PRIMAL/DUAL SPATIAL RELATIONSHIPS AND APPLICATIONS

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ABSTRACT

It is well known that extraction of the skeleton of a polygon from its outline may aid in the perception or classification of its form. It has also been suggested that the 'exoskeleton' may be used to express the relationships between objects in space. A new algorithm has been developed that extracts both the boundary and the skeleton of the spatial representation of an object in one easy step, based on local properties of the Delaunay/Voronoi diagram, without requiring additional information, such as point order or polygon labelling. This displays and preserves the fundamental relationships between the boundary and the skeleton that helps considerably in many cartographic problems. Illustrations include contour map input and terrain visualization; watershed and flow estimation from river network input, and drainage network estimation from basin boundaries; topological reconstruction from scanned map input, and text recognition and placement in cadastral maps. The concept of preservation of the 'form' of the skeleton suggests methods for map generalization without significant loss of meaning. Spatial uncertainty may also be addressed in terms of the boundary sampling requirements and permissible locational error without loss of the ability to interpret the basic form, spatial relationships and meaning of the map.

Introduction - Blum's Medial Axis Transform

In a previous paper (Gold, 1992) we argued that the medial axis transform/skeleton might form part of our perceptual processes. Here we would like to extend that to suggest that - given the skeleton and the boundary - we can reconstruct many aspects of spatial relationships and surface form from incomplete sampling. These computer reconstructions, based on the Delaunay/Voronoi diagram, may relate to the way people perceive spatial relationships.

The thrust of the earlier work was that current raster and vector algorithms were clearly not closely related to the mechanisms of human spatial perception, and that “A spatial model in the computer that is at least not counter-intuitive may well be a step in the right direction.” We suggested that in the process of the conversion of raw information into useable knowledge, there must be a step involving the recognition of the spatial relationships among the objects perceived, and that, of the necessary properties of a useable spatial modelling system, there must be at least the recognition of the property of object adjacency. The most uncontroversial definition of adjacency is that of polygons sharing a common boundary.

Figure 1: A forest map

In the discussion of computer representations, we emphasized that a process of spatial subdivision, rather than construction, would be much more robust, in that no holes or invalid topology would exist. The simple incremental Voronoi diagram satisfies these conditions, whereas the line-intersection model used in commercial polygon overlay packages does not.

[illegible] side, [illegible] ially with optical analogue equipment, was fundamental in developing the techniques for the description of shape that are now well known under the description of “Medial Axis Transform.” Based on the “grass-fire analogy” he showed that cusps appeared in the travelling wave-front at the centres of the local minima of curvature, and thereafter the wave-front moves along the generated medial axis,

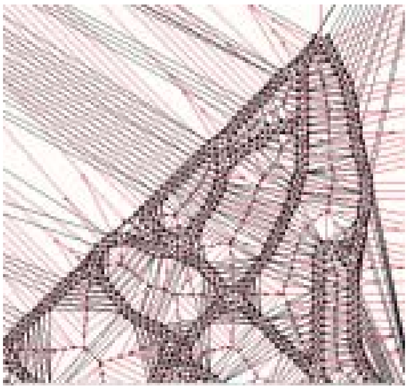

Figure 2: Fringe points of part of Figure 1.

making it a directed graph. In later work (Blum, 1973, Blum and Nagle, 1978) he attempted to perform shape classification based on the in-valence and out-valence of the nodes of the medial axis. Since modern work is all performed on digital computers, the significance of his original physical (optical) methods has perhaps been overlooked - they could indeed be models of human perception.

Applications to Scanned Maps

The approach of Gold (1992) has been pursued with some success in the specific context of generating digitized maps, either by manual digitizing or from scanned images. In Gold *et al.*, (1996) points were digitized around the "interior" of each forest polygon, and given the label of the appropriate polygon. The Voronoi diagram and Delaunay triangulation were generated for all these points, and then the irrelevant edges eliminated. In our case "relevant" edges were those Voronoi edges that separated Delaunay vertices with different labels - in other words, whose dual Delaunay edges connected vertices with different labels. An efficient one-pass algorithm was designed to achieve this, and to connect the segments together. Okabe *et al.* (1992) gives a good summary of Voronoi methods, and Worboys (1995) gives a good survey of GIS techniques from a Computer Science perspective.

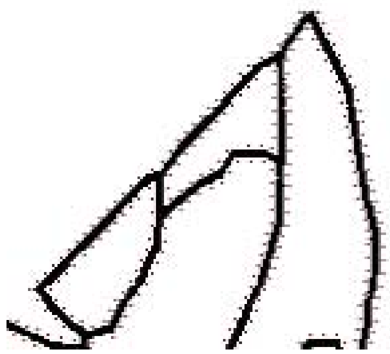

Figure 3: Voronoi/Delaunay tessellation.

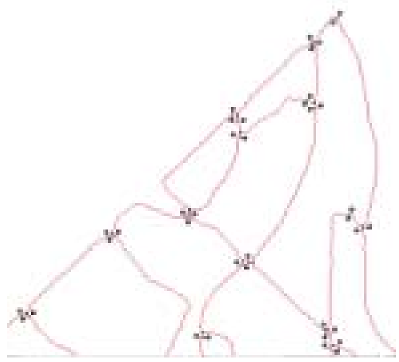

Figure 4: Extracted arcs.

This method could be thought of as generating the "boundaries" of the line work defining the polygons. This was made more explicit in the procedure for scanned maps of Gold (1997), where edges of the black pixels were extracted using image filters. A flood-fill algorithm was used to give all edge pixels belonging to the same "white space" the same label, and then the previous algorithm applied to extract the skeleton of the black line work. Figures 1-5 illustrate this procedure. Thus, the interior skeleton of the black connected "shape" of connected line work gave the polygon topology.

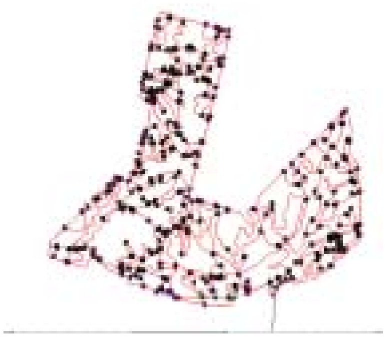

Figure 5: Topologically complete map.

An improvement to this method was described in Gold (1998) where the topological structuring (which was guaranteed to be complete because of its relationship with the original Voronoi diagram) could be saved in a simple form useable within small computer programs.

This takes the Quad-Edge data structure of Guibas and Stolfi (1985) and modifies it to handle complete arcs between polygons. The small "dumbbells" in Figures 4 and 5 indicate the connectivity of each arc: loops around each polygon and around each node. The Quad-Edge approach is very attractive as it can handle any connected planar graph (or, more correctly, on orientable manifolds) with a simple "edge algebra" and only two construction functions: Make-Edge and Splice.

The problem with the previous scanned-map approach is that it was functional only for closed polygons, due to the flood-fill labelling process: unclosed line work was lost, even though the Quad-Arc structure could manage it.

Figure 6: Voronoi diagram of sample points on a curve.

The "Crust"

These limitations are overcome due to the work of Amenta, Bern and Eppstein (1998), who showed that the "crust" of a curve or polygon boundary can be extracted from unstructured (and unlabelled) input data points if the original curve is sufficiently well sampled. Their intuition was that, as the vertices of the Voronoi diagram approximate the medial axis (or skeleton) of a set of sample points from a smooth curve (Figure 6, after Amenta, Bern and Eppstein (1998)) then by inserting the original vertices plus the Voronoi vertices into a Delaunay triangulation (Figure 7, after Amenta, Bern and Eppstein (1998)). Figure 8 shows the desired crust and skeleton.

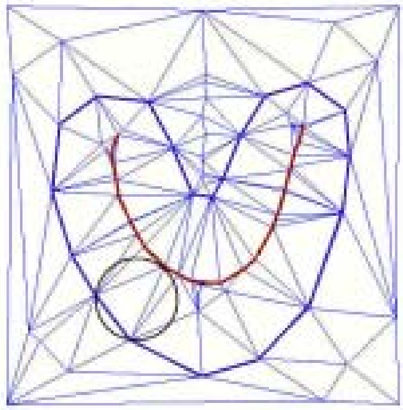

Figure 7: Skeleton and crust segment.

The circumcircles of this new triangulation approximate empty circles between the original smooth curve and its medial axis. Thus any Delauany edge connecting a pair of the original sample points forms a portion of the sampled curve - the "crust." In subsequent papers (Amenta and Bern, 1998, Amenta, Bern and Kamvysselis, 1998) they extended this to three dimensions, extracting triangulations of the surface based purely on the x, y, z coordinates of surface sample points.

This solves admirably the problem of extracting the crust - but in our work on scanned maps we wished to extract the skeleton between our rows of fringe points. Experimentation with the crust algorithm showed that, while crust edges (connecting pairs of original sample points) are extracted correctly, connecting pairs of Voronoi vertices do not necessarily produce a good approximation to the skeleton, as in the construction of the second Delaunay triangulation additional edges are added between Voronoi vertices - see Figures 9 and 10. Our objective was to extract both the crust and the skeleton, to process various types of map input.

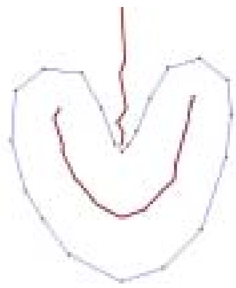

Figure 8: Crust and skeleton extracted together.

The One-Step Algorithm

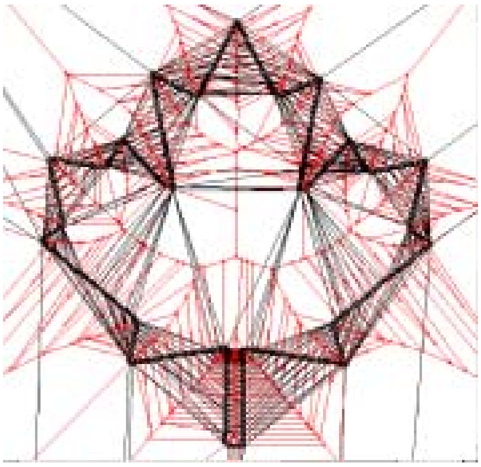

Figure 9: Maple leaf.

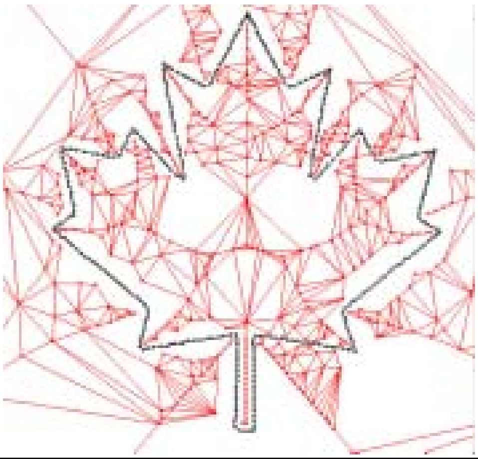

Figure 10: Crust and residual edges - the two-step approach.

Further consideration led to the examination of the relationship between Voronoi edges and Delaunay edges in the original Voronoi-Delaunay construction. This was made simpler by the use of the Quad-Edge data structure (Guibas and Stolfi, 1985), where two of the pointers refer to Delaunay vertices, and two to the dual Voronoi vertices. Our intuition was to apply the crust test to individual Quad-Edges on the original diagram, rather than creating a second structure. The circle in Figure 7 above, for example, contains a crust edge but no skeleton edges. Thus, for each Quad-Edge, we wished to determine if the Delaunay edge had a circle that was empty of Voronoi vertices (and hence of a portion of the skeleton), and which intersected the Delaunay edge.

Each Delaunay edge is adjacent to two triangles whose circumcircles are centred at Voronoi vertices. In fact, the Voronoi edge between these circle centres is the dual to the Delaunay edge. The original crust essentially tests each Delaunay edge to make sure that it has a circle that contains the edge, but does not contain any Voronoi vertices. We make this a local test - testing only the two Voronoi vertices that are the endpoints of the dual Voronoi edge.

The "Skeleton"

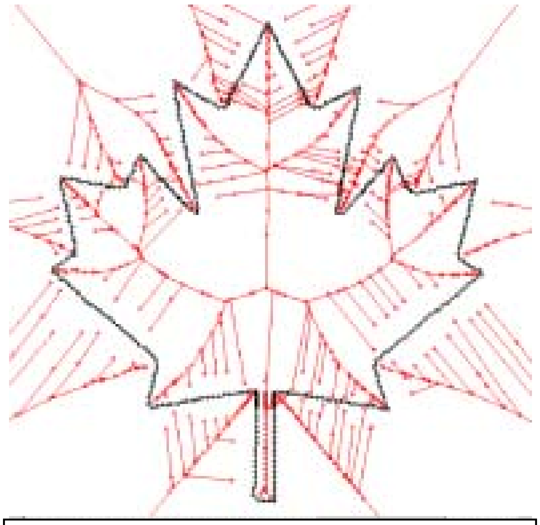

Figure 11: One-step crust and skeleton.

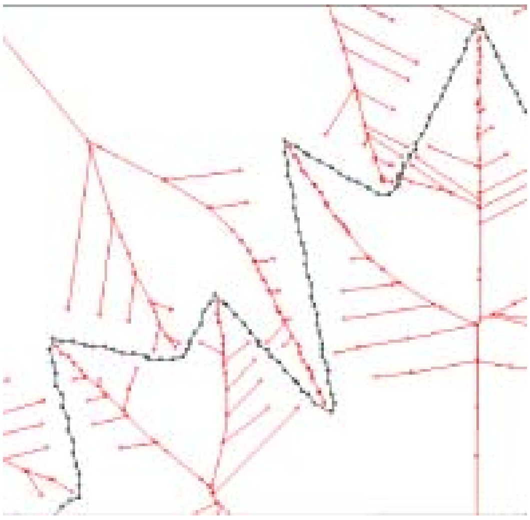

Figure 12: Enlargement of part of Figure 11.

A second idea was that Quad-Edges that failed the crust criterion were part of the skeleton or "anti-crust." This term was mentioned briefly in the conclusions of Amenta, Bern and Eppstein (1998), citing Robinson *et al.*, 1992, Brandt and Algazi, 1992 and Ogniewicz, 1994. This is based on the idea that the dual of a crust edge is a Voronoi edge that intersects the crust - and has been rejected.

The remaining Voronoi edges form a "tree" structure that extends towards the crust but does not cross it. (Indeed, with the Quad-Edge structure, each leaf of the skeleton is associated with a particular crust vertex.) The one-step algorithm consists of *assigning* Quad-Edges either to the crust or the skeleton, instead of constructing a second diagram. The results for Figure 6 are shown in Figure 8, and the results for Figure 9 are shown in Figures 11 and 12. Mis-assignments occur where the sampling conditions of Amenta, Bern and Eppstein (1998) are not met - especially at acute angles.

The leaf vertices or “hairs” on this skeleton exist where there are three adjacent sample points generating an empty circumcircle - at the end of a major branch of the skeleton, or at a minor perturbation of the sample points. These reach out to every minimum of curvature (Alt and Schwartzkopf, 1995). For a true curve, not a sampled curve, they would only occur at the ends of major branches, as the skeleton is formed wherever a circle can touch two (not three) points on the curve.

Of course, in order to detect the correct curve, we must have sufficient samples of it. Not coincidentally, Amenta, Bern and Eppstein (1998) found that the required sampling is a function of the distance from the curve to the skeleton, as the further the crust is from the skeleton, the larger the circle may be through a Delaunay edge before it includes any skeleton points. Thus, the sample spacing along the curve must be ***t*** times the circumcircle at the closest skeleton point. The necessary range of values of ***t*** is still being refined, but less than 0.25 is always satisfactory, and 1.0 or greater is impossible. Under normal circumstances problems only arise at acute angles in the crust, where the radius approaches zero.

The Crust and Skeleton Diagram - Properties and Applications

This gives us a diagram, as in Figures 11 and 12, which is half Delaunay and half Voronoi, with many interesting properties and applications. Firstly we have the crust, which is now a connected sequence of the originally unordered points. Secondly we have the skeletons (one for each skeleton region) which give information about the region’s shape.

Generalization and Adjacency

In the case of the simple maple-leaf polygon of Figure 11, both the crust and the skeleton are extracted essentially correctly by the new algorithm. In the case of the crust, sharp corners fail to satisfy the crust sampling criterion and, as expected under the Amenta, Bern and Eppstein (1998) sampling theorem, an occasional Delaunay edge crosses the tip. In the case of the skeleton, the form is correct but some extraneous branches are generated where perturbations of the boundary data points are treated as incipient salients. This also is a well-known situation. Indeed, the “hairs” on this skeleton form an excellent tool for curve generalization or simplification, as they represent minima of curvature, as described above.

Simplification of the skeleton by removing individual hairs is achieved by perturbing or removing individual crust points so as to remove these minima of curvature. These can be detected as their ends are formed from three adjacent samples - this condition must be due to either perturbations in the sampling, or because we are at the end of a skeleton branch. (In a true curve, each circle will only touch - be tangent with - one point on a local portion of a curve, or else represents a minimum of curvature). The skeleton here may here be simplified by perturbing crust points until "hairs" have been removed, and perhaps the medial axis itself smoothed - in this case we are smoothing while keeping our "shape." The resulting curve has a simple skeleton, and may be generalized in the sense of Ogniewicz and Ilg (1990) or an equivalent.

Perhaps more important, we also have a skeleton *between* objects, and these express their spatial relationships. Two adjacent objects have a skeleton boundary between them - really just the Voronoi boundary between two complex objects. Thus we have, for example on a page of text, a spatial structure expressing the form of the letters as well as the relationships between them.

Terrain Visualization from Contours

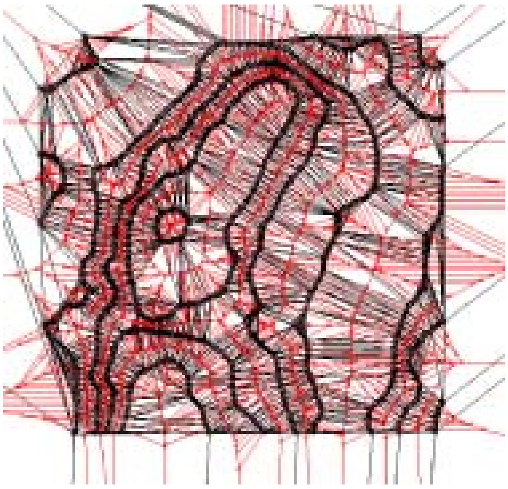

Figure 13: A contour map.

Nevertheless, there is more information within the relationships between objects than merely adjacency. A particularly interesting example, much studied, is the set of contour lines representing a topographic surface. With practice, one can visualize many details of the surface from these symbols - and the process can be hard to explain. Indeed, one of the problems with many contouring programs is that generalization of one contour curve both loses significant information in itself, and impinges negatively on the relationships with the two adjacent contours.

Figure 14: Crust and skeleton.

The skeleton helps to explain why, in Figures 13, 14 and 15. There are three types of information visible in the crust/skeleton diagram. Firstly the main curve passes mid-way between each contour pair, as expected. Secondly, where there is a salient in one curve, a branch of the skeleton is generated, indicating the mid-line of the minor ridge or valley. This completely follows experience, which says that such an indent in the level curve must be due to a "groove" (or ridge) in the surface - and here the skeleton generates the base of this groove mid-way between the two sides of the single contour line generating it. Thirdly, there are a number of "hairs" - small skeleton branches that exist only due to minor perturbations in the sampling of the curve, and which may be removed by a small amount of generalization, without affecting the primary forms.

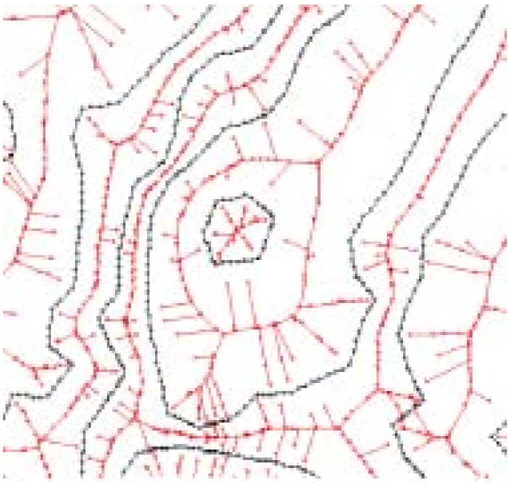

Figure 15: Enlargement of part of Figure 14.

This, however, does not exhaust the implications of the diagram. All Quad-Edges are parts of either crust or skeleton. Contour segments have a known elevation. Skeleton segments have an implicit elevation - halfway between the curves in the simple case, or up some gully in the case of a salient. But the interpretation of a contour map involves more than merely the estimation of elevation - the form of the surface involves visualization of at least slopes, as well. The slope is zero along contours, and therefore a maximum perpendicular to any contour segment. But the dual (Voronoi) segment is also perpendicular to this portion of the crust, and already has implicit elevations attached in most cases. Thus, producing a good slope model from contours simplifies to producing reasonable elevation values at skeleton points. In most cases this is easy, but in some cases - gullies, saddles, summits and depressions, for example - a little more work is required.

The Crust and Skeleton Diagram therefore gives a very powerful tool for interpreting the terrain on the basis of the relationships between contours, and not merely by individual contours themselves. The Voronoi/Delaunay relationships are particularly useful here, as there is zero slope along contour lines defined by Delaunay (crust) edges, and thus the direction of runoff or maximum downhill slope is perpendicular to this - exactly as expressed by the associated dual Voronoi edge. These interrelationships readily permit the reconstruction of a meaningful terrain model, as they mimic the natural processes involved. Both the Delaunay and the Voronoi relationships are part of the visualization process.

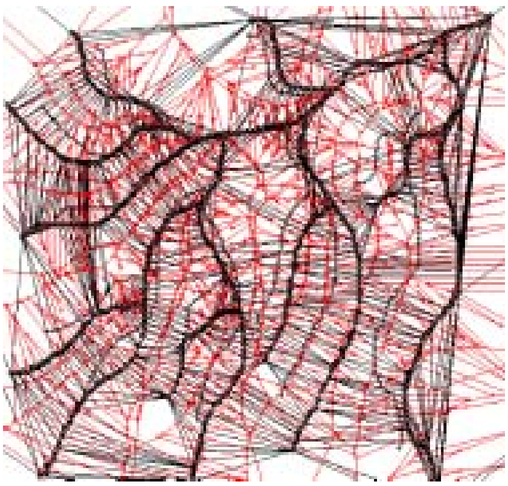

Figure 16: A river network.

River Networks

The skeleton of the polygon of Figure 11 forms an excellent approximation of drainage network development in a homogeneous terrain, given the watershed. While obviously not conforming to the non-homogeneous reality, it may well be a useful approximation. An equivalent application is the design of the road network for access to all parts of a homogeneous terrain - perhaps for forest harvesting.

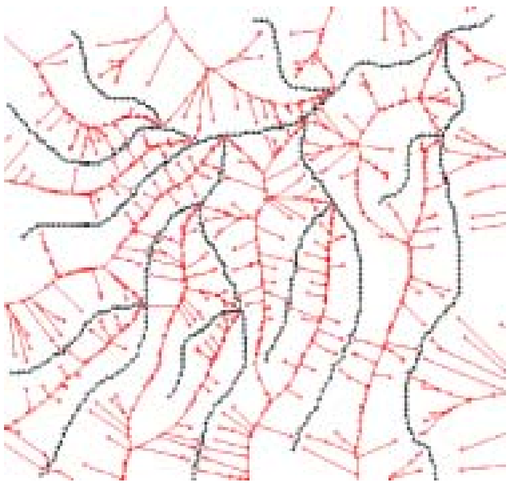

Figure 17: Estimated watersheds.

Figures 16, 17 and 18 show the reverse operation: a river network is represented as a set of sample points, and the one-step algorithm applied. Following the original idea of Blum (1967), an "elevation" is imagined as a linear function of distance from the nearest object. This gives watershed boundaries equidistant from river segments. It also automatically segments the drainage area into sub-basins, and can be used to segment the originally-unstructured sample points on the river network into a topological network, with Strahler numbers derived, and flow (as a function of catchment area) assigned to each point.

The crust and the skeleton are recognized with the one-step algorithm, and the connected components linked with the algorithm described in Gold *et al.* (1996) for rapid digitizing, and in Gold (1998) for scanned maps using the Quad-Edge data structure. The estimated sub-basins may be clearly seen, and extracted, on the basis of the skeleton. This approach has been used successfully in British Columbia, Canada, for preliminary watershed estimation, with later correction by direct observation.

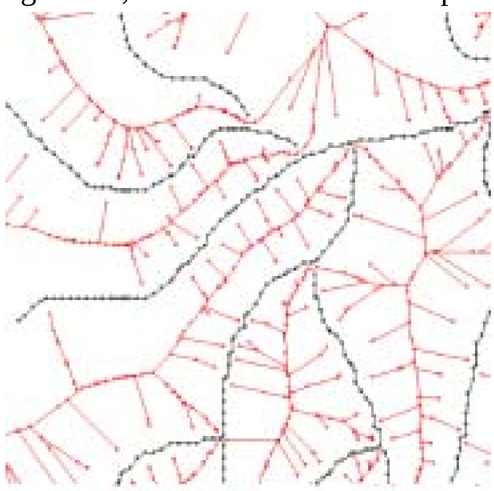

Figure 18: Enlargement of part of Figure 17.

Notice the cases of inadequate sampling according to the Amenta, Bern and Eppstein (1998) criterion, especially at junctions, causing the skeleton to break through the river network. The same approach may be used for transportation networks. In both cases the catchment area may be estimated for any point on the network, permitting the development of flow or transportation capacity maps.

Scanned Maps

The above work was performed using the idea of a single line of samples delineating the desired input curve. However, the early motivation for the work came from the need to process scanned line work. In this case we have a binary image, from which we can extract samples of the boundary between the black regions and the white regions. Intuition interprets a band of black pixels as a centreline along the band - exactly the same as the skeleton previously described. (If the line width is of interest, the circles associated with the skeleton points give that also.) Thus, line work may rapidly be converted to a topologically connected structure. (This applies for any line work, not just closed polygons as in Gold *et al.*, 1996) This "black" skeleton gives connected centrelines,

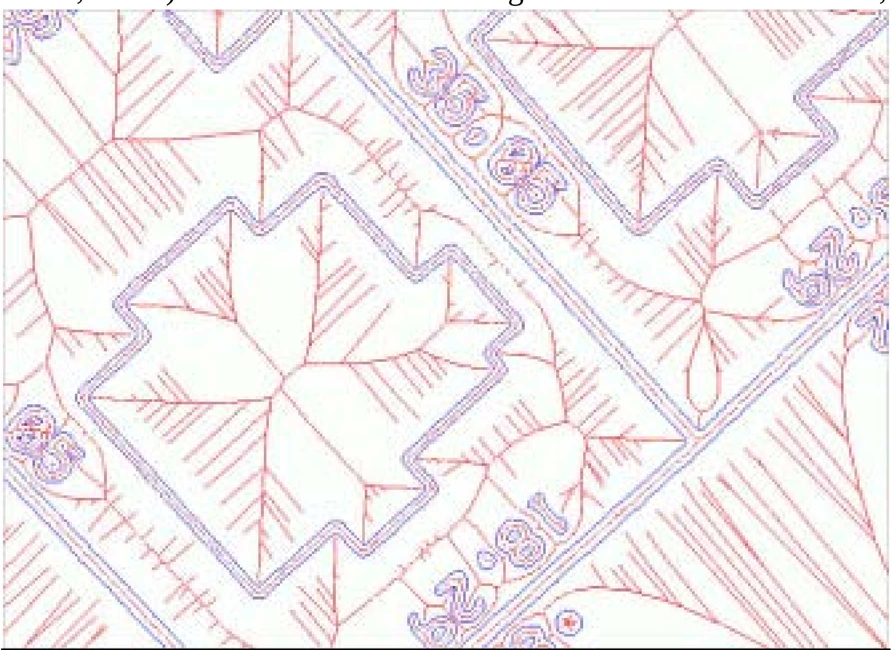

Figure 19: Crust and skeleton of part of a scanned cadastral map.

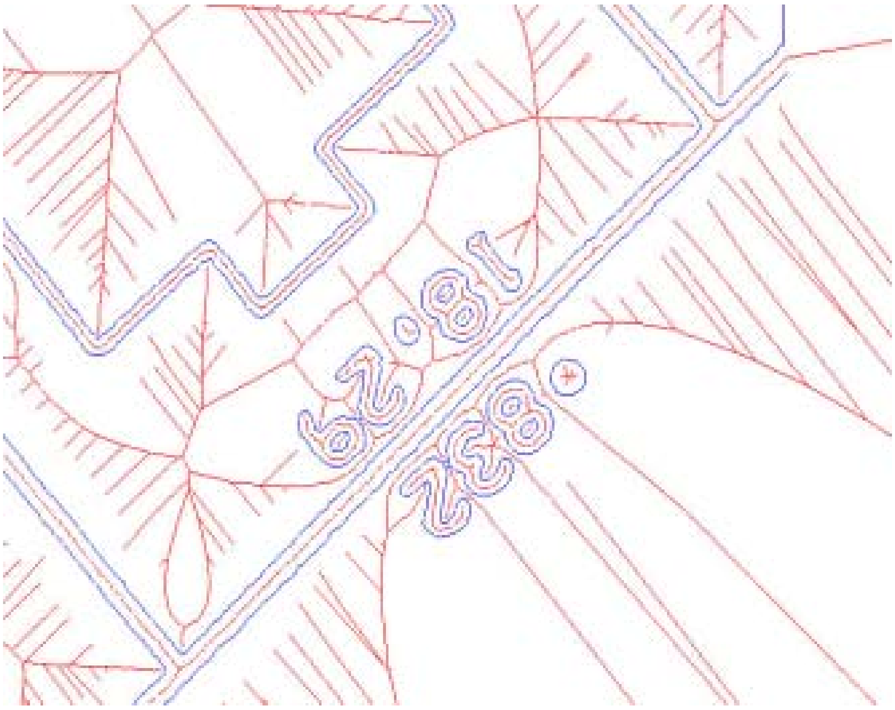

Figure 20: An enlargement of part of the scanned map of Figure 19.

assuming that black space is just line work. If we have black regions, then this becomes the region skeleton - exactly equivalent to the "white" skeletons of the enclosed polygons. The extracted crusts are probably not of interest in this application, but the suppressed dual Voronoi edges give the connectivity between the white skeleton and the black one.

Scanned maps, similar to those shown in the first section, may be processed using the crust criterion instead of vertex colouring. Thus edge pixels for each black line are extracted, as previously, but they are not labelled by floodfilling polygons. Instead, the crust criterion is applied to extract the crust of each black/white boundary. As this is purely geometric, it is applicable to closed polygons, connected networks and unconnected black lines, such as text. These crusts separate the black/white portions of the scanned map. However, the skeletons are generated by the same algorithm - but for the white portions of the map as well as for the black portions, as in Figures 19 and 20. If the black line work forms a connected graph of a polygon set, or a network, then the skeleton of this black region forms the connected topology representing the line work - as in Gold *et al.* (1996) and Figures 1 to 5. The skeletons of the white regions express the relationships between disconnected black line work, or else express the form of the white shapes, as in Blum (1967). The crusts may or may not be preserved. As in Figure 18, the crust may break if the sampling criterion of Amenta, Bern and Eppstein (1998) is not preserved - especially at sharp junctions. This will cause the "white" skeleton to break through the crust and connect with the "black" skeleton.

Several alternatives exist in this case: we may revert to the labelled vertex algorithm; we may improve the sampling; or we may take advantage of the fact that, as all Voronoi vertices fall on one side or the other of the line of the crust, they may each be labelled as "black" or "white" by reference back to the original image. Links are then broken between black and white vertices.

As shown in Figures 19 and 20, the skeleton generation algorithm allows for the extraction of the topology of the line work as well as for the detection and placement of text. Burge and Monagan (1995a, 1995b) also worked on the extraction of text from scanned maps, without the topology emphasis. For scanned maps the crust is the boundary between black and white pixels, whereas there is a skeleton for the white regions ("polygons") as well as for the black regions (line work). These last form the centrelines of the scanned line work: skeletonization in the Euclidean, rather than the raster, sense, and form a topologically complete graph of the original input map. It should be noted that the algorithm using labelled fringe points may do better for a simple polygon map, if sampling does not achieve the crust criterion, as more information is supplied about the desired boundary connectivity. Application of a "height" to the distance from the boundary may be of interest here as well. 3D urban models may be derived from the results of Figures 19 and 20, where (ignoring text) closed unconnected loops may be interpreted as buildings and given an arbitrary height, whereas their interior skeletons may be treated as roof lines.

Processing scanned text

Scanned text may equally well be treated in the same way as in the scanned map example, after processing with an edge-detection filter, as shown in Figures 21 and 22. The interior skeleton of the character can detect the characteristic form of the connected graph, essentially for the cost of constructing the Voronoi diagram. The exterior skeleton may be used to express the relationships between

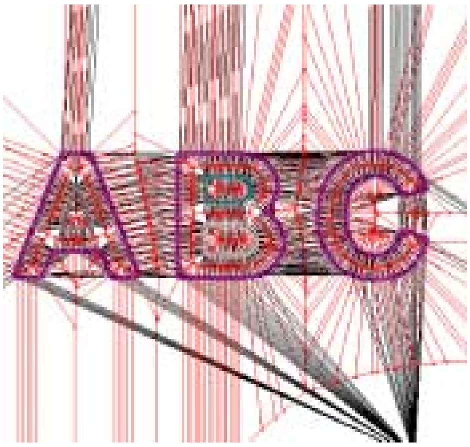

Figure 21: Character outlines.

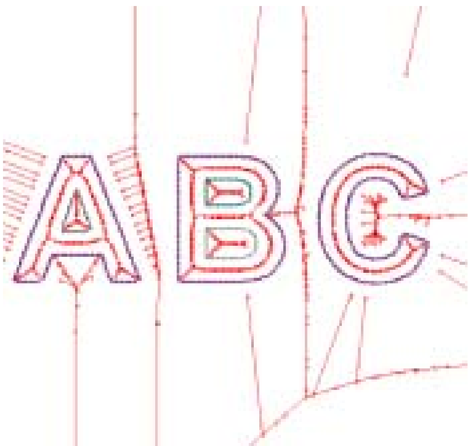

Figure 22: Interior and exterior skeletons.

the characters. These relationships are also useful for languages using non-connected components, or diacritical marks or accents. Burge and Monagan's (1995a, 1995b) methods involved extracting the text from cadastral maps using Voronoi diagrams - adjacent pixels were given the same label, and the exterior skeletons were used to connect the letters together to form complete words. As in the original work by Blum (1967) the interior skeletons of arbitrary shapes may be used as a shape descriptor.

Conclusions and Future Work

From the algorithmic viewpoint, we have shown that crust/skeleton extraction requires only a few lines of code beyond the standard Voronoi diagram, together with the necessary structures to extract the desired connected components (Quad-Arc). From the application viewpoint, we can see that the resulting skeletons and crusts serve well for questions of spatial adjacency relationships. From the perceptual point of view, we can see that crust extraction, and skeleton extraction, are useful tools in visualizing the "space" represented by our original map, whether it is drainage pattern/watershed representation, topographic relief models given by contour lines, or urban cadastral maps. Indeed, in each of these cases the relationship "maximum distance = skeleton = maximum height" can give useful three-dimensional intuitions of the space represented by the data. This is perhaps to be expected, given that the resulting skeleton represents the Voronoi diagram of complex objects. Two things however are new: the idea that generalization may be performed in such a way as to simplify the skeleton, but not to destroy it; and that *all* Voronoi/Delaunay pairs are necessary for understanding spatial relationships - the question is simply whether the particular pair represents the crust or the skeleton. (In some cases both are needed, as when the Delaunay edge represents the contour and the Voronoi edge represents the slope across that contour.)

Thus, the Delaunay/Voronoi dualism appears to be closely related to our visualization of the spatial model indicated by our two-dimensional representations. We use it to imagine space. We use it to reconstruct relationships. We use it to generalize or simplify the model. We need not just the Delaunay triangulation; not just the Voronoi diagram; but both of them simultaneously, together with the intuition to determine which of the relationships is relevant for the particular local situation. And all of these functions are based purely on the geometric properties of equidistance and cocircularity.

We have shown that a simple one-step algorithm may generate the crust and anti-crust simultaneously, and that these may be extracted to form topologically structured maps. The results are equivalent to those of the crust algorithm of Amenta, Bern and Eppstein (1998). We have shown how this resolves a variety of issues in map input and analysis, and we expect to address individual applications in more detail in the near future.

The Quad-Arc approach may be of significant use in Geomatics as it provides a simple data structure that suits the needs of the data input process, preserves the map topology, and may be implemented within small-scale PC-based software for topological map querying. The combination of the crust criterion, the one-step algorithm and the Quad-Arc data structure are sufficient to put cartographic topology within the range of simple mapping programs.

Acknowledgments

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