The *n* Extra Element Theorem

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The *n* Extra Element Theorem (*n*EET) is an extension of Middlebrook's Extra Element Theorem, to the case when multiple extra elements are added simultaneously to a circuit. Its major application is to write transfer functions directly as rational fractions, without need to perform loop an node analysis and algebraic manipulations. This is accomplished by treating each reactive component as an "extra" element that is added to the dc gain of the system. The method gives a physical interpretation to the coefficients of L and C in the standard normalized form of the transfer function, and it allows complex transfer functions to be derived nearly by inspection.

These results were derived by S. Sabharwal, who was a Caltech undergraduate who extended the basic (single) Extra Element Theorem. The results were never published until many years later [1].

The basic *n*EET is described here without proof, and several examples are worked. Extensions involving special cases are described in another document.

1 PRELIMINARIES

Given a linear network containing n inductors and m capacitors, it is desired to find the transfer function $G(s) = y(s)/u(s)$. It is assumed that this transfer function can be written as a rational fraction as follows:

$$G(s) = G_{dc} \frac{1 + a_1 s + a_2 s^2 + \cdots + a_{n+m} s^{n+m}}{1 + b_1 s + b_2 s^2 + \cdots + b_{n+m} s^{n+m}} \tag{1}$$

Extensions allow derivation of $G(s)$ in other forms that involve frequency inversion. The method used employs a generalization of the Extra Element Theorem, in which all of the inductors and capacitors are treated as "extra" elements, and are added simultaneously. The zeroes of $G(s)$ are found with the output nulled in the presence of the input, while the poles are found with the input set to zero. The method allows the coefficients $a_1, a_2,...,a_{n+m}, b_1, b_2,...,b_{n+m}$ to be found by evaluating the resistances seen looking into the ports under various conditions.

1.1 Definitions

DC state. The DC state of an inductor is a short circuit, and the DC state of a capacitor is an open circuit.

HF state. The high-frequency (HF) state of an inductor is an open circuit, and the HF state of a capacitor is a short circuit.

The above definitions are extended later to handle the case of frequency inversion, via definition of *normal* and *abnormal* states of reactive elements.

The DC gain $G_{dc} = G(0)$ is found with all dynamic elements set to their DC states. The transfer function s-coefficients depend on how the reactive elements change to their HF states, as explained below.

1.2 General Form of the Coefficients

The general form of the coefficient of s^k has dimensions $(\text{Hz})^{-k}$, and is a sum of all combinations of terms of the form $R_x C_i$ and L_j/R_y which contain the proper dimensions. The R_x and R_y terms are found by application of the n-EET, with injection at the terminals of the corresponding reactive element. In the case of denominator coefficients, the input source is set to zero. For numerator coefficients, the transfer function output is nulled. For example, consider the low-pass filter of Fig. 1. It is desired to compute the transfer

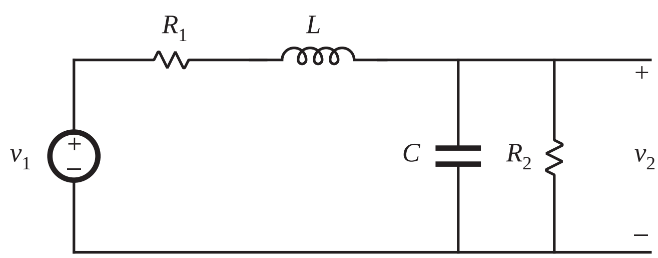

Fig. 1 *R-L-C* circuit example.

function $G(s) = v_2(s)/v_1(s)$. This transfer function contains two poles and no zeroes (why?), and can be written in the following form:

$$G(s) = G_{dc} \frac{1}{1 + b_1 s + b_2 s^2} \tag{2}$$

The dimensions of b_1 are $(\text{Hz})^{-1}$. The two possible terms in b_1 are:

$$\frac{L}{R_a} \qquad \text{and} \qquad R_b C \tag{3}$$

The dimensions of b_2 are $(\text{Hz})^{-2}$. The only possible term is of the form

$$\left(\frac{L}{R_c}\right)\left(R_d C\right) \tag{4}$$

2 PROCEDURE

In general, the numerator and denominator polynomials are of the following form:

$$1 + s\left(\sum_{i=1}^{n} \frac{L_i}{R_i} + \sum_{j=1}^{m} R_j C_j\right) + s^2\left(\sum\sum \frac{L_i L_j}{R_i R_j'} + \sum\sum \frac{L_i}{R_i} C_j R_j' + \sum\sum C_i R_i C_j R_j'\right) +$$
$$s^3\left(\sum\sum\sum \frac{L_i L_j L_k}{R_i R_j' R_k''} + \sum\sum\sum \frac{L_i}{R_i}\frac{L_j}{R_j'} C_k R_k'' + \sum\sum\sum \frac{L_i}{R_i} C_j R_j' C_k R_k'' + \sum\sum\sum C_i R_i C_j R_j' C_k R_k''\right) + s^4 \cdots \tag{5}$$

The orders of terms in the above equation are irrelevant; for example, it can be shown by reciprocity that $R_i'R_j = R_j'R_i$. The coefficients are determined as follows.

Coefficients of s^1: R_i and R_j are the resistances seen at port *i* (or *j*) with all other ports set to their DC states.

Coefficients of s^2: R_i and R_j (without prime) are the same terms in the coefficients of s^1; i.e., they are the resistances seen at port *i* (or *j*) with all other ports set to their DC states. R_j' (with prime) is the resistance seen at port *j*, with all other ports *except port i* set to their DC states. Port *i* is set to its HF state.

Coefficients of s^3: R_i, R_j (without prime), R_i', and R_j' (with prime) are the same terms in the coefficients of s^2. R_k'' (with double-prime) is the resistance seen at port *k*, with all other ports *except ports i and j* set to their DC states. Ports *i* and *j* are set to their HF states.

etc.

Each term (e.g., R_i) is found by current injection at the connections to the corresponding reactive element (e.g., in place of L_i). For denominator terms, the transfer function input is set to zero. For numerator terms, the transfer function output is nulled, in the presence of the input. For each coefficient, it is necessary to derive only one new term; the other terms are identical to the corresponding terms in a previous lower-order coefficient.

By following the above rules, the transfer function can be written directly, without need for algebra. Admittedly, some practice is required to become facile with these rules; nonetheless, the effort required to write exact expressions for complex circuits can be considerable reduced.

3 EXAMPLES

Consider first the simple *R-L-C* circuit example of Fig. 1. The DC gain G_{dc} is found with both reactive elements set to their DC states; i.e., the inductor is set to a short-circuit and the capacitor is set to an open-circuit. Solution of the resulting voltage divider leads to

$$G_{dc} = \frac{R_2}{R_1 + R_2} \tag{6}$$

The terms in the denominator polynomial are found with the input source v_1 set to zero. The circuit of Fig. 2 is then obtained. Since the circuit contains two reactive elements, the denominator is a second-order polynomial. It can therefore be of the following form:

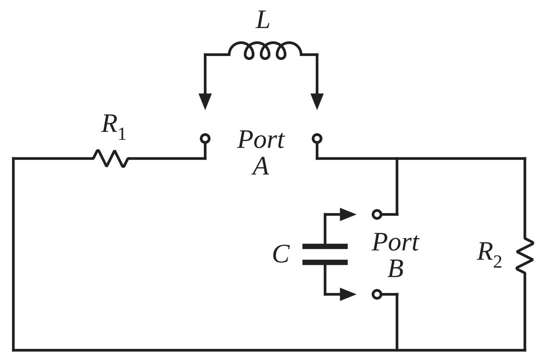

Fig. 2 Finding the denominator terms, *R-L-C* circuit example.

$$denominator = 1 + s\left(\frac{L}{R_a} + R_b C\right) + s^2\left(\frac{L}{R_c} R_d C\right) \quad (7)$$

The term R_a is the resistance seen at the inductor port (Port A), when the capacitor is set to its DC state, or open-circuited. It can be seen that R_a is the series combination of R_1 and R_2:

$$R_a = R_1 + R_2 \quad (8)$$

The term R_b is the resistance seen at the capacitor port (Port B), when the inductor is set to its DC state, or short-circuited. It can be seen that R_b is the parallel combination of R_1 and R_2:

$$R_b = R_1 \| R_2 \quad (9)$$

For the coefficient of s^2, we can choose one of the terms (either R_c or R_d) to be the same as the corresponding s^1 term. The other term is then found using the procedure for "prime" terms. For example, let us select the term associated with the inductor port, R_c, to be the same as in the s_1 coefficient:

$$R_c = R_a = R_1 + R_2 \quad (10)$$

Then R_d is given by the resistance looking into the capacitor port (Port B), with the inductor set to its high-frequency state, or open-circuited. It can be seen from Fig. 2 that

$$R_d = R_2 \quad (11)$$

Therefore, the transfer function $G(s)$ is

$$G(s) = \frac{R_2}{R_1 + R_2} \frac{1}{1 + s\left(\frac{L}{R_1 + R_2} + R_1 \| R_2 C\right) + s^2\left(LC \frac{R_2}{R_1 + R_2}\right)} \quad (12)$$

If desired, it can be verified that the numerator coefficients of s and s^2 are zero.

As a second example, consider the two-section *R-L-C* filter of Fig. 3. Since this circuit has four reactive elements, we expect the transfer function $G(s) = v_2(s)/v_1(s)$ to have four poles. The DC gain G_{dc} is equal to one. The denominator polynomial is found when the input $v_1(s)$ is set to zero. The resulting polynomial is:

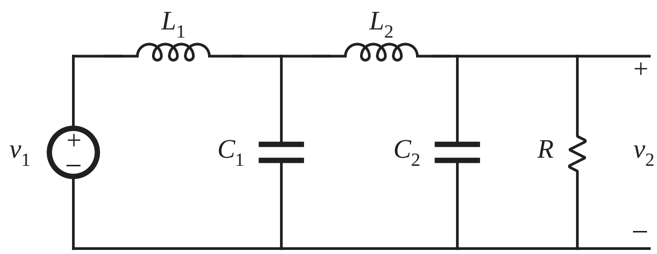

Fig. 3 Two-section R-L-C filter, Example 2.

$$
\begin{aligned}
&\textit{denominator} = 1 + s\left(\frac{L_1}{R} + \frac{L_2}{R} + C_1\cdot 0 + C_2\cdot 0\right) + \\
&s^2\left(\frac{L_1}{R}\,\frac{L_2}{\infty} + \frac{L_1}{R}\,C_1R + \frac{L_1}{R}\,C_2R + \frac{L_2}{R}\,C_1\cdot 0 + \frac{L_2}{R}\,C_2R + C_1\cdot 0\cdot C_2\cdot 0\right) + \\
&s^3\left(\frac{L_1}{R}\,C_1R\,\frac{L_2}{R} + \frac{L_1}{R}\,\frac{L_2}{\infty}\,C_2R + \frac{L_1}{R}\,C_2R\,C_1\cdot 0 + \frac{L_2}{R}\,C_2R\,C_1\cdot 0\right) + \\
&s^4\left(\frac{L_1}{R}\,\frac{L_2}{R}\,C_1R\,C_2R\right)
\end{aligned}
\tag{13}
$$

This transfer function contains no zeroes. Hence, $G(s)$ is given by:

$$
G(s) = \frac{1}{1 + s\left(\dfrac{L_1 + L_2}{R}\right) + s^2\left(L_1\left(C_1 + C_2\right) + L_2C_2\right) + s^3\left(\dfrac{L_1L_2C_1}{R}\right) + s^4\left(L_1L_2C_1C_2\right)}
\tag{14}
$$

A third example is given in Fig. 4. It is again desired to find $G(s) = v_2(s)/v_1(s)$. This example will be worked in class.

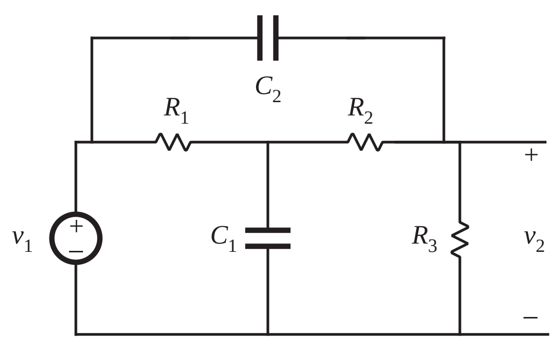

Fig. 4 Example 3: *R-C* circuit.

References

[1] R.D.Middlebrook, Vatché Vorpérian, and John Lindal, "The N Extra Element Theorem," *IEEE Trans. on Circuits and Systems I: Fundamental Theory and Applications*, vol. 45, no. 9, Sept. 1998; pp. 919-935.