Global stability for nonlinear difference equations with variable coefficients

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1 Introduction

Let us consider the following nonlinear difference equation with variable coefficients:

$$\begin{cases} x(n+1) = x(n) - \sum_{j=0}^{m} a_j(n) f(x(n-j)), & n = 0, 1, 2, \cdots, \\ x(j) = x_j \geq 0, & -m \leq j \leq 0, \quad \text{and} \quad x(0) = x_0 > 0, \end{cases} \tag{1.1}$$

where $f(x)$ is a strictly monotone increasing function on $(-\infty, +\infty)$ such that

$$f(0) = 0, \qquad \text{and if } f(x) \neq x, \text{ then } \lim_{x \to -\infty} f(x) \text{ is finite}, \tag{1.2}$$

and $\{a_j(n)\}_{n=0}^{\infty}$, $0 \leq j \leq m$ are sequences such that

$$a_j(n) \geq 0, \quad \sum_{j=0}^{m} a_j(n) > 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{j=0}^{m} a_j(n) = +\infty. \tag{1.3}$$

Note that for $f(x) = e^x - 1$, (1.1) becomes a logistic equation with piecewise constant arguments.

Definition 1.1 *The zero solution of* (1.1) *is said to be uniformly stable, if for any* $\varepsilon > 0$ *and nonnegative integer* n_0*, there is a* $\delta = \delta(\varepsilon) > 0$ *such that* $\max\{|x(n_0 - j)| \mid j = -k, -k+1, \cdots, 0\} < \delta$, *implies that the solution* $\{x(n)\}_{n=0}^{\infty}$ *of* (1.1) *satisfies* $|x(n)| < \varepsilon$, $n = n_0, n_0 + 1, \cdots$.

Definition 1.2 *The zero solution of* (1.1) *is called globally attractive, if every solution of* (1.1) *tends to zero as* $n \to \infty$.

Definition 1.3 *The zero solution of* (1.1) *is called globally asymptotically stable, if it is uniformly stable and globally attractive.*

For the case of $f(x) = e^x - 1$ and $a_j(n) = r a_j / (\sum_{j=0}^{m} a_j)$, $0 \leq j \leq m$, by Gopalsamy *et al.*[2], $r \leq \log 2/(m+1)$ is a sufficient condition of global asymptotic stability for the zero solution of (1.1). This condition is improved by So and Yu [8] as follows.

Theorem A (see So and Yu [8]). *Assume that* $f(x) = e^x - 1$ *and* $a_j(n) = r(n) a_j / (\sum_{j=0}^{m} a_j)$ (a_j, $0 \leq j \leq m$ *are constants*) *in* (1.1) *with* (1.3) *and*

$$\sum_{k=n-m}^{n} \sum_{j=0}^{m} a_j(k) = \sum_{k=n-m}^{n} r(k) \leq \frac{3}{2}, \qquad n \geq m. \tag{1.4}$$

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Then, the zero solution of (1.1) *is globally asymptotically stable.*

In this talk, for the zero solution of the above nonlinear difference equation (1.1) with (1.2) and (1.3), we establish conditions of the global asymptotic stability which are other conditions than "3/2 criteria" type and an extension of the result of Györi and Pituk [3] for $f(x) = x$ to the case (1.2) (see Theorem 1.1). Put

$$r_1 = \limsup_{n\geq 0} \sum_{j=0}^{m} a_j(n), \qquad r_2 = \limsup_{n\geq m} \sum_{k=n-m}^{n-1} \sum_{j=0}^{m} a_j(k), \qquad \varphi(x) = x - r_1 f(x). \quad (1.5)$$

Theorem 1.1 *Assume*

$$\varphi(x) < 0, \qquad x < 0, \quad (1.6)$$

and

$$\begin{cases} \varphi(-r_2 f(L)) - r_2 f(-r_2 f(L)) > L, & L < \hat{L}, \\ -r_2 f(-r_2 f(L)) > L, & \hat{L} \leq L < 0, \end{cases} \quad (1.7)$$

where if $\varphi(x) < 0$, *put* $\hat{L} = 0$, *and otherwise,* $\hat{L} < 0$ *is uniquely defined by* $\varphi(-r_2 f(\hat{L})) = 0$. *Then, the zero solution of* (1.1) *with* (1.2) *and* (1.3) *is globally asymptotically stable.*

In particular, for $f(x) = e^x - 1$, (1.7) *is satisfies by the condition*

$$r_1,\ r_2 \leq 1 \qquad \text{and} \qquad r_2 < \hat{R}(r_1) + \frac{-1+\sqrt{5}}{2}, \quad (1.8)$$

where $\hat{R}(1) = 0$ *and for* $r_1 < 1$, $\hat{R}(r_1) = -r_2 f(\hat{L}) > 0$ *is defined by* $\varphi(\hat{R}(r_1)) = 0$.

Note that the obtained condition (1.8) contains other cases than (1.4) such that $r_1,\ r_2 \leq 1$ but $r_1 + r_2 > \frac{3}{2}$ (see Figure 1). In particular case of $m = 0$, we can improve the results of Zhou and Zou [10].

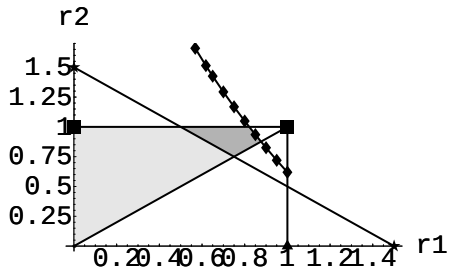

Figure 1: Condition (1.8)

In section 2, we prove Theorem 1.1. In section 3, under the condition $a_0 > \sum_{j=1}^{m} a_j$ for autonomous case of (1.1), applying similar monotone method, we establish new sufficient conditions to Theorem A and Theorem 1.1 (see Uesugi, Muroya and Ishiwata [7]).

2 Global stability for nonlinear difference equations

Applying the techniques used in So and Yu [8], we obtain the following lemmas (see Lemmas 2.1, 2.2 and Theorem 3.1 in So and Yu [8]):

Lemma 2.1 *Let $\{x(n)\}_{n=0}^{\infty}$ be the solution of (1.1) with (1.2). If $x(n)$ is eventually greater (respect. less) than 0, then $x(n)$ is eventually decreasing (respect. increasing), and $\lim_{n\to\infty} x(n)$ exists and it holds $\lim_{n\to\infty} x(n) = 0$.*

Lemma 2.2 *Let $f(x) \neq x$ and $\{x(n)\}_{n=0}^{\infty}$ be the solution of (1.1) with (1.2) and (1.3). If $x(n)$ is oscillatory about 0, and $\sup_{n\geq m} \sum_{k=n-m}^{n} \sum_{j=0}^{m} a_j(k) < +\infty$, then $x(n)$ is bounded above and below.*

Remark 2.1 If $f(x) \neq x$, then by Lemma 2.2, we see that any solution $x(n)$ of (1.1) which is oscillatory about 0, is bounded above and below.

Lemma 2.3 *Let $\{x(n)\}_{n=0}^{\infty}$ be the solution of (1.1) with (1.2) and (1.3). If $|x(n+1)| > |x(n)|$, then there exists an integer $g(n) \in [n-m, n]$ such that*

$$\begin{cases} x(g(n)) = \min\limits_{0\leq j\leq m} x(n-j) < 0, & \text{if } x(n+1) > 0, \\ x(g(n)) = \max\limits_{0\leq j\leq m} x(n-j) > 0, & \text{if } x(n+1) < 0. \end{cases} \tag{2.1}$$

From the property of g, $g(n) \geq 0$ for $n \geq m$.

Lemma 2.4 *Assume (1.6) and suppose that the solution $x(n)$ of (1.1) with (1.2) and (1.3), is oscillatory about 0. If for some real number $L < 0$, there exists an positive integer $n_L \geq m$ such that $x(n) \geq L$ for $n \geq n_L - m$, then for $n \geq n_L$,*

$$x(n+1) < R_L, \qquad \text{and} \qquad x(n+1) \geq S_L, \tag{2.2}$$

where

$$\begin{cases} R_L = -(\sup\limits_{n\geq m} \sum\limits_{k=n-m}^{n-1} \sum\limits_{j=0}^{m} a_j(k)) f(L) \qquad \text{and} \\ S_L = \min(R_L - (\sup\limits_{n\geq m} \sum\limits_{j=0}^{m} a_j(n)) f(R_L), 0) - (\sup\limits_{n\geq m} \sum\limits_{k=n-m}^{n-1} \sum\limits_{j=0}^{m} a_j(k)) f(R_L). \end{cases} \tag{2.3}$$

In particular, for a sufficiently large $n \geq m$,

$$x(n+1) < R_{-\infty} \qquad \text{and} \qquad x(n+1) > S_{-\infty}, \tag{2.4}$$

where

$$\begin{cases} R_{-\infty} = -(\limsup\limits_{n\to\infty} \sum\limits_{k=n-m}^{n-1} \sum\limits_{j=0}^{m} a_j(k))(\lim\limits_{x\to-\infty} f(x)) < +\infty, \\ S_{-\infty} = \min(R_{-\infty} - (\limsup\limits_{n\to\infty} \sum\limits_{j=0}^{m} a_j(n)) f(R_{-\infty}), 0) - (\limsup\limits_{n\to\infty} \sum\limits_{k=n-m}^{n-1} \sum\limits_{j=0}^{m} a_j(k)) f(R_{-\infty}) > -\infty. \end{cases} \tag{2.5}$$

Corollary 2.1 (cf. Györi and Pituk [3]). *For $f(x) = x$, if for (1.5), $r_1,\ r_2 < 1$, then the zero solution of (1.1) with $f(x) = x$ and (1.3), is globally asymptotically stable.*

3 An autonomous case of (1.1)

For autonomous case of (1.1), that is, each $a_j(n) = a_j$(constant), there are many literatures, for example, Gopalsamy, Kulenovic and Ladas [2], So and Yu [8], Wang *et al.* [9], Muroya [4, 5, 6] and Uesugi, Muroya and Ishiwata [7] (see Figure 2). Put

$$r_1 = a_0 > 0, \quad r_2 = \sum_{j=1}^{m} a_j \geq 0, \quad \varphi(x) = x - r_1 f(x). \tag{3.1}$$

Then, $x(n+1) = \varphi(x(n)) - \sum_{j=0}^{n} a_j f(x(n-j))$.

Under the condition $a_0 > \sum_{j=1}^{m} a_j$, applying the similar monotone iterative method as in Section 2, but without using the condition $x(n+1)x(n) < 0$, Uesugi, Muroya and Ishiwata [7] obtained the following lemmas.

Lemma 3.1 *Assume that $\varphi(x)$ attains a unique local maximum at $L^* < 0$, and for $L \leq 0$, put*

$$F(L) \equiv \min\{\varphi(L), \varphi(\varphi(\max\{L^*, L\}) - r_2 f(L))\} - r_2 f(\varphi(\max\{L^*, L\}) - r_2 f(L)). \tag{3.2}$$

If $F(L) > L$ for any $L < 0$, then it holds that $\lim_{n\to\infty} x_n = 0$.

Lemma 3.2 *Assume that $\varphi(x)$ attains a unique local maximum at $R^* > 0$ and $R^* \geq \varphi(R^*) + r_2$. Put $F(L) \equiv \varphi(L) + r_2 f(\varphi(R^*) - r_2 f(L))$ for $L \leq 0$. If $r_1 > r_2 \geq 0$ and $\lim_{L\to-\infty} F(L) - L > 0$, then $F(L) > L$ for any $L < 0$, and $\lim_{n\to\infty} x_n = 0$.*

Lemma 3.3 *Assume that $\varphi(x)$ attains a unique local maximum at $R^* > 0$ and $R^* < \varphi(R^*) + r_2$. Then, there exists a unique $\bar{L} < 0$ such that $R^* = \varphi(R^*) - r_2 f(\bar{L})$, and it holds that $R^* > \varphi(R^*) - r_2 f(L) > 0$, $\bar{L} < L \leq 0$. Put*

$$F(L) \equiv \min\{\varphi(L), \varphi(\varphi(R^*) - r_2 f(L))\} - r_2 f(\varphi(R^*) - r_2 f(L)), \quad L \leq \bar{L}. \tag{3.3}$$

If $r_1 > r_2$, and $F(L) > L$ for any $L \leq \bar{L}$, then it holds $\lim_{n\to\infty} x_n = 0$.

In particular case $f(x) = e^x - 1$, conditions in Lemmas are simplified as follows (see Uesugi, Muroya and Ishiwata [7]).

Theorem 3.1 *Assume the condition* (3.1). *Then, for r_1 and r_2 such that*

$$r_1 > r_2 \geq 0, \ r = r_1 + r_2 \leq 2, \ \text{and } r_1 + r_2 - \frac{r_2}{r_1} e^{r_1 + r_2 - 1} \geq 0, \quad \text{if } r_1 + r_2 > 1, \tag{3.4}$$

it holds that for any solution $x(n)$ of (1.1), $\lim_{t\to\infty} x(n) = 0$, *and the zero solution is globally asymptotically stable.*

Figure 2 implies several conditions of Wang *et al.*[9] and Muroya [5] for $r_1 + r_2 \leq 1$, Muroya [4] for contractivity of solutions, and Muroya [6] for a Lyapunov-like method.

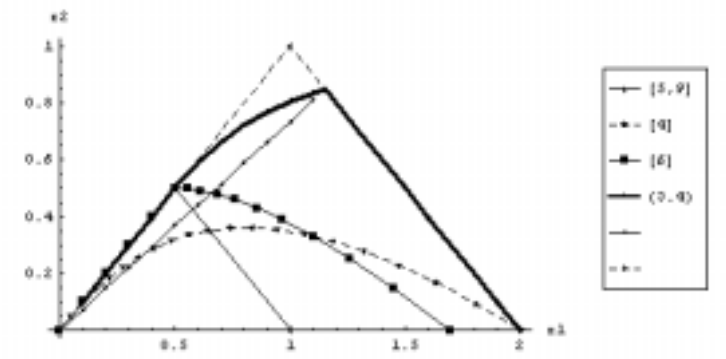

Figure 2: Results of several papers

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