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Fourier Transforms and the Method of Stationary Phase

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Some theorems are derived for the asymptotic behaviour of the Fourier transform of a generalized function under conditions which are likely to occur in applied mathematics and which should be capable of relatively straightforward verification. These theorems are then applied to integrals of the type $\int_{-\infty}^{\infty} g(x)\, e^{-i\alpha f(x)}\, dx$ when the derivative of f is non-vanishing. Further theorems are developed to cover the case when the derivative of f vanishes at a point. In this way the validity of the method of stationary phase, together with a useful estimate of the error, is established under circumstances of practical importance.

1. Introduction

THE method of stationary phase is well-known as a means of obtaining asymptotic expansions of integrals containing a large parameter. The principle, as generally enunciated by Kelvin (1887), that the main contribution comes from near the points where the phase is stationary has been modified so that now the contributions from critical points (van der Corput, 1934, 1936, 1938) must be considered. An extensive theory was developed by van der Corput but, for many purposes, a relatively simple approach using systematic integration by parts (Erdelyi, 1955, 1956, 1959; Jones & Kline, 1958) is sufficient for dealing with certain types of integrand.

The aim of the present paper is to provide a theory that will cover a wide variety of the integrals which occur in practical applications and yet keep the proof as uncomplicated as possible. It is not intended to establish the validity of the method of stationary phase under the weakest possible conditions but only for those circumstances which occur frequently in applied mathematics. Detailed evaluation of particular integrals is not given so as not to limit the scope of the theory unduly and so that the ways of obtaining expansions and estimating their errors under reasonably verifiable conditions can be indicated clearly. Quite often, it is simpler to work out a specific integral by following the prescribed rule than by substitution in a general formula. It is also anticipated that the methods can be generalized in a comparatively straightforward way to integrals of several variables.

In the following theory the presence of generalized functions in the integrand is permitted. This not only increases the range of validity of the principle, but also simplifies the proofs in some aspects, as has already been indicated for Fourier integrals by Lighthill (1958).

Section 2 is concerned with laying down reasonable conditions under which the integrals under consideration may be expected to exist and the principle of stationary phase to be applicable. The separation of the contributions from the critical points

is carried out in Section 3. Fourier integrals are discussed in Sections 4 and 5; Section 4 deals primarily with restrictions which ensure that the contribution from infinity is insignificant whereas Section 5 derives suitable expansions for critical points at a finite distance from the origin. This theory is applied to the general non-stationary critical point in Section 6. Finally, the contribution from a stationary critical point is evaluated in Section 7.

An Appendix gives a number of results; reference to equations here is indicated by the addition of the letter A.

2. Preliminaries

The purpose of the method of stationary phase is to supply an approximation to

$$\int g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$$

when $|\alpha|$ is large. In the integral f is a real-valued function of x and α is a real parameter. On the other hand g may be a generalized function provided that the integral then has a meaning.

It will be assumed that the path of integration is the whole real axis so that the integral can be written

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx.$$

Such a notation will be interpreted as

$$\lim_{M\to\infty}\int_{-M}^{M}$$

in cases of doubt. However, we are going to think of the integral as a generalized function in α (as defined by Lighthill (1958)) so that it may exist (in a generalized sense) when it would not do so according to the ordinary theory of convergence. If $H(x)$ is the Heaviside unit function and

$$\int_{-\infty}^{\infty} g(x)H(x-a)\,\mathrm{e}^{-i\alpha f(x)}\,dx,$$

a being finite, exists as a generalized function of α we shall write

$$\int_{-\infty}^{\infty} g(x)H(x-a)\,\mathrm{e}^{-i\alpha f(x)}\,dx = \int_{a}^{\infty} g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx. \tag{1}$$

Correspondingly, we write

$$\int_{a}^{b} g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = \int_{-\infty}^{\infty} g(x)H(x-a)H(b-x)\,\mathrm{e}^{-i\alpha f(x)}\,dx \tag{2}$$

when the right-hand side exists, a and b being finite. If g is an integrable complex-valued function on $[a,b]$ equation (2) always holds but it is not valid for every generalized function. For example, neither equation (1) nor (2) is valid if $g(x) = \delta(x-a)$ because the product $\delta(x-a)H(x-a)$ is not defined. Nevertheless equations (1) and (2) provide means of applying the method of stationary phase to integrals over a finite interval in such cases as are likely to arise in practice.

It will further be assumed that neither g nor f involves the parameter α.

Some additional restrictions on g and f are necessary in order that calculations may proceed. It is desirable that these restrictions should bear some relation to likely practical applications. While it is undoubtedly possible to construct functions which behave in a highly inconvenient manner, the g and f in common usage are infinitely differentiable almost everywhere. We therefore make the assumption:

I. *The points where g and f are not infinitely differentiable (in the ordinary sense) are finite in number. The number of points where $f'(x) = 0$ is finite. These points, together with $\pm\infty$, are called critical points.*

Expressed another way, the number of critical points is taken to be finite.

Next we consider possible g for which the integral has a meaning. Since we are regarding the integral as a generalized function it will have a meaning whenever

$$\int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} e^{-i\alpha f(x)}\gamma(\alpha)\,d\alpha\,dx$$

has a definite finite value for every good function γ. This requires that a meaning be ascribed to

$$\int_{-\infty}^{\infty} g(x)\Gamma[f(x)]\,dx$$

where

$$\Gamma(y) = \int_{-\infty}^{\infty} \gamma(x)\,e^{-iyx}\,dx$$

is the Fourier transform of γ. $\Gamma(y)$ is also a good function of y. There is one set of circumstances in which a meaning can be attached for any generalized function as follows:

(i) *If f is an infinitely differentiable real-valued function such that $f'(x) \neq 0$ for any x, if there are finite real $K, v_0, v_1, \ldots$ such that $|f^{(r)}(x)| < K|x|^{v_r} (r = 0,1,\ldots)$ and if there is a finite $v > 0$ such that $|f'(x)| > K_1|x|^{v-1}$ for $|x| > X$ then $\int_{-\infty}^{\infty} g(x)e^{-i\alpha f(x)}\,dx$ is a generalized function of α.*

Clearly, on these assumptions, $\Gamma[f(x)]$ is an infinitely differentiable function of x. Also, as $|x| \to \infty$, $|f| \to \infty$ because $|f| > K|x|^{v}$ and $v > 0$. Since Γ is good it follows that $|\Gamma^{(k)}[f(x)]| < |f|^{-s}$ for arbitrary integer s as $|x| \to \infty$. Therefore, taking account of (i),

$$\begin{aligned}|x|^{r}\left|\{\Gamma[f(x)]\}^{(n)}\right| &< K|x|^{r+M}\sum_{k=0}^{n}\left|\Gamma^{(k)}[f(x)]\right| \\ &< K|x|^{r+M-sv}\end{aligned}$$

for some finite M. On choosing s large enough the right-hand side tends to zero as $|x| \to \infty$. This demonstrates that

$$\Gamma[f(x)]$$

is a good function of x and, consequently, that

$$\int_{-\infty}^{\infty} g(x)\Gamma[f(x)]\,dx$$

has a meaning for any generalized function g.

To relax the restrictions on f we must be prepared to impose conditions on g. Let $g = v^{(N)}$ where v is absolutely integrable over any finite interval; because g is a generalized function v is bounded by a polynomial at infinity (see Jones (1966)). The derivatives in the specification of g and all those used from now on are generalized derivatives, to be identified with ordinary derivatives when the latter exist. Now the formula

$$\int_{-\infty}^{\infty} g(x)\Gamma\{f(x)\}\,dx = [v^{(N-1)}(x)\Gamma\{f(x)\}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v^{(N-1)}(x)[f'(x)\Gamma'\{f(x)\}]\,dx \qquad (3)$$

attaches a significance to the left-hand side whenever the right-hand side has a meaning. This certainly occurs when each of the terms on the right-hand side of equation (3) has a meaning. Now, if we apply a similar integration of parts $(N-1)$ times to the integral on the right-hand side of equation (3) we eventually arrive at an integral of the type $\int_{-\infty}^{\infty} v(x)\{\Gamma[f(x)]\}^{(N)}dx$ which will involve only ordinary functions if f possesses N ordinary derivatives. It may therefore be handled as an ordinary integral. Furthermore, assumption I implies that $v^{(N-1)}(x)$, $v^{(N-2)}(x), \ldots, v(x)$ will be continuous functions for large enough $|x|$. Therefore their values near infinity are well-defined. Hence, provided that $\Gamma(f)$ and such derivatives as are generated by the integration by parts behave satisfactorily at infinity, $[\ ]_{-\infty}^{\infty}$ (which uses only values near infinity) exists and the left-hand side of equation (3) is defined. In particular this is certainly true if f behaves at infinity as in (i) so that

(ii) *If f possesses N derivatives continuous on the real line and if, for $|x| > X$ $|f^{(r)}(x)| < K|x|^r (r = 0,1,\ldots,N)$ and $|f'(x)| > K|x|^{\nu-1}$ $(\nu > 0)$ then*

$$\int_{-\infty}^{\infty} g(x)\,e^{-i\alpha f(x)}\,dx$$

exists when $g = v^{(N)}$, v being absolutely integrable over any finite interval.

For, under these conditions, Γ behaves at infinity in the same way as in (i) and

$$\int_{-\infty}^{\infty} g(x)\Gamma[f(x)]\,dx = (-)^N\int_{-\infty}^{\infty} v(x)\{\Gamma[f(x)]\}^{(N)}\,dx. \qquad (4)$$

The integral on the right-hand side is convergent because of the properties of v and f for finite x and the behaviour of v, f and Γ as $|x| \to \infty$.

Most applications are governed by (i) and (ii) but it is obviously possible to consider specific integrals outside their scope along the general lines indicated before (ii).

As far as integrals over finite intervals are concerned f may only be specified over $[a,b]$. Outside this interval it can be defined in any convenient manner and, clearly, this can be done in such a way that the conditions of (ii) are fulfilled, provided that f has N continuous derivatives on $[a,b]$. Accordingly, if $g(x)H(x-a)H(b-x) = v^{(N)}(x)$,

$$\int_{a}^{b} g(x)\Gamma[f(x)]\,dx = (-1)^N\int_{-\infty}^{\infty} v(x)\{\Gamma[f(x)]\}^{(N)}\,dx. \qquad (5)$$

The limits of integration on the right-hand side cannot be replaced immediately by a and b because, although $g(x)H(x-a)H(b-x)$ vanishes outside $[a,b]$, v is not

necessarily zero outside the interval. Naturally, in $x > b$, v is a polynomial of degree $(N-1)$, say $\sum_{m=0}^{N-1} a_m(x-b)^m$; thus

$$\int_b^\infty v(x)\{\Gamma[f(x)]\}^{(N)}\,dx = \sum_{m=0}^{N-1} m!a_m(-1)^{m-1}\{\Gamma[f(b)]\}^{(N-m-1)}.$$

If $v,v',\ldots,v^{(N-1)}$ are continuous at $x = b$ we can put $m!a_m = v^{(m)}(b)$. With a similar procedure for the integration from $-\infty$ to a equation (5) would go over into the standard rule for integration by parts. However, the right-hand side can be calculated even when $v',\ldots,v^{(N-1)}$ are not continuous at $x = b$, e.g.

$$g(x) = (x-a)^{-\frac{1}{2}}H(x-a)H(b-x),$$
$$v(x) = -4(x-a)^{\frac{1}{2}}H(x-a)H(b-x)-2b^{-\frac{1}{2}}(x+b)H(x-b);$$

consequently the form given in equation (5) is preferable.

Correspondingly, we have, if $g(x)H(x-a) = v^{(N)}(x)$ and f satisfies (ii) at $+\infty$,

$$\int_a^\infty g(x)\Gamma[f(x)]\,dx = (-)^N\int_{-\infty}^\infty v(x)\{\Gamma[f(x)]\}^{(N)}\,dx. \tag{6}$$

In view of the structure of Γ it can be asserted that, *when one of* (*i*), (*ii*), *equation* (5) *or equation* (6) *is valid,*

$$\int_{-\infty}^\infty g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = (-1)^N\int_{-\infty}^\infty v(x)[\mathrm{e}^{-i\alpha f(x)}]^{(N)}\,dx. \tag{7}$$

A simple application of this result is that, if $\int_{-\infty}^\infty g'(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$ exists under one of these sets of conditions,

$$\int_{-\infty}^\infty g'(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = i\alpha\int_{-\infty}^\infty g(x)f'(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx. \tag{8}$$

There is another point of view which is sometimes relevant when $f'(x) \neq 0$ anywhere. Then

$$\int_{-\infty}^\infty g(x)\Gamma[f(x)]\,dx = \int_{-\infty}^\infty g[f^{-1}(y)]\left|\frac{df^{-1}}{dy}\right|\Gamma(y)\,dy$$

where $x = f^{-1}(y)$ means $y = f(x)$; therefore

$$\int_{-\infty}^\infty g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = \int_{-\infty}^\infty g[f^{-1}(y)]\left|\frac{df^{-1}}{dy}\right|\mathrm{e}^{-i\alpha y}\,dy.$$

Thus, so long as $g[f^{-1}(y)]|\,df^{-1}/dy\,|$ is a generalized function of y, calculations can be carried out in terms of it.

3. Separation of the Critical Points

In this section the separation of the contributions of the critical points is accomplished. This is done by the introduction of *fine functions* or, as they were called by van der Corput, neutralizers. A fine function is an infinitely differentiable function which, together with all its derivatives, vanishes outside a finite interval. The fine functions form a sub-class of the good functions.

Let A_j be an interval of length 2δ which has the jth critical point (supposed not a infinity) as mid-point. Because of assumption I, 2δ can be chosen small enough for A_j to contain no other critical point. If there are M critical points not at infinity j runs through $1, \ldots, M$. Let A_0 be the interval which extends from $-\infty$ to $-1/\delta$ and A_{M+1} the interval from $1/\delta$ to ∞, δ being chosen small enough for all the $A_j (j = 0,1, \ldots, M+1)$ to be disjoint.

Let $\phi_j (j = 1, \ldots, M)$ be a fine function which vanishes identically outside A_j and which equals 1 on the interval A_j' of length $2\delta_1$ $(< 2\delta)$ with the jth critical point as mid-point. On $A_j - A_j'$ let $0 \leqslant \phi_j(x) \leqslant 1$. A possible choice for ϕ_j if the critical point is at $x = x_j$ is $\phi_j(x - x_j)$ where

$$\begin{aligned} \phi_j(x) &= 1 && (|x| \leqslant \delta_1) \\ &= \frac{\int_{|x|}^{\delta} \exp\left(\frac{1}{\delta_1 - t} - \frac{1}{\delta - t}\right) dt}{\int_{\delta_1}^{\delta} \exp\left(\frac{1}{\delta_1 - t} - \frac{1}{\delta - t}\right) dt} && (\delta_1 < |x| < \delta) \\ &= 0 && (|x| \geqslant \delta). \end{aligned}$$

Let η_0, η_{M+1} be infinitely differentiable functions such that $0 \leqslant \eta_0 \leqslant 1$, $0 \leqslant \eta_{M+1} \leqslant 1$ and

$$\begin{aligned} \eta_0(x) &= 1 \quad (x \leqslant -1/\delta_1), & &= 0 \quad (x \geqslant -1/\delta), \\ \eta_{M+1}(x) &= 1 \quad (x \geqslant 1/\delta_1), & &= 0 \quad (x \leqslant 1/\delta). \end{aligned}$$

The intervals from $-\infty$ to $-1/\delta_1$ and $1/\delta_1$ to ∞ are denoted by A_0' and A_{M+1}' respectively.

Consider the function η defined by

$$\eta(x) = 1 - [1 - \phi_1(x)] \ldots [1 - \phi_M(x)][1 - \eta_0(x)][1 - \eta_{M+1}(x)].$$

It is infinitely differentiable, takes the value 1 on $A_0' \cup A_1' \cup \ldots \cup A_{M+1}'$ and vanishes outside $A_0 \cup A_1 \cup \ldots \cup A_{M+1}$. Also $0 \leqslant \eta \leqslant 1$ everywhere. Consequently $1 - \eta(x)$ is a fine function which vanishes on $A_0' \cup A_0' \cup \ldots \cup A_{M+1}'$.

On account of the properties of η

$$\int_{-\infty}^{\infty} g(x)\, e^{-i\alpha f(x)}\, dx = \int_{-\infty}^{\infty} g(x)\eta(x)\, e^{-i\alpha f(x)}\, dx + \int_{-\infty}^{\infty} g(x)[1 - \eta(x)]\, e^{-i\alpha f(x)}\, dx$$

under conditions (i) or (ii).

Now, by assumption I, g is infinitely differentiable away from the critical points and therefore $(1-\eta)g$ is a fine function, so that the last integral involves only ordinary functions. Number the A_j so that A_j lies to the left of $A_{j+1} (j = 0,1, \ldots, M)$ and let (c_j, d_j) be the interval which extends from the right-hand end of A_j to the left-hand end of A_{j+1}. On $c_j \leqslant x \leqslant d_j$ f' is one-signed by assumption I and there is no loss of generality in taking it to be positive. Then the substitution $y = f(x)$ gives

$$\int_{-\infty}^{\infty} g(x)[1 - \eta(x)]\, e^{-i\alpha f(x)}\, dx = \sum_{j=0}^{M} \int_{f(c_j)}^{f(d_j)} [1 - \eta(x)] \frac{g(x)}{f'(x)}\, e^{-i\alpha y}\, dy \qquad (9)$$

where $f(c_j)$ and $f(d_j)$ are finite on account of the assumptions made.

The function $(1-\eta)g/f'$ is infinitely differentiable with respect to y and it and all its derivatives vanish at $y = f(c_j)$ and $y = f(d_j)$ because of the properties of η. Hence, there is a fine function $\phi_0(y)$, given by

$$\begin{aligned}\phi_0(y) &= [1-\eta(x)]g(x)/f'(x) && [f(c_j) < y < f(d_j)]\\ &= 0 && [y < f(c_j) \text{ or } y < f(d_j)]\end{aligned}$$

such that the integral with respect to y can be written as

$$\int_{-\infty}^{\infty} \phi_0(y)\, \mathrm{e}^{-i\alpha y}\, dy.$$

But this is the Fourier Transform of a fine function and so, since a fine function is also good, must be a good function of α. Consequently

$$\lim_{|\alpha|\to\infty} |\alpha|^r \int_{-\infty}^{\infty} \phi_0(y)\, \mathrm{e}^{-i\alpha y}\, dy = 0$$

for any finite r. Therefore the right-hand side of equation (9) is $O(|\alpha|^{-r})$ for any finite r as $|\alpha| \to \infty$. Hence, *under conditions* (*i*) *and* (*ii*) (*including equations* (5) *and* (6))

$$\int_{-\infty}^{\infty} g(x)\, \mathrm{e}^{-i\alpha f(x)}\, dx = \int_{-\infty}^{\infty} g(x)\eta(x)\, \mathrm{e}^{-i\alpha f(x)}\, dx + O(|\alpha|^{-r}) \qquad (10)$$

for any finite r as $|\alpha| \to \infty$.

This result shows that, unless the integral on the right-hand side of equation (10) is exponentially decreasing (roughly speaking), the main contribution to the asymptotic development will be from it. This means that we can expect neighbourhoods of the critical points to provide the dominant part of the asymptotic development.

It is evident that the preceding analysis permits considerable liberty of choice in δ, δ_1 and ϕ_j so that it may be anticipated that the asymptotic contribution from any critical point is independent of δ, δ_1 and η.

Finally, it should be observed that in the analysis leading to equation (10), no great use has been made of the fact that there is only one critical point inside A_j—its main application has been to allow a free choice of δ. The analysis, and consequently equation (10), would still be valid if the critical points were infinite in number provided that they could be enclosed in a finite number of disjoint A_j'. This would permit, for example, $f'(x) = 0$ on an interval of finite positive length. However, in such a case, the asymptotic contribution would be $\int g(x)\, dx$ and could not, in general, be simplified in the way that usually occurs for the isolated critical point. The possibility will not be discussed further; an example for double integrals is given by Jones and Kline (1958).

The formula (10) demonstrates that the determination of the asymptotic behaviour our original integral turns upon the evaluation of integrals of the type

$$\int_{-\infty}^{\infty} g(x)\phi_f(x)\, \mathrm{e}^{-i\alpha f(x)}\, dx.$$

This evaluation will be investigated in detail in subsequent sections.

4. Fourier Integrals—Contribution from Infinity

All that has been proved in the preceding section applies, *a fortiori*, to Fourier transforms in which $f(x) \equiv x$. For such an f conditions (i) are obviously complied with. Hence the dominant part of the asymptotic behaviour of $G(\alpha) = \int_{-\infty}^{\infty} g(x)\, e^{-i\alpha x}\, dx$ will come from the critical points where g is not infinitely differentiable and possibly from $\pm\infty$.

First, we shall examine the contribution to be expected from $\pm\infty$ in suitable circumstances, i.e., the asymptotic development of $\int_{-\infty}^{\infty} g(x)\eta_0(x)\, e^{-i\alpha x}\, dx$ and $\int_{-\infty}^{\infty} g(x)\eta_{M+1}(x)\, e^{-i\alpha x}\, dx$ will be investigated. It is assumed, as before, that g is infinitely differentiable on A_0 and A_{M+1}.

A generalized function g is said to *behave at positive infinity* if $g = \sum_{l=1}^{L} g_l$ (L finite) for $x \geq 1/\delta$ where $g_l (l = 1, \ldots, L)$ is infinitely differentiable and $[e^{ib_l x} g_l(x)]^{(Q)}$ is absolutely integrable over $(1/\delta, \infty)$ for some finite real b_l and some non-negative integer Q. A generalized function with similar properties on $(-\infty, -1/\delta)$ is said to *behave at negative infinity*. A generalized function which behaves at both positive and negative infinity with the same value of Q is said to *behave at infinity*.

For example, the generalized function $e^{ibx} \mid x \mid^{\beta} \ln \mid x \mid$ behaves at infinity; in this case Q may be taken as any integer greater than $1 + \mathscr{R}(\beta)$. On the other hand, e^{ix^2} does not behave at infinity although $e^{ix^2}/(1+x^2)$ does (with $Q = 0$).

Now

$$e^{-ib_l x}[e^{ib_l x} g_l(x)\eta_{M+1}(x)]^{(Q)} = \sum_{q=0}^{Q} \frac{Q!\, e^{-ib_l x}}{q!(Q-q)!} \eta_{M+1}^{(q)}(x)[e^{ib_l x} g_l(x)]^{(Q-q)}.$$

Since $\eta_{M+1}^{(q)}$ $(q > 0)$ vanishes outside a finite interval, the right-hand side consists of the sum of $\eta_{M+1}\, e^{-ib_l x}[e^{-ib_l x} g_l]^{(Q)}$ and a finite number of fine functions. Hence the left-hand side is absolutely integrable over $(-\infty, \infty)$. Therefore, by the Riemann-Lebesgue lemma, its Fourier transform is $o(1)$ as $\mid \alpha \mid \to \infty$. But this Fourier transform is $[i(\alpha + b_l)]^Q$ times the Fourier transform of $g_l \eta_{M+1}$. Since b_l is finite and independent of α one deduces that

$$\int_{-\infty}^{\infty} g_l(x)\eta_{M+1}(x)\, e^{-i\alpha x}\, dx = o(|\alpha|^{-Q})$$

as $\mid \alpha \mid \to \infty$. Consequently, when g *behaves at positive infinity*,

$$\int_{-\infty}^{\infty} g(x)\eta_{M+1}(x)\, e^{-i\alpha x}\, dx = o(|\alpha|^{-Q}). \tag{11}$$

Similarity, if g behaves at a negative infinity

$$\int_{-\infty}^{\infty} g(x)\eta_0(x)\, e^{-i\alpha x}\, dx = o(|\alpha|^{-Q}) \tag{12}$$

the Q not necessarily being the same as in equation (11). By combining these two results we obtain: *if g behaves at infinity*

$$\int_{-\infty}^{\infty} g(x)[\eta_0(x)+\eta_{M+1}(x)]\,e^{-i\alpha x}\,dx = o(|\alpha|^{-Q}). \tag{13}$$

Another relevant result will be found at the end of the next section.

One more definition is useful. The generalized function g is said to be *well-behaved* if it behaves at infinity and if, in addition, $[e^{ib_1x}g_l(x)]^{(Q+1)}$, $[e^{ib_1x}g_l(x)]^{(Q+2)}, \ldots$ are absolutely integrable over $|x| > 1/\delta$. Examples of well-behaved generalized functions are $\delta(x)$, $e^{ibx}|x|^\beta \ln|x|$, $e^{i|x|^\frac{1}{2}}$, $e^{i/x}$; in contrast $e^{ix^2}/(1+x^2)$ is not well-behaved.

Corresponding definitions for a generalized function which is well-behaved at positive (negative) infinity can be formulated but will not be used here.

If g is well-behaved Q in equation (13) can be replaced by any larger positive integer. Hence *if g is well-behaved*

$$\int_{-\infty}^{\infty} g(x)[\eta_0(x)+\eta_{M+1}(x)]\,e^{-i\alpha x}\,dx = O(|\alpha|^{-r}) \tag{14}$$

for any finite $r > Q$ as $|\alpha| \to \infty$.

It follows, from formula (10), that, *if g satisfies assumption I and is well-behaved*

$$\int_{-\infty}^{\infty} g(x)\,e^{-i\alpha x}\,dx = \sum_{j=1}^{M}\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha x}\,dx + O(|\alpha|^{-r}) \tag{15}$$

for any finite $r > Q$ as $|\alpha| \to \infty$.

Thus a well-behaved generalized function is characterized by the fact that the asymptotic development of its Fourier transform is dominated by the contributions from the points where infinite differentiability fails.

The fact that it is only these points which matter leads to a formula which is very often valuable. Suppose that, in the interval where ϕ_j is non-zero, $[g(x)-\sum_{s=1}^{S} g_{js}(x)]^{(N_j)}$ is absolutely integrable and suppose that g_{js} is well-behaved and infinitely differentiable away from $x = x_j$. Then, by the same method as was used to prove equation (11),

$$\int_{-\infty}^{\infty} [g(x)-\sum_{s=1}^{S} g_{js}(x)]\phi_j(x)\,e^{-i\alpha x}\,dx = o(|\alpha|^{-N_j}).$$

Also, since g_{js} has no critical point except $x = x_j$ and is well-behaved, equation (15) (with g_{js} for g) gives

$$\int_{-\infty}^{\infty} g_{js}(x)\phi_j(x)\,e^{-i\alpha x}\,dx = \int_{-\infty}^{\infty} g_{js}(x)\,e^{-i\alpha x}\,dx + O(|\alpha|^{-r}).$$

Hence, under the stated conditions,

$$\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha x}\,dx = \sum_{s=1}^{S}\int_{-\infty}^{\infty} g_{js}(x)\,e^{-i\alpha x}\,dx + o(|\alpha|^{-N_j}). \tag{16}$$

This formula is of great importance in showing how the contribution of a critical point can be expressed in terms of Fourier transforms without the intervention of ϕ_j.

So long as the g_{js} can be chosen so as to give a sufficiently good approximation to g near $x = x_j$ and yet have relatively simple Fourier transforms an excellent estimate of asymptotic behaviour can be obtained.

It is of interest to notice that the Fourier transform of $g\phi_j$ is a fairly good function since $g\phi_j$ is a generalized function which vanishes outside a finite interval. Thus, if g is well-behaved, G acts at infinity as a fairly good function which must be independent of ϕ_j. To put it another way, the conduct of a well-behaved g at infinity is related to the performance of G for finite α.

Applying formula (16) to each of the critical points from equation (15) supplies

THEOREM 1: *If g satisfies assumption I and is well-behaved, if, in some interval including x_j, $\left[g(x)-\sum_{s=1}^{S_j} g_{js}(x)\right]^{(N_j)}$ is absolutely integrable and if $g_{js}(s = 1,2,\ldots,S_j)$ is well-behaved and infinitely differentiable away from $x = x_j$, then*

$$\int_{-\infty}^{\infty} g(x)\,e^{-i\alpha x}\,dx = \sum_{j=1}^{M}\sum_{s=1}^{S_j}\int_{-\infty}^{\infty} g_{js}(x)\,e^{-i\alpha x}\,dx + o(|\alpha|^{-N_0}) \tag{17}$$

as $|\alpha|\to\infty$, with $N_0 = \min(N_1,\ldots,N_M)$.

This constitutes a slight generalization of a theorem given by Lighthill (1958).

If g_{js} is not well-behaved but behaves at infinity with index Q_{js} then formula (13) demonstrates that equation (17) is still valid with N_0 the smallest of N_j, Q_{js} taken over $s = 1,\ldots,S_j$, $j = 1,\ldots,N_M$.

One further useful result can be obtained. First write equation (17) in the form

$$G(\alpha) = \sum_{j=1}^{M}\sum_{s=1}^{S_j} G_{js}(\alpha) + o(|\alpha|^{-N_0}).$$

Now assume that xg and xg_{js} are also well-behaved. (This is not implied by g being well-behaved as can be seen from the counter-example in which g has the form $(1+x)^{-\frac{1}{2}}\exp\left(i\int^x \frac{dt}{\ln t}\right)$ near $+\infty$.) Then g and g_{js} can be replaced by xg and xg_{js} respectively throughout equation (17) which gives rise to

THEOREM 2: *If, in addition to the requirements of Theorem 1, xg and xg_{js} are well-behaved*

$$G'(\alpha) = \sum_{j=1}^{M}\sum_{s=1}^{S_j} G'_{js}(\alpha) + o(|\alpha|^{-N_0}). \tag{18}$$

Thus, provided that the extra conditions are complied with, a derivative may be taken of an asymptotic expansion without worsening the error. Of course, the relative error may be increased since G'_{js} may be of smaller magnitude than G_{js}. If this happened direct consideration of xg or $(x-x_j)g$ might permit an increase of N_j.

5. Fourier Integral—Critical Point at Finite Distance

Suppose that g possesses N_j continuous derivatives at $x = x_j$. Then in Theorem 1 we can take $S_j = N_j - 1$ and $g_{js} = (x-x_j)^s g^s(x_j)/s!$. The Fourier transform of g_{js} is now a multiple of $\delta^{(s)}(\alpha)$ which is identically zero at infinity and so makes no contribution to the asymptotic expansion. Hence the dominant part of the asymptotic

expansion of $G(\alpha)$ comes from those critical points at which the order of the first derivative which is discontinuous is smallest.

More generally, we can expect the most important contributions to come from those critical points where g is most singular. It is the purpose of this section to derive some formulae which indicate this and which also have some useful applications.

Quite often the integrals near a critical point take the form

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\,\mathrm{e}^{-i\alpha x}\,dx$$

where k is continuous and has N continuous derivatives in a neighbourhood of the critical point. To simplify some of the subsequent analysis we shall assume that the variable of integration has been changed so that the critical point is at the origin and write ϕ for ϕ_j. The modifications necessary for an arbitrarily situated critical point will be given at the end.

In order that g_0k shall be a generalized function some restrictions must be placed on g_0. We shall require that on $-\delta < x < \delta$ (the occurrence of the equality here is permissible since g_0 is assumed to be infinitely differentiable away from the critical point) $g_0 = v^{(N)}$ where $v \in L_1(-\delta,\delta)$. The notation $f \in L_p(a,b)$ signifies that

$$\int_a^b |f(x)|^p\,dx < \infty.$$

Further progress is possible if it is also assumed that $x^N g_0(x) \in L_1(-\delta,\delta)$. Note that this assumption is not necessarily implied by $v \in L_1(-\delta,\delta)$ as can be seen from the counter-example $v(x) = \sin(1/x)$, $N = 1$.

If $x^N g_0(x) \in L_1(-\delta,\delta)$ then obviously $x^N[v(x)\phi(x)]^{(N)} \in L_1(-\delta,\delta)$ because any derivative of ϕ vanishes on $(-\delta_1,\delta_1)$. Hence, by the Riemann-Lebesgue lemma, the Fourier transform of $x^N[v(x)\phi(x)]^{(N)}$ is $o(1)$ as $|\alpha| \to \infty$, i.e.

$$\frac{d^N}{d\alpha^N}[\alpha^N V_1(\alpha)] = o(1)$$

where V_1 is the Fourier transform of $v\phi$. Integration gives

$$\frac{d^s}{d\alpha^s}[\alpha^N V_1(\alpha)] = o(|\alpha|^{N-s})$$

for $s = 0,1,\ldots,N$. In particular, $V_1 = o(1)$ consistent with the assumption that $v \in L_1(-\delta,\delta)$. By taking $s = 1$, we find $V_1'(\alpha) = o(|\alpha|^{-1})$ and then by giving s the values $2,\ldots,N$ successively we have $V_1^{(s)} = o(|\alpha|^{-s})$ for $s = 0,1,\ldots,N$. When $s > N$ we can say no more than $V_1^{(s)} = o(|\alpha|^{-N})$. It is evident that, if $N \geqslant 2$, V behaves at infinity.

Since $v^{(k)}\phi^{(j)}$ is a fine function for $j \geqslant 1$, it follows that

$$\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,\mathrm{e}^{-i\alpha x}\,dx = o(|\alpha|^{N-s}) \qquad (s = 0,1,\ldots,N) \tag{19}$$

as $|\alpha| \to \infty$. If $s > N$ replace $N-s$ on the right by 0.

If, in addition, g_0 has no other critical point and $x^N g_0(x)$ is well-behaved, (obviously, then $x^s g_0(x)$ is well-behaved for $s = 0,1,\ldots,N$) equation (15) gives

$$\int_{-\infty}^{\infty} x^s g_0(x)\,\mathrm{e}^{-i\alpha x}\,dx = o(|\alpha|^{N-s}) \qquad (s = 0,1,\ldots,N). \tag{20}$$

Now the assumptions made on k imply that

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \int_{-\infty}^{\infty} g_0(x)\phi(x)\left[\sum_{s=0}^{N-1}\frac{x^s}{s!}k^{(s)}(0)+\frac{x^N}{N!}k^{(N)}(\xi x)\right]e^{-i\alpha x}\,dx$$

where ξ lies between 0 and 1. Since $x^N g_0(x)\in L_1(-\delta,\delta)$ and $k^{(N)}$ is continuous we obtain

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=1}^{N-1}\frac{k^{(s)}(0)}{s!}\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,e^{-i\alpha x}\,dx+o(1). \quad (21)$$

On account of equation (19) we expect the terms in equation (21) to be of decreasing order of importance as s increases.

If $x^{N-1}g_0(x)$ is well-behaved, ϕ can be replaced by unity on the right. Then equation (21) can be written in the form

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N-1}\frac{i^s}{s!}k^{(s)}(0)G_0^{(s)}(\alpha)+o(1). \quad (22)$$

Further, *if* $x^N g_0(x)$ *is well-behaved and satisfies the other assumptions made about* g_0 when g_0 does,

$$\int_{-\infty}^{\infty} xg_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N-1}\frac{i^{s+1}}{s!}k^{(s)}(0)G_0^{(s+1)}(\alpha)+o(1). \quad (23)$$

Translating the critical point back to x_j equations (22) and (23) give the following

THEOREM 3: *If* $g_0 = v^{(N)}$ *where* $v\in L_1(x_j-\delta,x_j+\delta)$ *if* $(x-x_j)^N g_0(x)\in L_1(x_j-\delta,x_j+\delta)$, *if* g_0 *has no other critical point than* $x = x_j$, *if* $x^{N-1}g_0(x)$ *is well-behaved and if* k *is continuous with* N *continuous derivatives on* $[x_j-\delta,x_j+\delta]$ *then*

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\,e^{-i\alpha x}\,dx = e^{-i\alpha x_j}\sum_{s=0}^{N-1}\frac{i^s}{s!}k^{(s)}(x_j)\frac{d^s}{d\alpha^s}[e^{i\alpha x_j}G_0(\alpha)]+o(1) \quad (24)$$

as $|\alpha|\to\infty$, *and, if* $x^N g_0(x)$ *is well-behaved*

$$\frac{d}{d\alpha}e^{i\alpha x_j}\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N-1}\frac{i^s}{s!}k^{(s)}(x_j)\frac{d^{s+1}}{d\alpha^{s+1}}[e^{i\alpha x_j}G_0(\alpha)]+o(1). \quad (25)$$

Note that $v\in L_1(x_j-\delta,x_j+\delta)$ is not necessary to the validity of equations (24) and (25), as remarked in connection with V_1, but is included because N will often be chosen in this way.

The above development is most appropriate when g_0 is highly singular so that N cannot be taken less than unity. If $g_0\in L_1(x_j-\delta,x_j+\delta)$ other assumptions are more convenient.

Once again the critical point is moved to the origin and it is now assumed that $x^s g_0^{(s)}(x)\in L_1(-\delta,\delta)$ for $s = 0,1,\ldots,P$. It is an immediate consequence that G_1, the Fourier transform of $g_0\phi$, satisfies

$$\frac{d^s}{d\alpha^s}[\alpha^s G_1(\alpha)] = o(1)$$

for $s = 0,1,\ldots,P$. One conclusion is that

$$\frac{d^r}{d\alpha^r}[\alpha^s G_1(\alpha)] = o(|\alpha|^{s-r})$$

for $0 < r < s$, from which can be deduced that $G_1^{(s)}(\alpha) = o(|\alpha|^{-s})$ for $s = 0,1,\ldots,P$. For $s > P$, all that one can say is $G_1^{(s)}(\alpha) = o(|\alpha|^{-P})$. In other words

$$\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,e^{-i\alpha x}\,dx = o(|\alpha|^{-s}) \tag{26}$$

as $|\alpha| \to \infty$ for $s = 0,1,\ldots,P$.

If, in addition, $x^P g_0(x)$ is well-behaved

$$\int_{-\infty}^{\infty} x^s g_0(x)\,e^{-i\alpha x}\,dx = o(|\alpha|^{-s}). \tag{27}$$

With the assumptions made the Pth derivative of $g_0(x)\phi(x)\left[k(x) - \sum_{s=0}^{P-1} \frac{x^s}{s!}k^{(s)}(0)\right]$ is absolutely integrable over $(-\delta,\delta)$ provided that $P < N$. Hence

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{P-1} \frac{k^{(s)}(0)}{s!}\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,e^{-i\alpha x}\,dx + o(|\alpha|^{-P}) \tag{28}$$

for $P < N$. Once again, formula (26) indicates that the terms of the series will be of decreasing order of magnitude in many cases.

Formulae (22) and (28) can be combined in the following way. Suppose that $g_0 = v^{(N_1)}$ where $N_1 < N$ and $x^s v^{(s)}(x) \in L_1(-\delta,\delta)$ for $s = N_1, N_1+1, \ldots, N$. Now

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N_1-1} \frac{k^{(s)}(0)}{s!}\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,e^{-i\alpha x}\,dx + \int_{-\infty}^{\infty} x^{N_1} g_0(x)\phi(x)\left[\frac{k(x) - \sum_{s=0}^{N_1-1} \frac{k^{(s)}(0)}{s!}x^s}{x^{N_1}}\right] e^{-i\alpha x}\,dx.$$

The quantity in [ ] on the right is continuously differentiable $(N-N_1)$ times and $x^s[x^{N_1}g_0(x)]^{(s)} \in L_1(-\delta,\delta)$ for $s = 0,1,\ldots,N-N_1$. Hence equation (28) can be applied with $P = N-N_1$, and so

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N-1} \frac{k^{(s)}(0)}{s!}\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\,e^{-i\alpha x}\,dx + o(|\alpha|^{N_1-N}). \tag{29}$$

On converting the critical point back to its original position we obtain

THEOREM 4: *If* $g_0 = v^{(N_1)} (0 < N_1 < N)$ *where* $(x-x_j)^s v^{(s)}(x) \in L_1(x_j-\delta, x_j+\delta)$ *for* $s = N_1, N_1+1, \ldots, N$, *if* g_0 *has no other critical point than* $x = x_j$, *if* $x^{N-1}g_0(x)$ *is well-behaved and if* k *is continuous with* N *continuous derivatives on* $[x_j-\delta, x_j+\delta]$ *then*

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\,e^{-i\alpha x}\,dx = e^{-i\alpha x_j}\sum_{s=0}^{N-1} \frac{i^s}{s!}k^{(s)}(x_j)\frac{d^s}{d\alpha^s}[e^{i\alpha x_j}G_0(\alpha)] + o(|\alpha|^{N_1-N}) \tag{30}$$

as $|\alpha| \to \infty$ *and, if* $x^N g_0(x)$ *is well-behaved,*

$$\frac{d}{d\alpha}e^{i\alpha x_j}\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\,e^{-i\alpha x}\,dx = \sum_{s=0}^{N-1} \frac{i^s}{s!}k^{(s)}(x_j)\frac{d^{s+1}}{d\alpha^{s+1}}[e^{i\alpha x_j}G_0(\alpha)] + o(|\alpha|^{N_1-N}). \tag{31}$$

The error estimates in expressions (30) and (31) can be appreciably improved if one is prepared to assume slightly more about g_0. We use the notation $f \in \hat{L}_p(a,b)$ if $f \in L_p(a-\varepsilon, b+\varepsilon)$ for some $\varepsilon > 0$ and if (a,b) can be split up into a finite number of sub-intervals $(a,c), (c,d), \ldots, (z,b)$ in each of which f has the form $f_1 - f_2$ where f_1, f_2 are non-decreasing and belong to L_p.

Functions which are in $\hat{L}_\infty(a,b)$ will be of bounded variation almost everywhere.

The usefulness of the space $\hat{L}_p$ stems from

LEMMA 1: *If $f \in \hat{L}_p(a,b)$, a and b finite, then*

$$\lim_{h\to 0} \int_a^b |f(x+h)-f(x)|\,dx = o(|h|^{1-1/p}) \qquad (1 \leqslant p < \infty) \tag{32}$$
$$= O(|h|) \qquad (p = \infty).$$

To prove this observe firstly that, as a and b are finite, $f \in L_p (p \geqslant 1)$ implies, via the Hölder inequality, that $f \in L_1$. Choose a typical member, say (c,d), of the sub-intervals, into which (a,b) can be split according to the definition of $\hat{L}_p$. Then, if $\varepsilon > h > 0$ and if h is smaller than the length of the smallest sub-interval,

$$\int_c^d |f(x+h)-f(x)|\,dx = \left(\int_c^{d-h} + \int_{d-h}^d\right) |f(x+h)-f(x)|\,dx$$
$$\leqslant \int_{d-h}^d |f(x+h)-f(x)|\,dx + \int_c^{d-h} [f_1(x+h)-f_1(x)]\,dx +$$
$$\int_c^{d-h} [f_2(x+h)-f_2(x)]\,dx$$
$$\leqslant \int_{d-h}^{d+h} |f(x)|\,dx + \int_{d-h}^d [f_1(x)+f_2(x)]\,dx - \int_c^{c+h} [f_1(x)+f_2(x)]\,dx.$$

Now

$$\int_{d-h}^d |f(x)|\,dx \leqslant \left[\int_{d-h}^d |f(x)|^p\,dx\right]^{1/p} h^{1-1/p} \tag{33}$$

by the Hölder inequality when $1 \leqslant p < \infty$. When $p = \infty$

$$\int_{d-h}^d |f(x)|\,dx \leqslant h \sup |f(x)|.$$

Similarly inequalities hold for f_1 and f_2. Since the integral in equation (33) tends to zero as $h \to 0$, the lemma is proved for a sub-interval when $h \to +0$. An analogous analysis holds if $h \to -0$ and so, since the number of sub-intervals is finite, the lemma is proved.

It is to be remarked that the estimate in the lemma for $p = \infty$ implies that f is of bounded variation almost everywhere and, if O were replaced by o, f would be a constant almost everywhere by a theorem of Titchmarsh (1927).

LEMMA 2: *If $f \in \hat{L}_p(a,b)$, a and b finite, and f_0 is of bounded variation on $(a-\varepsilon, b+\varepsilon)$ then f may be replaced by $f f_0$ in* (32).

As in the proof of Lemma 1, consider a typical sub-interval (c,d) and split it into $(c,c+h)$, $(c+h,d-h)$, $(d-h,d)$. The interval $(d-h,d)$ may be dealt with as before and

$(c,c+h)$ may be handled in a similar way. On $(c+h,d-h)$ we can put $f_0 = f_{01} - f_{02}$ where f_{01}, f_{02} are bounded, positive and non-decreasing. Now

$$f_{01}(x+h)f_1(x+h) - f_{01}(x)f_1(x) = f_{01}(x+h)[f_1(x+h) - f_1(x)] + f_1(x)[f_{01}(x+h) - f_{01}(x)].$$

The first term on the right obviously gives the same contribution as in Lemma 1 since f_{01} is bounded. With regard to the second term note that, since f_1 is non-decreasing, there are x_1 and x_2 such that $c+h \leqslant x_1 \leqslant x_2 \leqslant d-h$ and $f_1 \geqslant 0$ for $x \geqslant x_2$, $f_1 \leqslant 0$ for $x \leqslant x_1$. Hence

$$\int_{c+h}^{d-h} |f_1(x)[f_{01}(x+h) - f_{01}(x)]|\,dx = \int_{c+h}^{d-h} |f_1(x)|\,[f_{01}(x+h) - f_{01}(x)]\,dx$$

$$= \int_{x_2}^{d-h} f_1(x)[f_{01}(x+h) - f_{01}(x)]\,dx - \int_{c+h}^{x_1} f_1(x)[f_{01}(x+h) - f_{01}(x)]\,dx.$$

But

$$\int_{x_2}^{d-h} f_1(x)[f_{01}(x+h) - f_{01}(x)]\,dx = \int_{x_2+h}^{d} f_1(x-h)f_{01}(x)\,dx - \int_{x_2}^{d-h} f_1(x)f_{01}(x)\,dx$$

$$= \int_{x_2+h}^{d-h} [f_1(x-h) - f_1(x)]f_{01}(x)\,dx + \int_{d-h}^{d} f_1(x-h)f_{01}(x)\,dx - \int_{x_2}^{x_2+h} f_1(x)f_{01}(x)\,dx.$$

Each of these terms gives the same order of contribution as in Lemma 1 when $h \to 0$, account being taken of the boundedness of f_{01}.

The evaluation over $(c+h, x_1)$ is dealt with similarly, but writing

$$\int_{c+h}^{x_1} f_1(x)f_{01}(x)\,dx = \int_{c}^{x_1-h} f_1(x+h)f_{01}(x+h)\,dx.$$

Since the analysis remains valid if f_{01} is changed to f_{02} or f_1 to f_2 the proof of the lemma is complete.

It is now possible to demonstrate

THEOREM 5: *If* $f \in \hat{L}_p(a,b)$, *a and b being finite, and vanishes outside* (a,b) *then*

$$F(\alpha) = o(|\alpha|^{-1+1/p}) \qquad (1 < p < \infty)$$
$$= O(|\alpha|^{-1}) \qquad (p = \infty)$$

as $|\alpha| \to \infty$.

For

$$F(\alpha) = \int_a^b f(x)\,e^{-i\alpha x}\,dx$$
$$= -\int_a^b f(x)\,e^{-i\alpha(x+\pi/\alpha)}\,dx$$
$$= -\int_{a+\pi/\alpha}^{b+\pi/\alpha} f(x-\pi/\alpha)\,e^{-i\alpha x}\,dx.$$

Hence

$$|F(\alpha)| = \left|\tfrac{1}{2}\int_a^b f(x)\,e^{-i\alpha x}\,dx - \tfrac{1}{2}\int_{a+\pi/\alpha}^{b+\pi/\alpha} f(x-\pi/\alpha)\,e^{-i\alpha x}\,dx\right|$$

$$\leqslant \tfrac{1}{2}\int_{a+\pi/\alpha}^{b} |f(x) - f(x-\pi/\alpha)|\,dx + \tfrac{1}{2}\int_{b}^{b+\pi/\alpha} |f(x-\pi/\alpha)|\,dx + \tfrac{1}{2}\int_{a}^{a+\pi/\alpha} |f(x)|\,dx$$

there being no loss of generality in taking $\alpha > 0$. On account of Lemma 1 the first term on the right is consistent with the estimate in the theorem as $|\alpha| \to \infty$. The remaining two terms are of types which occurred in the proofs of Lemmas 1 and 2. The proof is terminated.

COROLLARY 5: *If $f \in \hat{L}_p(a,b)$, a and b being finite, and vanishes outside (a,b), if f_0 is of bounded variation on $(a-\varepsilon, b+\varepsilon)$ then the Fourier transform of ff_0 behaves, as $|\alpha| \to \infty$, in the same way as F in Theorem 5.*

The same proof as for Theorem 5 applies, with ff_0 for f and Lemma 2 for Lemma 1.

The analysis leading to Theorems 3 and 4 may now be repeated with $\hat{L}_p$ instead of L_1. Thus, if $x^N g_0(x) \in \hat{L}_p(-\delta,\delta)$, Corollary 5 shows that

$$\int_{-\infty}^{\infty} x^s g_0(x)\phi(x)\, e^{-i\alpha x}\, dx = o(|\alpha|^{N-s-1+1/p}) \qquad (s = 0,1,\ldots,N) \tag{34}$$

for $1 \leqslant p < \infty$, since ϕ is certainly of bounded variation. If $p = \infty$ the right-hand side should be changed to $O(|\alpha|^{N-s-1} \ln |\alpha|)(s < N)$, $O(|\alpha|^{-1})(s = N)$. In this way we derive

THEOREM 6: *If $g_0 = v^{(N_1)}(N_1 < N)$ where $(x-x_j)^s v^{(s)}(x) \in \hat{L}_p(x_j-\delta, x_j+\delta)$ for $s = N_1, N_1+1, \ldots, N$, if g_0 has no other critical point than $x = x_j$, if $x^{N-1}g_0(x)$ is well behaved, if $k^{(s)}(s = 0,1,\ldots,N-1)$ is continuous and $k^{(N)}$ of bounded variation on $[x_j-\delta, x_j+\delta]$ then*

$$\int_{-\infty}^{\infty} g_0(x)k(x)\phi_j(x)\, e^{-i\alpha x}\, dx = e^{-i\alpha x_j} \sum_{s=0}^{N-1} \frac{i^s}{s!} k^{(s)}(x_j) \frac{d^s}{d\alpha^s}[e^{i\alpha x_j} G_0(\alpha)] + R_N$$

as $|\alpha| \to \infty$, where $R_N = o(|\alpha|^{N_1-N-1+1/p})$ $(1 < p < \infty)$, and $R_N = O(|\alpha|^{N_1-N-1})$ $(p = \infty)$.

Remark: If $k^{(N+1)} \in L_1$, $k^{(N)}$ is absolutely continuous and therefore of bounded variation so that in this case k would satisfy the conditions of Theorem 6.

An illustration of the theorems is provided by $g_0(x) = (x-x_j)^{\lambda-n} H(x-x_j)$ where $0 < \lambda < 1$, n is a positive integer and H is the Heaviside unit function. The conditions of Theorem 6 are satisfied with $N_1 = n-1$ and $1/p > 1-\lambda$. Hence if k satisfies the conditions of Theorem 6, as $\alpha \to +\infty$,

$$\int_{-\infty}^{\infty} (x-x_j)^{\lambda-n} H(x-x_j)k(x)\phi_j(x)\, e^{-i\alpha x}\, dx$$

$$= e^{-i\alpha x_j} \sum_{s=0}^{N-1} \frac{k^{(s)}(x_j)}{s!} (\lambda+s-n)!\, e^{-\frac{1}{2}\pi i(\lambda-n+1+s)} \alpha^{n-\lambda-s-1} + R_N \tag{35}$$

where $$R_N = o(|\alpha|^{n-N-\lambda-1+\varepsilon}) \qquad (\varepsilon > 0,\ 0 < \lambda < 1)$$

and $$R_N = O(|\alpha|^{n-N-2}) \text{ if } \lambda = 1.$$

If $k^{(N)}$ were continuous but not of bounded variation Theorem 4 gives

$$R_N = o(|\alpha|^{n-N-1}).$$

A similar expansion is available when $g_0(x) = (x-x_j)^{\lambda-n} H(x-x_j) \ln (x-x_j)$. Formally, it can be obtained from equation (35) by taking a derivative with respect to λ without affecting the estimate of R_N, unless λ is put equal to unity after the operation in which case $R_N = o(|\alpha|^{n-N-1})$.

Yet another criterion for calculating the error may be obtained by using Theorem 9 below. It is especially appropriate when the generalized function vanishes on one side of the critical point.

6. Critical Points where $f' \neq 0$

This section is concerned with applying the results developed in the preceding two sections to the more general integral of Section 2. Attention will be confined to those critical points where $f' \neq 0$. In view of our assumptions $f' \neq 0$ can be taken to be true on the whole interval for which $\phi_j \neq 0$. Hence, if f and its first N derivatives are continuous, the substitution $y = f(x)$ gives an inverse which is N times continuously differentiable.

Then

$$\int_{-\infty}^{\infty} g(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = \int_{-\infty}^{\infty} g[f^{-1}(y)]\phi_j[f^{-1}(y)]\left|\frac{df^{-1}}{dy}\right|\mathrm{e}^{-i\alpha y}\,dy \qquad (36)$$

the limits of integration being taken as $\pm\infty$ with $\phi_j(f^{-1})$ identically zero outside $(f(x_j-\delta),f(x_j+\delta))$. We are thus led to a Fourier integral of the type which has been considered in the preceding sections. Although it seems, at first sight, that $\phi_j(f^{-1})$ may not be fine unless f is infinitely differentiable it should be noted that differentiability of f can fail only at $x = x_j$ according to assumption I and the choice of ϕ_j. But in a neighbourhood of $x = x_j$, and therefore of $y = f(x_j)$, ϕ_j is equal to unity and so all its derivatives are zero. Hence the lack of differentiability of f at $x = x_j$ does not prevent $\phi_j(f^{-1})$ being a fine function of y.

It follows that if $g(f^{-1})|\,df^{-1}/dy\,|$ is well-behaved (in y) ϕ_j may be placed equal to unity on the right of equation (36) as far as asymptotic behaviour is concerned.

On account of conditions (ii) the contributions of integrals where ϕ_j is replaced by η_0 or η_{M+1} can be dealt with in the same way.

Due to the wide variations permitted in f it is not easy to formulate conditions (in terms of x) which guarantee the applicability of the formulae of Sections 4 and 5 without making those conditions unnecessarily restrictive. For particular cases it is more appropriate to change to the variable y and then verify whether the conditions are complied with.

For example, we can see from equation (35) that, if $k^{(N)}(x)$ is continuous and f possesses $(N+1)$ continuous derivatives,

$$\int_{-\infty}^{\infty} (x-x_j)^{\lambda-n}H(x-x_j)k(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$$
$$= \mathrm{e}^{-i\alpha f(x_j)}\sum_{s=0}^{N-1}\frac{k_0^{(s)}[f(x_j)]}{s!}(\lambda+s-n)!\,\mathrm{e}^{-\frac{1}{2}\pi i(\lambda-n+1+s)}\alpha^{n-\lambda-s-1}+o(|\alpha|^{n-N-1}) \qquad (37)$$

where

$$k_0(y) = \frac{k(x)(x-x_j)^{\lambda-n}}{f'(x)[y-f(x_j)]^{\lambda-n}}.$$

There is an alternative way of calculating the asymptotic development which is often of value because it avoids making the change of variable $y = f(x)$. Since

$$e^{-ix} = \sum_{s=0}^{S-1}\frac{(-ix)^s}{s!} + \frac{(-ix)^S}{(S-1)!}\int_0^1 (1-t)^{S-1}\,\mathrm{e}^{-itx}\,dt$$

we have, putting

$$f_0(x) = f(x)-f'(x_j)(x-x_j)-f(x_j),$$

$$\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha f(x)}\,dx$$

$$= \exp\{-i\alpha[f(x_j)-x_jf'(x_j)]\}\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha f_0(x)}\,e^{-i\alpha f'(x_j)x}\,dx$$

$$= \exp\{-i\alpha[f(x_j)-x_jf'(x_j)]\}\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha f'(x_j)x}\left[\sum_{s=0}^{S-1}\frac{[-i\alpha f_0(x)]^s}{s!}+\right.$$

$$\left.\frac{[-i\alpha f_0(x)]^S}{(S-1)!}\int_0^1(1-t)^{S-1}\,e^{-i\alpha t f_0(x)}\,dt\right]dx. \qquad (38)$$

The terms arising from the series are Fourier integrals with $\alpha f'(x_j)$ for α.

With regard to the last term of equation (38) take as a new variable y where

$$y = tf_0(x)+f'(x_j)(x-x_j). \qquad (39)$$

If $f' > 0$, $tf_0'(x)+f'(x_j) \geqslant \frac{1}{2}f'(x_j)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $tf_0'(x)+f'(x_j) \geqslant \frac{1}{2}f'(x)$ for $\frac{1}{2} \leqslant t \leqslant 1$. Hence, if $f' > 0$, (39) defines x as a function $\psi(t,y)$ which is differentiable in t and y, as many times as f is differentiable, and has the value $x_j+y/f'(x_j)$ at $t = 0$ provided that δ is chosen small enough. A similar proof is valid when $f' < 0$. Hence the last term of equation (38) can be written as

$$\int_0^1(1-t)^{S-1}\int g(\psi)\phi_j(\psi)[-i\alpha f_0(\psi)]^S\frac{\partial\psi}{\partial y}\,e^{-i\alpha y}\,dy\,dt.$$

The limits on the inner integral can be taken as $\pm\infty$ because ϕ_j acts as a fine function of y just as earlier in this section.

Suppose now that $(x-x_j)^{N_1+s}g^{(s)}(x) \in L_1(x_j-\delta,x_j+\delta)$ for $s = 0,1,\ldots,N-N_1$ (it is assumed that $N \geqslant N_1 \geqslant 0$ and that f has $(N+1)$ continuous derivatives). Then it is clear that $y^{N_1+s}\,d^sg(\psi)/dy^s$ belongs to L_1 for all t in $[0,1]$. Also when $y \approx 0, f_0(\psi) = O(y^2)$ when $N \geqslant 1$. Hence the $(N-N_1)$th derivative with respect to y of the integrand (the factor $e^{-i\alpha y}$ being omitted) is absolutely integrable provided that $2S \geqslant N$. Therefore, as $|\alpha| \to \infty$ the integral with respect to y is $o(|\alpha|^{N_1-N})$ uniformly in t for $0 \leqslant t \leqslant 1$. Consequently the double integral is $o(|\alpha|^{S+N_1-N})$. This is least when S is chosen as the smallest integer which is not less than $\frac{1}{2}N$. Hence we have

THEOREM 7: *If $(x-x_j)^{N_1+s}g^{(s)}(x) \in L_1(x_j-\delta,x_j+\delta)$ for $s = 0,1,\ldots,N-N_1$ $(0 \leqslant N_1 \leqslant N)$, if f is continuous with $(N+1)$ continuous derivatives on $[x_j-\delta,x_j+\delta)$ and if $f' \neq 0$ then*

$$\int_{-\infty}^{\infty} g(x)\phi_j(x)\,e^{-i\alpha f(x)}\,dx = \exp\{-i\alpha[f(x_j)-x_jf'(x_j)]\}\times$$

$$\sum_{s=0}^{N_0-1}\frac{(-i\alpha)^s}{s!}\int_{-\infty}^{\infty} g(x)\phi_j(x)[f_0(x)]^s\,e^{-i\alpha f'(x_j)x}\,dx+o(|\alpha|^{N_0+N_1-N})$$

as $|\alpha| \to \infty$, N_0 being the smallest integer which is not less than $\frac{1}{2}N$.

The advantages of this theorem are (i) it reduces the calculation to the evaluation of Fourier integrals, with $\alpha f'(x_j)$ for α, of the type considered in earlier sections and (ii) the terms in the series are easily written down from the expansion of $e^{-i\alpha f(x)}$ after the linear and constant terms of f have been removed. Quite often, especially when

f is given as a power series, it is quicker to determine the first few terms of the asymptotic development in this way. If many terms are needed it may be more cumbersome because so many of the integrals may provide contributions of the same order.

7. Critical Points where $f' = 0$

Finally, we come to the contributions from the critical points which are determined by the method of stationary phase, namely the critical points where $f' = 0$. In this case the substitution $y = f(x)$ is not satisfactory and it becomes desirable to consider the intervals to the left and right of the critical point separately. For this reason equation (7) is employed and, if $g = v^{(N_1)} (N_1 < N)$,

$$\int_{-\infty}^{\infty} g(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx = (-)^{N_1}\int_{-\infty}^{\infty} v(x)[\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}]^{(N_1)}\,dx$$

$$= (-)^{N_1}\int_{-\infty}^{\infty} v(x)\phi_j(x)[\mathrm{e}^{-i\alpha f(x)}]^{(N_1)}\,dx \qquad (40)$$

ignoring contributions of good functions and noting that derivatives of ϕ_j vanish on a neighbourhood of the critical point. By splitting the interval of integration at the critical point we are led to consider integrals of the type, $\int_{x_j}^{\infty} v(x)\phi_j(x)k(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$ or

$$\int_{-\infty}^{\infty} v(x)H(x-x_j)\phi_j(x)k(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$$

where k is continuous on $x \geqslant x_j$.

Let us suppose that on $x_j+\delta \geqslant x \geqslant x_j$

$$f'(x) = (x-x_j)^{\mu-1}f_1(x)$$

where f_1 is positive and N times continuously differentiable, and $\mu \geqslant 1$. The class of f thereby covered is wider than that contemplated in deriving equation (40). Now make the change of variable $y = f(x)-f(x_j)$. The inverse mapping takes the form

$$x-x_j = \left[\frac{\mu y}{f_1(x_j)}\right]^{1/\mu}[1+a_1y^{1/\mu}+a_2y^{2/\mu}+\ldots] \qquad (41)$$

and in fact $\left(y^{1-1/\mu}\dfrac{d}{dy}\right)^s x$ is continuous on $y \geqslant 0$ for $s = 0,1,\ldots,N+1$.

We are thus led to integrals of the type

$$\int_{-\infty}^{\infty} v(x)H(y)\phi_j(x)k(x)\frac{dx}{dy}\,\mathrm{e}^{-i\alpha y}\,dy. \qquad (42)$$

Next we introduce the operator I (whose properties are considered more fully in the Appendix) and use the suffix y to indicate that quantities are to be expressed in terms of the variable y, i.e. if $x = h(y)$ is written for equation (41)

$$I_y^{\beta}(vt^{\theta}) = \int_0^y v[h(t)]t^{\theta}(y-t)^{\beta-1}\,dt/(\beta-1)!$$

when β is not zero or a negative integer.

Now, if $\rho < 0$ and $I_y^\rho f \in L_1$ (as defined in the Appendix but now in respect of the variable y), Lemma A3 shows that $I_y^\rho(f\phi_j) \in L_1$. It will now be shown that, in fact, $I_y^\rho(f\phi_j) \in L_1(-\infty,\infty)$. If $\rho = 0$ this is obviously true since $f\phi_j$ vanishes outside a finite interval. If $\rho < 0$ there is a positive integer p such that $\rho = \sigma - \phi$ with $0 \leqslant \sigma < 1$. Then, for large y,

$$I_y^\rho(f\phi_j) = \frac{d^{p+1}}{dy^{p+1}} \int_0^T (y-t)^\sigma f(t)\phi_j[h(t)]\, dt/\sigma!$$

since ϕ_j vanishes outside a finite interval, say $(0,T)$. If $\sigma = 0$ the right-hand side is zero and if $\sigma \neq 0$ it is

$$\int_0^T (y-t)^{\sigma-p-1} f(t)\phi_j[h(t)]\, dt/(\sigma-p-1)!.$$

Evidently, in either case $I_y^\rho(f\phi_j) = O(y^{\rho-1})$ as $y \to \infty$. Since $I_y^\rho(f\phi_j) \in L_1$, it follows that $I_y^\rho(f\phi_j) \in L_1(-\infty,\infty)$.

The considerations of the previous paragraph indicate that only the values of f on $[0,T]$ are of significance and, indeed, it is clear from the behaviour of $I_y^\rho(f\phi_j)$ outside this interval that the condition $I_y^\rho f \in L_1$ can be replaced by $I_y^\rho f \in L_1(a,b)$ where (a,b) is any finite interval which encloses $[0,T]$. Since $\phi_j[h(y)]$ can always be chosen to vanish outside $-\frac{1}{2}T \leqslant y \leqslant \frac{1}{2}T$ there is no loss of generality in taking (a,b) to be $(0,T)$.

Returning to (42) we shall assume, in order to permit a fairly wide variety of k, that

$$k(x) = (x-x_j)^\nu k_1(x)$$

where $\nu \geqslant 0$ and k_1 is continuous with N continuous derivatives. It follows that

$$\left(y^{1-1/\mu}\frac{d}{dy}\right)^s \frac{k_1(x)}{y^{\nu/\mu}}$$

is continuous on $y \geqslant 0$ for $s = 0,1,\ldots,N+1$. Consequently

$$y^{1-(\nu+1)/\mu} k(x)\frac{dx}{dy} = b_0 + b_1 y^{1/\mu} + b_2 y^{2/\mu} + \ldots + b_{N-1} y^{N-1/\mu} + k_0(y), \tag{43}$$

where

$$b_j = \frac{1}{j!}\left[\left(\mu y^{1-1/\mu}\frac{d}{dy}\right)^j y^{1-(\nu+1)/\mu} k(x)\frac{dx}{dy}\right]_{y=0}$$

and k_0 is such that $y^{s-N/\mu} k_0^{(s)}(y)$ $(s = 0,1,\ldots,N)$ are continuous.

Now, if we assume that $I_y^{-(\nu+s)/\mu}[v t^{-1+(\nu+1+s)/\mu}] \in L_1(0,T)$ for $s = 0,\ldots,N$ the above considerations and Lemma A5 demonstrate that

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)] y^{-1+(\nu+1+s)/\mu} H(y)\, e^{-i\alpha y}\, dy = o(|\alpha|^{-(\nu+s)/\mu}). \tag{44}$$

There remains consideration of

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)] k_0(y) y^{-1+(\nu+1)/\mu}\, e^{-i\alpha y}\, dy.$$

Suppose firstly that $\nu = 0$ and that $-N/\mu$ is not an integer. Then there is a positive integer p and a σ, with $0 < \sigma < 1$, such that $-N/\mu = \sigma - p$. Since $\mu \geqslant 1$, $p < N+1$.

Also $k_0(t) = t^{N/\mu}[t^{-N/\mu}k_0(t)]$ and $x^s[x^{-N/\mu}k_0(x)]^{(s)}$ is continuous for $s = 0, \ldots, N$ because of the properties of k_0. It follows from our assumptions and Lemma A4 that $I_y^{-N/\mu}(v\phi_j k_0 t^{-1+1/\mu}) \in L_1(0,T)$ and hence, from Lemma A5,

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]k_0(y)y^{-1+1/\mu}\,e^{-i\alpha y}\,dy = o(|\alpha|^{-N/\mu}). \tag{45}$$

If $v = 0$ and $-N/\mu$ is a negative integer it seems just as easy to add supplementary conditions so that $y^{-1+1/\mu}v[h(y)]$, $y^{1/\mu}v[h(y)]$, $\ldots$, $y^{-1+(N+1)/\mu}v^{(N/\mu)}[h(y)]$ all belong to L_1, since such conditions can be verified as easily as the others. It is then clear that formula (45) will be valid.

If $v \neq 0$, $(v+N)/\mu$ may exceed $1+N$ so that k_0 and h may not possess sufficient derivatives to permit the application of Lemma A4. If r is the integer such that $-(N+1)-1/\mu \leqslant -(v+N-r)/\mu < -(N+1)$ the right-hand side of formula (45) becomes $o(|\alpha|^{-(v+N)/\mu})$ if $r \leqslant 0$ and $o(|\alpha|^{-(v+r-N)/\mu})$ if $r > 0$. The former alternative arises if $0 \leqslant v < 1-N+\mu(N+1)$; if the second alternative occurs it can always be said that the left-hand side of formula (45) is $o(|\alpha|^{-N-1})$. However, this may not be sufficient to ensure that it is less significant than earlier terms which are retained in explicit form.

We may summarize these results as

THEOREM 8: *If equation* (43) *holds, if* $I_y^{-(v+s)/\mu}[vt^{-1+(v+1+s)/\mu}] \in L_1(0,T)$ *for* $s = 0, \ldots, N$, *if, when* $-(v+N)/\mu$ *is an integer* $-M$, $y^{-1+1/\mu}v$, $y^{1/\mu}\dfrac{dv}{dy}$, $\ldots$, $y^{-1+1/\mu+M}\dfrac{d^M v}{dy^M}$ *all belong to* $L_1(0,T)$ *and if* $k(x) = (x-x_j)^\nu k_1(x) (\nu > 0)$ *where* k_1 *is continuous with* N *continuous derivatives then*

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]k(x)\frac{dx}{dy}H(y)\,e^{-i\alpha y}\,dy$$

$$= \sum_{s=0}^{N-1} b_s \int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]y^{-1+(v+1+s)/\mu}H(y)\,e^{-i\alpha y}\,dy + R_N$$

and if $v[h(y)]y^{-1+(v+1+N)/\mu}$ *is well-behaved*

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]k(x)\frac{dx}{dy}H(y)\,e^{-i\alpha y}\,dy$$

$$= \sum_{s=0}^{N-1} b_s \int_{-\infty}^{\infty} v[h(y)]y^{-1+(v+1+s)/\mu}H(y)\,e^{-i\alpha y}\,dy + R_N$$

where $R_N = o(|\alpha|^{-(v+N)/\mu})$ *if* $0 \leqslant v < \mu(N+1)+1-N$ *and* $R_N = o(|\alpha|^{-(v+N-r)/\mu})$ *otherwise.*

On account of formula (44) we expect the dominant terms to come from the smaller values of s in most circumstances.

A variation of this theorem, which is of some importance, arises if we assume slightly more about v. Consider the case $v = 0$ and suppose that $I^{-(\lambda+s)/\mu}[vt^{-1+(1+s)/\mu}] \in L_1(0,T)$ with $0 < \lambda \leqslant 1$. Then we may proceed as before and obtain formulae (44) and (45) with the extra factor $|\alpha|^{-\lambda/\mu}$ on the right-hand side. Thus we have

THEOREM 9: *If* $I_y^{-(\lambda+s)/\mu}[vt^{-1+(1+s)/\mu}] \in L_1(0,T)$ $(0<\lambda<1)$ *for* $s = 0,1,\ldots,N$ *and* k *is continuous with* N *continuous derivatives*

$$\int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]k(x)\frac{dx}{dy}H(y)\,e^{-ixy}\,dy$$

$$= \sum_{s=0}^{N-1} b_s \int_{-\infty}^{\infty} v[h(y)]\phi_j[h(y)]y^{-1+(1+s)/\mu}H(y)\,e^{-ixy}\,dy + o(|\alpha|^{-(\lambda+N)/\mu}). \tag{46}$$

As an example of this theorem consider $\int_{x_j}^{\infty}(x-x_j)^{\lambda-1}v_1(x)\phi_j(x)\,e^{-i\alpha f(x)}\,dx$ where $0<\lambda<1$ and v_1 is N times continuously differentiable for $x_j \leqslant x \leqslant x_j+\delta$. Then if we identify $v[h(y)]$ with $y^{(\lambda-1)/\mu}$ and $k(x)$ with $y^{(1-\lambda)/\mu}(x-x_j)^{\lambda-1}v_1(x)$ we have (see the definition (A.1))

$$I_y^{-(\lambda'+s)/\mu}[vt^{-1+(1+s)/\mu}] = y_{-(\lambda'+s)/\mu} * y_{(\lambda+s)/\mu}[(\lambda+s)/\mu-1]!$$
$$= y_{(\lambda-\lambda')/\mu}[(\lambda+s)/\mu-1]!$$

which belongs to L_1 if $\lambda' < \lambda$. Hence, from Theorem 9,

$$\int_{x_j}^{\infty}(x-x_j)^{\lambda-1}v_1(x)\phi_j(x)\,e^{-i\alpha f(x)}\,dx$$

$$= \sum_{s=0}^{N-1}\left(\frac{\lambda+s}{\mu}-1\right)!c_s\alpha^{-(\lambda+s)/\mu}\,e^{-\pi i(\lambda+s)/2\mu} + o(|\alpha|^{-(\lambda+N-s)/\mu}) \tag{47}$$

as $\alpha \to +\infty$. Here $s>0$ and

$$c_s = \frac{1}{s!}\left[\left(\mu y^{1-1/\mu}\frac{d}{dy}\right)^s\left\{y^{1-\lambda/\mu}(x-x_j)^{\lambda-1}v_1(x)\frac{dx}{dy}\right\}\right]_{y=0}$$
$$= \frac{1}{s!\mu}\left[\left(\frac{d}{du}\right)^s\left\{u^{1-\lambda}(x-x_j)^{\lambda-1}v_1(x)\frac{dx}{du}\right\}\right]_{u=0}$$

where $y = u^{\mu}$.

There is a similar result when $(x-x_j)^{\lambda-1}\ln(x-x_j)$ is put for $(x-x_j)^{\lambda-1}$.

It is often convenient to have the conditions expressed in the original variable x. To this end, note that if $\rho > 0$

$$I_y^{\rho}f_0 = \int_0^y (y-t)^{\rho-1}f_0[h(t)]\,dt$$
$$= \int_{x_j}^{x}[f(x)-f(u)]^{\rho-1}f_0(u)\frac{dt}{du}\,du$$

on putting $y = f(x)-f(x_j)$, $t = f(u)-f(x_j)$. Since we are only concerned with x near x_j we can, in most circumstances replace $f(x)-f(u)$ by $[(x-x_j)^{\mu}-(u-x_j)^{\mu}]f_1(x_j)/\mu$. Then the condition $I_y^{-(\nu+s)/\mu}[vt^{-1+(\nu+1+s)/\mu}] \in L_1(0,T)$ becomes, if $-(\nu+s)/\mu = -p+\rho$ with $0<\rho<1$,

$$\frac{d}{dx}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^{p-1}H(x-x_j)\int_{x_j}^{x}[f(x)-f(u)]^{\rho-1}v(u)(u-x_j)^{\nu+s}\,du \in L_1(x_j,x_j+\delta).$$

There is an analogue to Theorem 7. It is obtained by writing

$$f(x) = f(x_j)+(x-x_j)^{\mu}f_1(x_j)/\mu+f_0(x)$$

and proceeding in the same way but with the substitution $y = tf_0(x)+(x-x_j)^{\mu}f_1(x_j)/\mu$ instead of equation (39). Then we have

THEOREM 10: *If* $I_y^{-(1+1/\mu)S}[vt^{S-1+(S+1)\mu}]\in L_1(0,T)$ *and* $(1+1/\mu)S < N$

$$\int_{x_j}^{\infty} v(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$$

$$= \mathrm{e}^{-i\alpha f(x_j)}\sum_{s=0}^{S-1}\frac{(-i\alpha)^s}{s!}\int_{x_j}^{\infty} v(x)\phi_j(x)[f_0(x)]^s \exp\left[-i\alpha(x-x_j)^{\mu}f_1(x_j)/\mu\right]dx+o(|\alpha|^{-S/\mu})$$

as $|\alpha|\to\infty$.

This formula is often useful when the asymptotic behaviour of the integrals in the series can be calculated relatively simply.

Formulae for $\int_{-\infty}^{x_j} v(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$ follow from the preceding by easy changes of variable and hence asymptotic developments of $\int_{-\infty}^{\infty} g(x)\phi_j(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$ are obtained from equation (40). Expressions for $\int_{-\infty}^{\infty} g(x)\,\mathrm{e}^{-i\alpha f(x)}\,dx$ can then be derived from equations (10) and (15).

Finally we remark that it may be possible to obtain sharper estimates of the error if it is assumed that $v\in L_p$ or $\hat{L}_p$ by using the results of Hardy and Littlewood (1928) on I^{β}.

Appendix

In this appendix some of the properties of the operator I^{ρ} used in Section 7 are given.

The generalized function $I^{\rho}g$ is defined for all complex ρ and all $g\in K_+$, K_+ being the space of generalized functions which are zero for $x< 0$, by the convolution

$$I^{\rho}g = x_{\rho}*g \tag{A.1}$$

where

$$x_{\rho} = \frac{x^{\rho-1}H(x)}{(\rho-1)!} \qquad (\rho \neq 0,-1,-2,\ldots)$$

$$= \delta^{(-\rho)}(x) \qquad (\rho = 0,-1,-2,\ldots).$$

It has the property that $I^{\rho}g\in K_+$. Also

$$I^{\rho}(I^{\sigma}g) = (I^{\rho}I^{\sigma})g = I^{\rho+\sigma}g, \tag{A.2}$$

$$I^{-m}g = g^{(m)} \qquad (m = 0,1,\ldots), \tag{A.3}$$

$$I^{\rho-m}g = \frac{d^m}{dx^m}I^{\rho}g = I^{\rho}g^{(m)} \qquad (m = 0,1,\ldots), \tag{A.4}$$

generalized derivatives being employed. If ρ is neither zero nor a negative integer

$$I^{\rho}g = \int_{-\infty}^{\infty} \frac{(x-t)^{\rho-1}}{(\rho-1)!} H(x-t)g(t)\,dt$$

$$= \frac{H(x)}{(\rho-1)!} \int_0^x (x-t)^{\rho-1} g(t)\,dt.$$

The notation $f \in L_1$ will be used to signify a function which belongs to K_+ and is absolutely integrable over every finite interval. If $f \in L_1$ then $I^{\rho}f \in L_1$ for $\rho > 0$. As a matter of fact this is also true if $\mathscr{R}(\rho) > 0$ but we shall be concerned exclusively with real ρ from now on.

Suppose now that $0 \leqslant \sigma < 1$, that p is a positive integer, and that $I^{\sigma-p}g \in L_1$. On account of equation (A.4) $I^{\sigma}g^{(p)} \in L_1$ and a further application of equations (A.4) and (A.2) gives $g^{(p-1)} = I^{1-\sigma}f$ where $f \in L_1$. It follows at once that $I^{\sigma-p}g \in L_1$ implies that $g^{(p-1)} \in L_1$.

More can be said because $I^{\sigma}g = I^{p}f$ so $I^{\sigma}g$ and its first $(p-1)$ derivatives must be absolutely continuous and vanish at the origin. Conversely if this be true the derivative of the absolutely continuous $I^{\sigma+1-p}g$ must be integrable which, from equation (A.4), mean that $I^{\sigma-p}g \in L_1$. Therefore we have

LEMMA A.1: *Necessary and sufficient conditions for $I^{\sigma-p} \in L_1$ are that $I^{\sigma}g$, $I^{\sigma-1}g, \ldots,$ $I^{\sigma+1-p}g$ be absolutely continuous and vanish at the origin. It is necessary that $g^{p-1} \in L_1$ and sufficient that $g^{(p)} \in L_1$.*

The second half of the last sentence comes immediately from the properties in the last paragraph but two.

We now wish to prove the following lemma:

LEMMA A.2: *If $I^{\sigma-1}f \in L_1 (0 < \sigma \leqslant 1)$ then $I^{\sigma-1}(fk) \in L_1$ and $I^{\sigma-1}|fk| \in L_1$ where k is a bounded function in K_+.*

Proof: To prove the first part it is sufficient to show that $I^{\sigma}(fk)$ is absolutely continuous.

Since $I^{\sigma-1}f \in L_1$, $f = I^{1-\sigma}f_1$ where $f_1 \in L_1$. Hence, if $0 \leqslant x \leqslant x+h$,

$$\left| \int_0^{x+h} (x+h-t)^{\sigma-1} k(t) f(t)\,dt - \int_0^x (x-t)^{\sigma-1} k(t) f(t)\,dt \right|$$

$$\leqslant B \int_0^x [(x-t)^{\sigma-1} - (x+h-t)^{\sigma-1}] |f(t)|\,dt + B \int_x^{x+h} (x+h-t)^{\sigma-1} |f(t)|\,dt$$

where B is an upper bounded for k. The right-hand side does not exceed

$$B \int_0^x [(x-t)^{\sigma-1} - (x+h-t)^{\sigma-1}] \int_0^t (t-u)^{-\sigma} |f_1(u)|\,du\,dt +$$

$$B \int_x^{x+h} (x+h-t)^{\sigma-1} |f(t)|\,dt$$

$$\leqslant B \left[\int_0^x |f_1(t)|\,dt - \int_0^{x+h} |f_1(t)|\,dt \right] + 2B \int_x^{x+h} (x+h-t)^{\sigma-1} |f(t)|\,dt$$

$$\leqslant B \int_x^{x+h} |f_1(t)|\,dt + 2B \int_x^{x+h} (x+h-t)^{\sigma-1} |f(t)|\,dt.$$

The first term tends to zero with h and so does the second provided that it is finite. However,

$$\int_x^{x+h}(x+h-t)^{\sigma-1}|f(t)|\,dt<\int_0^{x+h}(x+h-t)^{\sigma-1}|f(t)|\,dt<\int_0^{x+h}|f_1(t)|\,dt$$

so that finiteness is assured. Consequently, for every set of non-overlapping intervals $(x_\nu,x_\nu+h_\nu)$ we can make

$$\sum_\nu\left|\int_0^{x_\nu+h_\nu}(x_\nu+h_\nu-t)^{\sigma-1}f(t)k(t)\,dt-\int_0^{x_\nu}(x_\nu-t)^{\sigma-1}k(t)f(t)\,dt\right|$$

as small as we like by taking Σh_ν small enough. Thus $I^\sigma(fk)$ is absolutely continuous.

The same process applies to $I^\sigma|fk|$ and the proof is terminated.

More generally we can demonstrate the following lemma:

LEMMA A.3: *If $I^{\sigma-p}f\in L_1$ $(0<\sigma<1$, p a positive integer) then $I^{\sigma-p}(fk)\in L_1$ provided that $k,k',\ldots,k^{(p-1)}$ are bounded functions in K_+.*

For $I^{\sigma-p}f\in L_1$ implies that $I^{\sigma-1}f^{(p-1)}\in L_1$ so that, from Lemma A.2, $I^{\sigma-1}(kf^{(p-1)})\in L_1$. Also $I^{\sigma-p}f\in L_1$ means that $I^{\sigma-p+1}f\in L_1$ and a similar argument gives $I^{\sigma-1}(k'f^{(p-2)})\in L_1$. Proceeding in this way we find that $I^{\sigma-1}(kf)^{(p-1)}\in L_1$ and the lemma is proved.

Indeed the lemma is true under lighter restrictions on k as is shown by

LEMMA A.4: *If $I^{\sigma-p}f\in L_1$ $(0<\sigma<1, p$ a positive integer) then $I^{\sigma-p}(fk)\in L_1$, provided that $x^sk_1^s(x)$ are bounded functions in K_+ for $s=0,1,\ldots,p-1$.*

As in Lemma A.3, $I^{\sigma-1}(kf^{(p-1)})\in L_1$. Consider now $(1/x)f^{(p-2)}=(1/x)If^{(p-1)}$. Since $f^{(p-1)}=I^{1-\sigma}f_1$ where $f_1\in L_1$, $|(1/x)f^{(p-2)}|<(1/x^\sigma)I|f_1|$ which demonstrates compliance with the necessary condition of Lemma A.1 for $I^{\sigma-1}[(1/x)f^{(p-2)}]\in L_1$, since $\sigma\neq 1$. Further, $I^\sigma[(1/x)f^{(p-2)}]$ may be shown to be absolutely continuous by the method used in Lemma A.2, $|f(t)|$ being replaced by

$$\left|\frac{1}{t}\int_0^t(t-u)^{1-\sigma}f_1(u)\,du\right|<\int_0^t(t-u)^{-\sigma}|f_1(u)|\,du.$$

Hence $I^{\sigma-1}[(1/x)f^{(p-2)}]\in L_1$ and it follows from the boundedness of $xk'(x)$ and Lemma A.2 that $I^{\sigma-1}(k'f^{(p-2)})\in L_1$. Similarly, it may be shown that $I^{\sigma-1}(k^{(s)}f^{(p-s-1)})\in L_1$ and the lemma is proved.

Next we turn to the question of the Fourier transform of $I^\beta g$. We would like to say that at infinity its behaviour is something like $|\alpha|^{-\beta}G(\alpha)$ but since it is not true in general that the Fourier transform of a convolution is the product of the separate transforms we have to proceed somewhat indirectly.

Let $a>0$ and let f be a continuous function which is zero for $x<0$ and which is $O(x^k)$ for some finite k as $x\to\infty$. The Fourier transform of $e^{-ax}f(x)$ then exists and will be denoted by $\hat{F}(\alpha)$. Now, if γ is a good function with Fourier transform Γ,

$$\left|\int_{-\infty}^{\infty}[\hat{F}(\alpha)-F(\alpha)]\Gamma(\alpha)d\alpha\right|=\left|\int_0^\infty 2\pi(e^{-ax}-1)f(x)\gamma(-x)dx\right|$$

$$<2\pi a\int_0^\infty x|f(x)\gamma(-x)|\,dx\to 0$$

s $a\to 0$. Hence $\lim_{a\to+0}\hat{F}=F$ in the generalized sense.

Also any $g \in K_+$ can be expressed in the form $g = f^{(r)}$ (with r finite) where f has the properties described above. Hence

$$\lim_{a \to +0} \hat{G}(\alpha) = \lim_{a \to +0} (i\alpha + a)^r \hat{F}(\alpha) = (i\alpha)^r F(\alpha) = G(\alpha). \tag{A.5}$$

Consequently the Fourier transform can be evaluated by a limiting process from that of $e^{-ax} I^\beta g$ or of $\int_0^x e^{-at} t_\beta \, e^{-a(x-t)} g(x-t)\, dt$. Hence the Fourier transform of $I^\beta g$ is $\lim_{a \to +0} \hat{X}_\beta \hat{G}$ where X_β is the Fourier transform of x_β. If α is not near zero there is no difficulty about putting $a = 0$ in $\hat{X}_\beta$. Hence, taking note of (A.5), we have

$$\int_{-\infty}^{\infty} e^{-i\alpha x} I^\beta g \, dx = C \, |\alpha|^{-\beta} G(\alpha) \tag{A.6}$$

as $|\alpha| \to \infty$, C being a known finite constant.

On the other hand, if $I^\beta g \in L_1(-\infty, \infty)$, its Fourier transform is $o(1)$ as $|\alpha| \to \infty$. Combining this result with equation (A.6) we obtain

LEMMA A.5: *If* $I^\beta g \in L_1(-\infty, \infty)$, $G(\alpha) = o(|\alpha|^\beta)$ *as* $|\alpha| \to \infty$.

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