our condition (7.1). We conclude that the number N identified in (5.12) is 14 and it follows from (5.14) that

$$h_0 = \lim_{t\to\infty} \frac{\mathcal{J}(t)}{\mathcal{H}(t)} = N/2 = 7$$

which is not an eigenvalue of the Stokes operator A.

Part II. Fix $\varepsilon > 0$, let $\mathbf{k}_1 = (0,2,0)$ and $a_{\mathbf{k}_1} = \varepsilon(1,0,0)$ and define

$$\xi_4 = a_{\mathbf{k}_1}(e^{i\mathbf{k}_1\cdot\mathbf{x}} + e^{-i\mathbf{k}_1\cdot\mathbf{x}}) \in R_4H. \tag{7.12}$$

We choose $\mathbf{m}_1 = (2,1,0)$, $\mathbf{m}_2 = (-2,1,0)$, $b_{\mathbf{m}_1} = \varepsilon(-1,2,0)$, and $b_{\mathbf{m}_2} = \varepsilon(0,0,1)$ and let

$$\xi_5 = b_{\mathbf{m}_1}(e^{i\mathbf{m}_1\cdot\mathbf{x}} + e^{-i\mathbf{m}_1\cdot\mathbf{x}}) + b_{\mathbf{m}_2}(e^{i\mathbf{m}_2\cdot\mathbf{x}} + e^{-i\mathbf{m}_2\cdot\mathbf{x}}) \in R_5H. \tag{7.13}$$

Since the Fourier coefficients $a_{\mathbf{k}_1}$, $b_{\mathbf{m}_1}$ and $b_{\mathbf{m}_2}$ belong to $\mathbb{R}^3$, we have

$$\langle \mathfrak{C}\xi_4, \xi_4\rangle = 0 = \langle \mathfrak{C}\xi_5, \xi_5\rangle, \tag{7.14}$$

due to Remark 6.1. It is also obvious that $|\xi_4|$ and $|\xi_5|$ can be made as small as necessary in Part I by choosing $\varepsilon > 0$ suitably small. By writing $B(\xi_5,\xi_5) = \sum B_{\mathbf{m}} e^{i\mathbf{m}\cdot\mathbf{x}}$ and using (6.8) for $u = \xi_4$ and $v = B(\xi_5,\xi_5)$ we obtain

$$\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle = 2L^3\,\mathrm{Re}[i(\mathbf{k}_1 \times a_{\mathbf{k}_1})\cdot B^*_{\mathbf{k}_1}].$$

Note that $\mathbf{m}_1 + \mathbf{m}_2 = \mathbf{k}_1$ and, by (6.6) and (6.7),

$$B_{\mathbf{k}_1} = Q_{\mathbf{k}_1} - \frac{Q_{\mathbf{k}_1}\cdot\mathbf{k}_1}{|\mathbf{k}_1|^2}\mathbf{k}_1,$$

where

$$Q_{\mathbf{k}_1} = i[(b_{\mathbf{m}_1}\cdot\mathbf{m}_2)b_{\mathbf{m}_2} + (b_{\mathbf{m}_2}\cdot\mathbf{m}_1)b_{\mathbf{m}_1}].$$

Therefore,

$$\begin{aligned}\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle &= 2L^3\operatorname{Re}[i(\mathbf{k}_1\times a_{\mathbf{k}_1})\cdot Q^*_{\mathbf{k}_1}]\\ &= 2L^3(\mathbf{k}_1\times a_{\mathbf{k}_1})\cdot[(b_{\mathbf{m}_1}\cdot\mathbf{m}_2)b_{\mathbf{m}_2}+(b_{\mathbf{m}_2}\cdot\mathbf{m}_1)b_{\mathbf{m}_1}].\end{aligned}$$

Finally, we have $\mathbf{k}_1\times a_{\mathbf{k}_1} = -2\varepsilon(0,0,1)$, and consequently also

$$\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle = -4L^3\varepsilon^3(-1,2,0)\cdot(-2,1,0) = -16L^3\varepsilon^3.$$

Hence $\langle \mathfrak{C}\xi_4, B(\xi_5,\xi_5)\rangle$ is not zero. With this and (7.14), we conclude that ξ_4 and ξ_5 specified in (7.12) and, respectively, (7.13) also satisfy condition (7.1). □

As far as the asymptotic behaviors (for $t\to\infty$) of $\mathcal{J}(t)$, $\mathcal{H}(t)$ and $|u(t)|^2$ are concerned in the case $u_0\in\mathcal{R}_1$, we know from (5.16) of Proposition 5.10 that those of $\mathcal{H}(t)$ and $\mathcal{J}(t)$ are essentially the same. Also in the case when the limit α_0 in (3.5) is not zero, the same is true for the asymptotic behaviors of the helicity $\mathcal{H}(t)$ and the energy $|u(t)|^2$. We will pay more attention to the case $\alpha_0=0$, for $u_0\in\mathcal{R}_1$. We recall from (3.1) and (5.17) that as $t\to\infty$, $|u(t)|^2$ behaves like e^{-2n_0t}, where $n_0\in\sigma(A)$ while $\mathcal{H}(t)$ behaves like $t^de^{-2h_0t}$. Example 7.2 shows that we can find $u_0\in\mathcal{R}_1$ such that $\lim_{t\to\infty}\mathcal{H}(t)e^{2h_0t}$ exists and is not zero, whereas $h_0=7\notin\sigma(A)$. This means that the exponents in the exponential decay of $\mathcal{H}(t)$ and $|u(t)|^2$ are quite different. The next example shows that the difference goes farther than that. More precisely, the degree d in (5.17) can be positive.

Example 7.3. There is $u_0\in\mathcal{R}$ such that $\lim_{t\to\infty}\mathcal{H}(t)t^{-1}e^{4t}$ exists and is not zero, that is, $d=1>0$.

Proof. *Part* I. We choose $\xi_1\in R_1H$ and $\xi_2\in R_2H$ satisfying the following conditions

$$\langle \mathfrak{C}\xi_1,\xi_1\rangle = 0 \quad\text{and}\quad \langle \mathfrak{C}B(\xi_1,\xi_1), B(\xi_1,\xi_1)\rangle = 0, \tag{7.15}$$

$$\langle \mathfrak{C}\xi_2,\xi_2\rangle = 0, \tag{7.16}$$

and

$$\langle \mathfrak{C}\xi_2, B(\xi_1,\xi_1)\rangle \neq 0. \tag{7.17}$$

Again we will explicitly specify such ξ_1 and ξ_2 in the second part of this proof. By scaling ξ_1 and ξ_2, we can assume in addition to conditions (7.15), (7.16) and (7.17) that $|\xi_1|$ and $|\xi_2|$ are small enough in order that $\xi=(\xi_1,\xi_2,0,0)\in\Pi_4S_A$ should be in $\Pi_4W(\mathcal{R})$ (see property (4.5)). Then $\xi=\Pi_4W(u_0)$ for some $u_0\in\mathcal{R}$. Let $u(t)$ be the regular solution with the initial data u_0. The first three polynomials $q_n(t)$ in the asymptotic expansions (4.1) and (5.1) of $u(t)$ are calculated below by using (4.8)–(4.10):

$$q_1=\xi_1,\quad \beta_2=B(q_1,q_1)=B(\xi_1,\xi_1)\in R_2H, \tag{7.18}$$

$$q_2=\xi_2-t\beta_2\in R_2H, \tag{7.19}$$

$$\begin{aligned}\beta_3 &= B(q_1,q_2)+B(q_2,q_1) = B(\xi_1,\xi_2-t\beta_2)+B(\xi_2-t\beta_2,\xi_1)\\ &= B(\xi_1,\xi_2)+B(\xi_2,\xi_1)-t[B(\xi_1,\beta_2)+B(\beta_2,\xi_1)] = \beta_3^0+t\beta_3^1,\end{aligned} \tag{7.20}$$

where

$$\beta_3^0 = B(\xi_1,\xi_2)+B(\xi_2,\xi_1) \quad\text{and}\quad \beta_3^1 = -[B(\xi_1,\beta_2)+B(\beta_2,\xi_1)], \tag{7.21}$$

and

$$\begin{aligned}q_3 = &-\int_0^t R_3\beta_3(\tau)\,d\tau - [(A-3)(I-R_3)]^{-1}(I-R_3)(\beta_3^0+t\beta_3^1)\\ &+[(A-3)(I-R_3)]^{-2}(I-R_3)\beta_3^1.\end{aligned} \tag{7.22}$$

By Lemma 5.8, in the asymptotic expansion (5.4) of $\mathcal{H}(t)$ we have

$$\phi_1 = \phi_3 = 0. \tag{7.23}$$

Condition (7.15) implies that

$$\phi_2 = \langle \mathfrak{C}q_1, q_1\rangle = \langle \mathfrak{C}\xi_1, \xi_1\rangle = 0. \tag{7.24}$$

The next polynomial in this expansion is

$$\phi_4 = 2\langle \mathfrak{C}q_1, q_3\rangle + \langle \mathfrak{C}q_2, q_2\rangle. \tag{7.25}$$

Using (7.19) and conditions (7.15) and (7.16), we have

$$\langle \mathfrak{C}q_2, q_2\rangle = \langle \mathfrak{C}(\xi_2 - t\beta_2), \xi_2 - t\beta_2\rangle = \langle \mathfrak{C}\xi_2, \xi_2\rangle - 2t\langle \mathfrak{C}\xi_2, \beta_2\rangle + t^2\langle \mathfrak{C}\beta_2, \beta_2\rangle = -2t\langle \mathfrak{C}\xi_2, \beta_2\rangle.$$

On the other hand, thanks to (7.22) and (4.10), one obtains

$$\begin{aligned}\langle \mathfrak{C}q_1, q_3\rangle &= -\langle \mathfrak{C}q_1, [(A-3)(I-R_3)]^{-1}(I-R_3)(\beta_3^0 + t\beta_3^1)\rangle \\ &\quad + \langle \mathfrak{C}q_1, [(A-3)(I-R_3)]^{-2}(I-R_3)\beta_3^1\rangle \\ &= \tfrac{1}{2}\langle \mathfrak{C}q_1, \beta_3^0 + t\beta_3^1\rangle + \tfrac{1}{4}\langle \mathfrak{C}q_1, \beta_3^1\rangle \\ &= \tfrac{1}{2}\langle \mathfrak{C}q_1, \beta_3^0\rangle + \tfrac{1}{4}\langle \mathfrak{C}q_1, \beta_3^1\rangle + \tfrac{1}{2}t\langle \mathfrak{C}q_1, \beta_3^1\rangle.\end{aligned}$$

Thus, (7.25) becomes

$$\phi_4 = \langle \mathfrak{C}q_1, \beta_3^0\rangle + \tfrac{1}{2}\langle \mathfrak{C}q_1, \beta_3^1\rangle + t[\langle \mathfrak{C}q_1, \beta_3^1\rangle - 2\langle \mathfrak{C}\xi_2, \beta_2\rangle].$$

By using (7.21) and the identity (4.12), we have

$$\langle \mathfrak{C}q_1, \beta_3^1\rangle = -\langle \mathfrak{C}\xi_1, B(\xi_1, \beta_2) + B(\beta_2, \xi_1)\rangle = \langle \mathfrak{C}\beta_2, B(\xi_1, \xi_1)\rangle = \langle \mathfrak{C}\beta_2, \beta_2\rangle = 0,$$

by condition (7.15). Therefore

$$\phi_4 = \langle \mathfrak{C}q_1, \beta_3^0\rangle - 2t\langle \mathfrak{C}\xi_2, \beta_2\rangle,$$

where $\langle \mathfrak{C}\xi_2, \beta_2\rangle \neq 0$ by condition (7.17). Hence, according to (5.12) and (5.15), we have $N = 4$ and $\lim_{t\to\infty} \mathcal{H}(t)t^{-d}e^{Nt}$ exists and is non-zero where $d = \deg\, \phi_4 = 1$.

Note. In fact, we have $\phi_4 = -(1+2t)\langle \mathfrak{C}\xi_2, \beta_2\rangle$, for, again by (7.21) and (4.12),

$$\langle \mathfrak{C}q_1, \beta_3^0\rangle = \langle \mathfrak{C}\xi_1, B(\xi_1, \xi_2) + B(\xi_2, \xi_1)\rangle = -\langle \mathfrak{C}\xi_2, B(\xi_1, \xi_1)\rangle = -\langle \mathfrak{C}\xi_2, \beta_2\rangle.$$

Part II (*a selection of* ξ_1 *and* ξ_2 *satisfying conditions* (7.15), (7.16) *and* (7.17)). Let ξ_1 be defined by

$$\xi_1 = \mathbf{a}e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{a}^*e^{-i\mathbf{k}\cdot\mathbf{x}} + \mathbf{b}e^{i\mathbf{k}'\cdot\mathbf{x}} + \mathbf{b}^*e^{-i\mathbf{k}'\cdot\mathbf{x}} \in R_1H, \tag{7.26}$$

where

$$\mathbf{k} = \mathbf{e}_1 = (1,0,0), \quad \mathbf{a} = \mathbf{e}_2 = (0,1,0),$$
$$\mathbf{k}' = \mathbf{e}_2 = (0,1,0), \quad \mathbf{b} = \mathbf{e}_1 + \mathbf{e}_3 = (1,0,1),$$

and ξ_2 be defined by

$$\xi_2 = \mathbf{c}e^{i\mathbf{l}\cdot\mathbf{x}} + \mathbf{c}^*e^{-i\mathbf{l}\cdot\mathbf{x}} \in R_2H, \tag{7.27}$$

where $\mathbf{l} = (1,1,0)$ and $\mathbf{c} = (1,-1,1)$. Since $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, we have

$$\langle \mathfrak{C}\xi_1, \xi_1\rangle = \langle \mathfrak{C}\xi_2, \xi_2\rangle = 0, \tag{7.28}$$

by Remark 6.1. In order to compute β_2, we first use (6.6) to find

$$(\xi_1 \cdot \nabla)\xi_1 = Q_{\mathbf{m}}e^{i\mathbf{m}\cdot\mathbf{x}} + Q_{\mathbf{m}}^*e^{-i\mathbf{m}\cdot\mathbf{x}} + Q_{\mathbf{m}'}e^{i\mathbf{m}'\cdot\mathbf{x}} + Q_{\mathbf{m}'}^*e^{-i\mathbf{m}'\cdot\mathbf{x}}, \tag{7.29}$$

where

$$\mathbf{m} = \mathbf{k} + \mathbf{k}' = (1,1,0) = \mathbf{l}, \quad \mathbf{m}' = \mathbf{k} - \mathbf{k}' = (1,-1,0),$$

and

$$Q_{\mathbf{m}} = i(\mathbf{a}\cdot\mathbf{k}')\mathbf{b} + i(\mathbf{b}\cdot\mathbf{k})\mathbf{a} = i(1,1,1) \in i\mathbb{R}^3, \tag{7.30}$$

$$Q_{\mathbf{m}'} = -i(\mathbf{a}\cdot\mathbf{k}')\mathbf{b} + i(\mathbf{b}\cdot\mathbf{k})\mathbf{a} \in i\mathbb{R}^3. \tag{7.31}$$

Then applying the Leray projection to (7.29), we obtain

$$\beta_2 = B(\xi_1,\xi_1) = P[(\xi_1\cdot\nabla)\xi_1] = B_{\mathbf{m}}e^{i\mathbf{m}\cdot\mathbf{x}} + B^*_{\mathbf{m}}e^{-i\mathbf{m}\cdot\mathbf{x}} + B_{\mathbf{m}'}e^{i\mathbf{m}'\cdot\mathbf{x}} + B^*_{\mathbf{m}'}e^{-i\mathbf{m}'\cdot\mathbf{x}},$$

where $B_{\mathbf{m}}$ and $B_{\mathbf{m}'}$ are calculated by using (6.7). As seen in (7.30) and (7.31) both $Q_{\mathbf{m}}$ and $Q_{\mathbf{m}'}$ are in $i\mathbb{R}^3$, and hence so are $B_{\mathbf{m}}$ and $B_{\mathbf{m}'}$. From Remark 6.1 we infer that $\langle\mathfrak{C}\beta_2,\beta_2\rangle = 0$. Together with (7.28), we have proved that ξ_1 and ξ_2 satisfy conditions (7.15) and (7.16). To verify the last condition (7.17) we derive from (6.8) that

$$\begin{aligned}\langle\mathfrak{C}\xi_2,\beta_2\rangle &= 2L^3\,\mathrm{Re}[i(\mathbf{l}\times\mathbf{c})\cdot B^*_{\mathbf{m}}] = 2L^3\,\mathrm{Re}[i(\mathbf{l}\times\mathbf{c})\cdot Q^*_{\mathbf{m}}]\\ &= 2L^3(1,-1,-2)\cdot(1,1,1) = -4L^3 \neq 0.\end{aligned}$$

□

From the construction of Examples 7.2 and 7.3 it easily follows that the corresponding limits $n_0 = \lim_{t\to\infty}\|u(t)\|^2/|u(t)|^2$ and $h_0 = \lim_{t\to\infty}\mathcal{J}(t)/\mathcal{H}(t)$ are given by $n_0 = 4$, $h_0 = 7$ and $n_0 = 2$, $h_0 = 4$, respectively. We will see that the quotient $\mathcal{H}(t)/|u(t)|^2$ between the helicity and the energy behaves even more differently than the two quotients above as $t\to\infty$. Its asymptotic behavior is stated in Theorems 3.4 and 3.5 which will be proved in a sequel.

First, we establish the existence of the limit α_0 in equation (3.5) and its connection to the normalization map $W(u_0)$ (see § 4).

PROPOSITION 7.4. *Let $u_0 \in \mathcal{R}\setminus\{0\}$ and $u(t)$ be the corresponding regular solution and n_0 be defined as in* (3.1). *Then*

$$\alpha_0 = \lim_{t\to\infty}\frac{\mathcal{H}(t)}{|u(t)|^2} = \frac{\langle\mathfrak{C}W_{n_0}(u_0), W_{n_0}(u_0)\rangle}{|W_{n_0}(u_0)|^2}. \tag{7.32}$$

Proof. From the general asymptotic expansion (5.1) of $u(t)$ and specific properties (3.1) and (4.6) we know that $u(t)$ has the following asymptotic expansion in $H^m(\Omega)$ for $m = 0,1,2,\ldots$:

$$u(t) \sim W_{n_0}(u_0)e^{-n_0t} + q_{n_0+1}e^{-(n_0+1)t} + \ldots, \quad \text{where } W_{n_0}(u_0) \in R_{n_0}H\setminus\{0\}. \tag{7.33}$$

This implies that

$$u(t) = W_{n_0}(u_0)e^{-n_0t} + O(e^{-(n_0+\varepsilon)t}) \quad \text{in } H, \text{ for some } \varepsilon > 0, \tag{7.34}$$

and

$$|u(t)|^2 = |W_{n_0}(u_0)|^2e^{-2n_0t} + O(e^{-(2n_0+\varepsilon)t}). \tag{7.35}$$

By the asymptotic expansion (5.4) of the helicity $\mathcal{H}(t) = \langle\mathfrak{C}u(t),u(t)\rangle$ and possibly some adjustment in the values of $\varepsilon > 0$ appearing in (7.33) and (7.34), we have

$$\langle\mathfrak{C}u(t),u(t)\rangle = \langle\mathfrak{C}W_{n_0}(u_0),W_{n_0}(u_0)\rangle e^{-2n_0t} + O(e^{-(2n_0+\varepsilon)t}). \tag{7.36}$$

Therefore,

$$\lim_{t\to\infty}\frac{\langle\mathfrak{C}u(t),u(t)\rangle}{|u(t)|^2} = \frac{\langle\mathfrak{C}W_{n_0}(u_0),W_{n_0}(u_0)\rangle}{|W_{n_0}(u_0)|^2}.$$

We can now pass to the proofs of Theorems 3.4 and 3.5.

Proof of Theorem 3.4. Let u_0, $u(t)$ and n_0 be as in Theorem 3.4. By Proposition 7.4, we have

$$\alpha_0 = \lim_{t\to\infty} \frac{\mathcal{H}(t)}{|u(t)|^2} = \frac{\langle \mathfrak{C}W_{n_0}(u_0), W_{n_0}(u_0)\rangle}{|W_{n_0}(u_0)|^2}.$$

Note that

$$\frac{|\langle \mathfrak{C}W_{n_0}(u_0), W_{n_0}(u_0)\rangle|}{|W_{n_0}(u_0)|^2} \leqslant \frac{|\mathfrak{C}W_{n_0}(u_0)||W_{n_0}(u_0)|}{|W_{n_0}(u_0)|^2} = \frac{\|W_{n_0}(u_0)\|}{|W_{n_0}(u_0)|} = \sqrt{n_0},$$

since $W_{n_0}(u_0) \in R_{n_0}H$. Hence $\alpha_0 \in [-\sqrt{n_0}, \sqrt{n_0}\,]$.

We now fix $n \in \sigma(A)$ and $\alpha \in [-\sqrt{n}, \sqrt{n}\,]$. Let $\mathbf{k} \in \mathbb{Z}^3$ satisfy $|\mathbf{k}|^2 = n$. Let $a^1_{\mathbf{k}}$ and $a^2_{\mathbf{k}}$ be two real vectors in the plane perpendicular to $\mathbf{k}$ such that $|a^1_{\mathbf{k}}| = |a^2_{\mathbf{k}}| \neq 0$. Moreover, since $\alpha/\sqrt{n} \in [-1, 1]$, we can adjust the angle between $a^1_{\mathbf{k}}$ and $a^2_{\mathbf{k}}$ to have

$$\mathbf{k}\cdot(a^2_{\mathbf{k}} \times a^2_{\mathbf{k}}) = \frac{\alpha}{\sqrt{n}}|\mathbf{k}||a^1_{\mathbf{k}}||a^2_{\mathbf{k}}| = \alpha|a^1_{\mathbf{k}}|^2.$$

Set $a_{\mathbf{k}} = a^1_{\mathbf{k}} + ia^2_{\mathbf{k}}$ and $u_0 = a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}$. Note that $u_0 \in R_nH$ and that $B(u_0, u_0) = 0$, by Remark 6.2. Therefore the solution of the Navier–Stokes equations (2.17) in this case is

$$u(t) = e^{-nt}u_0, \quad \text{for } t \geqslant 0.$$

Clearly, $\lim_{t\to\infty} \|u(t)\|^2/|u(t)|^2 = \|u_0\|^2/|u_0|^2 = n$. By (6.9), we also have

$$\frac{\mathcal{H}(t)}{|u(t)|^2} = \frac{\langle \mathfrak{C}u_0, u_0\rangle}{|u_0|^2} = \frac{4L^3\mathbf{k}\cdot(a^1_{\mathbf{k}} \times a^2_{\mathbf{k}})}{2L^3|a_{\mathbf{k}}|^2} = \frac{4L^3\alpha|a^1_{\mathbf{k}}|^2}{4L^3|a^1_{\mathbf{k}}|^2} = \alpha.$$

This completes our proof of Theorem 3.4. □

REMARK 7.5. Let n_0 and N be defined as in (3.1) and (5.12), respectively. The asymptotic expansion (7.36) of the helicity shows that $N \geqslant 2n_0$. From the proof of Proposition 5.10, the limit $h_0 = \lim_{t\to\infty} \mathcal{J}(t)/\mathcal{H}(t)$ is exactly $N/2$; hence $h_0 \geqslant n_0$. On the other hand, if $h_0 = n_0$ we have $N = 2n_0$ and therefore in the expansion (7.36), $\langle \mathfrak{C}W_{n_0}(u_0), W_{n_0}(u_0)\rangle \neq 0$. Then (7.32) implies that $\alpha_0 \neq 0$.

Concerning the case $\alpha_0 = 0$, we first notice that the last statement in Theorem 3.5 follows directly from Examples 7.2 and 7.3. It remains to show that h_0 can be arbitrarily larger than n_0; this means that for any given exponential rate of decay, there are initial data such that the corresponding quotient $\mathcal{H}(t)/|u(t)|^2$ decays exponentially faster than that given rate. So, the proof below establishes this remaining part of Theorem 3.5.

Proof of Theorem 3.5. Let $n \in \sigma(A)$ and $\mathbf{k}_n \in \mathbb{Z}^3$ such that $|\mathbf{k}_n|^2 = n$ and let $a_{\mathbf{k}_n} \in \mathbb{R}^3$ be a non-zero vector such that $a_{\mathbf{k}_n}\cdot\mathbf{k}_n = 0$. Set

$$\xi_n = a_{\mathbf{k}_n}(e^{i\mathbf{k}_n\cdot\mathbf{x}} + e^{-i\mathbf{k}_n\cdot\mathbf{x}}) \neq 0.$$

For $s \in \mathbb{N}$ with $s \geqslant 2$, take $m = s^2n$ and $\mathbf{k}_m = s\mathbf{k}_n$. Since $m = |\mathbf{k}_m|^2 \in \sigma(A)$ and by Lemma 6.3, we can find $a_{\mathbf{k}_m} \in \mathbb{C}^3$ such that

$$\xi_m = a_{\mathbf{k}_m}e^{i\mathbf{k}_m\cdot\mathbf{x}} + a^*_{\mathbf{k}_m}e^{-i\mathbf{k}_m\cdot\mathbf{x}} \in R_mH \setminus \{0\} \quad \text{and} \quad \mathfrak{C}\xi_m = \sqrt{m}\,\xi_m. \tag{7.37}$$

From Remarks 6.2 and 6.1 we have

$$B(\xi_n, \xi_n) = 0, \quad \langle \mathfrak{C}\xi_n, \xi_n\rangle = 0 \quad \text{and} \quad B(\xi_n, \xi_m) = B(\xi_m, \xi_n) = 0. \tag{7.38}$$

Take $\xi_j = 0$ for all $j \neq n, m$ and let $\xi = \bigoplus_{j=1}^{2m} \xi_j \in \Pi_{2m}S_A$. By rescaling to have small $|\xi_n|$ and $|\xi_m|$ we can assume that $\xi = \Pi_{2m}W(u_0)$ for some $u_0 \in \mathcal{R}$, by virtue of (4.5). Therefore

$$n_0 = n \quad \text{and} \quad W_{n_0}(u_0) = \xi_n. \tag{7.39}$$

Thanks to Proposition 7.4 and (7.38), we infer that

$$\alpha_0 = \frac{\langle \mathfrak{C}\xi_n, \xi_n\rangle}{|\xi_n|^2} = 0.$$

The asymptotic expansion (4.1) or (5.1) of the corresponding regular solution $u(t)$ in this case is

$$u(t) \sim \xi_n + \sum_{j>n} q_j(t)e^{-jt}. \tag{7.40}$$

This means that $q_1 = q_2 = \ldots = q_{n-1} = 0$ and $q_n = \xi_n$. Since $s \geqslant 2$ and $m = s^2 n > 2n$, we have $\beta_j = 0$ for all $j \in [1, 2n)$; hence $q_j = 0$ for $j \in (n+1, 2n)$. By (7.38), we have $\beta_{2n} = B(\xi_n, \xi_n) = 0$; hence $q_{2n} = 0$. Using (4.8) and (4.9) to calculate β_j and q_j for $j > 2n$ we obtain $q_j = 0$ for all $j = 1, 2, \ldots, m-1$ except for $j = n$ and $\beta_j = 0$ for all $j = 1, 2, \ldots, m$. In particular, $\beta_m = 0$ and hence $q_m = \xi_m$. Once again, using the formulas (4.8) and (4.9), one gets $q_{m+1} = q_{m+2} = \ldots = q_{m+n-1} = 0$, and also by (7.38), $\beta_{m+n} = B(\xi_m, \xi_n) + B(\xi_n, \xi_m) = 0$; thus, $q_{m+n} = 0$. Similar arguments lead us to the overall identification of the first $(2m-1)$ polynomials q_j in the asymptotic expansion of $u(t)$ as the following:

$$q_n = \xi_n \in R_n H, \quad q_m = \xi_m \in R_m H \quad \text{and} \quad q_j = 0 \quad \text{for } j \in [1, 2m) \setminus \{n, m\}.$$

Consequently, the polynomials in the asymptotic expansion (5.4) of $\mathcal{H}(t)$ are $\phi_j = 0$ for all $j < 2m$ with $j \neq 2n, m+n$. By (7.38) we have $\phi_{2n} = \langle \mathfrak{C}\xi_n, \xi_n\rangle = 0$. Since $q_n \in R_n H$ and $q_m \in R_m H$, $\phi_{n+m} = 2\langle \mathfrak{C}q_n, q_m\rangle = 0$. The next polynomial in the expansion of $\mathcal{H}(t)$ is

$$\phi_{2m} = \langle \mathfrak{C}q_m, q_m\rangle = \langle \mathfrak{C}\xi_m, \xi_m\rangle = \sqrt{m}\,|\xi_m|^2 \neq 0,$$

by virtue of (7.37). This shows that the number N in (5.13) is given by $N = 2m$. Therefore

$$\frac{\mathcal{H}(t)}{|u(t)|^2} = \frac{\phi_{2m}e^{-2mt} + O(e^{-(2m+\varepsilon)t})}{|\xi_n|^2 e^{-2nt} + O(e^{-(2n+\varepsilon)t})} = O(e^{-2(m-n)t}), \quad \text{as } t \to \infty, \tag{7.41}$$

where ε is some adequately positive number. For each $M > 0$, take $s > 0$ such that

$$h_0 = N/2 = m = s^2 n \geqslant n + M = n_0 + M.$$

Then (7.41) above implies that

$$\frac{\mathcal{H}(t)}{|u(t)|^2} = O(e^{-2Mt}) \quad \text{as } t \to \infty.$$

Remark 7.6. So far, we have identified all the possible values of

$$n_0 = \lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} \quad \text{as well as of} \quad \alpha_0 = \lim_{t\to\infty} \frac{\mathcal{H}(t)}{|u(t)|^2}.$$

However, we only know that the set

$$\left\{ h_0 = h_0(u_0) = \lim_{t\to\infty} \mathcal{J}(t)/\mathcal{H}(t) : u_0 \in \mathcal{R}_1 \right\} \tag{7.42}$$

is strictly larger than the spectrum of A. This leaves open the question of whether the set defined by (7.42) can be the set of all natural numbers.

8. *Analytic domains of solutions*

In this section we will study the case when all polynomials $\phi_n(t)$ in the asymptotic expansion (5.4) of the helicity are identically zero. This implies that

$$\mathcal{H}(t) = o(e^{-nt}) \quad \text{as } t \to \infty, \quad \text{for all } n \in \mathbb{N}. \tag{8.1}$$

Our goal is to prove that $\mathcal{H}(t) \equiv 0$ by using the analyticity with respect to time of the regular solutions as well as some appropriate results from complex analysis.

We first introduce the Navier–Stokes equations with complex times and their analytic solutions. Let X be a real Hilbert space with scalar product $(\cdot,\cdot)_X$. We define the complexification of X as the following:

$$X_{\mathbb{C}} = \{u + iv : u, v \in X\} \tag{8.2}$$

with the addition and scalar products defined by

$$(u_1 + iu_2) + (v_1 + iv_2) = (u_1 + v_1) + i(u_2 + v_2), \quad \text{for } u_1, u_2, v_1, v_2 \in X \tag{8.3}$$

and

$$(z_1 + iz_2)(u_1 + iu_2) = z_1u_1 - z_2u_2 + i(z_2u_1 + z_1u_2), \quad \text{for } z_1, z_2 \in \mathbb{R}, u_1, u_2 \in X, \tag{8.4}$$

respectively. The complexified space $X_{\mathbb{C}}$ is a Hilbert space with respect to the following inner product:

$$(u + iv, u' + iv')_{X_{\mathbb{C}}} = (u, u')_X + (v, v')_X + i[(v, u')_X - (u, v')_X], \quad \text{for } u, v, u', v' \in X. \tag{8.5}$$

When $X = H$ or $X = V$, we obtain the complexified spaces $H_{\mathbb{C}}$ and $V_{\mathbb{C}}$ and their corresponding inner products and norms. For the sake of simplicity, we use the same notations $|\cdot|$ and $\|\cdot\|$ to denote $|\cdot|_{H_{\mathbb{C}}}$ and $\|\cdot\|_{V_{\mathbb{C}}}$, respectively.

The Stokes operator A is extended to the operator $A_{\mathbb{C}}$ defined on $\mathcal{D}_{A_{\mathbb{C}}} = (\mathcal{D}_A)_{\mathbb{C}}$ by the formula

$$A_{\mathbb{C}}(u + iv) = Au + iAv, \quad \text{for } u, v \in \mathcal{D}_A. \tag{8.6}$$

The curl operator $\mathfrak{C}$ is extended to $\mathfrak{C}_{\mathbb{C}}$ defined in $V_{\mathbb{C}}$ by

$$\mathfrak{C}_{\mathbb{C}}(u + iv) = \mathfrak{C}u + i\mathfrak{C}v, \quad \text{for } u, v \in V. \tag{8.7}$$

Similarly, $B(\cdot,\cdot)$ can be extended to a bounded bilinear map from $V_{\mathbb{C}} \times \mathcal{D}_{A_{\mathbb{C}}}$ to $H_{\mathbb{C}}$ as follows:

$$B_{\mathbb{C}}(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')] \tag{8.8}$$

for $u, v \in V$ and $u', v' \in \mathcal{D}_A$. Note that, unlike the real case, we have

$$\langle B_{\mathbb{C}}(u, v), v\rangle_{H_{\mathbb{C}}} \not\equiv 0, \quad \text{for } u, v \in \mathcal{D}_{A_{\mathbb{C}}}. \tag{8.9}$$

The Navier–Stokes equations with complex times are defined as

$$\frac{du(\zeta)}{d\zeta} + B_{\mathbb{C}}(u(\zeta), u(\zeta)) + A_{\mathbb{C}}u(\zeta) = 0, \tag{8.10}$$

$$u(\zeta_0) = u_0, \tag{8.11}$$

where $\zeta_0 \in \mathbb{C}$ and $u_0 \in V_{\mathbb{C}}$ are given, and $d/d\zeta$ denotes the derivative of $H_{\mathbb{C}}$-valued functions.

We recall here a basic result on the analyticity of the complex-valued solutions of the Navier–Stokes equations [**11**], using the approach introduced in [**8**]. For our convenience, we define

$$E(\zeta_0, r_0) = \left\{\zeta_0 + se^{i\theta} : |\theta| < \pi/2,\ 0 < s < \frac{\cos^3\theta}{4c_0r_0^4},\right\}, \quad \text{for } \zeta_0 \in \mathbb{C},\ r_0 \geqslant 0, \tag{8.12}$$

where $c_0 = 3^3 \cdot 5^4 \cdot 4^{-5} \cdot \pi^{-6} \cdot (29 + 20\sqrt{6}\,)$.

THEOREM 8.1. *Given $u_0 \in V_{\mathbb{C}}$ and $\zeta_0 \in \mathbb{C}$, there exists a unique $V_{\mathbb{C}}$-valued function $u(\zeta)$ defined in $\bar{E}(\zeta_0, \|u_0\|)$ satisfying the following properties*:

(i) *$u(\zeta)$ is continuous from $\bar{E}(\zeta_0, \|u_0\|)$ to $V_{\mathbb{C}}$ and satisfies* (8.11);
(ii) *$u(\zeta)$ is analytic from $E(\zeta_0, \|u_0\|)$ to $\mathcal{D}_{A_{\mathbb{C}}}$ and satisfies* (8.10).

For completeness we provide a sketch of the proof for Theorem 8.1.

Proof of Theorem 8.1. For $\zeta_0 \in \mathbb{C}$ and $u_0 \in V_{\mathbb{C}}$, let $u(\zeta)$ be the solution to a Galerkin approximation of (8.10) and (8.11) (see, for example, [**3**]). Fix $\theta \in (-\pi/2, \pi/2)$; from (8.10) we have the following *a priori* estimate along the ray $\{\zeta = \zeta_0 + se^{i\theta} : s > 0\}$:

$$\frac{d}{ds}\|u(\zeta_0 + se^{i\theta})\|^2 + \cos\theta |A_{\mathbb{C}} u(\zeta_0 + se^{i\theta})|^2 \leqslant \frac{c_0}{\cos^3\theta}\|u(\zeta_0 + se^{i\theta})\|^6, \tag{8.13}$$

where c_0 defined above is a positive constant independent of u_0, ζ_0, s and θ and of the dimension of the Galerkin approximation. Consequently,

$$\frac{d}{ds}\|u(\zeta_0 + se^{i\theta})\|^2 \leqslant \frac{c_0}{\cos^3\theta}\|u(\zeta_0 + se^{i\theta})\|^6. \tag{8.14}$$

Therefore,

$$\|u(\zeta_0 + se^{i\theta})\|^2 \leqslant \|u_0\|^2 \left(1 - \frac{2c_0 s}{\cos^3\theta}\|u_0\|^4\right)^{-1/2} \leqslant \sqrt{2}\,\|u_0\|^2,$$

for

$$0 \leqslant s \leqslant \frac{\cos^3\theta}{4c_0\|u_0\|^4}.$$

We refer to [**8**] for further details. □

REMARK 8.2. The solution $u(\zeta)$ can be analytically extended beyond the boundary of $E(\zeta_0, \|u_0\|)$. However, with our limited choice of $E(\zeta_0, \|u_0\|)$ one also has

$$\|u(\zeta)\| \leqslant 2^{1/4}\|u_0\| \quad \text{for all } \zeta \in \bar{E}(\zeta_0, \|u_0\|). \tag{8.15}$$

For our purpose, we need to study the solutions' analytic domains in more detail. The following proposition will give explicit descriptions of some regions contained in those domains. These regions turn out to be large enough for us to continue our analysis of the solutions and their helicity.

PROPOSITION 8.3. *For* $\zeta_0 = 0$ *and* $u_0 \in V$ *such that* $\|u_0\|^2 < 1/(2\sqrt{2c_0}\,)$, *there exists a unique solution* $u(\zeta)$ *to* (8.10) *and* (8.11) *in the domain*

$$\mathcal{D} = \bigcup_{t \geqslant 0}\{\zeta = \tau + i\sigma : \tau > t,\ |\sigma| < \Gamma_t(\tau)\}, \tag{8.16}$$

where

$$\Gamma_t(\tau) = c(\tau - t)e^{\gamma(\tau + t)} \quad \text{for } \gamma = \tfrac{1}{8} \text{ and } c = \frac{1}{4\sqrt{2}(2c_0)^{1/4}\|u_0\|}. \tag{8.17}$$

Moreover,

$$\|u(\zeta)\| \leqslant \|u_0\| \quad \text{for all } \zeta \in \mathcal{D}. \tag{8.18}$$

Proof. For $t_0 \geqslant 0$ and $v_0 \in V$, let $v(\zeta) = v(\zeta; t_0, v_0)$ be the analytic solution to the following problem:

$$\begin{cases} \dfrac{dv(\zeta)}{d\zeta} + B_{\mathbb{C}}(v(\zeta), v(\zeta)) + A_{\mathbb{C}} v(\zeta) = 0, \\ v(t_0) = v_0. \end{cases} \tag{8.19}$$

Note that the existence and uniqueness of $v(\zeta)$ come from Theorem 8.1. Fix $\theta \in (-\pi/2, \pi/2)$; from (8.19) we have the following *a priori* estimate along the ray $\{\zeta = t_0 + se^{i\theta} : s > 0\}$:

$$\frac{d}{ds}\|v(t_0 + se^{i\theta})\|^2 + \cos\theta |A_{\mathbb{C}} u(t_0 + se^{i\theta})|^2 \leqslant \frac{c_0}{\cos^3\theta}\|v(t_0 + se^{i\theta})\|^6 \tag{8.20}$$

or

$$\frac{d}{ds}\|v(t_0+se^{i\theta})\|^2+\left(\cos\theta-\frac{c_0}{\cos^3\theta}\|v(t_0+se^{i\theta})\|^4\right)\|v(t_0+se^{i\theta})\|^2\leqslant 0.$$

If $\|v_0\|^2<c_1=1/\sqrt{2c_0}$ and θ satisfies

$$\frac{c_0\|v_0\|^4}{\cos^3\theta}\leqslant\frac{\cos\theta}{2},\quad \text{or equivalently,}\quad \|v_0\|^2\leqslant c_1\cos^2\theta, \tag{8.21}$$

then we have

$$\frac{d}{ds}\|v(t_0+se^{i\theta})\|^2\leqslant 0\quad\text{for all } s>0.$$

Thus $\|v(t_0+se^{i\theta})\|$ is decreasing in s and

$$\|v(t_0+se^{i\theta})\|^2\leqslant\|v_0\|^2\leqslant c_1\cos^2\theta\quad\text{for all } s>0. \tag{8.22}$$

Therefore an *a priori* estimate of $\|v(\zeta)\|$ is

$$\|v(\zeta)\|\leqslant\|v_0\|,\quad \zeta\in D(t_0,v_0):=\left\{\zeta=t_0+se^{i\theta}\in\mathbb{C}: s>0,\,\cos\theta>\frac{\|v_0\|}{\sqrt{c_1}}\right\}. \tag{8.23}$$

Then by Theorem 8.1, the analytic domain of $v(\zeta)$ contains $D(t_0,v_0)$. We now fix $u_0\in\mathcal{R}$ and let $u(t)$ be the regular solution to the real Navier–Stokes equations (2.17). For each $t\geqslant 0$, we have shown that $u(\cdot)$ can be extended to the function $u(\zeta;t)=v(\zeta;t,u(t))$ which is analytic in the ζ variable where ζ belongs to

$$D_t=D(t,u(t))=\left\{\zeta=t+se^{i\theta}: s>0,\,\cos\theta>\frac{\|u(t)\|}{\sqrt{c_1}}\right\}. \tag{8.24}$$

By the uniqueness of analytic solutions, $u(\zeta,t,u(t))=u(\zeta,t',u(t'))$ for any $t,t'>0$ and $\zeta\in D_t\cap D_{t'}$. Therefore, $u(t)$, with $t>0$, has an analytic extension $u(\zeta)$ with $\zeta\in\bigcup_{t\geqslant 0}D_t$. Set $t_0=0$; we know that

$$\|u(t)\|^2<\|u_0\|^2e^{-t/2}=Me^{-\Lambda t}\quad\text{for all } t\geqslant t_0, \tag{8.25}$$

where $\Lambda=\frac{1}{2}$ and $M=\|u_0\|^2<c_1$ and

$$Me^{-\Lambda t_0}\leqslant c_1/2\quad\text{and hence}\quad \|u(t)\|^2<c_1\quad\text{for all } t\geqslant t_0. \tag{8.26}$$

Therefore the analytic domain of $u(\zeta)$ contains

$$D=\bigcup_{t\geqslant t_0}D_t=\left\{\zeta=t+se^{i\theta}: s>0,\, t\geqslant t_0,\,\cos\theta>\frac{\|u(t)\|}{\sqrt{c_1}}\right\}.$$

Moreover, according to (8.22) and (8.26), for each $\zeta=t+se^{i\theta}\in D_t$, we have

$$\|u(\zeta)\|\leqslant\|u(t)\|<\sqrt{c_1}\quad\text{and hence}\quad \|u(\zeta)\|<\sqrt{c_1}\quad\text{for all }\zeta\in D. \tag{8.27}$$

Because of the decay of the solution along the real axis as in (8.25), for each $t\geqslant t_0$, the set D_t defined in (8.24) above contains a subset

$$D_t'=\{\zeta=t+se^{i\theta}: s>0,\,\cos\theta>c_2e^{-\Lambda t/2}\},\quad\text{where } c_2=\sqrt{M/c_1}.$$

Thus, $D\supset\bigcup_{t\geqslant t_0}D_t'$. Fix $t\in\mathbb{R}$ with $t\geqslant t_0$. Given $\tau\in(t,\infty)$, consider $\zeta=\tau+i\sigma\in D_{t'}'$, where $t'=\frac{1}{2}(t+\tau)$. We have $\zeta=t'+s'e^{i\theta'}$, where $s'=\sqrt{\sigma^2+(\tau-t')^2}>0$ and $\cos\theta'>c_2e^{-\Lambda t'/2}$. The last condition can be written in terms of τ and σ as

$$\left|\frac{\sigma}{\tau-t'}\right|=|\tan\theta'|=\sqrt{\frac{1}{\cos^2\theta'}-1}<\sqrt{\frac{e^{\Lambda t'}}{c_2^2}-1},$$

that is

$$|\sigma|<\frac{\tau-t}{2}\sqrt{\frac{c_1e^{\Lambda(\tau+t)/2}}{M}-1}. \tag{8.28}$$

On the other hand, from (8.26) we infer that for all $\tau \geqslant t \geqslant t_0$, we have

$$\frac{c_1 e^{\Lambda(\tau+t)/2}}{M} \geqslant \frac{c_1 e^{\Lambda t}}{M} \geqslant \frac{c_1 e^{\Lambda t_0}}{M} \geqslant 2.$$

Then

$$\frac{\tau-t}{2}\sqrt{\frac{c_1 e^{\Lambda(\tau+t)/2}}{M}-1} \geqslant \frac{\tau-t}{2}\sqrt{\frac{c_1 e^{\Lambda(\tau+t)/2}}{2M}} = c_3(\tau-t)e^{\gamma(\tau+t)}, \tag{8.29}$$

where

$$\gamma = \tfrac{1}{4}\Lambda = \tfrac{1}{8} \quad \text{and} \quad c_3 = \frac{\sqrt{c_1}}{2\sqrt{2M}} = \frac{1}{4\sqrt{2}(2c_0)^{1/4}\|u_0\|}. \tag{8.30}$$

For each $t \geqslant t_0$, let us denote

$$\Gamma_t(\tau) = c_3(\tau-t)e^{\gamma(\tau+t)}, \quad \text{for } \tau > t. \tag{8.31}$$

Then from (8.28) and (8.29), we infer that the set $\bigcup_{\tau>t} D'_\tau$ contains

$$\mathcal{D}_t = \{\zeta = \tau + i\sigma : \tau > t,\, |\sigma| < \Gamma_t(\tau)\}. \tag{8.32}$$

Therefore the set

$$\mathcal{D} = \bigcup_{t\geqslant t_0} \mathcal{D}_t = \bigcup_{t\geqslant t_0} \{\zeta = \tau + i\sigma : \tau > t,\, |\sigma| < \Gamma_t(\tau)\}$$

is contained in D and hence in the analytic domain of $u(\zeta)$. Moreover, by (8.27), $\|u(\zeta)\|$ is bounded by $\sqrt{c_1}$ in D, and hence also in $\mathcal{D}$. □

A small change in the above proof will give us another description of the domain which is connected to the normalization map.

PROPOSITION 8.4. *For $\zeta_0 = 0$ and $u_0 \in \mathcal{R}$, let n_0 be as defined in* (3.1) *and $u(\zeta)$ be the unique solution to* (8.10) *and* (8.11). *Then the analytic domain of $u(\zeta)$ contains*

$$\left[\bigcup_{0\leqslant t\leqslant t_0} E(t, r_0)\right] \cup \mathcal{D}, \tag{8.33}$$

for some $t_0 \geqslant 0$, $r_0 = \max\{\|u(t)\| : 0 \leqslant t \leqslant t_0\}$ and

$$\mathcal{D} = \bigcup_{t\geqslant t_0} \{\zeta = \tau + i\sigma : \tau > t,\, |\sigma| < \Gamma_t(\tau)\}, \tag{8.34}$$

where $\Gamma_t(\tau) = c(\tau-t)e^{\gamma(\tau+t)}$ for

$$\gamma = \tfrac{1}{2}n_0 \quad \textit{and} \quad c = \frac{1}{4(2c_0)^{1/4}\|W_{n_0}(u_0)\|}. \tag{8.35}$$

Moreover,

$$\|u(\zeta)\| \leqslant (2c_0)^{-1/4} \quad \textit{for all } \zeta \in \mathcal{D}. \tag{8.36}$$

Proof. Instead of having $t_0 = 0$ together with (8.25) and (8.26), we use (3.1) and (4.6) to infer that with $\Lambda = 2n_0$ and $M = 2\|W_{n_0}(u_0)\|^2$, there is some $t_0 \geqslant 0$ such that

$$Me^{-\Lambda t_0} \leqslant c_1/2 \quad \text{and} \quad \|u(t)\|^2 < Me^{-\Lambda t} \quad \text{for all } t \geqslant t_0.$$

With the new values of Λ and M, (8.30) now becomes

$$\gamma = \tfrac{1}{4}\Lambda = \tfrac{1}{2}n_0 \quad \text{and} \quad c_3 = \frac{\sqrt{c_1}}{2\sqrt{2M}} = \frac{1}{4(2c_0)^{1/4}\|W_{n_0}(u_0)\|}.$$

Combining with Theorem 8.1, we see that the analytic domain of $u(\zeta)$ contains the subset defined by (8.33). □

Our next goal is to convert the analytic domain of $u(\zeta)$ to a more standard one in the complex plane, say, the right half plane $\mathbb{H}_0 = \{\zeta = \tau + i\sigma : \tau > 0,\ \sigma \in \mathbb{R}\}$, such that the property (8.1) of the helicity still holds. This is done in the following lemma by using an explicit analytic transformation.

LEMMA 8.5. *Let $u_0 \in \mathcal{R}$ and $\mathcal{D}$ be defined by* (8.34)*. Then there are a subdomain $\mathcal{D}^*$ of $\mathcal{D}$ and a conformal map $\varphi(\zeta)$ of $\mathcal{D}^*$ onto the right half plane $\mathbb{H}_0$ such that $\varphi^{-1}((0,\infty)) = (a,\infty)$, for some $a \geqslant 0$, and for any complex function $f(\zeta)$ defined in $\mathcal{D}$ we have*

$$\limsup_{0<\eta\to\infty} e^{\alpha\eta}|(f\circ\varphi^{-1})(\eta)| \leqslant \limsup_{0<\zeta\to\infty} e^{\alpha\zeta}|f(\zeta)|, \quad \textit{for all } \alpha > 0. \tag{8.37}$$

For the proof of this lemma, see Appendix B. Using Lemma 8.5 we now present the proof of Proposition 5.11.

Proof of Proposition 5.11. Suppose that the asymptotic expansion (5.4) of the helicity $\mathcal{H}(t)$ is identically zero. If the initial data $u_0 = u(0)$ is zero, then the solution $u(t)$ is identically zero, and hence so is its helicity $\mathcal{H}(t)$. In the case when $u_0 \neq 0$, let $\mathcal{D}$ be the domain in (8.34) and $\mathcal{H}(\zeta)$, with $\zeta \in \mathcal{D}$, be the analytic extension of the real helicity $\mathcal{H}(t)$ with $t > t_0$. More precisely, if

$$u(\zeta) = u_1(\zeta) + iu_2(\zeta), \quad \text{where } u_1(\zeta), u_2(\zeta) \in V \text{ and } \zeta \in \mathcal{D},$$

then

$$\mathcal{H}(\zeta) = \langle \mathfrak{C}u_1(\zeta), u_1(\zeta)\rangle - \langle \mathfrak{C}u_2(\zeta), u_2(\zeta)\rangle + i[\langle \mathfrak{C}u_1(\zeta), u_2(\zeta)\rangle + \langle \mathfrak{C}u_2(\zeta), u_1(\zeta)\rangle].$$

Let φ be the conformal map introduced in Lemma 8.5 with domain $\mathcal{D}^*$. Then $F = \mathcal{H}\circ\varphi^{-1}$ is a bounded analytic function on the right half plane $\mathbb{H}_0$ and satisfies

$$\lim_{\eta>0,\eta\to\infty} e^{\alpha\eta}F(\eta) = 0, \quad \text{for all } \alpha > 0, \tag{8.38}$$

by virtue of (8.1) and (8.37). From Proposition C.1 in Appendix C, we obtain $F(\eta) \equiv 0$ in $\mathbb{H}_0$ and hence $\mathcal{H}(\zeta) \equiv 0$ in $\mathcal{D}^*$. Since $\mathcal{D}^* \supset (a,\infty)$ for some $a > 0$, the analyticity of $\mathcal{H}(\zeta)$ in a neighborhood of $(0,\infty)$ (see (8.33) and (8.34)) implies that $\mathcal{H}(t) = 0$ for all $t > 0$. By the continuity of the helicity $\mathcal{H}(t)$ at $t = 0$, we also have $\mathcal{H}(0) = 0$. □

We recall that Theorem 3.1 is now just an obvious consequence of Propositions 5.10 and 5.11.

Appendix A

In this appendix, we present the asymptotic expansions of the functions valued in general normed spaces.

DEFINITION A.1. Let $(X, \|\cdot\|_X)$ be a normed space and $u(t)$ be a map from $(0,\infty)$ to X. We say that $u(t)$ has the *asymptotic expansion*

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots, \tag{A.1}$$

where $q_j(t)$, for $j = 1, 2, 3, \ldots$, are polynomials in t, if for each N,

$$v_N(t) = u(t) - [q_1(t)e^{-t} + q_2(t)e^{-2t} + \ldots + q_N(t)e^{-Nt}] \tag{A.2}$$

satisfies

$$\|v_N(t)\|_X = O(e^{-(N+\varepsilon)t}), \quad \text{as } t\to\infty, \text{ for some } \varepsilon = \varepsilon_N > 0. \tag{A.3}$$

For the convenience of the reader, we will present several immediate consequences of Definition A.1.

LEMMA A.2. *Suppose that $u(t)$ has an asymptotic expansion* (A.1). *Then the polynomials $q_j(t)$, for $j = 1, 2, 3, \dots$, are unique.*

Proof. Let $u(t)$ have two asymptotic expansions

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \dots \tag{A.4}$$

and

$$u(t) \sim p_1(t)e^{-t} + p_2(t)e^{-2t} + p_3(t)e^{-3t} + \dots. \tag{A.5}$$

Suppose that there is $j \geqslant 1$ such that $q_j \neq p_j$. Let N be the first of such orders, that is, $q_N \neq q_N$ and $q_j = p_j$ for $j = 1, 2, \dots, N-1$. We denote

$$U_N(t) = u(t) - [q_1(t)e^{-t} + q_2(t)e^{-2t} + \dots + q_N(t)e^{-Nt}], \tag{A.6}$$

$$V_N(t) = u(t) - [p_1(t)e^{-t} + p_2(t)e^{-2t} + \dots + p_N(t)e^{-Nt}]. \tag{A.7}$$

According to Definition A.1, these two remainders of $u(t)$ satisfy

$$U_N(t) = O(e^{-(N+\varepsilon)t}) \quad \text{and} \quad V_N(t) = O(e^{-(N+\varepsilon)t}), \tag{A.8}$$

as $t \to \infty$, for some $\varepsilon > 0$. Hence

$$q_N(t) - p_N(t) = e^{Nt}(V_N(t) - U_N(t)) = O(e^{-\varepsilon t}) \quad \text{as } t \to \infty. \tag{A.9}$$

Since $q_N(t) - p_N(t)$ is a non-zero polynomial, it cannot decay exponentially as $t \to \infty$. Hence (A.9) is contradicted. Therefore, $q_j = p_j$ for all $j \in \mathbb{N}$. □

LEMMA A.3. *Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces and S is a bounded linear map from X to Y. Assume that $u(t)$ has the asymptotic expansion*

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \dots \quad \text{in } X. \tag{A.10}$$

Then the asymptotic expansion of $Su(t)$ in Y is

$$Su(t) \sim Sq_1(t)e^{-t} + Sq_2(t)e^{-2t} + Sq_3(t)e^{-3t} + \dots. \tag{A.11}$$

Proof. Given $N \in \mathbb{N}$, we have $u(t) = \sum_{j=1}^N q_j(t)e^{-jt} + v_N(t)$, with $\|v_N(t)\|_X = O(e^{-(N+\varepsilon)t})$ for $t \to \infty$ and some $\varepsilon > 0$. By the linearity of S,

$$Su(t) = \sum_{j=1}^N Sq_j(t)e^{-jt} + Sv_N(t),$$

where the last term decays exponentially as

$$\|Sv_N(t)\|_Y \leqslant \|S\| \|v_N(t)\|_Y = O(e^{-(N+\varepsilon)t}), \quad \text{for } t \to \infty;$$

here $\|S\|$ denotes the operator norm of S. Therefore, according to Definition A.1, $Su(t)$ has asymptotic expansion (A.11). □

LEMMA A.4. *Suppose that $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are normed spaces and $M(\cdot,\cdot)$ is a bounded bilinear map from $X \times Y$ to Z. Assume that $u : (0,\infty) \to X$ and $v : (0,\infty) \to Y$*

have corresponding asymptotic expansions

$$u(t) \sim \sum_{j=1}^{\infty} u_j(t)e^{-jt} \quad \text{in } X \quad \text{and} \quad v(t) \sim \sum_{j=1}^{\infty} v_j(t)e^{-jt} \quad \text{in } Y. \tag{A.12}$$

Then $M(u,v)$ has the following asymptotic expansion in Z:

$$M(u(t),v(t)) \sim \sum_{j=1}^{\infty} M_j(t)e^{-jt}, \tag{A.13}$$

where $M_1 = 0$ and for $j > 1$,

$$M_j(t) = \sum_{k+l=j} M(u_k(t), v_l(t)). \tag{A.14}$$

Proof. By the boundedness of $M(\cdot,\cdot)$, there is $C > 0$ such that

$$\|M(x,y)\|_Z \leqslant C\|x\|_X\|y\|_Y \quad \text{for all } x \in X \text{ and } y \in Y. \tag{A.15}$$

According to Definition A.1, we have

$$u(t) = \sum_{j=1}^{N} u_j(t)e^{-jt} + U_N(t) \quad \text{and} \quad v(t) = \sum_{j=1}^{N} v_j(t)e^{-jt} + V_N(t), \tag{A.16}$$

where

$$\|U_N(t)\|_X = O(e^{-(N+2\varepsilon)t}) \quad \text{and} \quad \|V_N(t)\|_Y = O(e^{-(N+2\varepsilon)t}), \tag{A.17}$$

for $t \to \infty$ and some $\varepsilon \in (0, \frac{1}{4})$. Thanks to the linearity of $M(\cdot,\cdot)$ with respect to each component, we have

$$\begin{aligned} M(u(t),v(t)) = \sum_{k,l=1}^{N} e^{-(k+l)t} M(u_k(t), v_l(t)) + \sum_{k=1}^{N} e^{-kt} M(u_k(t), V_N(t)) \\ + \sum_{l=1}^{N} e^{-lt} M(U_N(t), v_l(t)) + M(U_N(t), V_N(t)). \end{aligned} \tag{A.18}$$

Since $u_k(t)$ and $v_l(t)$ are polynomials, their corresponding norms cannot exceed exponential functions with any positive exponents as $t \to \infty$. Therefore finite sums of their norms can be bounded as

$$\sum_{k=1}^{N} \|u_k(t)\|_X = O(e^{\varepsilon t}) \quad \text{and} \quad \sum_{l=1}^{N} \|v_l(t)\|_Y = O(e^{\varepsilon t}), \tag{A.19}$$

and hence

$$\sum_{k,l=1}^{N} \|u_k(t)\|_X \|v_l(t)\|_Y = O(e^{2\varepsilon t}) = O(e^{t/2}), \tag{A.20}$$

for $t \to \infty$. Combining (A.15), (A.17), (A.19) and (A.20), we derive

$$\|M(U_N(t), V_N(t))\|_Z \leqslant C\|U_N(t)\|_X \|V_N(t)\|_Y = O(e^{-2(N+2\varepsilon)t}),$$

$$\sum_{k=1}^{N} e^{-kt} \|M(u_k(t), V_N(t))\|_Z \leqslant C \sum_{k=1}^{N} \|u_k(t)\|_X \|V_N(t)\|_Y = O(e^{-(N+\varepsilon)t}),$$

and

$$\sum_{l=1}^{N} e^{-lt} \|M(U_N(t), v_l(t))\|_Z \leqslant C \sum_{l=1}^{N} \|U_N(t)\|_X \|v_l(t)\|_Y = O(e^{-(N+\varepsilon)t}),$$

for $t \to \infty$. Thus the last three terms on the right-hand side of (A.18) are of $O(e^{-(N+\varepsilon)t})$ as $t \to \infty$. The remaining sum can be rewritten as

$$\sum_{k,l=1}^{N} e^{-(k+l)t} M(u_k(t), v_l(t)) = \sum_{j=1}^{N} M_j(t) e^{-jt} + R_N(t),$$

where

$$R_N(t) = \sum_{1 \leqslant k,l \leqslant n, k+l \geqslant N+1} e^{-(k+l)t} M(u_k(t), v_l(t)).$$

We bound $\|R_N(t)\|_Z$ by

$$\begin{aligned}\|R_N(t)\|_Z &\leqslant C e^{-(N+1)t} \sum_{k,l=1}^{N} \|u_k(t)\|_X \|v_l(k)\|_Y \\ &= e^{-(N+1)t} O(e^{t/2}) = O(e^{-(N+1/2)t}) = O(e^{-(N+\varepsilon)t}),\end{aligned}$$

for $t \to \infty$. In summary, we have

$$M(u(t), v(t)) = \sum_{j=1}^{N} M_j(t) e^{-jt} + O(e^{-(N+\varepsilon_N)t}) \quad \text{in } Z \text{ as } t \to \infty,$$

for all $N = 1, 2, 3, \ldots$. This proves the asymptotic expansion (A.13) of $M(u(t), v(t))$. □

Appendix B

We will prove Lemma 8.5 in this section.

Proof of Lemma 8.5. Using the time translation $\varphi_0(\zeta) = \zeta - t_0$, we can transform the subdomain $\mathcal{D}_{t_0}$ of $\mathcal{D}$ (see (8.32) and (8.34)) into $\mathcal{E}_0 = \varphi_0(\mathcal{D}_{t_0})$ with

$$\mathcal{E}_0 = \{\zeta = \tau + i\sigma : \tau > 0,\, |\sigma| < \tilde{c}\tau e^{\tilde{\gamma}\tau}\},$$

for some constants $\tilde{c}, \tilde{\gamma} > 0$. We define

$$\varphi_1(\zeta) = \zeta - \kappa \log \zeta, \tag{B.1}$$

where κ is a positive constant specified later and $\log \zeta$ is the logarithm function on $\mathbb{C} \setminus (-\infty, 0]$ given by

$$\log(re^{i\theta}) = \log r + i\theta, \quad \text{for } r > 0 \text{ and } \theta \in (-\pi, \pi).$$

Denote the right half planes in $\mathbb{C}$ by

$$\mathbb{H}_a = \{\zeta = \tau + i\sigma \in \mathbb{C} : \tau > a,\, \sigma \in \mathbb{R}\}, \quad \text{for } a \in \mathbb{R}. \tag{B.2}$$

Fix $\kappa > 1/\tilde{\gamma}$ and then take $a > \max\{1, \kappa, 1/\tilde{c}\}$ such that

$$b = a - \kappa \log a > 0. \tag{B.3}$$

By an elementary computation, it can be shown that

$$\varphi_1(\partial(\mathcal{E}_0 \cap \mathbb{H}_a)) \subset \mathbb{C} \setminus \mathbb{H}_b. \tag{B.4}$$

Therefore $\mathbb{H}_b$ does not intersect $\varphi_1(\partial(\mathcal{E}_0 \cap \mathbb{H}_a))$ and it contains the half real line (b, ∞) which clearly is also in $\varphi_1(\mathcal{E}_0 \cap \mathbb{H}_a)$. On the other hand, since $\varphi_1'(\zeta) \neq 0$ for all $\zeta \in \mathcal{E}_0 \cap \mathbb{H}_a$, the inverse mapping theorem implies that $\varphi_1(\mathcal{E}_0 \cap \mathbb{H}_a)$ is open and that $\partial\varphi_1(\mathcal{E}_0 \cap \mathbb{H}_a) \subset \varphi_1(\partial(\mathcal{E}_0 \cap \mathbb{H}_a))$. Therefore, since $\mathbb{H}_b$ is connected, we have $\mathbb{H}_b \subset \varphi_1(\mathcal{E}_0 \cap \mathbb{H}_a)$. Consequently, we can apply the monodromy theorem to infer that the inverse function of φ_1 (which is locally defined) is a univalent analytic function on $\mathbb{H}_b$. Let

$$\mathcal{D}^{**} = (\varphi_1|_{\mathbb{H}_a \cap \mathcal{E}_0})^{-1}(\mathbb{H}_b) \subset \mathbb{H}_a \cap \mathcal{E}_0 \subset \varphi_0(\mathcal{D}_{t_0})$$

and

$$\mathcal{D}^* = \varphi_0^{-1}(\mathcal{D}^{**}) \subset \mathcal{D}_{t_0} \subset \mathcal{D}.$$

Then $\varphi_1(\zeta)$ is a conformal map from $\mathcal{D}^{**}$ to $\mathbb{H}_b$ and $(\varphi_1 \circ \varphi_0)(\zeta)$ is a conformal map from $\mathcal{D}^*$ to $\mathbb{H}_b$. Let $\varphi = (\varphi_1 \circ \varphi_0)\big|_{\mathcal{D}^*}$; then $\varphi^{-1}((b,\infty)) = (t_0 + a, \infty)$. Suppose that $f(\zeta)$ is a complex analytic function defined and bounded on $\mathcal{D}$. Then the function $F = f \circ \varphi^{-1}$ is analytic and bounded in $\mathbb{H}_b$. Moreover, for any $\alpha > 0$, we have

$$\begin{aligned}\limsup_{b<\eta\to\infty} e^{\alpha\eta}|F(\eta)| &= \limsup_{a<\zeta\to\infty} e^{\alpha\varphi_1(\zeta)}|F(\varphi_1(\zeta))| = \limsup_{a<\zeta\to\infty} \frac{1}{\zeta^{\alpha\kappa}} e^{\alpha\zeta}|(f\circ\varphi_0^{-1})(\zeta)| \\ &= \limsup_{a<\zeta\to\infty} \frac{1}{\zeta^{\alpha\kappa}} e^{\alpha\zeta}|f(\zeta+t_0)| = \limsup_{t_0<\zeta\to\infty} \frac{1}{e^{\alpha t_0}(\zeta-t_0)^{\alpha\kappa}} e^{\alpha\zeta}|f(\zeta)| \\ &\leqslant \limsup_{t_0<\zeta\to\infty} e^{\alpha\zeta}|f(\zeta)|.\end{aligned}$$

Finally, by using once more the translation $\varphi_2(\eta) = \eta - b$ and replacing φ by $\varphi_2 \circ (\varphi_1 \circ \varphi_0)\big|_{\mathcal{D}^*}$, we can assume that $b = 0$. This completes the proof of Lemma 8.5. □

Appendix C

In our proof of Proposition 5.11 given in § 8, we used the following theorem in complex analysis.

PROPOSITION C.1. *Let $f(\zeta)$ be a bounded analytic function on the right half plane*

$$\mathbb{H}_0 = \{\zeta = \tau + i\sigma : \tau > 0,\, \sigma \in \mathbb{R}\}$$

satisfying

$$\lim_{0<\tau\to\infty} e^{\alpha\tau} f(\tau) = 0 \quad \textit{for all } \alpha > 0. \tag{C.1}$$

Then $f(\zeta)$ is identically zero.

Proof. Let us recall the following result in complex analysis (see [**21**, problem 325, p. 168]). *If g is holomorphic in the closed right half plane, if there are positive constants A and B such that $|g(\zeta)| \leqslant Ae^{B|\zeta|}$ for ζ in the right half plane, if $|g(\zeta)| \leqslant 1$ for ζ on the boundary of the half plane, and if $\limsup_{r\to\infty} r^{-1}\log|g(r)| \leqslant 0$, then $|g(\zeta)| \leqslant 1$ for ζ in the right half plane.*

Let $M = \sup_{\zeta\in\mathbb{H}_0} |f(\zeta)|$. Apply the above theorem to $g(\zeta) = e^{\alpha\zeta} f(\zeta+\varepsilon)/M$ and obtain

$$|f(\zeta+\varepsilon)| \leqslant M|e^{-\alpha\zeta}|,$$

for all $\zeta \in \mathbb{H}_0$ and $\varepsilon, \alpha > 0$. Letting $\alpha \to \infty$ and then $\varepsilon \to 0$, we have $f(\zeta) \equiv 0$ for $\zeta \in \mathbb{H}_0$. □

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