Effective dimensions in Computational Complexity, Information Theory, and Fractal Geometry

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Fractal dimensions

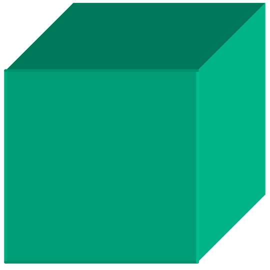
- I will talk especifically about
  - Hausdorff dimension
  - Packing dimension
- These two concepts are widely used in fractal geometry
- They can be defined in any metric space. Here I will stick to these spaces:

  R^n the euclidean space

  Σ^∞ the set of inifinite sequences over finite Σ

The intuition in fractal dimension

- Let $A \subseteq R^3$
- We consider balls of the form



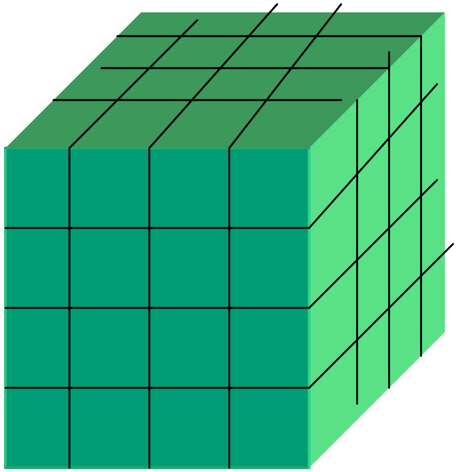
Fractal dimension

A line can be covered with N balls of diameter 1/N

A surface can be covered with cN^2 balls of diameter 1/N

A has dimension D

A can be covered with around N^D balls of diameter 1/N



A can be s-covered if

Number-of-balls * diameter^s

is bounded for arbitrarily small diameter

Hausdorff measure

Let $s \in [0, \infty)$, $\delta > 0$

- $H^s_\delta(A) = \inf \Sigma |U_i|^s$ (infimum over all covers of A, $\{U_i\}$ with diameter $|U_i| < \delta$)

- $H^s(A) = \lim_{\delta \to 0} H^s_\delta(A)$

H^s is the **s-Hausdorff measure**

Hausdorff Dimension

For each A there is a value D, the **Hausdorff dimension of A** such that

- If $s < D$, $H^s(A) = \infty$
- If $s > D$, $H^s(A) = 0$

Effective dimension

- Effective dimension is based in a characterization of Hausdorff dimension on Σ^{∞} given by Lutz (2000)
- The characterization is a very clever way to deal with a single covering using gambling

Hausdorff dimension (Lutz characterization)

A gambling game

A player bets on each element of an unknown sequence $x \in \Sigma^\infty$

- When betting on x[i] he knows x[1..i-1].

- A strategy in this game is a function $b:\Sigma^* x\ \Sigma \rightarrow [0,1]$, b(w, a)=$\alpha$ means "if x[1..i-1]=w, bet a fraction α of present capital to x[i]=a".
- If he wins he multiplies by $|\Sigma|$, if he loses he gets 0.

- But the house takes a fixed percentage before each bet.

Hausdorff dimension (Lutz characterization)

A **martingale** is $d: \Sigma^* \rightarrow [0,\infty)$ such that

$$\frac{\Sigma_{a\in\Sigma}\, d(wa)}{|\Sigma|} = d(w)$$

It is the capital corresponding to a fixed strategy and a fair game (the house takes nothing)

Hausdorff dimension (Lutz characterization)

An **s-gale** is $d : \Sigma^* \to [0,\infty)$ such that

$$\frac{\Sigma_{a \in \Sigma}\, d(wa)}{|\Sigma|^s} = d(w)$$

It is the capital corresponding to a fixed strategy and a the house taking a fraction of $1 - |\Sigma|^{s-1}$

Hausdorff dimension (Lutz characterization)

- An s-gale d **succeeds** on $x \in \Sigma^{\infty}$ if
  $\limsup_{i \to \infty} d(x[1..i]) = \infty$
- d **succeeds** on $A \subseteq \Sigma^{\infty}$ if d succeeds on each $x \in A$
- $G(A) = \{s \mid$ there is an s-gale that succeeds on A$\}$

The smaller the s the harder to succeed

Hausdorff dimension (Lutz characterization)

Let $A \subseteq \Sigma^{\infty}$

Lemma: $H^s(A)=0$ iff there is an s-gale that succeeds on A.

Corollary: The Hausdorff dimension of A is

the infimum of G(A).

[Lutz 00]

Efectivizing Hausdorff dimension

- We restrict to constructive or effective strategies and get the corresponding "dimensions" that are meaningful in subsets of Σ^∞ we are interested in

Constructive dimension [Lutz00]

- A function f: $\Sigma^* \to R$ is **constructive** if there exists a non-decreasing computable function f': $\Sigma^* \times N \to R$ such that for every x

$$\lim_{n\to\infty} f'(x,n)=f(x)$$

- $G_{constr}(A) = \{s \mid$ there is a constructive s-gale that succeeds on A$\}$

- $\mathrm{cdim}(A) = \inf G_{constr}(A)$

Effective dimensions

- Restricting to effectively computable strategies we have:
  - computable by a finite automata $\dim_{FS}$
  - computable in polynomial time $\dim_{p}$
  - computable in polynomial space $\dim_{pspace}$

- Each of this effective dimensions is “the right one” for a set of sequences (complexity class)

[Lutz00, DaiLatLutMay04]

Some properties

- $\dim_\Delta(X)$ is defined for every set X
- $X \subseteq Y$ implies $\dim_\Delta(X) <= \dim_\Delta(X)$

- $\dim_\Delta(\cup X_i) = \sup_i \dim_\Delta(X_i)$
  for "suitable" effective unions

where $\dim_\Delta$ is any of the effective dimensions

Before getting into those

- We are still interested in effective dimensions in R^n, they are easy to characterize through the identification
  - Fix base k, $\Sigma=\{0, \ldots, k-1\}$
  - $x=(x_1, \ldots, x_n)$ corresponds to the sequence seq(x) in Σ^∞ result of interleaving base-k representations of $x_1, \ldots, x_n$

$$\dim(A)= n \dim(seq(A))$$

- Or just redefine betting game

Before getting into those

- Packing dimension corresponds to the idea of "packing" a set in the place of covering
- With a more involved definition, still gale characterization is dual: [Athreya et al]
  - Strong success: $\liminf_{i \to \infty} d(x[1..i]) = \infty$
  - $G^{str}(A)$ = {s | there is an s-gale that strongly succeeds on A}
  - **Dim (A)=** $\inf G^{str}(A)$
  - dim(A) <= Dim(A)

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In computational complexity

- Each language (decisional problem) L is represented with its characteristic sequence $\chi_L \in \{0,1\}^\infty$

- A complexity class is a set of languages (a set of inifinite sequences)

  P, NP

  E= DTIME (2^n)

  EXP = DTIME $(2^{p(n)})$

In computational complexity

- $\dim_p(E) = 1 = \dim_p(\{0,1\}^\infty)$
- $\dim_p(P) = 0$
- We use $\dim_p$ to estimate size of subclasses of E

Notice that $\dim_p(X) < 1$ implies $X \neq E$

Probabilistic method

- Hitchcock (2005) improved the best known result on the nonuniform complexity of E:

  $$E \not\subset P_{n-T}(\mathrm{DENSE}^c)$$

  by showing that

  $$\dim_p(P_{n-T}(\mathrm{DENSE}^c)) = 0$$

Size estimation

- We know (HiLuMa04) that there is a fine size distinction between all these classes

$E \cap \text{circuitsize}(1/2\ 2^n/n)$

$E \cap \text{circuitsize}(2^{n/2})$

$E \cap \text{circuitsize}(2^{\sqrt{n}})$

Alternative hypothesis and concepts

- If $\dim_p(NP)>0$ then Max3Sat is hard to approximate [Hit02]

MaxSat is the problem of computing the number of satisfied clauses in a 3SAT formula

………..

Alternative hypothesis and concepts

- The upper span of A is $P_m(A) = \{ B \mid A \leq_m^p B \}$
- The lower span of A is $P_m^{-1}(A) = \{ C \mid C \leq_m^p A \}$
- A **Small Span Theorem** states that for every language A in E, either $P_m(A)$ or $P_m^{-1}(A)$ is small

[Ambos-Spies et al, Hitchcock, Hitchcock el al]

- ....

More questions

- $\{A \mid P^A = NP^A\}$ has dimension 1

- $\{A \mid P^A \neq NP^A\}$ has dimension 1

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Constructive dim & information th.

- Remember that cdim (constructive dimension) was defined using constructible gales

- Constructible = lower-semicomputable

The dimension of a point

- If we take a single $x \in \Sigma^{\infty}$ we can have cdim({x})>0
- In fact constructive dimension can be defined in a pointwise way [Lutz03]

$$\mathrm{cdim}(A)=\sup_{x\in A} \mathrm{cdim}(\{x\})$$

Dimension is Kolmogorov complexity

- But the constructive dimension of a sequence is related to its information content

- [Mayordomo02] For each $x \in \Sigma^\infty$

$$\mathrm{cdim}(\{x\}) = \liminf_{n\to\infty} \frac{K(x[1..n])}{n \log|\Sigma|}$$

$K(w) = \min\{|p| \mid U(p)=w\}$

- So constructive dimension of a sequence is its (worst case) information content

- Remember the notion of Martin-Lof random sequence

x is Martin-Lof random iff

$$K(x[1..n]) > n-c \quad \text{for all } n$$

Randomness vs positive dimension

- It was known that for each Martin-Lof random x, BPP$\subseteq$ P^{x}

- Gu and Lutz (2006) have succeeded in using positive dimension sequences as random sources

  cdim(x)>0 implies BPP$\subseteq$ P^{x}

Randomness vs positive dimension

- Doty (2006) gets initial results in a more general setting

  cdim(x)>0 implies cDim(y)=1 for $y \equiv_T x$

- Constructive packing dimension also can be characterized in this fashion:

$$\mathrm{cDim}(\{x\}) = \limsup_{n\to\infty} \frac{K(x[1..n])}{n \log|\Sigma|}$$

- Even in R^n this has a useful interpretation …

In resource-bounded setting

$KS^p(w) = \min\{|p| \mid U(p)=w \text{ in space } p(|w|)\}$

- For each $x \in \Sigma^\infty$

$$\dim_{pspace}(A) = \inf_p \sup_{x \in A} \liminf_{n\to\infty} \frac{KS^p(x[1..n])}{n \log|\Sigma|}$$

Compression is dimension

- In the cases of Finite State and polynomial time bounds it has been proven ([Dai et al], [López el al]) that compression ratio is also dimension
- Normal sequences are exactly those of finite-state dimension 1

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The dimension of a point …

- Let's go back to R^n and see what we can do with effective dimension

- Important: For many sets (every union of Π^0_1-definable sets) [Hitchcock 04]

$$\mathrm{cdim}(A) = \dim(A)$$

- This is the case of the usual self-similar fractals, dimension is constructive dimension

- I mentioned that [Lutz03]

$$\mathrm{cdim}(A)=\sup_{x\in A} \mathrm{cdim}(\{x\})$$

- For A a union of Π^0_1-definable sets

$$\dim(A)=\mathrm{cdim}(A)=\sup_{x\in A} \mathrm{cdim}(\{x\})$$

- So dimension is a local property!!!!
- Even more, the dimension corresponds to the information content of the points …

Questions and anwers

- The dim(A)=cdim(A) needs to be understood, exploited
- Points in computable curves have dimension at most 1, points outside all of them have arbitrary dimension
- The base we use doesn’t matter unless resources are very low …
- ………..

Summary

- Effective fractal dimension has a wide set of connections and applications in:
  - Computational complexity
  - Information theory
  - Fractal geometry
  - Compression algorithms, prediction algorithms, …

Some achievements

- Solved open problems in CC
- Derandomization using only positive dimension in the place of Martin-Lof
- Dimension is a local property
- ……

Many questions

- Small Span theorems and weak notions of completeness
- How useful are positive dimension sequences?
- Can we simplify/get unknown dimensions of fractal sets?
- …….

Thanks

- To Barry Cooper and Dieter Spreen for their technichal help

Proceedings

- A couple of detailed open problems

References

- Most recent survey:

  J. M. Hitchcock, J. H. Lutz, and E. Mayordomo.
  The fractal geometry of complexity classes.
  SIGACT News, 36(3):24-38, September 2005.

- Effective Fractal Dimension Bibliography

  Maintained by John Hitchcock

  http://www.cs.uwyo.edu/~jhitchco/