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Littelmann paths and Brownian paths

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Abstract. We study some path transformations related to Littelmann path model and their applications to representation theory and Brownian motion in a Weyl chamber.

1. Introduction

Some transformations defined on continuous paths with values in a vector space have appeared in recent years, in two separate parts of mathematics. On the one hand Littelmann [**22**] developed his path model in order to give a unified combinatorial setup for representation theory, generalizing the theory of Young tableaux to semi-simple or Kac-Moody Lie algebras of type other than A. On the other hand, in probability theory, several path transformations have been introduced that yield a construction of Brownian motion in a Weyl chamber starting from a Brownian motion in the corresponding Cartan Lie algebra. The oldest and simplest of these transformations comes from Pitman's theorem [**30**] which states that if $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion, then the stochastic process $R_t := B_t - 2\inf_{0\leq s\leq t} B_s$ is a three dimension Bessel process, i.e. is distributed as the euclidean norm of a three dimensional Brownian motion (actually Pitman stated his theorem with the transformation $2\max_{0\leq s\leq t} B_s - B_t$, but thanks to the symmetry of Brownian motion this is clearly equivalent to the above statement). It turns out that the fact that, here, the dimension of the Brownian motion is equal to 1, the rank of the group $SU(2)$, while 3, the dimension of the Bessel process, is the dimension of the group $SU(2)$ is not a mere coincidence but a fundamental fact which we will clarify in the following. Pitman's theorem has been extended in several ways. The first step has been the result of Gravner, Tracy and Widom, [**16**] and of Baryshnikov [**1**] which states that the largest eigenvalue of a random $n\times n$ Hermitian matrix in the GUE is distributed as the random variable

$$\max_{1=t_n\geq t_{n-1}\geq\ldots\geq t_1\geq t_0=0}\sum_{i=1}^{n}(B_i(t_i)-B_i(t_{i-1}))$$

where $(B_1,\ldots,B_n)$ is a standard n-dimensional Brownian motion. This result in turn was generalized in [**7**] and [**29**]. These extensions involve path transformations which generalize Pitman's and are closely related to the Littelmann path model. One of the purposes of this paper is to clarify these connections as well as

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to settle a number of questions raised in these works. In the course of these investigations we will derive several applications to representation theory. These path transformations occur in quite different contexts, since the one in [**7**] is expressed by representation theoretic means, whereas the one in [**29**] is purely combinatorial, and arises from queuing theory considerations.

Let us describe more precisely the content of the paper. We start by defining the Pitman transforms which are the main object of study in this paper. These transforms operate on the set of continuous functions $\pi : [0, T] \to V$, with values in some real vector space V and are given by the formula

$$\mathcal{P}_\alpha \pi(t) = \pi(t) - \inf_{t \geq s \geq 0} \alpha^\vee(\pi(s))\alpha, \qquad t \in [0, T].$$

Here $\alpha \in V$ and $\alpha^\vee \in V^\vee$ (where $V^\vee$ is the dual space of V) satisfy $\alpha^\vee(\alpha) = 2$. These are multidimensional generalizations of the transform occuring in Pitman's theorem. They are related to Littelmann's operators as shown in section 2.2. We show that these transforms satisfy braid relations, i.e. if $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ are such that $\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2$, and $\alpha^\vee(\beta) < 0, \beta^\vee(\alpha) < 0$ and $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2 \frac{\pi}{n}$, where $n \geq 2$ is some integer, then one has

$$\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots = \mathcal{P}_\beta \mathcal{P}_\alpha \mathcal{P}_\beta \ldots$$

where there are n factors in each product. Consider now a Coxeter system (W, S). To each fundamental reflection we associate a Pitman transform $\mathcal{P}_{\alpha_i}$. The braid relations imply that if $w \in W$ has a reduced decomposition $w = s_1 \ldots s_n$, then the operator $\mathcal{P}_w = \mathcal{P}_{\alpha_1} \ldots \mathcal{P}_{\alpha_n}$ is well defined, i.e. it depends only on w and not on the reduced decomposition. We show that if W is a Weyl group, $w_0 \in W$ is the longest element, and π is a dominant path ending in the weight lattice, then for any path η in the Littelmann module generated by π, one has

$$\pi = \mathcal{P}_{w_0}\eta. \tag{1.1}$$

The path transformation introduced in [**29**] can be expressed as $\mathcal{P}_{w_0}$ where w_0 is the longest element in the Coxeter group of type A.

We derive a representation theoretic formula for $\mathcal{P}_w$, in the case of a Weyl group, expressed in terms of representations of the Langlands dual group, see Theorem 3.12. This formula is canonical, in the sense that it is independent of any choice of a reduced decomposition of w in the Weyl group. It is obtained by lifting the path to a path $g(t)$ with values in the Borel subgroup of the simply connected complex Lie group associated with the root system. Then one obtains integral transformations which relate the diagonal parts in the Gauss decompositions of the elements $\overline{w}g(t)$. The Pitman transforms are obtained by going down to the Cartan algebra by applying Laplace's method. By (1.1) we obtain in this way a new formula for the dominant path in some Littelmann module, in terms of any of the paths of the module, which is a generalization to arbitrary root systems of Greene's formula (see [**14**]). As a byproduct of this formula we also obtain a direct proof of the symmetry of the Littlewood-Richardson coefficients.

This formula appeared in [**7**] where it was conjectured that the associated map transforms a Brownian motion in the Cartan Lie algebra into a Brownian motion in the Weyl chamber. This conjecture was proved in [**7**] for some classical groups. Here we give a completely different proof, valid for all root systems.

This paper is organized as follows. In section 2 we define the elementary Pitman transformations operating on continuous paths with values in some real vector space V, taking the value 0 at 0. The first result is a formula for the repeated compositions of two Pitman transforms, which implies that they satisfy the braid relations. Then we define Pitman transformations $\mathcal{P}_w$ associated to a Coxeter system (W, S). In section 3 we prove our main result which is a representation theoretic formula for these operators $\mathcal{P}_w$ in the case where W is a Weyl group. This formula unifies the results of [**29**] and of [**7**]. Results of Lusztig [**25**] on canonical bases, and of Berenstein and Zelevinsky [**2**] and Fomin and Zelevinsky [**12**] on totally positive matrices play a crucial role in the proof. In section 4 we make some comments on a duality transformation naturally defined on paths, which generalizes the Schützenberger involution, and give an application to the symmetry of the Littelwood-Richardson rule. In section 5 we give two proofs of the generalization of the representation of Brownian motion in a Weyl chamber obtained in [**29**] and [**7**]. One of the proofs relies essentially on the duality properties, while the other uses Littelmann paths in the context of Weyl groups. Finally section 6 is an appendix where we have gathered several technical proofs which were postponed in the main body of the text.

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2. Braid relations for the Pitman transforms

2.1. Pitman transforms. Let V be a real vector space, with dual space $V^\vee$. Let $\alpha \in V$ and $\alpha^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = 2$.

DEFINITION 2.1. *The Pitman transform $\mathcal{P}_\alpha$ is defined on the set of continuous paths $\pi : [0, T] \to V$, satisfying $\pi(0) = 0$, by the formula:*

$$\mathcal{P}_\alpha \pi(t) = \pi(t) - \inf_{t \geq s \geq 0} \alpha^\vee(\pi(s))\alpha, \qquad T \geq t \geq 0.$$

This transformation seems to have appeared for the first time in [**30**] in the one-dimensional case. Note that $\mathcal{P}_\alpha$ actually depends on the pair $(\alpha, \alpha^\vee)$. For simplicity we shall use the notation $\mathcal{P}_\alpha$, it will be always clear from the context which $\alpha^\vee$ is involved.

When, for some $v \in V$, π is the linear path $\pi(t) = tv$ then $\mathcal{P}_\alpha \pi = \pi$ when $\alpha^\vee(v) \geq 0$ and $\mathcal{P}_\alpha \pi = s_\alpha \pi$ when $\alpha^\vee(v) \leq 0$ where s_α is the reflection on V

$$s_\alpha v = v - \alpha^\vee(v)\alpha \tag{2.1}$$

for $v \in V$.

We list a number of elementary properties of the Pitman transform below.

PROPOSITION 2.2. *(i) For any $\lambda > 0$ the Pitman transformation associated with the pair $(\lambda\alpha, \alpha^\vee/\lambda)$ is the same as the one associated with the pair $(\alpha, \alpha^\vee)$.*

(ii) One has $\alpha^\vee(\mathcal{P}_\alpha \pi(t)) \geq 0$ for all $t \in [0, T]$. Furthermore $\mathcal{P}_\alpha \pi = \pi$ if and only if $\alpha^\vee(\pi(t)) \geq 0$ for all $t \in [0, T]$.

(iii) The transformation $\mathcal{P}_\alpha$ is an idempotent, i.e. $\mathcal{P}_\alpha \mathcal{P}_\alpha \pi = \mathcal{P}_\alpha \pi$ for all π.

(iv) Let $\pi : [0, \infty[\to V$ be a path, then $-\inf_{0 \leq t \leq T} \alpha^\vee(\pi(t)) \in [0, \alpha^\vee(\mathcal{P}_\alpha \pi(T))]$. Conversely, given a path η satisfying $\eta(0) = 0$, $\alpha^\vee(\eta(t)) \geq 0$ for all $t \in [0, T]$

and $x \in [0, \alpha^\vee(\eta(T))]$, *there exists a unique path* π *such that* $\mathcal{P}_\alpha \pi = \eta$ *and* $x = -\inf_{T \geq t \geq 0} \alpha^\vee(\pi(t))$. *Actually* π *is given by the formula*

$$\pi(t) = \eta(t) - \min\left(x, \inf_{T \geq s \geq t} \alpha^\vee(\eta(s))\right)\alpha. \tag{2.2}$$

Proof. Items (i) and (ii) are trivial, and (iii) follows immediately from (ii). Hopefully the reader can give a formal proof of (iv), see section 6.1 for such a proof, but it is perhaps more illuminating to stare for a few minutes at Fig. 1, which shows, in the one dimensional case, with $\alpha = 1, \alpha^\vee = 2$, the graph of a function $g : [0,1] \to \mathbb{R}$ as well as those of $I, -I$ and $f = \mathcal{P}_\alpha g$ where $I(s) = \inf_{0 \leq u \leq s} g(u)$. $\diamondsuit$.

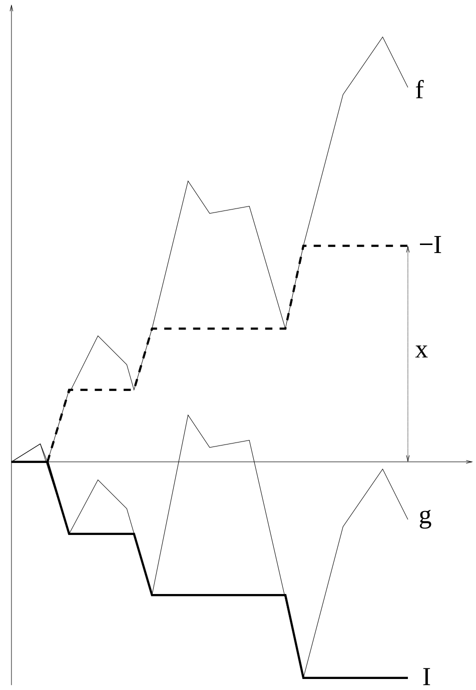

Fig. 1

2.2. Relation with Littelmann path operators. Using Proposition 2.2 (iv) we can define generalized Littelmann transformations. Recall that Littelmann operators are defined on paths with values in the dual space $\mathfrak{a}^*$ of some real Lie algebra $\mathfrak{a}$. The image of a path is either another path or the symbol $\mathbf{0}$ (actually the zero element in the $\mathbb{Z}$-module generated by all paths). We define continuous versions of these operators.

Definition 2.3. *Let* $\pi : [0,T] \to V$ *be a continuous path satisfying* $\pi(0) = 0$, *and* $x \in \mathbb{R}$, *then* $E_\alpha^x \pi$ *is the unique path such that*

$$E_\alpha^x \pi = \mathcal{P}_\alpha \pi \quad \textit{and} \quad \alpha^\vee(E_\alpha^x \pi(T)) = \alpha^\vee(\pi(T)) + x$$

if $-2\alpha^\vee(\pi(T)) + 2\inf_{0 \leq t \leq T} \alpha^\vee(\pi(t)) \leq x \leq -2\inf_{0 \leq t \leq T} \alpha^\vee(\pi(t))$ *and* $E_\alpha^x \pi = \mathbf{0}$ *otherwise.*

One checks easily that $E_\alpha^0 \pi = \pi$ and $E_\alpha^x E_\alpha^y \pi = E_\alpha^{x+y} \pi$ as long as $E_\alpha^y \pi \neq \mathbf{0}$. When α is a root and $\alpha^\vee$ its coroot, in some root system, then E_α^2 and E_α^{-2} coincide with the Littelmann operators e_α and f_α, defined in [**22**]. Note that Littelmann operators are in fact defined only for integral paths ([**22**]). Recall that a path π is called integral if its endpoint $\pi(T)$ is in the weight lattice and, for each simple root α, the minimum of the function $\alpha^\vee(\pi(t))$ is an integer. The class of integral paths is invariant under the Littelmann operators. For such paths, the action of a Pitman transform can be expressed through Littelmann operators by

$$\mathcal{P}_\alpha \pi = e_\alpha^{n_\alpha}(\pi) \tag{2.3}$$

where n_α is the largest integer n such that $e_\alpha^n(\pi) \neq \mathbf{0}$.

2.3. Braid relations. An important property of the Pitman transforms is the following result.

THEOREM 2.4. *Let $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\alpha) = \beta^\vee(\beta) = 2$, and $\alpha^\vee(\beta) < 0, \beta^\vee(\alpha) < 0$ and $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2\frac{\pi}{n}$, where $n \geq 2$ is some integer, then one has*

$$\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots = \mathcal{P}_\beta \mathcal{P}_\alpha \mathcal{P}_\beta \ldots$$

where there are n factors in each product.

We shall prove Theorem 2.4 as a corollary to the result of section 2.4. Note that if $\alpha^\vee(\beta) = \beta^\vee(\alpha) = 0$ then $\mathcal{P}_\alpha\mathcal{P}_\beta = \mathcal{P}_\beta\mathcal{P}_\alpha$ by a simple computation. For crystallographic angles (i.e. $n = 2, 3, 4, 6$) a proof of Theorem 2.4 could also be deduced from Littelmann's theory (see [**23**] or [**20**]). We shall provide still another (hopefully more conceptual) proof for these angles in section 3, see Remark 3.10. The general case seems to be new.

2.4. A formula for $\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta\ldots$. Let $\alpha, \beta \in V$ and $\alpha^\vee, \beta^\vee \in V^\vee$ be such that $\alpha^\vee(\beta) < 0$ and $\beta^\vee(\alpha) < 0$. By Proposition 2.2 (*i*) we can - and will - assume by rescaling that $\alpha^\vee(\beta) = \beta^\vee(\alpha)$, without changing $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$. We use the notations

$$\rho = -\frac{1}{2}\alpha^\vee(\beta) = -\frac{1}{2}\beta^\vee(\alpha), \quad X(s) = \alpha^\vee(\pi(s)), \quad Y(s) = \beta^\vee(\pi(s)).$$

THEOREM 2.5. *Let n be a positive integer, if $\rho \geq \cos\frac{\pi}{n}$, then one has*

$$\begin{aligned}(\underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots}_{n\ terms})\pi(t) \quad &= \quad \pi(t) - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-1}\geq 0}\Big(\sum_{i=0}^{n-1} T_i(\rho)Z^{(i)}(s_i)\Big)\alpha \\ &\quad - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-2}\geq 0}\Big(\sum_{i=0}^{n-2} T_i(\rho)Z^{(i+1)}(s_i)\Big)\beta \end{aligned} \tag{2.4}$$

where $Z^{(k)} = X$ if k is even and $Z^{(k)} = Y$ if k is odd. The $T_k(x)$ are the Tchebycheff polynomials defined by $T_0(x) = 1$, $T_1(x) = 2x$, and $2xT_k(x) = T_{k-1}(x) + T_{k+1}(x)$ for $k \geq 1$.

The Tchebycheff polynomials satisfy $T_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}$ and, in particular, under the assumptions on ρ and n, one has $T_k(\rho) \geq 0$ for all $k \leq n-1$.

Assuming Theorem 2.5 we obtain Theorem 2.4.

Proof of Theorem 2.4. Let $\alpha^\vee(\beta) = \beta^\vee(\alpha) = -2\cos\frac{\pi}{n}$, then one has $T_{n-1}(\rho) = 0$ and the last term in the coefficient of α in the right hand side of (2.4) vanishes. It follows by inspection that this term equals the coefficient of α in the analogous formula for $\underbrace{\mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta\ldots}_{n\text{ terms}}\pi(t)$. A similar argument works for the coefficient of β. $\diamondsuit$

The proof of Theorem 2.5 will be by induction on n. It is easy to check the formula for $n = 1$ or 2. We shall do the induction in sections 2.5 and 2.6.

2.5. Two intermediate lemmas.

LEMMA 2.6. *Let $X : [0, t] \to \mathbb{R}$ be a continuous functions with $X(0) = 0$ and let $t_0 = \sup\{s \geq 0 \,|\, X_s = \inf_{s \geq u \geq 0} X_u\}$, then for all $u \leq t_0$ one has*

$$\inf_{t \geq s \geq u} (X(s) - 2 \inf_{s \geq w \geq 0} X(w)) = - \inf_{u \geq v \geq 0} X(v).$$

Proof. This is obtained as a byproduct of the proof in section 6.1. Again it is perhaps more convincing to stare at Fig. 1 than to give a formal proof. ♢

Elaborating on this we obtain the next result.

LEMMA 2.7. *Let X and Y be continuous functions, such that $X(0) = Y(0) = 0$, then*

$$\inf_{t \geq s \geq 0}\big(X(s) + \inf_{s \geq u \geq 0} Y(s)\big) = \inf_{t \geq s \geq 0} X(s) +$$
$$\inf_{t \geq s \geq 0}\Big(X(s) - 2\inf_{s \geq u \geq 0} X(u) + \inf_{s \geq u \geq 0}\big(Y(u) + \inf_{u \geq v \geq 0} X(v)\big)\Big)$$

Proof. The first term is $I = \inf_{t \geq s \geq u \geq 0}(X(s) + Y(u))$. Let t_0 be, as in Lemma 2.7, the last time when X reaches its minimum over $[0, t]$, then

$$I = \inf\big(\inf_{t_0 \geq u \geq 0}(Y(u) + X(t_0)); \inf_{t \geq s \geq u \geq t_0 \geq 0}(Y(u) + X(s))\big)$$

Let J be the second term in the identity to be proved, then

$$J = \inf_{t \geq s \geq 0}\big[X(s) - 2\inf_{s \geq u \geq 0} X(u) + \inf_{s \geq u \geq 0}(Y(u) + \inf_{u \geq v \geq 0} X(v))\big] + X(t_0)$$

Introduce again the time t_0, then

$$\begin{aligned}
J &= \inf_{t \geq s \geq u \geq 0}\big(X(s) - 2\inf_{s \geq w \geq 0} X(w) + Y(u) + \inf_{u \geq v \geq 0} X(v)\big) + X(t_0) \\
&= \inf\Big(\inf_{\substack{t \geq s \geq u \geq 0 \\ t_0 \geq u}}\big(Y(u) + X(s) - 2\inf_{s \geq w \geq 0} X(w) + \inf_{u \geq v \geq 0} X(v) + X(t_0)\big); \\
&\qquad \inf_{t \geq s \geq u \geq t_0}\big(Y(u) + X(s) - 2\inf_{s \geq w \geq 0} X(w) + \inf_{u \geq v \geq 0} X(v) + X(t_0)\big)\Big)
\end{aligned}$$

but if $u \leq t_0$ then by lemma 2.6 one has $\inf_{t \geq s \geq u}(X(s) - 2\inf_{s \geq w \geq 0} X(w)) = -\inf_{u \geq v \geq 0} X(v)$. If $t_0 \leq u$ then $\inf_{s \geq w \geq 0} X(w) = X(t_0)$, therefore

$$\begin{aligned}
J &= \inf\Big(\inf_{t_0 \geq u \geq 0}\big(Y(u) + X(t_0)\big); \inf_{0 \geq u \geq t_0}\big(Y(u) + \inf_{t \geq s \geq u} X(s)\big)\Big) \\
&= \inf\big(\inf_{t_0 \geq u \geq 0}(Y(u) + X(t_0)); \inf_{t \geq s \geq u \geq t_0 \geq 0}(Y(u) + X(s))\big) \\
&= I.
\end{aligned}$$

2.6. End of proof of Theorem 2.5. Assume the result of the Theorem holds for some n with n even. Then $\underbrace{\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots}_{n+1 \text{ terms}} = \underbrace{\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\alpha \ldots}_{n \text{ terms}} \mathcal{P}_\alpha$, and one has

$$\begin{aligned}
\alpha^\vee(\mathcal{P}_\alpha \pi(s)) &= X(s) - 2\inf_{s \geq u \geq 0} X(u) \\
\beta^\vee(\mathcal{P}_\alpha \pi(s)) &= Y(s) + 2\rho \inf_{s \geq u \geq 0} X(u)
\end{aligned}$$

therefore, by induction hypothesis

$$\begin{aligned}
\underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots}_{n+1\text{ terms}}\pi(t) &= \underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots}_{n\text{ terms}}(\mathcal{P}_\alpha\pi)(t)\\
&= \pi(t) - \inf_{t\geq s\geq 0} X(s)\alpha - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-1}\geq 0}\Big(\sum_{i=0}^{n-1}\hat{Z}^{(i)}(s_i)\Big)\alpha\\
&\quad - \inf_{t\geq s_0\geq s_1\geq\ldots\geq s_{n-2}\geq 0}\Big(\sum_{i=0}^{n-2}\hat{Z}^{(i+1)}(s_i)\Big)\beta
\end{aligned}$$

where

$$\hat{Z}_\alpha^{(i)}(s) = \begin{cases} X(s) - 2\inf_{s\geq u\geq 0} X(u) & \text{for } i \text{ even}\\ Y(s) + 2\rho\inf_{s\geq u\geq 0} X(u) & \text{for } i \text{ odd.}\end{cases}$$

The coefficient of α in the above expression has the form

$$\begin{aligned}
&H_\alpha = -\inf_{t\geq s\geq 0} T_0(\rho)X(s)\\
&-\inf_{t\geq s\geq 0}\Big(T_0(\rho)X(s) - 2\inf_{s\geq u\geq 0} T_0(\rho)X(u) + \inf_{s\geq u\geq 0}\big(\Gamma(u) + \inf_{u\geq v\geq 0} T_0(\rho)X(v)\big)\Big)
\end{aligned}$$

where

$$\begin{aligned}
\Gamma(u) &= T_1(\rho)Y(u) + 2\rho T_1(\rho)\inf_{u\geq v\geq 0} X(v) +\\
&\quad \inf_{u\geq u_2\geq u_3\geq\ldots\geq u_{n-1}\geq 0}\Big(\sum_{i=2}^{n-1} T_i(\rho)\hat{Z}^{(i)}(u_i)\Big) - T_0(\rho)\inf_{u\geq v\geq 0} X(v)\\
&= T_1(\rho)Y(u) + T_2(\rho)\inf_{u\geq v\geq 0} X(v) +\\
&\quad \inf_{u\geq u_2\geq u_3\geq\ldots\geq u_{n-1}\geq 0}\Big(\sum_{i=2}^{n-1} T_i(\rho)\hat{Z}^{(i)}\Big)
\end{aligned}$$

so that we can apply lemma 2.7 to transform it into

$$H_\alpha = -\inf_{t\geq s\geq 0}\big(T_0(\rho)X(s) + \inf_{s\geq u\geq 0}\Gamma(u)\big)$$

Let us prove by induction on k that

$$H_\alpha = -\inf_{t\geq u_0\geq u_1\geq\ldots\geq u_{2k}}\Big(\sum_{i=0}^{2k} T_i(\rho)Z^{(i)}(u_i) + W_k(u_{2k-1})\Big)$$

with

$$W_k(v) = \inf_{v\geq u_{2k}\geq u_{2k+1}\geq\ldots\geq u_{n-1}\geq 0}\Big(\sum_{i=2k}^{n-1} T_i(\rho)(\hat{Z}^{(i)}(u_i)\Big)$$

Indeed the formula holds for $k=1$ by the computation above. Assume this holds for some k then one has

$$\begin{aligned}
H_\alpha &= -\inf_{t\geq u_0\geq u_1\geq\ldots\geq u_{2k}}\Big(\sum_{i=0}^{2k} T_i(\rho)Z^{(i)}(u_i)+W_k(u_{2k-1})\Big)\\
&= -\inf_{t\geq u_1\geq u_2\geq\ldots\geq u_{2k-1}}\Bigg(\sum_{i=0}^{2k-1} T_i(\rho)Z^{(i)}(u_i)+\inf_{u_{2k-1}\geq v\geq 0} T_{2k}(\rho)X(v)+\\
&\quad \inf_{u_{2k-1}\geq v\geq 0}\Big((T_{2k}(\rho)X(v)-2\inf_{v\geq w\geq 0}T_{2k}(\rho)X(w))+\\
&\quad \inf_{w\geq z\geq 0}\big(R_k(z)+\inf_{z\geq\tau\geq 0}T_{2k}(\rho)X(\tau)\big)\Big)\Bigg)
\end{aligned}$$

where

$$\begin{aligned}
R_k(z) &= T_{2k+1}(\rho)Y(z)+2\rho T_{2k+1}(\rho)\inf_{z\geq\tau\geq 0}X(\tau)+\\
&\qquad \inf_{z\geq u_{2k+2}\geq\ldots u_{n-1}}\big(\sum_{i=2k+2}^{n-1}\hat{Z}^{(i)}(u_i)\big)-\inf_{z\geq\tau\geq 0}T_{2k}(\rho)X(\tau)\\
&= T_{2k+1}(\rho)Y(z)+T_{2k+2}(\rho)\inf_{z\geq\tau\geq 0}X(\tau)+\inf_{z\geq u_{2k+2}\geq\ldots u_{n-1}}\big(\sum_{i=2k+2}^{n-1}\hat{Z}^{(i)}(u_i)\big)
\end{aligned}$$

where we used $2\rho T_{2k+1}(\rho)-T_{2k}(\rho)=T_{2k+2}(\rho)$. Applying Lemma 2.1. we get

$$\begin{aligned}
H_\alpha &= -\inf_{t\geq u_1\geq u_2\geq\ldots\geq u_{2k-1}}\Big(\sum_{i=0}^{2k-1} T_i(\rho)Z^{(i)}(u_i)+\\
&\qquad \inf_{u_{2k-1}\geq v\geq 0}\big(T_{2k}(\rho)X(v)+\inf_{w\geq z\geq 0}R_k(z)\big)\Big)\\
&= -\inf_{t\geq u_0\geq u_1\geq\ldots\geq u_{2k+2}}\Big(\sum_{i=0}^{2k+2} T_i(\rho)Z^{(i)}(u_i)+W_{k+1}(u_{2k+1})\Big)
\end{aligned}$$

Taking $k=n$ gives the required formula for H_α. For the coefficient of β, remark that

$$\underbrace{\mathcal{P}_\alpha\mathcal{P}_\beta\mathcal{P}_\alpha\ldots}_{n+1\text{ terms}}\pi(t)=\mathcal{P}_\alpha(\underbrace{\mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta\ldots}_{n\text{ terms}}\pi)(t)$$

and the formula for $n+1$ follows immediately from the formula at step n for $\underbrace{\mathcal{P}_\beta\mathcal{P}_\alpha\mathcal{P}_\beta\ldots}_{n\text{ terms}}$. The case where n is odd is treated in a similar way. $\diamondsuit$

2.7. Pitman transformations for Coxeter and Weyl groups. Let W be a Coxeter group, i.e. W is generated by a finite set S of reflections of a real vector space V, and (W,S) is a Coxeter system (see [**8**], [**19**]). For each $s\in S$, let $\alpha_s\in V$ and $\alpha_s^\vee\in V^\vee$, where $V^\vee$ is the dual space of V, such that $s=s_\alpha$ where s_α is the reflection associated to α (see (2.1)). Then α_s is called the simple root associated with $s\in S$ and $\alpha_s^\vee$ its coroot.

Denote by $\mathcal{P}_s$ the Pitman transform associated with the pair $(\alpha_s,\alpha_s^\vee)$. Let H_s be the closed half space $H_s=\{v\in V|\alpha_s^\vee(v)\geq 0\}$. Let $w\in W$ and let $w=s_1\ldots s_l$ be a reduced decomposition of w, where $l=l(w)$ is the length of w. By Theorem 2.4 and a fundamental result of Matsumoto ([**8**] Ch. IV, n^o 1.5, Proposition 5) the operator $\mathcal{P}_{s_1}\dots\mathcal{P}_{s_l}$ depends only on w, and not on the chosen reduced decomposition. We shall denote by $\mathcal{P}_w$ this operator.

PROPOSITION 2.8. *Let $w \in W$, $L_w = \{s \in S \,|\, l(sw) < l(w)\}, R_w = \{s \in S \,|\, l(ws) < l(w)\}$. For any path π, the path $\mathcal{P}_w\pi$ lies in the convex cone $\cap_{s\in L_w} H_s$, one has $\mathcal{P}_s\mathcal{P}_w = \mathcal{P}_w$ for all $s \in L_w$ and $\mathcal{P}_w\mathcal{P}_s = \mathcal{P}_w$ for all $s \in R_w$*

Proof. If $l(sw) < l(w)$ then w has a reduced decomposition $w = ss_1\dots s_k$ therefore $\mathcal{P}_w = \mathcal{P}_s\mathcal{P}_{s_1}\dots\mathcal{P}_{s_k}$ and $\mathcal{P}_w\pi = \mathcal{P}_s(\mathcal{P}_{s_1}\dots\mathcal{P}_{s_k}\pi)$ lies in H_s by Proposition 2.2 (*ii*). Furthermore one has $\mathcal{P}_s\mathcal{P}_w = \mathcal{P}_s$ since $\mathcal{P}_s$ is an involutiion (see Proposition 2.2 (*ii*)). Similarly $\mathcal{P}_w\mathcal{P}_s = \mathcal{P}_s$ when $l(ws) < l(w)$. ♢

COROLLARY 2.9. *If W is finite and w_0 is the longest element, then $\mathcal{P}_{w_0}\pi$ takes values in the closed Weyl chamber $\overline{C} = \cap_{s\in S}H_s$, furthermore $\mathcal{P}_{w_0}$ is an idempotent and $\mathcal{P}_w\mathcal{P}_{w_0} = \mathcal{P}_{w_0}\mathcal{P}_w = \mathcal{P}_{w_0}$ for all $w \in W$.*

Assume now that W is a finite Weyl group, associated with a weight lattice in V. Recall that paths taking values in the Weyl chamber $\overline{C}$ are called dominant paths in [**22**], and that the set of all (nonzero) paths obtained by applying products of Littelmann operators to a dominant path π is called $B\pi$. From the connection between Pitman's and Littelmann's operators, given in section 2.2, we deduce the following (see also [**23**]).

COROLLARY 2.10. *Let π be a dominant path, and let $\eta \in B\pi$, then one has $\pi = \mathcal{P}_{w_0}\eta$.*

Conversely, given a dominant path π, a path η belongs to the Littelmann module $B\pi$ if and only if η is integral and $\pi = \mathcal{P}_{w_0}\eta$.

Indeed for any path η and x such that $E^x_\alpha\eta \neq \mathbf{0}$ one has $\mathcal{P}_\alpha E^x_\alpha\eta = \mathcal{P}_\alpha\eta$, therefore $\mathcal{P}_{w_0}E^x_\alpha\eta = \mathcal{P}_{w_0}\mathcal{P}_\alpha E^x_\alpha\eta = \mathcal{P}_{w_0}\eta$. It follows that the set of paths whose image by $\mathcal{P}_{w_0}$ is π is stable under the action of Littelmann operators. ♢.

Let us come back to the general case of a finite Coxeter group. We shall now study the set of all paths η such that $\mathcal{P}_w\eta$ is a given dominant path. Let $w = s_1\dots s_q$ be a reduced decomposition. Let η be a path such that $\eta(0) = 0$ and $\pi = \mathcal{P}_w\eta$ is a dominant path. Denote $\eta_0 = \pi, \eta_q = \eta$, and $\eta_j = \mathcal{P}_{s_{j+1}}\dots\mathcal{P}_{s_q}\eta_q$ for $j = 1, 2, \dots, q-1$, then by Proposition 2.2 (*iv*) for all $j = 1, 2, \dots, q$ the path η_j is uniquely specified among paths γ such that $\mathcal{P}_{s_j}\gamma = \eta_{j-1}$, by the number $x_j = -\inf_{0\le t\le T}\alpha^\vee_{s_j}(\eta_j(t)) \in [0, \alpha^\vee_{s_j}(\eta_{j-1}(T))]$. It follows that $\eta = \eta_q$ is uniquely specified, among all paths γ such that $\mathcal{P}_{w_0}\gamma = \pi$ by the sequence $x_1, x_2, \dots, x_q$. These coordinates are subject to the inequalities $0 \le x_j \le \alpha^\vee_{s_j}(\eta_{j-1}(T))$. From

$$\eta_{j-1}(T) = \eta_j(T) + x_j\alpha_{s_j}$$

one obtains

$$\pi(T) = \eta_0(T) = \eta_j(T) + \sum_{l=1}^{j} x_l\alpha_{s_l}$$

therefore the inequality $0 \le x_j \le \alpha^\vee_{s_j}(\eta_{j-1}(T))$ reads

$$0 \le x_j \le \alpha^\vee_{s_j}(\pi(T)) - \sum_{l=1}^{j-1} x_l\alpha^\vee_{s_j}(\alpha_{s_l}).$$

It follows that the set of all paths η such that $\mathcal{P}_w\eta = \pi$ can be parametrized by a subset of the convex polytope

$$K_\pi = \{(x_1, \dots, x_q) \in \mathbb{R}^q \,|\, 0 \le x_j \le \alpha_{s_j}^\vee(\pi(T)) - \sum_{l=1}^{j-1} x_l \alpha_{s_j}^\vee(\alpha_{s_l}); j = 1, \dots, q\}.$$

The path η corresponding to the point $(x_1, \dots, x_q)$ is specified by the equalities

$$\eta_{j-1}(T) = \eta_j(T) + x_j \alpha_{s_j}$$

where $\eta_j = \mathcal{P}_{s_{j+1}} \dots \mathcal{P}_{s_q}\eta$. In the case of a Weyl group, it follows from [**23**] that the subset of K_π corresponding to paths η such that $\mathcal{P}_w\eta = \pi$ is the intersection of K_π with a certain convex cone which does not depend on π. This convex cone is quite difficult to describe. Also we do not know if a similar result holds for other finite Coxeter groups. We hope to come back to these questions in future work.

3. A representation theoretic formula for $\mathcal{P}_w$

3.1. Semisimple groups. We recall some standard terminology. We consider a simply connected complex semisimple Lie group G, associated with a root system R. Let H be a maximal torus, and B^+, B^- be corresponding opposite Borel subgroups with unipotent radicals N^+, N^-. Let $\alpha_i, i \in I$, and $\alpha_i^\vee, i \in I$, be the simple positive roots and coroots, and s_i the corresponding reflections in the Weyl group W. Let $e_i, f_i, h_i, i \in I$, be Chevalley generators of the Lie algebra of G. One can choose representatives $\overline{w} \in G$ for $w \in W$ by putting $\overline{s_i} = \exp(-e_i)\exp(f_i)\exp(-e_i)$ and $\overline{vw} = \overline{v}\,\overline{w}$ if $l(v) + l(w) = l(vw)$ (see [**12**] (1.8), (1.9)). The Lie algebra of H, denoted by $\mathfrak{h}$ has a Cartan decomposition $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$ such that the roots α_i take real values on the real vector space $\mathfrak{a}$. Thus $\mathfrak{a}$ is generated by $\alpha_i^\vee, i \in I$ and its dual $\mathfrak{a}^*$ by $\alpha_i, i \in I$. The set of weights is the lattice $P = \{\lambda \in \mathfrak{a}^*; \lambda(\alpha_i^\vee) \in \mathbb{Z}, i \in I\}$ and the set of dominant weights is $P^+ = \{\lambda \in \mathfrak{a}^*; \lambda(\alpha_i^\vee) \in \mathbb{N}, i \in I\}$. For each $\lambda \in P^+$, choose a representation space V_λ with a highest weight vector v_λ, and an invariant inner product on V_λ for which v_λ is a unit vector.

LEMMA 3.1. *For any dominant weight λ, and indices $i_1, \dots, i_n \in I$ one has*

$$\langle e_{i_1} \dots e_{i_n} \overline{w} v_\lambda, v_\lambda \rangle \ge 0$$

Proof. Unfortunately we do not know a simple proof of this Lemma. Clearly it is enough to prove it when the root system is irreducible. We use the theory of the canonical basis, see [**25**]. In the simply laced case, Lusztig has proved that the matrix coefficients of Chevalley generators in the canonical basis of a highest weight module have nonnegative coefficients, see [**25**] Ch. 19. The claim follows from this and the fact that each $\overline{w}v_\lambda$ is a positive multiple of an element of the canonical basis. In the general case the positivity property is not true any more, however we learned from Éric Vasserot [**31**] that the following result can be obtained, using Lusztig's geometric approach to the canonical basis. Consider the expansion of any monomial $f_{i_1} \dots f_{i_n}$ in the canonical basis of the U^- part of the enveloping algebra. then this expansion has nonnegative coefficients, i.e. one has $f_{i_n} \dots f_{i_1} = \sum_{r \in \Gamma} z_r U_r$ where $(U_r; r \in \Gamma)$ is the canonical basis and the z_r are nonnegative. Let now v_λ be the highest weight vector, then one has

$$\langle e_{i_1} \dots e_{i_n} \bar{w} v_\lambda, v_\lambda \rangle = \langle \bar{w} v_\lambda, f_{i_n} \dots f_{i_1} v_\lambda \rangle = \langle \bar{w} v_\lambda, \sum_r z_r U_r v_\lambda \rangle.$$

By a fundamental property of the canonical basis, for any $r \in \Gamma$ the element $U_r v_\lambda$ is either 0 or belongs to the canonical basis of V_λ, therefore one has $\langle \bar{w} v_\lambda, U_r v_\lambda \rangle \geq 0$, and the positivity property follows. Since this proof uses unpublished results, for the convenience of the reader we also give a proof which relies on a case by case analysis (worked out for all cases except F_4), which we postpone to the Appendix, see section 6.2. $\diamondsuit$

Let $(\omega_i, i \in I) \in P^I$ be the fundamental weights, characterized by the relations $\omega_i(\alpha_j^\vee) = \delta_{i,j}, j \in I$. The principal minor associated with ω_i is the function on G given by

$$\Delta^{\omega_i}(g) = \langle g v_{\omega_i}, v_{\omega_i} \rangle$$

see [**2**] and [**12**]. If $g \in G$ has a Gauss decomposition $g = [g]_-[g]_0[g]_+$ with $[g]_- \in N^-$, $[g]_0 \in H$, $[g]_+ \in N^+$, then one has

$$\Delta^{\omega_i}(g) = [g]_0^{\omega_i} = e^{\omega_i(\log[g]_0)}. \tag{3.1}$$

3.2. Some auxiliary path transformations. We shall now introduce some path transformations.

Definition 3.2. *Let $n_i : [0,T] \to \mathbb{R}^+, i \in I$, be a family of strictly positive continuous functions, and let $a : (0,T] \to \mathfrak{a}$ be a continuous map such that*

$$\int_{0^+} e^{-\alpha_i(a(s))} n_i(s) ds < \infty$$

we define, for $0 < t \leq T$,

$$\mathcal{T}_{i,n} a(t) = a(t) + \log \left(\int_0^t e^{-\alpha_i(a(s))} n_i(s) ds \right) \alpha_i^\vee.$$

Observe that in general the maps $t \mapsto a(t)$ and $t \mapsto \mathcal{T}_{i,n} a(t)$ need not be continuous at 0. For all that follows, consideration of the case $n_i \equiv 1$ in the above definition would be sufficient for our purposes, but the proofs would be the same as the general case.

Let $R^\vee$ be the root system dual to R, namely the roots of $R^\vee$ are the coroots of R and vice versa, and denote by $\mathcal{P}_{\alpha_i^\vee}, i \in I$, the corresponding Pitman transformations on $\mathfrak{a}$. Let π be a continuous path in $\mathfrak{a}$, with $\pi(0) = 0$. For $\varepsilon > 0$, let D_ε be the dilation operator $D_\varepsilon \pi(t) = \varepsilon \pi(t)$. A simple application of Laplace method yields the following

$$\mathcal{P}_{\alpha_i^\vee} \pi = \lim_{\varepsilon \to 0} D_\varepsilon \mathcal{T}_{i,n} D_\varepsilon^{-1} \pi \tag{3.2}$$

We shall establish, in section 3.4, a representation theoretic formula for a product $\mathcal{T}_{i_k,n} \dots \mathcal{T}_{i_1,n}$ corresponding to a minimal decomposition $w = s_{i_1} \dots s_{i_k}$ in the Weyl group. Using this formula we shall use (3.2) to get a formula for the Pitman transform.

3.3. A group theoretic interpretation of the operators $\mathcal{T}_{i,n}$. Let a be a smooth path in $\mathfrak{a}$ and let b be the path in the Borel subgroup $B^+ = HN^+$ solution to the differential equation

$$\frac{d}{dt} b(t) = \left(\frac{d}{dt} a(t) + \sum_i n_i(t) e_i \right) b(t); \qquad b(0) = id.$$

The following expression is easy to check.

LEMMA 3.3.

$$b(t) = e^{a(t)} + e^{a(t)} \sum_{k\geq 1} \sum_{i_1,\dots,i_k \in I^k} \left(\int_{t\geq t_1\geq t_2\geq \dots \geq t_k\geq 0} e^{-\alpha_{i_1}(a(t_1))} n_{i_1}(t_1)\dots e^{-\alpha_{i_k}(a(t_k))} n_{i_k}(t_k) dt_1 \dots dt_k \right) e_{i_1}\dots e_{i_k} \tag{3.3}$$

Observe that this expression is well defined in each finite dimensional representation of G since the operators e_i are nilpotent and this sum has only a finite number of nonzero terms. It is always in this context that we shall use this formula.

LEMMA 3.4. *For any $t > 0$ and $w \in W$ one has*

$$\Delta^{\omega_i}(b(t)\overline{w}) > 0.$$

Proof. By eq. (3.3) one has

$$\begin{aligned} \Delta^{\omega_i}(b(t)\overline{w}) = \langle e^{a(t)} \overline{w} v_{\omega_i}, v_{\omega_i}\rangle + \\ \sum_{r\geq 1} \sum_{i_1,\dots,i_r\in I^r} \int_{t\geq t_1\geq t_2\geq\dots\geq t_r\geq 0} \langle e^{a(t)} e^{-\alpha_{i_1}(a(t_1))} n_{i_1}(t_1)\dots \\ \dots e^{-\alpha_{i_r}(a(t_r))} n_{i_r}(t_r) e_{i_1}\dots e_{i_r} \overline{w} v_{\omega_i}, v_{\omega_i}\rangle\, dt_1\dots dt_r \end{aligned} \tag{3.4}$$

which is a sum of nonegative terms by Lemma 3.1. Furthermore, since v_{ω_i} is a highest weight vector, there exists some sequence $i_1, \dots, i_r$, such that $e_{i_1}\dots e_{i_r}\overline{w} v_{\omega_i}$ is a nonzero multiple of v_{ω_i}, and the n_i do not vanish, therefore the sum is positive. $\diamondsuit$

It follows in particular that, according to the terminology of [**12**], $b(t)$ belongs to the double Bruhat cell $B_+ \cap B_- w_0 B_-$, and that $b(t)\overline{w}$ has a Gauss decomposition $b(t)\overline{w} = [b(t)\overline{w}]_- [b(t)\overline{w}]_0 [b(t)\overline{w}]_+$ for all $t > 0$.

Now comes the main result of this section.

THEOREM 3.5. *Let $w \in W$ and $w = s_{i_1}\dots s_{i_k}$ be a reduced decomposition, then the H part in the Gauss decomposition of $b(t)\overline{w}$ is equal to*

$$\exp(\mathcal{T}_{i_k,n}\dots\mathcal{T}_{i_1,n} a(t)).$$

The fact that the path $\mathcal{T}_{i_k,n}\dots\mathcal{T}_{i_1,n} a(t)$ is well defined is part of the Theorem. By the uniqueness of the Gauss decomposition the preceding result implies

COROLLARY 3.6. *The path*

$$\mathcal{T}_{i_k,n}\dots\mathcal{T}_{i_1,n} a(t)$$

depends only on w and n and not on the chosen reduced decomposition of w.

We shall denote by $\mathcal{T}_w a$ the resulting path (it depends on n). We thus have

$$[b(t)\overline{w}]_0 = e^{\mathcal{T}_w a(t)}. \tag{3.5}$$

Proof of Theorem 3.5. The proof is by induction on the length of w. Let s_i be such that $l(ws_i) = l(w)+1$. We assume that the H part of the Gauss decomposition of $b(t)\overline{w}$ is $\mathcal{T}_{i_k,n}\dots\mathcal{T}_{i_1,n} a(t)$ as required. By (3.1) it is then enough to prove that for all $t > 0$ and $i, j \in I$ one has

$$\Delta^{\omega_j}(b(t)\overline{ws_i}) = \Delta^{\omega_i}(b(t)\overline{w})$$

if $i \neq j$ and

$$\Delta^{\omega_j}(b(t)\overline{ws}_i) = \Delta^{\omega_i}(b(t)\overline{w}) \int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds.$$

The claim for $i \neq j$ follows from Proposition 2.3 in [**12**], it remains to check the case $i = j$.

Lemma 3.7.

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})} \to_{t \to 0} 0.$$

Proof. From the decomposition (3.4), the fact that all terms are positive and that the n_i are positive continuous functions, we see that as $t \to 0$ one has $\Delta^{\omega_i}(b(t)\overline{w}) \sim c_1 t^{l_1}$ and $\Delta^{\omega_i}(b(t)\overline{ws}_i) \sim c_2 t^{l_2}$ for some $c_1, c_2 > 0$, where l_1 (resp. l_2) is the number of terms in the decomposition of $\omega_i - w(\omega_i)$ (resp. $\omega_i - ws_i(\omega_i)$) as a sum of simple roots. Since $l(ws_i) > l(w)$ the weight $w(\omega_i) - ws_i(\omega_i)$ is positive, and one has $l_2 > l_1$. ♢

Lemma 3.8. *Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition, and let $b^w(t) = [b(t)\overline{w}]_0 [b(t)\overline{w}]_+$, then one has*

$$\frac{d}{dt} b^w(t) = \left(\frac{d}{dt} \mathcal{T}_{i_k,n} \dots \mathcal{T}_{i_1,n} a(t) + \sum_{j \in I} n_j(t) e_j \right) b^w(t).$$

Proof. We do this by induction on the length of w. Assume this is true for w and let s_i be such that $l(ws_i) = l(w) + 1$, then one has

$$\frac{d}{dt} b^w(t) = \left(\frac{d}{dt} \mathcal{T}_w a(t) + \sum_j n_j e_j \right) b^w(t)$$

therefore

$$\frac{d}{dt} b^w(t)\overline{s}_i = \left(\frac{d}{dt} \mathcal{T}_w a(t) + \sum_j n_j(t) e_j \right) b^w(t)\overline{s}_i$$

Since $b^w(t) \in B^+$, by [**2**], [**12**], the Gauss decomposition of $b^w(t)\overline{s}_i$ has the form

$$b^w(t)\overline{s}_i = \exp(\beta(t) f_i) b^{ws_i}(t)$$

with $\beta(t) > 0$ for $t > 0$, and one has, since f_i commutes with all e_j for $j \neq i$.

$$\begin{aligned}
\frac{d}{dt} b^{ws_i}(t) &= \frac{d}{dt}\left[\exp(-\beta(t) f_i) b^w(t) \overline{s}_i\right] \\
&= -\left(\frac{d}{dt}\beta(t)\right) f_i \exp(-\beta(t) f_i) b^w(t)\overline{s}_i + \\
&\qquad \exp(-\beta(t) f_i)\left(\frac{d}{dt}\mathcal{T}_w a(t) + \sum_j n_j(t) e_j\right) b^w(t)\ \overline{s}_i \\
&= -\frac{d}{dt}\beta(t) f_i b^{ws_i}(t) + \\
&\quad \left(\frac{d}{dt}\mathcal{T}_w a(t) + \sum_j n_j(t) e_j + n_i(t)\beta(t) h_i + n_i(t)\beta^2(t) f_i\right) b^{ws_i}(t) \\
&= \Bigg[\left(\frac{d}{dt}\beta(t) + \frac{d}{dt}\alpha_i(\mathcal{T}_w a(t)) + n_i(t)\beta^2(t)\right) f_i + \\
&\qquad \frac{d}{dt}\mathcal{T}_w a(t) + n_i(t)\beta(t) h_i + \sum_j n_j(t) e_j\Bigg] b^{ws_i}(t)
\end{aligned}$$

Since $b^{ws_i}(t) \in B_+$, one has $\frac{d}{dt}\beta(t) + \frac{d}{dt}\alpha_i(T_w a(t)) + \beta^2(t) = 0$ therefore

$$\beta(t) = \frac{e^{-\alpha_i(\mathcal{T}_w a(t))}}{C + \int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds}$$

for some constant $C \geq 0$. Integrating the H part of the Gauss decomposition of $b^{ws_i}(t)$ we see that this part is equal to

$$\exp(\mathcal{T}_w a(t)) \exp(C' + \log(C + \int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds)) h_i \tag{3.6}$$

therefore

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})} = \exp(C')(C + \int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds)$$

and $C = 0$ by Lemma 3.7. We conclude that

$$\beta(t) = \frac{e^{-\alpha_i(\mathcal{T}_w a(t))}}{\int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds}.$$

This implies that

$$\begin{aligned}
\frac{d}{dt} b^{ws_i}(t) &= \left[\frac{d}{dt}\mathcal{T}_w a(t) + n_i(t)\frac{e^{-\alpha_i(\mathcal{T}_w a(t))}}{\int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds} h_i + \sum_j n_j(t) e_j\right] b^{ws_i}(t) \\
&= \left[\frac{d}{dt}\mathcal{T}_{i,n}\mathcal{T}_w a(t) + \sum_j n_j(t) e_j\right] b^{ws_i}(t)
\end{aligned}$$

as required. ◊

From (3.6) we obtain

$$\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})} = \Delta^{\omega_i}(e^{-\mathcal{T}_w a(t)} b^w(t)\overline{s}_i) = \exp(C')\int_0^t e^{-\alpha_i(\mathcal{T}_w a(s))} n_i(s) ds$$

Differentiating with respect to t we get

$$\begin{aligned}\frac{d}{dt}e^{-\mathcal{T}_w a(t)}b^w(t)\overline{s}_i &= e^{-\mathcal{T}_w a(t)}\sum_j n_j(t)e_j e^{\mathcal{T}_w a(t)}e^{-\mathcal{T}_w a(t)}b^w(t)\overline{s}_i\\ &= \left(\sum_i e^{-\alpha_j(\mathcal{T}_w a(t))}n_j(t)e_j\right)e^{-\mathcal{T}_w a(t)}b^w(t)\overline{s}_i\end{aligned}$$

where $e^{-\mathcal{T}_w a(t)}b^w(t)\in N$. It follows that

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\Delta^{\omega_i}(b(t)\overline{ws}_i)}{\Delta^{\omega_i}(b(t)\overline{w})}\right\} &= \left\langle\left(\sum_j e^{-\alpha_j(\mathcal{T}_w a(t))}n_j(t)e_j\right)e^{-\mathcal{T}_w a(t)}b^w(t)\overline{s}_i v_{\omega_i}, v_{\omega_i}\right\rangle\\ &= e^{-\alpha_i(\mathcal{T}_w a(t))}n_i(t)\langle e_i\overline{s}_i v_{\omega_i}, v_{\omega_i}\rangle\\ &= e^{-\alpha_i(\mathcal{T}_w a(t))}n_i(t)\end{aligned}$$

therefore $C'=0$. This proves the claim for $i=j$ and finishes the proof of Theorem 3.5. ◊

Corollary 3.9. *The transformations $\mathcal{T}_{i,n}$ satisfy the braid relations,*

$$\underbrace{\mathcal{T}_{i,n}\mathcal{T}_{j,n}\dots}_{m(i,j)\ terms}=\underbrace{\mathcal{T}_{j,n}\mathcal{T}_{i,n}\dots}_{m(i,j)\ terms}$$

where $m(i,j)$ is the Cartan integer $\alpha_i(\alpha_j^\vee)$.

Remark 3.10. In the case of rank two groups, the braid relations of the above corollary and an application of Laplace method yield the braid relations for Pitman operators as in Theorem 2.4, in the case of cristallographic angles $(\pi/m, m=2,3,4,6)$. It is instructive to give an elementary derivation of the braid relations for the $\mathcal{T}_{i,n}$ in the simplest nontrivial case namely type A_2 (i.e., $m=3$). In this case the relations amount to

$$\int_0^t ds\int_0^s dr\ F(r)\frac{G(s)}{G(r)}\frac{H(t)}{H(s)}=\int_0^t ds\int_0^s dr\ F(r)\frac{\tilde{G}(s)}{\tilde{G}(r)}\frac{\tilde{H}(t)}{\tilde{H}(s)}, \tag{3.7}$$

for some positive continuous functions F,G,H, where

$$\tilde{G}(s)=\left(\int_0^s G(r)H(r)^{-1}dr\right)^{-1}G(s)$$

and

$$\tilde{H}(s)=\left(\int_0^s G(r)H(r)^{-1}dr\right)H(s).$$

This can be checked directly by an application of Fubini's theorem, or an integration by parts. Similar but more complicated formulas correspond to the other crystallographic angles $\pi/4$ and $\pi/6$.

3.4. Representation theoretic formula for $\mathcal{P}_w$. Let $w\in W$, and let λ be a dominant weight, then $\lambda-w\lambda$ can be decomposed as a linear combination of simple positive roots $\lambda-w\lambda=\sum_{i\in I}u_i\alpha_i$ where u_i are nonnegative integers. If $(j_1,\dots,j_r)\in I^r$ is a sequence such that $\langle e_{j_1}\dots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle\neq 0$, then the number of k's in the sequence $j_1,\dots,j_r$ is equal to u_k. In particular the number r depends only on w and λ. We let $S(\lambda,w)$ denote the set of sequences $(j_1,\dots,j_r)\in I^r$ such that $\langle e_{j_1}\dots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle \neq 0$. Using (3.5), (3.4) and Theorem 3.5 we obtain the following expression

PROPOSITION 3.11. *Let a be a path in $\mathfrak{a}$, and λ a dominant weight, then one has*

$$\langle e^{\mathcal{T}_w(a)(t)}v_\lambda, v_\lambda\rangle = e^{\lambda(a(t))}\sum_{(j_1,\dots,j_r)\in S(\lambda,w)}\int_{t\geq t_1\geq\dots\geq t_r\geq 0}$$
$$e^{-\alpha_{j_1}(a(t_1))-\dots-\alpha_{j_r}(a(t_r))}n_{j_1}(t_1)\dots n_{j_r}(t_r)dt_1\dots dt_r\langle e_{j_1}\dots e_{j_r}\overline{w}v_\lambda, v_\lambda\rangle$$

Let $w\in W$ and let $\mathcal{P}_w^\vee$ denote the Pitman transformation on $\mathfrak{a}$ for the dual root system $R^\vee$, by (3.2), one has

$$\mathcal{P}_w^\vee\pi = \lim_{\varepsilon\to 0} D_\varepsilon\mathcal{T}_w D_\varepsilon^{-1}\pi.$$

Using Laplace method, Lemma 3.1 and Proposition 3.11 applied to fundamental weights, we now obtain the following expression for the Pitman transform (notice that W acts on $\mathfrak{a}^*$ and on $\mathfrak{a}$ by duality).

THEOREM 3.12. *(Representation theoretic formula for the Pitman transforms). Let $w\in W$, for each path π on $\mathfrak{a}$, one has*

$$\mathcal{P}_w^\vee\pi(t) = \pi(t) - \sum_{i\in I}\inf_{\substack{j_1,\dots,j_r\in S(\omega_i,w)\\ t\geq t_1\geq t_2\dots\geq t_r\geq 0}}\left(\alpha_{j_1}(\pi(t_1))+\dots+\alpha_{j_r}(\pi(t_r))\right)\alpha_i^\vee \tag{3.8}$$

This formula can be seen as a generalization of the formula in Theorem 2.5. Observe that sequences $j_1,\dots j_r$ such as the ones occuring in the theorem have appeared already in [**3**] under the name of **i**-trails. It is interesting to note that such sequences appear here naturally by an application of the Laplace method (sometimes called "tropicalization" in the algebraic litterature).

By Corollary 1, we see that Theorem 3.12 provides a representation theoretic formula for the dominant path in some Littelmann module, which is independent of any choice of a reduced decomposition of w_0.

REMARK 3.13. As noted before, formula 3.8 has a similar structure as formula 2.4 (when $\rho = \cos\frac{\pi}{n}$). We conjecture that such formulas exist for arbitrary Coxeter groups, i.e. for $w\in W$ there exists r and a set $S(s,w)\subset S^r$ such that

$$\mathcal{P}_w\pi(t) = \pi(t) - \sum_{s\in S}\inf_{\substack{s_1,\dots,s_r\in S(s,w)\\ t\geq t_1\geq t_2\dots\geq t_r\geq 0}}\left(\alpha_{s_1}^\vee(\pi(t_1))+\dots+\alpha_{s_r}^\vee(\pi(t_r))\right)\alpha_s. \tag{3.9}$$

However we do not know how to interpret these sets $S(s,w)$.

4. Duality

4.1. An involution on dominant paths. As in section 2.7, we consider a Coxeter system (W,S) generated by a set S of reflections of V. We assume now that the group W is finite and let w_0 be the longest element. We fix some $T>0$ and for any continuous path $\pi:[0,T]\to V$ such that $\pi(0)=0$ we let

$$\hat{\pi}(t) = -w_0(\pi(T-t)-\pi(T))$$

for $0\leq t\leq T$. We will show that the transformation

$$I(\pi) = \mathcal{P}_{w_0}\hat{\pi}$$

is an involution on the set of dominant paths, which generalizes the Schützenberger involution (see section 4.3 for the connection).

LEMMA 4.1. *Let v be an element of the closed Weyl chamber $\overline{C}$ and η be the linear path $\eta(t) = -tv$. Then $\mathcal{P}_w\eta = w\eta$ for any $w \in W$.*

Proof. We use induction on the length $l(w)$ of w. This is obvious when $l(w) = 0$. If $l(w) = n+1$, write $w = sw'$ where $s \in S$ and $l(w') = n$. We know ([**8**], V.4.4) that $\alpha_s^\vee(w'v) \geq 0$ when $v \in \overline{C}$, hence $\min_{0\leq r\leq t} \alpha_s^\vee(w'\eta)(r) = \alpha_s^\vee(w'\eta)(t)$ and

$$\mathcal{P}_s(w'\eta)(t) = (w'\eta)(t) - \alpha_s^\vee(w'\eta(t))\alpha_s = sw'\eta(t) = w\eta(t).$$

If, by induction hypothesis, one has $\mathcal{P}_{w'}\eta = w'\eta$ then $\mathcal{P}_w\eta = \mathcal{P}_s\mathcal{P}_{w'}\eta = \mathcal{P}_s(w'\eta) = w\eta$. $\diamondsuit$

Let $w \in W$ we now consider the following assumption on w.

HYPOTHESIS H(w): *There exists a map $\mathcal{H}_w : V \to \bar{C}$ such that, for any path π such that $\hat{\pi}$ is dominant, one has $\mathcal{P}_w\pi(T) = \mathcal{H}_w(\pi(T))$.*

PROPOSITION 4.2. *If W is a product of dihedral groups and Weyl groups, then H(w) holds for any $w \in W$.*

Proof. When $\hat{\pi}$ is dominant, then for any linear form Γ, nonnegative on the Weyl chamber, one has $\Gamma(\pi(T)) \leq \Gamma(\pi(s)) \leq 0$, when $0 \leq s \leq T$. In the representation formula given in Theorem 2.5 in the dihedral case, and in Theorem 3.12 when W is a Weyl group, we see that the inf is always reached for $t_0 = t_1 = \ldots = t_r = T$. This implies that $\mathcal{P}_w\pi(T)$ can be computed knowing only the value of $\pi(T)$, and implies the result for dihedral or Weyl groups. Of course the results extends trivially to products. $\diamondsuit$

We conjecture that H(w) holds for any element of a finitely generated Coxeter group. As the above proof shows, this would follow from the conjectured formula in 3.13.

PROPOSITION 4.3. *If H(w) holds, then one has $\mathcal{P}_w\pi(T) = w\pi(T)$ whenever $\hat{\pi}$ is a dominant path.*

Proof. It suffices to apply Lemma 4.1 to the linear path $\eta(t) = \frac{t}{T}\pi(T)$, since $\mathcal{P}_w\pi(T) = \mathcal{P}_w\eta(T)$. $\diamondsuit$

LEMMA 4.4. *Let J be the transformation on paths $J\pi(t) = \pi(T-t) - \pi(T)$, then $\mathcal{P}_{w_0} = \mathcal{P}_{w_0}J\mathcal{P}_{w_0}J$ and $(-w_0)\mathcal{P}_{w_0} = \mathcal{P}_{w_0}(-w_0)$.*

Proof. When α is a simple root then one checks easily that $\mathcal{P}_\alpha J\mathcal{P}_\alpha J = \mathcal{P}_\alpha$. Since $\mathcal{P}_{w_0} = \mathcal{P}_{w_0}\mathcal{P}_\alpha$ one has

$$\mathcal{P}_{w_0}J\mathcal{P}_\alpha J = \mathcal{P}_{w_0}\mathcal{P}_\alpha J\mathcal{P}_\alpha J = \mathcal{P}_{w_0}\mathcal{P}_\alpha = \mathcal{P}_{w_0}.$$

This implies the first equality since J is an involution. If α is a simple root, then $\tilde{\alpha} = -w_0\alpha$ is also a simple root and $\tilde{\alpha}^\vee = -\alpha^\vee w_0$. For any path π, since $w_0^2 = id$, one has

$$\begin{aligned}
\mathcal{P}_\alpha(-w_0\pi)(t) &= -w_0\pi(t) - \min_{0\leq s\leq t} \alpha^\vee((-w_0)\pi(s))\alpha \\
&= -w_0\pi(t) - \min_{0\leq s\leq t} \tilde{\alpha}^\vee(\pi(s))\alpha \\
&= -w_0[\pi(t) - \min_{0\leq s\leq t} \tilde{\alpha}^\vee(\pi(s))\tilde{\alpha}] \\
&= -w_0\mathcal{P}_{\tilde{\alpha}}\pi(t).
\end{aligned}$$

If $w_0 = \alpha_1 \cdots \alpha_r$ is a reduced expression we thus have

$$\mathcal{P}_{w_0}(-w_0) = \mathcal{P}_{\alpha_1} \cdots \mathcal{P}_{\alpha_r}(-w_0) = (-w_0)\mathcal{P}_{\tilde{\alpha}_1} \cdots \mathcal{P}_{\tilde{\alpha}_r} = (-w_0)\mathcal{P}_{w_0}$$

since $w_0 = \tilde{\alpha}_1 \cdots \tilde{\alpha}_r$. $\diamondsuit$

THEOREM 4.5. *The transformation $I\pi = \mathcal{P}_{w_0}\hat{\pi}$ has the following properties:*
(i) $I^2 = \mathcal{P}_{w_0}$;
(ii) The restriction of I to dominant paths is an involution;
(iii) $I\mathcal{P}_{w_0} = I$;
(iv) (Duality relation) If $H(w_0)$ holds, then for all path π, one has

$$\mathcal{P}_{w_0}\hat{\pi}(T) = \mathcal{P}_{w_0}\pi(T);$$

in particular, one has $I\pi(T) = \pi(T)$ when π is dominant.

Proof. By definition $I = \mathcal{P}_{w_0}J(-w_0)$, hence by Lemma 4.4,

$$I^2 = \mathcal{P}_{w_0}J(-w_0)\mathcal{P}_{w_0}(-w_0)J = \mathcal{P}_{w_0}J\mathcal{P}_{w_0}J = \mathcal{P}_{w_0}$$

this proves (i) and implies (ii) since $\mathcal{P}_{w_0}\pi = \pi$ when π is dominant. This also give

$$I\mathcal{P}_{w_0} = I^3 = I^2 I = I$$

since the image by I of any path is dominant. This relation can be written as $\mathcal{P}_{w_0}\hat{\pi} = \mathcal{P}_{w_0}\widehat{\mathcal{P}_{w_0}\pi}$. Applying Proposition 4.2 to the path $\widehat{\mathcal{P}_{w_0}\pi}$, we find that

$$\mathcal{P}_{w_0}\hat{\pi}(T) = \mathcal{P}_{w_0}\widehat{\mathcal{P}_{w_0}\pi}(T) = w_0\widehat{\mathcal{P}_{w_0}\pi}(T) = -J(\mathcal{P}_{w_0}\pi)(T) = (\mathcal{P}_{w_0}\pi)(T),$$

which proves (iv). $\diamondsuit$

Property (iv) will be important for the first proof of the Brownian motion property.

4.2. Symmetry of a Littelwood-Richardson construction. The concatenation $\pi \star \eta$ of two paths $\pi : [0,T] \to V$ $\eta : [0,T] \to V$ is defined in Littelmann [**22**] as the path $\pi \star \eta : [0,T] \to V$ given by $\pi \star \eta(t) = \pi(2t)$, when $0 \leq t \leq T/2$ and $\pi \star \eta(t) = \pi(T) + \eta(2(t - T/2))$ when $T/2 \leq t \leq T$.

LEMMA 4.6. *For all $w \in W$ one has $\mathcal{P}_w(\pi \star \eta) = \mathcal{P}_w(\pi) \star \eta'$, where $\mathcal{P}_{w_0}(\eta') = \mathcal{P}_{w_0}(\eta)$.*

Proof. One uses induction on the length $l(w)$ of w. When $l(w) = 1$ it is easy to see that $\mathcal{P}_w(\pi \star \eta) = \mathcal{P}_w(\pi) \star \eta'$ where $\mathcal{P}_w(\eta) = \mathcal{P}_w(\eta')$. Since $\mathcal{P}_{w_0}\mathcal{P}_w = \mathcal{P}_{w_0}$ the claim is thus true in this case. Suppose that it holds for elements of length n. Let $w = w_1 s$ where $l(w) = n+1, l(w_1) = n$, then one has

$$\mathcal{P}_w(\pi \star \eta) = \mathcal{P}_{w_1}\mathcal{P}_s(\pi \star \eta) = \mathcal{P}_{w_1}(\mathcal{P}_s(\pi) \star \eta')$$

where $\mathcal{P}_{w_0}\eta' = \mathcal{P}_{w_0}\eta$. Now by induction hypothesis

$$\mathcal{P}_{w_1}(\mathcal{P}_s(\pi) \star \eta') = (\mathcal{P}_{w_1}\mathcal{P}_s)(\pi) \star \eta''$$

where $\mathcal{P}_{w_0}\eta'' = \mathcal{P}_{w_0}\eta'$, and therefore $\mathcal{P}_{w_0}\eta'' = \mathcal{P}_{w_0}\eta$. $\diamondsuit$

In the case of Weyl groups, Littelmann has given the following analogue of the Littelwood-Richardson construction: Let π and η be two integral dominant paths defined on $[0,T]$, then the set

$$LR(\pi, \eta) = \{\pi \star \mu \,|\, \mu \in B\eta, \pi \star \mu \text{ is dominant}\}$$

gives a parametrization of the decomposition into irreducible representations of the tensor product of the representations with highest weights $\pi(T)$ and $\eta(T)$.

By Theorem 4.5 (*iii*), one has $I(\eta)(T) = \eta(T)$ and $I(\pi(T)) = \pi(T)$, therefore $LR(I(\eta), I(\pi))$ gives a parametrization of the decomposition of the tensor product of the representations with highest weights $\eta(T)$ and $\pi(T)$.

PROPOSITION 4.7. *Let $I(\pi) = \mathcal{P}_{w_0}\hat{\pi}$. Then*

$$I : LR(\pi, \eta)) \to LR(I(\eta), I(\pi))$$

is a bijective involution, which preserves the end points.

Proof. Let $\pi \star \mu \in LR(\pi, \eta)$. By Lemma 4.6 there is a path ξ such that

$$I(\pi \star \mu) = \mathcal{P}_{w_0}(\widehat{\pi \star \mu}) = \mathcal{P}_{w_0}(\hat{\mu} \star \hat{\pi}) = \mathcal{P}_{w_0}(\hat{\mu}) \star \xi$$

and $\mathcal{P}_{w_0}\xi = \mathcal{P}_{w_0}(\hat{\pi}) = I(\pi)$. By (ii) of Theorem 4.5, $\mathcal{P}_{w_0}(\hat{\mu}) = I(\mu) = I(\eta)$ thus $I(\pi \star \eta) \in LR(I(\eta), I(\pi))$. One checks easily that I preserves integrality, and the other properties follow from Theorem 4.5. ♢

4.3. Connection with the Schützenberger involution. In the case of a Weyl group of type A_{d-1} the transform $\mathcal{P}_{w_0}$ is connected with the Robinson, Schensted and Knuth (RSK) correspondence : Let us consider a word $v_1 v_2 \cdots v_n$ written with the alphabet $\{1, 2, \cdots, d\}$. Let $(P(n), Q(n))$ be the pair of tableaux associated with this word by RSK with column insertion (see, e.g., [**14**]). Let $\mathfrak{a} = \{(x_1, \cdots, x_d) \in \mathbb{R}^d; \sum_{i=1}^d x_i = 0\}$ and let (e_i) be the image in $\mathfrak{a}$ of the canonical basis of $\mathbb{R}^d$. We identify v_i with the path $\eta_i : t \mapsto te_{v_i}, 0 \le t \le 1$, and we consider the path $\pi = \eta_1 \star \eta_2 \cdots \star \eta_n$. Then $\mathcal{P}_{w_0}\pi$ is the path obtained by taking the succesives shapes of $Q(1), Q(2), \cdots, Q(n)$ (see Littelmann [**22**],[**24**]). Let us consider the pair $(\tilde{P}(n), \tilde{Q}(n))$ associated by the RSK algorithm to the word $v_n^* \cdots v_1^*$ where $v^* = d + 1 - v$. The Schützenberger involution is the map which associates the tableau $\tilde{Q}(n)$ to the tableau $Q(n)$ (see [**13**], [**14**], [**21**]). The path associated with the word $v_n^* \cdots v_1^*$ is $I(\pi)$. Thus I is a generalization of this involution. Note that I makes sense not only for Weyl groups, but for any finite Coxeter group.

5. Representation of Brownian motion in a Weyl chamber

5.1. Brownian motion in a Weyl chamber. In this section we recall some basic facts about Brownian motion in Weyl chambers.

We consider a Coxeter system (W, S) generated by a set S of reflections of an euclidean space V and we assume that W is finite. We shall denote by C the interior of a fundamental domain for the action of W on V (a Weyl chamber), and by $\overline{C}$ its closure.

If W is the Weyl group of a complex semi-simple Lie algebra $\mathfrak{g}$, with compact form $\mathfrak{g}_{\mathbb{R}}$, then V is identified with $\mathfrak{a}^*$, the dual space of the Lie algebra of a maximal torus T, and the Weyl chamber $\overline{C} = \overline{\mathfrak{a}}_+^*$ can be identified with the orbit space of $\mathfrak{g}_{\mathbb{R}}^*$ under the coadjoint action of the simply connected compact group K with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ (up to some identification of the walls). Let Z be a Brownian motion with values in $\mathfrak{g}_{\mathbb{R}}^*$, whose covariance is the Killing form. It is well known that the image of Z in the quotient space $\mathfrak{g}_{\mathbb{R}}^*/K$ remains in the interior of the Weyl chamber for all times $t > 0$, even if the starting point is inside some wall. Since the transition probabilities of Z are invariant under the coadjoint action it follows that this image, under the quotient map, is a Markov process on $\overline{C}$. A description of this Markov process can be done in terms of Doob's conditionning, namely the process is obtained from a Brownian motion X on $V = \mathfrak{a}^*$, killed at the boundary of the Weyl chamber, by means of a Doob transform with respect to the function

$$h(v) = \prod_{\alpha \in R^+} \alpha^\vee(v), \; v \in V,$$

(where R^+ is the set of positive roots) which is the unique, up to a scaling factor, positive harmonic function on $\overline{C}$ which vanishes on the boundary (see [**6**]). Recall that, by the reflection principle, the transition probabilities for the Brownian motion killed at the boundary of the Weyl chamber are

$$p_t^0(x,y)dy = \sum_{w \in W} \varepsilon(w) p_t(x, wy)dy, \; x, y \in \overline{C}, \tag{5.1}$$

where $p_t(x,y)dy$ are the transition probabilities for Brownian motion X, given by the Gaussian kernel on $\mathfrak{a}^*$ whose covariance is that of the Brownian motion. Thus the probability transitions for the Doob's process are

$$q_t(x,y)dy = \frac{h(y)}{h(x)} \sum_{w \in W} \varepsilon(w) p_t(x, wy)dy \tag{5.2}$$

for $x \in C$. These probability transitions can be continued by continuity to $x \in \overline{C}$, in particular to $x = 0$.

For a general finite Coxeter group, formula (5.1) still gives the probability transitions of Brownian motion killed at the boundary of the Weyl chamber. Let h be the product of the positive coroots, defined as the linear forms corresponding to the hyperplanes of the reflections in the group W, taking the signs so that they are positive inside the Weyl chamber, then the function h is still the only (up to a multiplicative constant) positive harmonic function vanishing on the boundary, and the equation (5.2) defines the semi-group of what we call the Brownian motion in the fundamental chamber $\overline{C}$ of V.

We shall prove that the Pitman operator $\mathcal{P}_{w_0}$ applied to Brownian motion in V yields a Brownian motion in the Weyl chamber. We shall give two very different proofs of this. The first one uses in an essential way the duality relation of Theorem 4.5 (*iv*) and a classical result in queuing theory. It is thus valid for dihedral groups or Weyl groups (or products of these). The second one uses a random walk approximation and relies on Littelmann theory and Weyl's character formula. It is valid only for Weyl groups. We have chosen to present this second proof because it emphazises the close connection between Brownian paths and Littelmann paths.

5.2. Brownian motion with a drift. We now consider a Brownian motion in V with invariant covariance, but with a drift $\xi \in C$. Its transition probabilities are now

$$p_{t,\xi}(x,y) = p_t(x,y) \exp(\langle \xi, y - x \rangle - \frac{\|\xi\|^2}{2} t)$$

Actually the distribution of this Brownian motion on the σ-field $\mathcal{F}_t$ generated by the coordinate functions $X_s, s \leq t$, on the canonical space, is absolutely continuous with respect to the one of the centered Brownian motion, with density

$$\exp(\langle \xi, X_t - X_0 \rangle - \frac{\|\xi\|^2}{2} t).$$

Consider such a Brownian motion in V with drift ξ, starting inside the chamber at point x, and killed at the boundary of C. The distribution of this process at time t is therefore given by the density, for $y \in C$,

$$\begin{aligned} p_t^0(x,y)\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t) &= \sum_{w\in W} \varepsilon(w) p_t(x,wy)\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t) \\ &= \sum_{w\in W} \varepsilon(w) p_t(0,y-wx)\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t) \end{aligned}$$

where we have used the invariance of p_t under the Weyl group. We now integrate this density over C, in order to get the probability that the exit time from C is larger than t. Denoting by T_C this exit time, one has

$$P(T_C > t) = \sum_{w\in W} \varepsilon(w) \int_C p_t(0,y-wx)\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t)\, dy$$

Since the drift ξ is in the chamber, for large t one has

$$\int_{V\setminus C} \exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t)\, dy \to 0$$

therefore

$$\int_C p_t(0,y-wx)\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t)\, dy \to_{t\to\infty} \exp(\langle \xi, w(x)-x\rangle)$$

and

$$\lim_{t\to\infty} P(T_C > t) = P(T_C = \infty) = \sum_{w\in W} \varepsilon(w)\exp(\langle \xi, w(x)-x\rangle)$$

We denote by $h_\xi(x)$ this function. It follows that, conditionally on $\{T_C = \infty\}$, the Brownian motion with drift ξ, starting in C and killed at the boundary of C, is a Markov process with transition probabilities

$$q_{t,\xi}(x,y) = p_t^0(x,y)\frac{h_\xi(y)}{h_\xi(x)}\exp(\langle \xi, y-x\rangle - \frac{\|\xi\|^2}{2}t).$$

Observe that $\frac{h_\xi(y)}{h_\xi(x)} \to \frac{h(y)}{h(x)}$ as $\xi \to 0$. Standard arguments now show that as $x \to 0$ and $\xi \to 0$ the distribution of this process approaches that of the Brownian motion in the Weyl chamber, starting from 0.

Finally we can rephrase this in the following way.

LEMMA 5.1. *The distribution of the Brownian motion with a drift $\xi \in C$, started at 0 and conditionned to stay forever in the cone $C - x$ (where $x \in C$) converges towards the distribution of the Brownian motion in the Weyl chamber when $x, \xi \to 0$.*

5.3. Some further path transformations. Let $w_0 = s_1 \dots s_q$ be a reduced decomposition and write $\alpha_i = \alpha_{s_i}$. Let $\eta : [0,T] \to V$ be a path with $\eta(0) = 0$. Recall that η is dominant if $\eta(t) \in \overline{C}$ for all $t \le T$. Set $\eta_q = \eta$ and, for $j = 1, \dots, q$,

$$\eta_{j-1} = \mathcal{P}_{s_j} \dots \mathcal{P}_{s_q}\eta_q, \qquad x_j = -\inf_{0\le t\le T} \alpha_j^\vee(\eta_j(t)).$$

Then

$$\mathcal{P}_{w_0}\eta(T) = \eta(T) + \sum_{j=1}^q x_j\alpha_j$$

and η is dominant if, and only if, $x_j = 0$ for all $j \leq q$. We now introduce some new path transformations and give an alternative characterisation of dominant paths.

In this section we assume that $\mathrm{H}(w_0)$ holds, and therefore, by Theorem 4.5, that $\mathcal{P}_{w_0}\eta(T) = \mathcal{P}_{w_0}\hat{\eta}(T)$ for all paths η.

Let $w \in W$ be a reflection, i.e. w is conjugate to some element in S. We choose a non zero element α of V such that $w\alpha = -\alpha$, then w is the reflection s_α given by (2.1) where $\alpha^\vee(v) = 2(\alpha, v)/(\alpha, \alpha)$. As in [**19**] we call α a positive root when $\alpha^\vee$ is positive on the Weyl chamber C, it is a simple root when $s_\alpha \in S$. Observe that one has $\mathcal{P}_\alpha = \mathcal{P}_{s_\alpha}$ for all positive roots (the left hand side is defined by Definition 2.1, and the second by Matsumoto's lemma, since $s_\alpha \in W$).

Let β be a positive root, and s_β the associated reflection. For any positive root α, one has

$$s_\beta \circ \mathcal{P}_\alpha \circ s_\beta = \mathcal{P}_{s_\beta(\alpha)}.$$

Consider the transformation $\mathcal{Q}_\beta = \mathcal{P}_\beta \circ s_\beta$. One has

$$\mathcal{Q}_\beta\eta(t) = s_\beta\eta(t) + \sup_{0 \leq s \leq t} \beta^\vee(\eta(s))\beta.$$

Furthermore if $w_0 = s_1 \dots s_q$ is a reduced decomposition ($s_i \in S$), then

$$\mathcal{Q}_{w_0} := \mathcal{P}_{w_0} \circ w_0 = \mathcal{Q}_{\beta_1} \circ \dots \circ \mathcal{Q}_{\beta_q}$$

where $\beta_1 = \alpha_1$ and $\beta_j = s_1 \dots s_{j-1}\alpha_j$.

Now define transformations $\mathcal{D}_\alpha = \iota \circ \mathcal{Q}_\alpha \circ \iota$ where ι is the involution defined, for $\eta : [0, T] \to V$, by $\iota\eta(t) = \eta(T) - \eta(T - t)$. One has

$$\mathcal{D}_\alpha\eta = \eta(t) + \inf_{T \geq u \geq t} \alpha^\vee(\eta(u) - \eta(t))\alpha - \inf_{T \geq u \geq 0} \alpha^\vee(\eta(u))\alpha. \tag{5.3}$$

Set

$$\mathcal{D}_{w_0} = \mathcal{D}_{\beta_1} \circ \cdots \circ \mathcal{D}_{\beta_q} = \iota \circ \mathcal{Q}_{w_0} \circ \iota$$

and note that $\mathcal{D}_{w_0}\eta = \iota\mathcal{P}_{w_0}\hat{\eta}$. Set $\rho_q = \eta$ and , for $j \leq q$,

$$\rho_{j-1} = \mathcal{D}_{\beta_j} \dots \mathcal{D}_{\beta_q}\rho_q, \qquad y_j = -\inf_{T \geq u \geq 0} \beta_j^\vee(\rho_j(u)).$$

LEMMA 5.2. *For all paths η one has*

$$\mathcal{P}_{w_0}\eta(T) = \eta(T) + \sum_{j=1}^{q} y_j\beta_j. \tag{5.4}$$

In particular, η is dominant if, and only if, $y_j = 0$ for all $j \leq q$.

Proof. By construction,

$$\mathcal{D}_{w_0}\eta(T) = \eta(T) + \sum_{j=1}^{q} y_j\beta_j.$$

Since $\mathcal{D}_{w_0}\eta = \iota\mathcal{P}_{w_0}\hat{\eta}$ and $\mathcal{P}_{w_0}\eta(T) = \mathcal{P}_{w_0}\hat{\eta}(T)$, this implies (5.4). The path η is dominant if, and only if, $\mathcal{P}_{w_0}\eta(T) = \eta(T)$. By (5.4), this holds if, and only if, $\sum_j y_j\beta_j = 0$ and, since the y_j and β_j are all positive, this is equivalent to the statement that $y_j = 0$ for all $j \leq q$. $\diamondsuit$

5.4. The representation theorem, first proof. The definitions of transformations $\mathcal{P}_\alpha$, $\mathcal{P}_{w_0}$, $\mathcal{Q}_\alpha$, $\mathcal{Q}_{w_0}$ extend naturally to paths π defined on $\mathbb{R}^+$. In this section we will prove (assuming again H(w_0)) that, if X is a Brownian motion in V (started from the origin), then $\mathcal{Q}_{w_0}X$ is a Brownian motion in the fundamental chamber $\overline{C}$. Since w_0 leaves the distribution of Brownian motion invariant, this implies that $\mathcal{P}_{w_0}X$ is a Brownian motion in $\overline{C}$.

To prove this, we first extend the definition of the $\mathcal{D}_\beta$. Let β be a positive root. For paths $\pi : [0, +\infty) \to V$ with $\pi(0) = 0$ and $\alpha^\vee(\pi(t)) \to +\infty$ as $t \to +\infty$ for all simple roots α, define

$$\mathcal{D}_\beta\pi(t) = \pi(t) + \inf_{s\geq t} \beta^\vee(\pi(s) - \pi(t))\beta - \inf_{s\geq 0} \beta^\vee(\pi(s))\beta. \tag{5.5}$$

Now set $\mathcal{D}_{w_0} = \mathcal{D}_{\beta_1} \circ \cdots \circ \mathcal{D}_{\beta_q}$ as before. Since $\mathcal{D}_{w_0}$ does not depend on the chosen reduced decomposition of w_0 we can also write $\mathcal{D}_{w_0} = \mathcal{D}_{\beta_q} \circ \cdots \circ \mathcal{D}_{\beta_1}$.

In the rest of this section we assume that H(w_0) holds.

Lemma 5.3. *If π is a dominant path, one has $\mathcal{Q}_{w_0}\mathcal{D}_{w_0}\pi = \pi$.*

Proof. It is easy to see that for any positive root β and path $\xi : [0, \infty) \to V$ with $\xi(0) = 0$ and $\inf_{t\geq 0} \beta^\vee(\xi(t)) = 0$ we have $\mathcal{Q}_\beta\mathcal{D}_\beta\xi = \xi$. Let $\eta_0 = \pi$ and

$$\eta_j = \mathcal{D}_{\beta_j} \ldots \mathcal{D}_{\beta_1}\pi, \qquad v_j(t) := -\inf_{u\geq t} \beta_j^\vee(\eta_{j-1}(u) - \eta_{j-1}(t)).$$

Since π is dominant we have, by lemma 5.2 (with $T \to \infty$) that $v_j(0) = 0$ for each $j \leq q$ and hence

$$\mathcal{Q}_{w_0}\mathcal{D}_{w_0}\pi = \mathcal{Q}_{\beta_1} \ldots \mathcal{Q}_{\beta_q}\mathcal{D}_{\beta_q} \ldots \mathcal{D}_{\beta_1}\pi = \pi$$

as required. ♢

Lemma 5.4. *If X is a Brownian motion with drift in C, then $\mathcal{D}_{w_0}X$ has the same distribution as X and, moreover, is independent of the collection of random variables $\{\inf_{t\geq 0} \alpha^\vee(X(t)),\ \alpha$ simple root$\}$.*

Proof. To prove this, we first need to extend the definitions of $\mathcal{D}_\beta$ and $\mathcal{Q}_\beta$ to paths π defined on $\mathbb{R}$ with $\pi(0) = 0$ and $\alpha^\vee(\pi(t)) \to \pm\infty$ as $t \to \pm\infty$ for all simple α. For $t \in \mathbb{R}$, set

$$\mathcal{Q}_\beta\pi(t) = s_\beta\pi(t) + \sup_{s\leq t} \beta^\vee(\pi(s))\beta - \sup_{s\leq 0} \beta^\vee(\pi(s))\beta$$

and define $\mathcal{D}_\beta\pi$ by (5.5) allowing $t \in \mathbb{R}$. Then, if ι denotes the involution

$$\iota\pi(t) = -\pi(-t)$$

one has $\mathcal{D}_\beta = \iota \circ \mathcal{Q}_\beta \circ \iota$ and $\mathcal{D}_{w_0} := \mathcal{D}_{\beta_q} \circ \cdots \circ \mathcal{D}_{\beta_1} = \iota \circ \mathcal{Q}_{w_0} \circ \iota$ as before. Note that $\mathcal{D}_{w_0}$ does not depend on the particular reduced decomposition of w_0, and also that $\mathcal{D}_\beta(\pi(t), t \geq 0) = (\mathcal{D}_\beta\pi(t), t \geq 0)$ and $\mathcal{D}_{w_0}(\pi(t), t \geq 0) = (\mathcal{D}_{w_0}\pi(t), t \geq 0)$. We will use the following auxillary lemma.

Lemma 5.5. *Let $\pi : \mathbb{R} \to V$ with $\pi(0) = 0$, and $\alpha(\pi(t)) \to \pm\infty$ as $t \to \pm\infty$ for all simple roots α. Then, for all $t \in \mathbb{R}$,*

$$-\inf_{u\geq t} \beta^\vee(\pi(u) - \pi(t)) = -\inf_{s\leq t} \beta^\vee(\mathcal{D}_\beta\pi(u) - \mathcal{D}_\beta\pi(t)).$$

Proof. This can be checked directly, or deduced from (2.2). ♢

Introduce a Brownian motion Y indexed by $\mathbb{R}$ such that $X = (Y(t), t \geq 0)$ and $(\iota Y(t), t \geq 0)$ is an independent copy of X. For any positive root β, the distribution of $\mathcal{D}_\beta Y$ is the same as that of Y. This is a one-dimensional statement which can be checked directly, or can be seen as a consequence of the classical output theorem on the $M/M/1$ queue (see, for example, [**28**]). In particular, the distribution of $\mathcal{D}_\beta X$ is the same as that of X. It follows that $\mathcal{D}_{w_0} Y$ has the same distribution as Y, and $\mathcal{D}_{w_0} X$ has the same distribution as X. Let $Y_0 = Y$ and

$$Y_j = \mathcal{D}_{\beta_j} \dots \mathcal{D}_{\beta_1} Y, \qquad V_j(t) := -\inf_{u \geq t} \beta_j^\vee (Y_{j-1}(u) - Y_{j-1}(t)).$$

Note that $Y_q = \mathcal{D}_{w_0} Y$ and recall that, for $t \geq 0$, $\mathcal{D}_{w_0} Y(t) = \mathcal{D}_{w_0} X(t)$. According to 5.5 one has

$$V_q(t) = -\inf_{s \leq t} \beta_j^\vee (Y_q(s) - Y_q(t))$$

$$Y_{q-1}(t) = Y_q(t) + (V_q(t) - V_q(0))\beta_q$$

and by induction on k,

$$V_{q-k}(t) = -\inf_{s \leq t} \beta_j^\vee (Y_{q-k}(s) - Y_{q-k}(t))$$

$$Y_{q-k-1}(t) = Y_{q-k}(t) + (V_{q-k}(t) - V_{q-k}(0))\beta_{q-k}$$

It follows that the $V_j(t); t \leq 0$ are measurable with respect to the σ-field generated by $\mathcal{D}_{w_0} Y(s); s \leq 0$. In particular, the random variable $V_1(0) = \inf_{t>0} \beta_1^\vee (X(t))$ is measurable with respect to the σ-field generated by $\mathcal{D}_{w_0} Y(s); s \leq 0$. Now, for each $\alpha \in S$, there is a reduced decomposition of w_0 with $\beta_1 = \alpha$, so we see that the random variables $\{\inf_{t \geq 0} \alpha^\vee (X(t)),\ \alpha \text{ simple}\}$ are all measurable with respect to the σ-field generated by $\mathcal{D}_{w_0} Y(s); s \leq 0$, and therefore independent of $\mathcal{D}_{w_0} Y(s); s \geq 0$, as required. ♢

THEOREM 5.6. *If $H(w_0)$ is satisfied and X is a Brownian motion in V then $\mathcal{P}_{w_0} X$ is a Brownian motion in $\overline{C}$.*

Proof. Let $x, \xi \in C$ and let X be a Brownian motion with drift ξ. From the above lemma the event 'X remains in the cone $C - x$ for all times' is independent of $\mathcal{D}_{w_0} X(t); t \geq 0$. Thus, if R has the same distribution as that of X conditioned on this event, then $\mathcal{D}_{w_0} R$ has the same distribution as X. Now we can let $x, \xi \to 0$ so that X is a Brownian motion with no drift and R is a Brownian motion in $\overline{C}$; by continuity, $\mathcal{D}_{w_0} R$ has the same distribution as X. Now, by lemma 5.3 $\mathcal{Q}_{w_0} \mathcal{D}_{w_0} R = R$ almost surely. It follows that $\mathcal{Q}_{w_0} X$, and hence $\mathcal{P}_{w_0} X$, is a Brownian motion in $\overline{C}$, as required. ♢

REMARK 5.7. This proof actually extends to the case of geometric brownian motion, generalizing results of Matsumoto and Yor [**26**], [**27**]. We hope to come back to this in future work.

5.5. Random walks and Markov chains on the weight lattice. We will now present the second proof of the Brownian motion property. We assume that W is the Weyl group of the semisimple Lie algebra $\mathfrak{g}$ as in sections 3.1, 5.1, and $V = \mathfrak{a}^*$. As in section 5.1, let T be a maximal torus of the compact group K, the simply connected compact group with Lie algebra $\mathfrak{g}_\mathbb{R}$, a compact form of $\mathfrak{g}$. Let $\omega \in P_+$ be a nonzero dominant weight and let χ_ω be the character of the associated highest weight module. As a function on T this is the Fourier transform of the positive measure R_ω on P, which puts a weight m_μ^ω on a weight μ where m_μ^ω is the multiplicity of μ in the module with highest weight ω. In other words

$$\chi_\omega = \sum_{\mu \in P_+} m_\mu^\omega e(\mu)$$

where $e(\mu)(\theta) = e^{2i\pi\langle \mu, \theta\rangle}$ is the character on T. We can divide this measure R_ω by $\dim \omega$ to get a probability measure

$$\nu_\omega = \frac{1}{\dim \omega} R_\omega.$$

Consider the random walk $(X_n; n \geq 0)$, on the weight lattice, whose increments are distributed according to this probability measure, started at zero. Thus the transition probabilities of this random walk are given by

$$p_\omega(\mu, \lambda) = \frac{m^\omega_{\lambda-\mu}}{\dim \omega}.$$

Donsker's theorem and invariance of m^ω under the Weyl group implies

THEOREM 5.8. *The stochastic process* $\frac{X_{[Nt]}}{\sqrt{N}}$ *converges, as* $N \to \infty$, *to a Brownian motion on* $\mathfrak{a}^*$ *with correlation invariant under* W.

Let us define a probability transition function q_ω on P_+ by the formula

$$\frac{\chi_\mu}{\dim \mu} \frac{\chi_\omega}{\dim \omega} = \sum_{\lambda \in P_+} q_\omega(\mu, \lambda) \frac{\chi_\lambda}{\dim \lambda}.$$

Thus $q_\omega(\mu, \lambda)$ is equal to $\frac{M^\lambda_{\omega,\mu} \dim \lambda}{\dim \omega \dim \mu}$ where $M^\lambda_{\omega,\mu}$ is the multiplicity of the module with highest weight λ in the decomposition of the tensor product of the modules with highest weights ω and μ, see, e.g. [**11**],[**5**].

LEMMA 5.9. *One has*

$$q_\omega(\mu, \lambda) = \frac{\dim \lambda}{\dim \mu} \sum_{w \in W} \varepsilon(w) p_\omega(\mu + \rho, w(\lambda + \rho)).$$

Proof. Let dk be the normalized Haar measure on K. By the orthogonality relations for characters, one has

$$M^\lambda_{\omega,\mu} = \int_K \overline{\chi_\lambda}(k) \chi_\mu(k) \chi_\omega(k) dk$$

therefore

$$q_\omega(\mu, \lambda) = \frac{M^\lambda_{\omega,\mu} \dim \lambda}{\dim \omega \dim \mu} = \frac{\dim \lambda}{\dim \mu \dim \omega} \int_K \overline{\chi_\lambda}(k) \chi_\mu(k) \chi_\omega(k) dk$$

Now we can use the Weyl integration formula as well as Weyl's character formula to rewrite the formula as an integral over T, the maximal torus of K. Thus

$$q_\omega(\mu, \lambda) = \frac{|W| \dim \lambda}{\dim \mu \dim \omega} \int_T \sum_{w_1, w_2 \in W} \varepsilon(w_1)\varepsilon(w_2) \overline{e(w_1(\lambda + \rho)}(\theta) e(w_2(\mu+\rho))(\theta) \chi_\omega(\theta) d\theta$$

where $e(\gamma)(\theta) = e^{2i\pi\langle\gamma,\theta\rangle}$ and ρ is half the sum of positive weights. Now using the invariance of χ_ω under the Weyl group we can rewrite this as

$$\begin{aligned} q_\omega(\mu,\lambda) &= \frac{\dim\lambda}{\dim\mu\dim\omega}\int_T \sum_{w\in W} \varepsilon(w)\overline{e(\lambda+\rho)}(\theta)e(w(\mu+\rho))(\theta)\chi_\omega(\theta)d\theta \\ &= \frac{\dim\lambda}{\dim\mu}\sum_{w\in W} p_\omega(\mu+\rho, w(\lambda+\rho)). \end{aligned}$$

$\diamondsuit$

From (5.2), Theorem 5.8, Lemma 5.9, and standard arguments, we deduce

PROPOSITION 5.10. *Let Y be a Markov chain on P_+ started at 0, with transition probabilities $q_\omega(\mu,\lambda)$, then $\frac{Y([Nt])}{\sqrt{N}}$ converges in distribution, as $N\to\infty$ to a Brownian motion in the Weyl chamber $\overline{C}$.*

5.6. Pitman operators and the Markov chain on the weight lattice. We choose a nonzero dominant weight ω, and a dominant path π^ω defined on $[0,1]$ with $\pi^\omega(1) = \omega$. Let $B\pi^\omega$ be the set of paths in the Littelmann module generated by π^ω. We now construct a stochastic process with values in P. Choose independent random paths $\eta_n \in B\pi^\omega; n = 1, 2, \ldots$, each with uniform distribution on $B\pi^\omega$, and define the stochastic process Z as the random path obtained by the usual concatenations $\eta_1 * \eta_2 * \cdots$ of the $\eta_i; i = 1, 2, \ldots$. In other words, one has $Z(t) = \eta_1(1) + \eta_2(1) + \ldots + \eta_{n-1}(1) + \eta_n(t-n)$ if $t \in [n, n+1]$. Beware that this concatenation does not coincide with Littelmann's definition, recalled in section (4.2), since we do not rescale the time. Littelmann's theory then implies that $\eta_n(1)$ is a random weight in P with distribution ν_ω, and $(Z(n); n = 0, 1, \ldots)$ is the random walk in $\mathfrak{a}^*$ with this distribution of increments.

THEOREM 5.11. *The stochastic process $\mathcal{P}_{w_0}Z(n); n = 0, 1, \ldots$ is a Markov chain on P_+, with probability transitions q_ω.*

Proof. First note that the set of paths of the form $\eta_1 * \eta_2 * \ldots * \eta_n$ where $\eta_i \in B\pi^\omega$ is stable under Littelmann operators, by [**22**], therefore by (2.3) it is also stable under Pitman transformations. Consider a dominant path of the form $\gamma_1 * \gamma_2 * \ldots * \gamma_n$, with all $\gamma_i \in B\pi^\omega$. We shall compute the conditional probability distribution of $\mathcal{P}_{w_0}Z(n+1)$ knowing that $\mathcal{P}_{w_0}Z(t) = \gamma_1 * \gamma_2 * \ldots * \gamma_n(t)$ for $t \le n$. Let $\mu = \gamma_1 * \gamma_2 * \ldots * \gamma_n(1)$. By Corollary 2.10 the set of all paths of the form $\eta_1 * \eta_2 * \ldots * \eta_n$ such that $\mathcal{P}_{w_0}(\eta_1 * \eta_2 * \ldots * \eta_n) = \gamma_1 * \gamma_2 * \ldots * \gamma_n$ coincides with the Littelmann module $B(\gamma_1 * \gamma_2 * \ldots * \gamma_n)$. Now consider a path $\eta_{n+1} \in B\pi$ and the concatenation $\eta_1 * \eta_2 * \ldots * \eta_n * \eta_{n+1}$, then $\mathcal{P}_{w_0}(\eta_1 * \eta_2 * \ldots * \eta_n * \eta_{n+1})$ will be the dominant path in the Littelmann module generated by $\eta_1 * \eta_2 * \ldots * \eta_n * \eta_{n+1}$. By Littelmann's version of the Littlewood-Richardson rule (section 10 in [**22**]), the number of pairs of paths $(\eta_1 * \eta_2 * \ldots * \eta_n, \eta_{n+1})$ such that $\mathcal{P}_{w_0}(\eta_1 * \eta_2 * \ldots * \eta_n) = \gamma_1 * \gamma_2 * \ldots * \gamma_n$ and $\mathcal{P}_{w_0}(\eta_1 * \eta_2 * \ldots * \eta_n * \eta_{n+1})(1) = \lambda$ is equal to the dimension of the isotypic component of type λ in the module which is the tensor product of the highest weight modules μ and ω, in particular this depends only on μ, and is equal to $M^\lambda_{\omega,\mu}\dim\lambda$. Since the total number of pairs $(\eta_1 * \eta_2 * \ldots * \eta_n, \eta_{n+1})$ with $\mathcal{P}_{w_0}(\eta_1 * \eta_2 * \ldots * \eta_n) = \gamma_1 * \gamma_2 * \ldots * \gamma_n$ is $\dim\mu\dim\omega$, we see that the conditional probability we seek is $\frac{M^\lambda_{\omega,\mu}\dim\lambda}{\dim\omega\dim\mu} = q_\omega(\mu,\lambda)$. This proves the claim. $\diamondsuit$

5.7. Second proof of the representation theorem for Weyl groups. Putting together Proposition 5.10 and Theorem 5.11 we get another proof of Theorem 5.6. Indeed, by Donsker's theorem, the process $\frac{Z([Nt])}{\sqrt{N}}$ gives as limit the Brownian motion in $\mathfrak{a}^*$. The process $\mathcal{P}_{w_0}Z(n); n \geq 0$, is distributed as the Markov process of Proposition 5.10, by Theorem 5.11. Applying the scaling of Proposition 5.10 to the stochastic process $(\mathcal{P}_{w_0}Z(t); t \geq 0)$ yields for limit process the Brownian motion on the Weyl chamber. Since $\mathcal{P}_{w_0}$ is a continuous map, which commutes with scaling we get the proof of Theorem 5.6, when W is the Weyl group of a complex semisimple Lie algebra. $\diamondsuit$

5.8. A remark on the Duistermaat-Heckman measure. The distribution of the path $t \in [0, n] \mapsto Z(t)$ is uniform on the set

$$B(\pi^\omega)^{*n} = \{\eta_1 * \eta_2 * \cdots * \eta_n; \eta_i \in B\pi^\omega\}.$$

Therefore, for any path $\eta \in B(\pi^\omega)^{*n}$, the distribution of $(Z(s))_{0\leq s\leq n}$ conditionally on $\{\mathcal{P}_{w_0}Z(s) = \eta(s), 0 \leq s \leq n\}$, is uniform on the set $\{\gamma \in B(\pi^\omega)^{*n}; \mathcal{P}_{w_0}\gamma = \eta\}$. It thus follows from Littelmann theory [**22**] that the conditional distribution of the terminal value Z_n is the probability measure ν_η. It has been proved by Heckman [**18**] (see also [**17**], [**10**]) that if $\gamma_\varepsilon \to \infty$ in $\mathfrak{a}_+^*$ and $\varepsilon\gamma_\varepsilon \to v$ then $D_\varepsilon \nu_{\gamma_\varepsilon}$ converges to the so called Duistermaat-Heckman measure associated to v, i.e. the projection of the normalized measure on the coadjoint orbit of K through v, by the orthogonal projection on $\mathfrak{a}^*$. This follows from Kirillov's character formula for K. From the preceding section we deduce that if X is the Brownian motion on $\mathfrak{a}^*$, then the law of $X(T)$ conditionally on $\mathcal{P}_{w_0}X = \gamma$ on $[0, T]$ is the Duistermaat-Heckman measure associated with $\gamma(T)$.

6. Appendix

6.1. Proof of Proposition 2.2 (*iv*). Let η be a path, let $\pi = \mathcal{P}_\alpha\eta$, let $x = -\inf_{T\geq t\geq 0} \alpha^\vee(\eta(t))$ and $t_0 = \sup\{t | \alpha^\vee(\eta(t)) = -x\}$, we shall check that equation 2.2 is valid.

If $t \geq t_0$ then one has $\inf_{0\leq s\leq t} \alpha^\vee(\eta(s)) = -x$ therefore

$$\begin{aligned}
\alpha^\vee(\pi(t)) &= \alpha^\vee(\eta(t)) + 2x \\
&= x + (\alpha^\vee(\eta(t)) + x) \\
&\geq x
\end{aligned}$$

for all $t \geq t_0$. It follows that $\inf(x, \inf_{T\geq s\geq t} \alpha^\vee(\pi(s))) = x$ for $t \geq t_0$. Formula 2.2 follows for $t \geq t_0$.

If $t < t_0$, let $u = \inf\{s \geq t | \alpha^\vee(\eta(s)) = \inf_{0\leq v\leq t} \alpha^\vee(\eta(v))\}$. Then $t \leq u \leq t_0$. One has

$$\begin{aligned}
\alpha^\vee(\pi(u)) &= \alpha^\vee(\eta(u)) - 2\inf_{0\leq v\leq u} \alpha^\vee(\eta(v)) \\
&= -\alpha^\vee(\eta(u))
\end{aligned}$$

which implies that $\inf_{T\geq v\geq t}\alpha^\vee(\pi(v)) \leq -\inf_{0\leq v\leq t}\alpha^\vee(\eta(v))\} \leq x$. On the other hand, for $v \geq t$ one has

$$\begin{aligned}\alpha^\vee(\pi(v)) &= \alpha^\vee(\eta(v)) - 2\inf_{0\leq s\leq v}\alpha^\vee(\eta(s)) \\ &\geq (\alpha^\vee(\eta(v)) - \inf_{0\leq s\leq v}\alpha^\vee(\eta(s))) - \inf_{0\leq s\leq t}\alpha^\vee(\eta(s)) \\ &\geq -\inf_{0\leq s\leq t}\alpha^\vee(\eta(s))\end{aligned}$$

therefore $\inf_{T\geq v\geq t}\alpha^\vee(\pi(v)) = -\inf_{0\leq s\leq t}\alpha^\vee(\eta(s))$ and Formula 2.2 for $t < t_0$ follows. The existence and uniqueness in Proposition 2.2 follows. $\diamondsuit$

6.2. Proof of Lemma 3.1. We first recall the following known facts (see [**25**] Ch. 28.1). Let $w \in W$ with reduced decomposition $w = s_{i_1}\dots s_{i_n}$, let $b_k = \alpha^\vee_{j_k}(s_{j_{k-1}}\dots s_{j_1}\lambda)$, which are nonnegative integers, then there exists a unique element $e^{(b_1)}_{j_1}\dots e^{(b_n)}_{j_n}$ depends only on w and not on the reduced decomposition. If V_λ is a highest weight module with highest weight λ and highest weight vector v_λ, then one has $e^{(b_1)}_{j_1}\dots e^{(b_n)}_{j_n}\overline{w}(v_\lambda) = v_\lambda$. It follows from these facts that if in some module V there exists a basis $\{v_0,\dots,v_q\}$ of weight vectors in which the highest weight vector is v_0 and the matrices of the Chevalley generators e_i are given by linear combinations, with nonnegative coefficients, of the matrices $E_{i-1,i}; i \geq 1$, then the statement of the lemma is true for this module. Furthermore, the statement is also true for the submodule of $\wedge^k V$ generated by $v_0\wedge v_1\wedge v_2\wedge\dots\wedge v_{k-1}$, since in the basis $v_{i_0}\wedge\dots\wedge v_{i_{k-1}}$ of $\wedge^k V$, with $i_0 < \dots < i_{k-1}$ the matrices of the e_i have nonnegative coefficients.

For a highest weight module to have such a basis, it is enough that all weights have multiplicity 1, and there exists a unique string $i_1,\dots,i_q$ such that the sequence

$$\lambda, \lambda-\alpha_{i_1}, \lambda-\alpha_{i_1}-\alpha_{i_2}, \dots, \lambda-\alpha_{i_1}-\dots-\alpha_{i_q}$$

is the sequence of all weights in V_λ. For then the corresponding ordered basis satisfies the required property. The module corresponding to the first fundamental weight for the root systems of types B_l, C_l and G_2, satisfies this property (we follow the notation of Bourbaki [**8**] Ch. VI, §4 for the fundamental weights).

For type G_2, the module with highest weight ω_2 is obtained as the submodule of $\wedge^2 V_{\omega_1}$ generated by $v_0\wedge e_1 v_0$, therefore the Lemma holds for the fundamental weights of G_2.

Using exterior powers of the basic representation we get all fundamental weights in the case of type C_l, and all fundamental weights, except ω_l in case B_l. This last weight corresponds to the spinor representation for type B_l, which is a minuscule representation, i.e. all weights are conjugate to ω_l. Let $w_0 = s_{i_1}\dots s_{i_q}$ be a reduced decomposition with partial products $w_k = s_{i_{k+1}}\dots s_{i_q}$, this induces an ordering of the weight vectors such that in this basis the Chevalley generators have an upper triangular matrix with nonnegative coefficients. The lemma follows for this module.

Actually the above argument works for any minuscule weight, thus this gives an elementary proof for the type A_l when all fundamental weights are minuscule.

We now observe that if the statement of the lemma is true for two highest weight modules V_{λ_1} and V_{λ_2} then in the module $V_{\lambda_1}\otimes V_{\lambda_2}$, the submodule generated by $v_{\lambda_1}\otimes v_{\lambda_2}$ is a highest weight module of highest weight $\lambda_1+\lambda_2$, and one has $\overline{w}(v_{\lambda_1}\otimes v_{\lambda_2}) = \overline{w}v_{\lambda_1}\otimes\overline{w}v_{\lambda_2}$ therefore $\langle e_{i_1}\dots e_{i_n}\overline{w}(v_{\lambda_1}\otimes v_{\lambda_2}), v_{\lambda_1}\otimes v_{\lambda_2}\rangle$ is a sum of nonnegative terms, and the lemma holds for $\lambda_1+\lambda_2$.

Having proved the lemma for fundamental weights, this property extends its validity to all weights.

It remains to check the property for type F_4. The Lie algebra F_4 has 4 fundamental modules, of dimensions 26, 52, 273 and 1274. We have checked, using the Quagroup package of the GAP [**15**] software that the Chevalley generators have nonnegative coefficients in the canonical basis, for the dimensions 26, 52 and 273. However this is not true for the representation of dimension 1274 (the computation took 6 days on a 2.6 Ghz Unix station, using 1.6 Go of memory), where two of the Chevalley generators happen to have some negative coefficients. It should be possible, in principle, to finish the check of the Lemma by using the explicit matrices of the generators in this basis, however it seems that we would not be able to do this in a reasonable time.

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