Distributed Channel Allocation for PCN with Variable Rate Traffic

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***Abstract*—We consider the design of efficient channel allocation algorithms in personal communication networks (PCN) where the cells have varying traffic loads. A common communication channel is to be dynamically shared between the cells. We propose a distributed intercell channel allocation policy that is easy to implement through the use of simple signaling between neighboring cells. For cells arranged in a line, we show that the proposed policy achieves maximum throughput. The same is true when the cells are arranged in a circle and the frequency reuse distance is 2, while for larger reuse distances and planar hexagonal arrays, the policy may not always achieve maximal throughput. For general circular arrays, we enhance the policy to achieve maximal throughput asymptotically as the number of cells increases. For planar hexagonal arrays, we show that the policy can guarantee throughputs which are fairly close to maximal.**

***Index Terms*—Distributed decision making, dynamic wireless channel allocation, wireless network.**

I. INTRODUCTION

THE field of personal wireless communications is expanding rapidly as a result of advances in digital communications, portable computers, cellular networks, wireless LAN's, and personal communication systems. Wireless spectrum is a scarce resource, making efficient bandwidth allocation a critical problem.

The problem of bandwidth allocation has been extensively studied in the context of cellular telephone systems. Most of the proposed bandwidth allocation schemes employ an FDMA approach for sharing the spectrum between cells, including many of the schemes used by the second-generation cellular systems. The entire spectrum is divided into a number of frequency channels, and a given channel is exclusively allocated to a voice call within a cell. A channel assigned to a cell can be reused in a spatially disjoint cell, subject to the reuse constraints which are formed by taking the various interference conditions into consideration. The channel-assignment schemes can be static, dynamic, or hybrid [12], [13].

With the proliferation of portable computers and personal digital assistants, it is envisioned that services and applications including file server access, client–server execution including remote graphical windowing applications such as Web browsers, and multimedia mail will require wireless access in the near future [1]. In contrast to voice traffic, the traffic generated by these applications is expected to have varying bandwidth requirements and/or be bursty in nature. Hence, FDMA schemes designed for cellular telephone systems with predominantly voice traffic are inappropriate, and bandwidth-sharing schemes of a different nature are needed to efficiently accommodate this traffic. The high bit-rate requirements force the use of large-bandwidth channels, and the limited spectrum implies that there are few such channels available. Therefore, large-bandwidth channels need to be time-shared dynamically among adjacent cells.

In this paper, we explore the design of efficient channel allocation algorithms in personal communications networks (PCN) where the mobile users have variable data rate requirements, the traffic load at the various cells may differ significantly, and cells share a common channel. We assume a cellular architecture where each cell has a base station that is a component of an underlying wired network infrastructure. We focus on the problem of intercell channel allocation in this paper as there are several efficient schemes for doing intracell allocation using the base station as a central node [7], [9]. Given the channel reuse constraints, the issue is how to share the channel among the various cells in a distributed manner so as to maximize throughput.

As a first step toward solving the general problem, we consider in Section II the situation in which all of the cells are arranged in a line. This topology is natural in certain environments [6], [8], [12]. We provide a simple distributed channel allocation policy that provides throughput guarantees at the various cells whenever this is possible. Each cell needs to communicate a signaling message to only two other cells, among those that are within its reuse distance, once every T time units where T is a global constant. Although the maximal throughput policy is studied under the requirement of zero propagation delays, we show how this requirement can be relaxed with a small penalty in achievable throughput. The implementability of the policy in practical wireless systems is also discussed. In Section III, we consider the topology in which the cells are arranged in a circle. Besides being a generalization of linear arrays representing certain natural topologies, it is a basic step in understanding the channel allocation problem in planar arrays. For the special case where a channel can be reused in every other cell, we show that the policy for the linear array, with a minor modification, achieves

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maximal throughput. For the general case, however, the policy may not achieve maximal throughput. We enhance the policy to achieve maximal throughput asymptotically as the number of cells increases. Finally, we consider planar hexagonal arrays in Section IV, and provide fairly weak conditions under which cell throughputs can be guaranteed.

Frodigh [6] considered a certain greedy policy for allocating channels in a linear array of cells that can also be adapted to provide maximum throughput for our linear array model. The policy allocates time slots starting from one end cell (say, on the extreme left) and progressing successively toward the other end cell. Compared to that policy, the policy described here has the advantages of requiring fewer information exchanges and easy adaptability to load fluctuations. More specifically, in the policy described here, the cells learn the optimal allocation on-line requiring only two simple signaling messages per cell in every T time units. If the traffic requirements change, then the cells start using their new parameters in the algorithm, and an automatic readjustment takes place without requiring any special action. Furthermore, in contrast to the policy in [6], the policy described in this paper always results in a compact allocation, i.e., a cell with required throughput β transmits continuously for βT units in every T units. A noncompact allocation increases the intercell communication complexity and complicates the intracell allocation problem.

Frequency allocation policies proposed by Cimini [4] and Raymond [12] are described in the context of FDMA with the objective of minimizing blocking probabilities. However, if one thinks of frequency channels and time slots as interchangeable, their policies do not achieve the same capacity as the policy described by Frodigh [6] or the distributed policies described in this paper.

II. Cells Arranged in a Linear Array

In this section, we consider a linear array of N cells numbered $0, \cdots, N-1$. A single channel is available for use by all of the cells. The reuse distance is R, that is, the channel can be used simultaneously in two cells i and j only if $|i-j| \geq R$. We provide a distributed policy for sharing the channel among the cells, and show that it achieves maximal throughput.

A. A Channel Allocation Policy

Assign color 0 to cells $0, R, 2R, \cdots$; color 1 to cells $R-1, 2R-1, 3R-1, \cdots$; color 2 to cells $R-2, 2R-2, 3R-2, \cdots$; and so on. Formally,

$$\text{Color}(i) := (R-i) \bmod R, \qquad 0 \leq i \leq N-1. \quad (1)$$

Let Neighbor(i, c) denote the collection of cells having color c that are within the reuse distance R of cell i, that is, Neighbor$(i, c) := \{j \neq i\text{: Color}(j) = c, |j-i| \leq R-1\}$.

To describe the policy, we assume initially that all color 0 cells can be synchronized to start transmission at time 0. This implies zero intercell message propagation delays between the cells. In Section II-D, we consider the effect of nonzero propagation delays.

The policy is parameterized by a global constant T and parameters $\{\beta_i\}_{i=0}^{N-1}$, where β_i is associated with cell i. The policy requires the cells to coordinate transmissions in such a way that the distance between two simultaneously transmitting cells is greater than R; therefore, there are no conflicts. Whenever cell i gets an opportunity to transmit, it transmits continuously for a duration no more than $\beta_i T$ time; its nth such transmission is said to constitute its nth transmission cycle, $n = 0, 1, 2, \cdots$. After completing a transmission cycle, cell i sends a message Done(i) to its neighbors of color $[\text{Color}(i)+1] \bmod R$. All color 0 cells begin their first transmission cycle simultaneously at time 0. A color $c > 0$ cell begins its first transmission cycle only after it has received Done$(\cdot)$ message from each of its color $c-1$ neighboring cells. A color c cell is allowed to begin its next cycle only after it has received Done$(\cdot)$ from each of its color $(c-1) \bmod R$ neighbor since its previous transmission cycle ended. The formal algorithm for cell i is described below.

First Transmission:

```
If Color(i) = 0
  Transmit(i);
else
  Wait until received Done(·) from each
           cell in Neighbor(i,Color(i)-1);
  Transmit(i);
```

Subsequent Transmissions:

```
Wait until received Done(·) from each cell
    in Neighbor(i,(Color(i) - 1) mod R) since
    LastTransmitTime(i);
Transmit(i);

procedure Transmit(i) {
  Transmit for at most βiT time;
  LastTransmitTime(i) := current local time;
  Send a message Done(i) to each cell in
        Neighbor(i,(Color(i) + 1) mod R);
}
```

The operation of the policy is illustrated in Fig. 1 for an array with reuse distance $R = 3$. Let $T_i(n)$ denote the time at which cell i begins its nth transmission cycle and let $S_i(n)$ denote the duration for which cell i transmits during its nth transmission cycle. Cells 0, 3, 6 start their first transmission cycle at time 0 and transmit for $S_0(0)$, $S_3(0)$, and $S_6(0)$ time, respectively. Cell 2 starts whenever both cells 0 and 3 complete. Similarly, cell 5 starts after both cells 3 and 6 complete. Cells 1, 4, and 7 transmit similarly next. This completes the first transmission cycle of all cells, and the process starts all over again. We will be concerned with the throughput properties of this policy in the rest of this section.

B. Discussion on the Implementability of the Policy

We describe how the policy can be appropriately adapted in a practical PCN environment such as wireless LAN. Other technologies for which the policy can be applied include

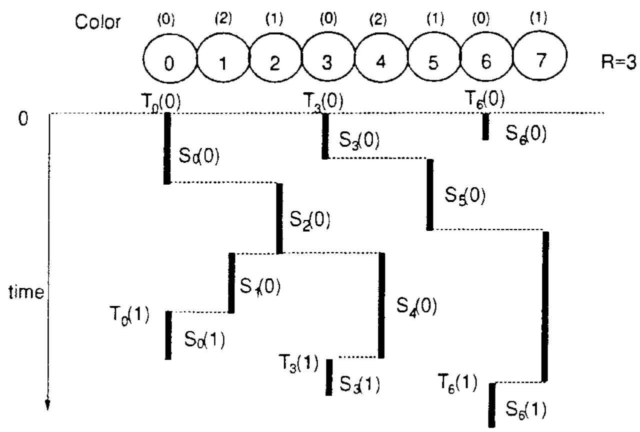

Fig. 1. Illustration of the policy.

the emerging high-speed (128 kbits/s and higher) wide area cellular services and the 5 GHz unlicensed data services (the so-called NII band). The current generation of wireless LAN's [2] operates within the ISM radio spectrum, with typical link rates of few megabits per second at a typical range of 50–100 m. Every cell has a base station that relays packets between the mobile users and the wired backbone infrastructure. The wireless access methods can be based on carrier-sense multiple access (CSMA) or dynamic time division multiple access (TDMA) or a combination of these schemes.

For concreteness, we consider a linear array of cells, 50 m in diameter. The frequency reuse distance is $R = 3$. The maximum raw radio link speed is 1 Mbit/s, and the media access method is dynamic TDMA. With dynamic TDMA, time is divided into equal size frames (typical length 100 ms), and the base station dynamically allocates time slots within a frame to its currently registered users. There are well-known methods [9] for doing intracell bandwidth allocation. To implement the intercell bandwidth allocation according to the policy described in Section II-A, we need a superframe. This is the parameter T of the policy. A good choice of T in this case is $T = 300$ ms. Since frequency reuse distance is $R = 3$, it is clear that the aggregate arrival rates (in Mbits/s) at three adjacent cells must be smaller than 1. Let the policy parameter β_i denote the aggregate arrival rate at cell i. It is now easy to see how things work. The mobile users register their arrival rates with a base station of their choice. Base stations compute the aggregate arrival rate $\{\beta_i\}$ of all of their registered users. In each transmission cycle, a base station transmits for a duration no more than the fraction of the superframe size given by $\beta_i T$. After completing a cycle, a base station sends a message over the wired network to the two base stations described in the algorithm.

The synchronization required to start the basic algorithm can be relaxed as shown in Section II-D. Since the two communicating base stations are never more than $R - 1$ wireless hops away, the propagation delay is small, and the reduction in throughput due to propagation delay is also small.

The policy has some important properties that we establish in the next section. First, it provides maximal throughput. In the context of the above example, this means that for any set of cell arrival rates satisfying the constraint that the sum of the rates of three adjacent cells be less than 1 Mbit/s, the policy guarantees cell throughput equal to its offered arrival rate (see Corollary 1 in Section II-C). For common burstiness constrained arrival processes, a bound on queue lengths can also be derived (see Corollary 2 in Section II-C). Second, the policy provides isolation; that is, the delays of the packets of one cell are independent of the potentially misbehaving traffic in other cells. This is captured in the delay bound in Corollary 2 in Section II-C.

The question arises: Is it possible to know the arrival rates? The answer is affirmative for some applications such as internet telephony, audio, and video conferencing. For these applications, protocols such as RSVP [3] are currently being designed to reserve bandwidth at various intermediate nodes in a wired network. To obtain predictable end-to-end behavior for a session encompassing both wireless and wired links, appropriate wireless bandwidth should also be reserved. The algorithm described in this paper, together with some intracell bandwidth allocation scheme, is a step in that direction. At the same time, for most other applications such as mail, file transfer, etc., available bandwidth should be used in a best effort manner. Consequently, a hybrid of CSMA and TDMA wireless access schemes can be used, and in the TDMA part, bandwidth can be allocated by using the intercell algorithm described in this paper together with some well-known intracell bandwidth allocation scheme.

C. Throughput Maximizing Properties of the Policy

In the rest of this paper, we assume, for simplicity in the discussion, that the unit of time is chosen so that the transmission rate of the cells is 1 bit/(unit of time). Let $B_j[t, t+\tau]$ be the amount of traffic transmitted by cell j in the interval $[t, t+\tau]$. The throughput $\hat{\beta}_j[t, t+\tau]$ of cell j in the interval $[t, t+\tau]$ is defined as the ratio

$$\hat{\beta}_j[t, t+\tau] := \frac{B_j[t, t+\tau]}{\tau}.$$

The long-term throughput (or simply "throughput") of cell j is defined as

$$\hat{\beta}_j := \lim_{\tau\to\infty} \inf \hat{\beta}_j[t, t+\tau].$$

Note that because of the reuse constraints, only one cell from the set of cells $\{i, i+1, \cdots, i+R-1\}$ can transmit at a time. Therefore, the achievable cell throughputs under any policy must satisfy the following relation.

$$\hat{\beta} := \max_{0\le i\le N-R} \sum_{j=i}^{i+R-1} \hat{\beta}_j \le 1. \tag{2}$$

Our main objective in this section is to show that whenever the parameters $\{\beta_i\}$ of the proposed channel allocation policy satisfy (2), cell i is guaranteed throughput β_i; that is, as long as cell i has enough input traffic, the throughput of cell i will be at least β_i irrespective of the traffic at the other nodes. Moreover, under certain burstiness constraints on the input traffic, it can be guaranteed that the queues at every cell are bounded. These properties are consequences of Theorem 1 below. The first part of the theorem provides a bound on the time it takes for any cell to begin its nth cycle. The second part provides a similar, but weaker bound on the time needed to start n consecutive cycles, after any time $t_0 \geq 0$.

We define the parameter β, which is used heavily in the sequel.

$$\beta := \max_{0 \leq i \leq N-R} \sum_{j=i}^{i+R-1} \beta_j.$$

Theorem 1: Let the policy defined in Section II-A operate with parameters $\{\beta_j\}_{j=0}^{N-1}$ and T. The following statements hold independent of the initial queue sizes at each cell.

a) For every cell j,

$$T_j(n) \leq n\beta T + \beta(\lceil j/R \rceil + 1)\lceil R/2 \rceil T, \qquad n \geq 0. \quad (3)$$

b) Let $t_0 \geq 0$, and let $n_j(t_0)$ be the first cycle of cell j that starts at or after time t_0. Then

$$T_j[n_j(t_0) + n] - t_0 \leq n\beta T + \beta[(\lceil j/R \rceil + 1)\lceil R/2 \rceil + (1 + 1/R)N/2]T.$$

Before proceeding with the proof of this theorem, we present two important implications.

Corollary 1: If the policy parameters $\{\beta_j\}_{j=0}^{N-1}$ satisfy (2), then every cell j is guaranteed throughput β_j.

Proof: From the definitions of $\hat{\beta}_j$ and $n_j(t)$, it follows that the throughput of cell j is given by

$$\hat{\beta}_j \geq \lim_{t \to \infty} \inf \frac{B_j\{0, T_j[n_j(t) - 1]\}}{T_j[n_j(t)]} = \lim_{n \to \infty} \inf \frac{B_j[0, T_j(n-1)]}{T_j(n)}.$$

Assume now that cell j has enough input traffic so that it transmits for $\beta_j T$ time units in each cycle. Then the amount of traffic that it transmits in n cycles is $n\beta_j T$. Using part a) of Theorem 1 and the fact that $\beta \leq 1$, we conclude that, in this case,

$$\hat{\beta}_j \geq \lim_{n \to \infty} \frac{(n-2)\beta_j T}{n\beta T + (\lceil j/R \rceil + 1)\beta T \lceil R/2 \rceil} \geq \beta_j.$$

□

Theorem 1 can be also used to provide bounds on the queue sizes. We need some assumptions on the arrival processes. Specifically, we assume that the amount of traffic arriving at cell j in the interval $[t, t+\tau]$, given by $A_j[t, t+\tau]$, satisfies the inequality

$$A_j[t, t+\tau] \leq \beta_j \tau + \sigma_j, \qquad \text{for all } t, \tau \geq 0.$$

In this case, we say that the arrivals to each cell j are (β_j, σ_j) constrained. This model has received considerable attention recently in the study of communication networks [5]. Let $Q_j(t)$ be the queue size at cell j at time t.

Corollary 2: Let $Q_j(0) = 0$, and assume that the arrivals at cell j are (β_j, σ_j) constrained, where $\{\beta_j\}_{j=0}^{N-1}$ satisfy (2). Then the queue sizes under the proposed policy operating with parameters $\{\beta_j\}_{j=0}^{N-1}$ and T satisfy the bound

$$\sup_{t \geq 0} Q_j(t) \leq \sigma_j + \beta_j T[(\lceil j/R \rceil + 1)\lceil R/2 \rceil + (1 + 1/R)N/2 + 1]. \quad (4)$$

Observe that, according to (4), if a guaranteed upper bound on the queue size is desired, then a bound on the number of cells should be imposed. However, we feel that the dependence on the number of cells, parameterized by j on the right-hand side of (3) and (4), is only a facet of the analysis, and not truly required, as has been verified by our simulations. On the other hand, it is also important to note that the bound on the queue size in a cell is not affected by the burstiness parameters $\{\sigma_i\}_{i=0}^{N-1}$ of the other cells. This is a very important isolation property of the proposed policy.

Proof of Corollary 2: Assume that $Q_j(t) > 0$. Let $t_0 < t$ denote the instant at which the current busy period (i.e., transition from empty to nonempty queue) of cell j started; t_0 is well defined since $Q_j(0) = 0$. Recall the notation that $B_j[t_0, t]$ denotes the amount of traffic transmitted by cell j in $[t_0, t]$. Since the arrivals at cell j are (β_j, σ_j) constrained and the queue of cell j is empty just before time t_0, we have

$$Q_j(t) \leq \beta_j(t - t_0) + \sigma_j - B_j[t_0, t]. \quad (5)$$

Recall the definition of $n_j(t)$ from Theorem 1, and let $n = n_j(t) - n_j(t_0) \geq 0$. From part b) of Theorem 1, we have

$$t - t_0 \leq T_j[n_j(t)] - t_0 \leq n\beta T + \beta T[(\lceil j/R \rceil + 1)\lceil R/2 \rceil + (1 + 1/R)N/2]. \quad (6)$$

Also, since cell j is busy during $[t_0, t]$, it transmits for $\beta_j T$ time in each of $(n-1)^+ := \max\{n-1, 0\}$ cycles. Therefore,

$$B_j[t_0, t] \geq (n-1)^+ \beta_j T. \quad (7)$$

Using the bounds for $t - t_0$ and $B_j[t_0, t]$ from (6) and (7), respectively, and also the fact $\beta \leq 1$, in (5), the desired bound (4) follows. □

The rest of this section is devoted to the proof of Theorem 1. The proof is quite lengthy, and it is worthwhile to outline the main ideas first. Following the operation of the policy, we first develop in Proposition 1 a recursive expression for cycle start times of color 0 cells. It turns out that it is easier to prove results in the situation where every cell i transmits for $\beta_i T$ time during *every* cycle, i.e., $S_i(n) = S_i := \beta_i T$. This means that even if cell i has nothing to transmit, it keeps the channel for $\beta_i T$ time during its transmission cycle. Let $\tilde{T}_i(n)$ denote the start time of the nth transmission cycle of cell i in this special case. We then establish an upper bound on the difference between successive cycle start times (Lemma 2) which, together with some bounds (Lemma 1), yields Theorem 1(a). We take the route via $\tilde{T}_i(n)$ instead of $T(n)$ since Lemma 2 may not hold for $T(n)$. The proof of Lemma 2 is the heart of the paper, and heavily uses the properties of the policy as captured by the recursive expression developed in Proposition 1. In part b) of Theorem 1, we

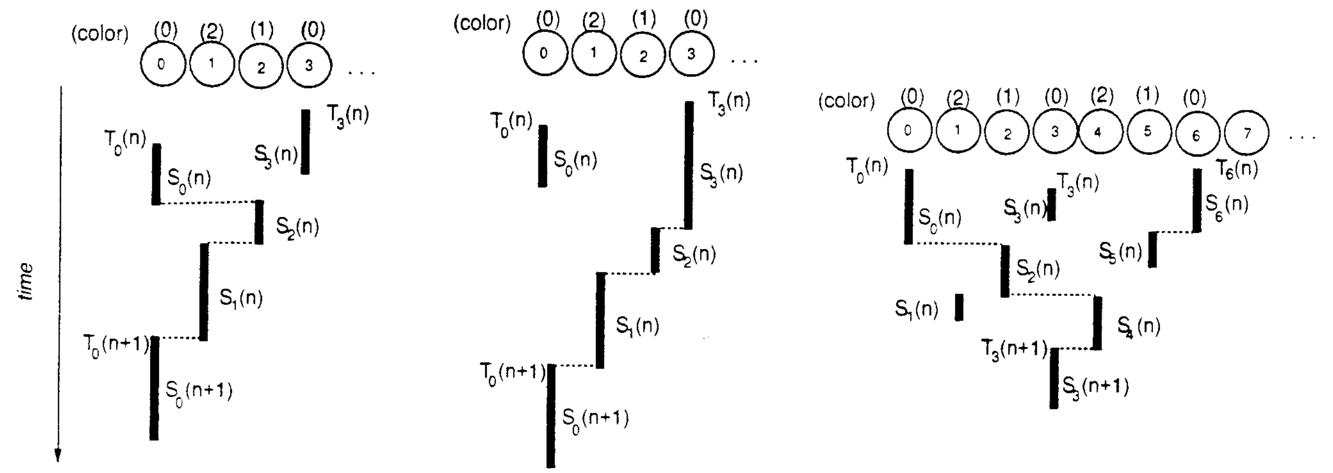

Fig. 2. Illustrations for Proposition 1: $R = 3$. (a) $T_0(n+1) = T_0(n) + S_0(n) + S_2(n)$. (b) $T_0(n+1) = T_3(n) + S_3(n) + S_2(n) + S_1(n)$. (c) $T_3(n+1) = T_0(n) + S_0(n) + S_2(n) + S_4(n)$..

need to consider the system after arbitrary time t_0. Since the color 0 cells may not transmit at time t_0, the situation here is different from that in part a). We bound the effect of nonzero first cycle start times of color 0 cells (Lemma 4), and then use the results in Theorem 1a) to develop the desired bound.

For simplicity in notation, we will assume for the rest of Section II that N is a multiple of R. Otherwise, we can add an appropriate number of cells with zero arrival rates. It can be easily seen that under the assumption of zero propagation delays, this modification does not alter the operation of the original cells.

Combining the basic relations (11) and (12), we obtain the following recursive expression for the start time of the $n+1$st cycle of a color 0 cell in terms of the start times of the nth cycle of an appropriate set of color 0 cells. Let $\Re^R$ denote the set of R-dimensional vectors, and let $\mathcal{K}_l$ be the subset of $\Re^R$ whose components take values 0 or 1 and sum up to l, i.e.,

$$\mathcal{K}_l = \left\{\mathbf{k} \in \Re^R \colon k_j \in \{0, 1\}, \sum_{j=1}^{R} k_j = l\right\}, \quad 0 \le l \le R.$$

Proposition 1: The following recursive relation holds for $i = 0, 1, 2, \cdots, N/R$, and $n \ge 0$:

$$T_{iR}(n+1) = \max_{0 \le l \le R} \left\{ T_{(i+1-l)R}(n) + \max_{\mathbf{k} \in \mathcal{K}_l} \left[\sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}(n) \right] \right\}. \tag{8}$$

We first illustrate the result with a simple example with reuse distance $R = 3$. Proposition 1 with $i = 0$ yields

$$T_0(n+1) = \max\{T_3(n) + S_3(n) + S_2(n) + S_1(n), T_0(n) + S_0(n) + S_2(n) + S_1(n)\}. \tag{9}$$

The two terms inside the maximum in (9) correspond to, respectively, $i = l = 0$ and $i = 0, l = 1$ in (8). Observe that, since $S_j(n) \equiv 0$ for $j < 0$, two additional terms that could have resulted from (8) are excluded from (9) since they do not contribute to the maximum. The relation (9) can be understood from the timing diagram shown in Fig. 2. There are two cases, depending on whether $T_0(n) + S_0(n)$ is greater than or less than $T_3(n) + S_3(n)$.

Similarly, Proposition 1 with $i = 1$ yields (10), shown at the bottom of the page. The first term inside the maximum in (10) corresponds to $i = l = 0$ in (8), the next three terms correspond to $i = 0, l = 1$ in (8), and the last two terms correspond to $i = 0, l = 2$ in (8). Observe that one term from (8) is excluded since it does not contribute to the maximum. The scenario in which the final term inside the maximum in (10) determines $T_3(n+1)$ is shown in Fig. 2; the other cases can be understood similarly. In the illustrated case in Fig. 2, $T_0(n) + S_0(n) > T_3(n) + S_3(n)$, $T_2(n) + S_2(n) > T_5(n) + S_5(n)$, and $T_4(n) + S_4(n) > T_1(n) + S_1(n)$ so that $T_3(n+1) = T_0(n) + S_0(n) + S_2(n) + S_4(n)$.

The specific expressions (9) and (10) bring out an interesting phenomenon. Consider (9) and (10) for $n = 0$. Provided that $\{\beta_i\}_{i=0}^{N-1}$ satisfy (2), it follows from (9) and the initial conditions $T_{iR}(0) \equiv 0$, $0 \le i \le N/R$, that $T_0(1) \le \beta T \le T$. This means that cell 0 gets its guaranteed throughput in the first cycle. But the same may not be true for other cells. For example, if the last term in (10) determines $T_3(1)$, then it is

$$T_3(n+1) = \max\{T_6(n) + S_6(n) + S_5(n) + S_4(n), T_3(n) + S_3(n) + S_5(n) + S_4(n), T_3(n) + S_3(n) + S_2(n) + S_4(n), T_3(n) + S_3(n) + S_2(n) + S_1(n), T_0(n) + S_0(n) + S_2(n) + S_1(n), T_0(n) + S_0(n) + S_2(n) + S_4(n)\}. \tag{10}$$

possible that

$$T_3(1) = S_0(n) + S_2(n) + S_4(n) = T(\beta_0 + \beta_2 + \beta_4) > T$$

implying that cell 3 may not get its required throughput in the first cycle. In general, it can be seen that cell iR may not get its guaranteed throughput in the first i cycles. The important point, however, is that eventually the schedules get adjusted in such a way that cell iR starts receiving its guaranteed throughput from ith cycle onward, followed by cells $(i-1)R < j \le iR$. This is the subject of Lemma 2, and captures the essence of the operation of the policy.

Before proceeding to the proof of Proposition 1, we observe some useful relations between the cycle start times that are direct consequences of the specified policy. Since color 0 cells start transmission at time 0, $T_{iR}(0) = 0$ for every i. For notational convenience, when $j < 0$, we will set $T_j(n) = S_j(n) = 0$, $n = 0, 1, \cdots$. Note that cell $iR-(c+1)$ (having color $c+1$) can start its nth transmission cycle exactly after both the color c cells $iR - c$ and $(i-1)R - c$ have completed their nth cycle. Therefore, for each $0 \le c < R-1$, $T_{iR-(c+1)}(n)$ can be written as

$$\max\{T_{iR-c}(n) + S_{iR-c}(n), T_{(i-1)R-c}(n) + S_{(i-1)R-c}(n)\} = \max_{k\in\{0,1\}} \{T_{(i-k)R-c}(n) + S_{(i-k)R-c}(n)\}. \quad (11)$$

Similarly, observe also that cell iR can begin its $n+1$th cycle exactly after cells $(i+1)R-(R-1)$ and $iR-(R-1)$ have both completed their nth cycle. Therefore, $T_{iR}(n+1)$ is given by

$$\max_{k\in\{0,1\}} \{T_{(i+1-k)R-(R-1)}(n) + S_{(i+1-k)R-(R-1)}(n)\}. \quad (12)$$

This equation will be repeatedly used in the rest of this paper.

Proof of Proposition 1: We first show that for $1 \le c \le R-1$ and $0 \le i \le N/R$, $T_{iR-c}(n)$ can be expressed as

$$\max_{k_j\in\{0,1\}} \left\{ T_{\left(i-\sum_{j=1}^{c} k_j\right)R}(n) + \sum_{m=1}^{c} S_{\left(i-\sum_{j=m}^{c} k_j\right)R-(m-1)}(n) \right\}. \quad (13)$$

The proof of (13) is by an induction on c. For $c = 1$, (13) is obtained by setting $c = 0$ in (11) obtained in Section II-A. The induction step is similar. Assume that (13) holds for c. Using the induction hypothesis in (11), $T_{iR-(c+1)}(n)$ given by

$$\max_{k_{c+1}\in\{0,1\}} \{T_{(i-k_{c+1})R-c}(n) + S_{(i-k_{c+1})R-c}\}$$

can also be written as

$$\max_{k_{c+1}\in\{0,1\}} \left\{ \max_{\substack{k_j\in\{0,1\}\\ 1\le j\le c}} \left\{ T_{\left(i-\sum_{j=1}^{c+1} k_j\right)R}(n) + \sum_{m=1}^{c} S_{\left(i-\sum_{j=m}^{c+1} k_j\right)R-(m-1)}(n) \right\} + S_{(i-k_{c+1})R-c} \right\}$$

which reduces to

$$\max_{\{k_j\}} \left\{ T_{\left(i-\sum_{j=1}^{c+1} k_j\right)R}(n) + \sum_{m=1}^{c+1} S_{\left(i-\sum_{j=m}^{c+1} k_j\right)R-(m-1)}(n) \right\}.$$

This completes the proof of (13). To show (8), first note from (12) that $T_{iR}(n+1)$ can be written as

$$\max_{k_R\in\{0,1\}} \{T_{(i+1-k_R)R-(R-1)}(n) + S_{(i+1-k_R)R-(R-1)}(n)\}.$$

From (13), it follows that $T_{(i+1-k_R)R-(R-1)}(n)$ in the above expression equals

$$\max_{\{k_j\}} \left\{ T_{\left(i+1-k_R-\sum_{j=1}^{R-1} k_j\right)R}(n) + \sum_{m=1}^{R-1} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}(n) \right\},$$

so that $T_{iR}(n+1)$ can be rewritten as

$$\max_{\{k_j\}} \left\{ T_{\left(i+1-\sum_{j=1}^{R} k_j\right)R}(n) + \sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}(n) \right\}. \quad (14)$$

The desired result (8) follows from (14) by collecting all terms such that $\sum_{j=1}^{R} k_j = l$, $0 \le l \le R$. □

Next, we collect some elementary technical results. Part a) provides an upper bound on a term that appears inside the maximum in the recursive expression (8). Part b) provides an upper bound on the start of the nth transmission cycle. Recall that $\{\tilde{T}_i(n)\}$ denotes the cycle start times when cell i transmits for $S_i(n) = S_i = \beta_i T$ time in every cycle. Contrast this to $T_i(n)$, which is the corresponding quantity for the case in which cell i transmits for at most $\beta_i T$ time, but gives up if it has nothing to transmit. Part c) states the intuitive result that $T_i(n) \le \tilde{T}_i(n)$ for every i, n. The proof of these technical results appear in the Appendix.

Lemma 1:

a) For every i, $0 \le i \le N/R$, and $0 \le l \le R$,

$$\max_{\mathbf{k}\in\mathcal{K}_l} \left(\sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}(n) \right) \le \beta T\lceil R/2\rceil. \quad (15)$$

b) For every $n \ge 0$, $T_{iR}(n) \le n\beta T\lceil R/2\rceil$, $0 \le i \le N/R$, and $T_j(n) \le (n+1)\beta T\lceil R/2\rceil$, $0 \le j \le N-1$.

c) For every cell j, $T_j(n) \le \tilde{T}_j(n)$ for every n.

The following result yields a bound on the difference between successive cycle start times in the special case in which a cell keeps the channel for $\beta_i T$ time during every transmission cycle. Intuitively, the bound means that after a certain number of cycles, the successive times at which a cell gets an opportunity to transmit are no more that βT time apart. Together with the condition $\beta \le 1$, this result implies that eventually cell i transmits at least for $\beta_i T$ time in every interval of duration T, thereby obtaining throughput at least β_i. The proof of this important result is somewhat lengthy, and therefore appears in the Appendix.

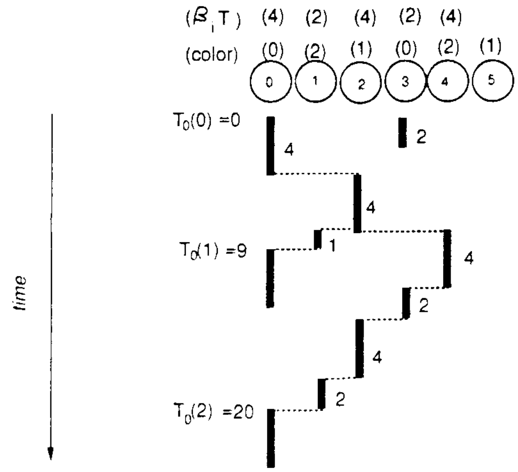

Fig. 3. Example: $R = 3$, $T = 10$, $\beta = 1$.

Lemma 2: For every cell j,

$$\tilde{T}_j(n+1) - \tilde{T}_j(n) \leq \beta T \qquad \forall n \geq \lceil j/R \rceil. \tag{16}$$

It is inefficient for a cell to keep the channel even if it has nothing to transmit, and hence the general rule $S_i(n) \leq \beta_i T$ in the definition of the policy. It is interesting to note that for the general case, an equivalent of Lemma 2 may not hold for $\{T_i(n)\}$. For an example, see Fig. 3, where $T_0(2) - T_0(1) > \beta T$. This results from cell 1 transmitting for $1 < \beta_1 T = 2$ units in the first cycle. To handle the general case, we first prove the following result that provides an upper bound on the difference in successive cycle start times of any two color 0 cells. The proof of this result also appears in the Appendix.

Lemma 3: For every i, j

$$T_{iR}(n+1) - T_{jR}(n) \leq (\beta NT/2)(1+1/R), \qquad n \geq 0. \tag{17}$$

So far, we have assumed that at time 0, all of the color 0 cells can start transmission. In the course of proving Theorem 1b), we need to relax this assumption. Letting $\Delta = \max_i T_{iR}(0) > 0$, let $T_j^{\Delta}(n)$ denote the start time of the nth cycle of cell j. In relation to the already introduced notation, $T_j^0(n) = T_j(n)$ and $\tilde{T}_j^0(n) = \tilde{T}_j(n)$. The following result provides a bound to the difference between $T_j^{\Delta}(n)$ and $\tilde{T}_j(n)$. For a proof, see the Appendix.

Lemma 4:

For every cell j,

$$T_j^{\Delta}(n) \leq \tilde{T}_j(n) + \Delta, \qquad n \geq 0. \tag{18}$$

A proof of Theorem 1 can now be given.

Proof of Theorem 1:

a) In view of Lemma 1c), it suffices to show that

$$\tilde{T}_j(n) \leq n\beta T + \beta T(\lceil j/R \rceil + 1)\lceil R/2 \rceil, \qquad n \geq 0. \tag{19}$$

If $n \geq \lceil j/R \rceil$, then we write

$$\begin{aligned} \tilde{T}_j(n) &= \sum_{t=\bar{j}}^{n-1} [\tilde{T}_j(t+1) - \tilde{T}_j(t)] + \tilde{T}_j(\lceil j/R \rceil) \\ &\leq (n - \bar{j})\beta T + \beta T(\lceil j/R \rceil + 1)\lceil R/2 \rceil \\ &\leq n\beta T + \beta T(\lceil j/R \rceil + 1)\lceil R/2 \rceil. \end{aligned} \tag{20}$$

The bounds for the two terms in (20) follow from Lemma 2 and Lemma 1b), respectively. For the case $n < \lceil j/R \rceil$, the desired bound follows from Lemma 1b): $\tilde{T}_j(n) \leq (n+1)T\lceil R/2 \rceil \leq \beta T(\lceil j/R \rceil + 1)\lceil R/2 \rceil$.

b) Let i^* denote the color 0 cell with the largest $n_i(t_0)$. Note that by the definition of i^*, we have $T_j[n_{i^*}(t_0)] \geq t_0$, for every j. Consider the operation of cell j, after time $T_j[n_{i^*}(t_0)]$. This is identical to the operation of cell j in a system that operates under the same channel allocation policy, with the same arrival pattern after time $T_j[n_{i^*}(t_0)]$, cycle start times $\hat{T}_j(0) = T_j[n_{i^*}(t_0)] - t_0 \geq 0$, and queue sizes $\hat{Q}_j[\hat{T}_j(0)] = Q_j\{T_j[n_{i^*}(t_0)]\}$. Applying Lemma 4 to the new system, together with the bound (3) from part a), we obtain

$$\begin{aligned} \hat{T}_j(n) \leq\, & n\beta T + (\lceil j/R \rceil + 1)\beta T\lceil R/2 \rceil \\ & + \max_{0 \leq i \leq N/R} \hat{T}_{iR}(0). \end{aligned} \tag{21}$$

Let $n_{i^*}(t_0) \geq 1$. From the definition of i^*, it follows that $T_{i^*R}[n_{i^*}(t_0) - 1] < t_0$, and using this with Lemma 3, we obtain that $\hat{T}_{iR}(0) = T_{iR}[n_{i^*}(t_0)] - t_0$ is bounded from above by

$$T_{iR}[n_{i^*}(t_0)] - T_{i^*R}[n_{i^*}(t_0) - 1].$$

Since the above expression is no more than $(\beta NT/2)(1 + 1/R)$, we obtain

$$\hat{T}_{iR}(0) \leq (\beta NT/2)(1 + 1/R), \qquad 0 \leq i \leq N/R. \tag{22}$$

Also, the same bound holds when $n_{i^*}(t_0) = 0$ since then $\hat{T}_{iR}(0) = T_{iR}(0) = 0$. Using (22) in (21), we obtain

$$\begin{aligned} \hat{T}_j(n) \leq\, & n\beta T + \beta T[(\lceil j/R \rceil + 1)\lceil R/2 \rceil \\ & + (N/2)(1 + 1/R)]. \end{aligned}$$

The desired bound follows since $n_i(t_0) \leq n_{i^*}(t_0)$ implies $T_j[n_i(t_0) + n] \leq T_j[n_{i^*}(t_0) + n] = \hat{T}_j(n)$. □

Remark: From the proof of Theorem 1b), it follows that, if the start times of the first transmission cycles of color 0 cells are nonzero but finite, then the channel allocation policy can still guarantee maximal throughput and bounded queue lengths.

D. Extension to Nonzero Propagation Delays

The policy in Section II-A required that all color 0 cells start transmission synchronously at time 0, and that intercell propagation delay be zero. In this section, we study the effect of nonzero propagation delay on the performance of the policy.

We first describe a simple extension for approximate synchronization of color 0 cells. Assume that all cells are idle at time 0. Cell 0 starts transmission at time 0, and simultaneously sends a message to cell R, requesting it to begin transmission. Upon receiving this message, cell R starts transmission, and further, sends a message to cell $2R$ to start transmission, and so on. Let δ be a bound on the message propagation delay between two cells R cells apart, that is, between two successive cells of the same color. Since base stations are typically connected to the wired infrastructure, this delay is

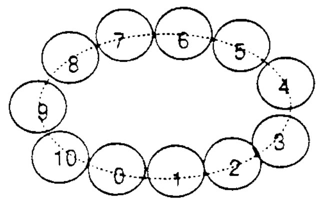

Fig. 4. Circular array: $N = 11$, $R = 3$.

typically small. The farthest color 0 cell gets started by time $(N/R)\delta$. Other than this, the policy is exactly as before.

We claim that if the policy parameters $\{\beta_j\}_{j=0}^{N-1}$ satisfy

$$\max_{0\le i\le N/R} \sum_{j=i}^{i+R-1} \beta_j \le 1 - \delta R/T \tag{23}$$

then cell j is guaranteed throughput β_j. Since δR time is wasted in communication in each cycle of length at most T, this result is rather intuitive. It is also rather easy to show using the results of the previous section. First, observe that (23) can be rewritten as

$$\max_{0\le i\le N-R} \sum_{j=i}^{i+R-1} (\beta_j T + \delta) \le T.$$

The propagation delay can be lumped together with the transmission time of the cell that is communicating its cycle completion. Then this system behaves exactly as the one described in Section II-C, but with $S_i(n) \le \beta_i T + \delta$, $T_{iR}(0) \le i\delta$, $0 \le i \le N/R$ and no communication delay. Using these facts and the remark at the end of Section II-C, the desired result follows. Appropriate bounds on delays can be derived similarly.

It is not difficult to see that the reduction in throughput is minimal in practical environments. Consider the scenario described in Section II-B with $R = 3$ and $T = 100$ ms. If the wired distance between cells that are R cells apart is, say, 1 km, then the propagation delay δ is about 3 μs. The reduction in throughput is from $\max_{0\le i\le N/R} \sum_{j=i}^{i+R-1} \beta_j \le 1$ to $\max_{0\le i\le N/R} \sum_{j=i}^{i+R-1} \beta_j \le 1 - 3\times 10^{-5}$.

III. Extensions to Cells Arranged on a Circle

In this section, we consider the topology in which N cells are arranged on a circle, i.e., cell $N-1$ is adjacent to cell 0. Given a reuse distance R, cells i and j, $i < j$, can use the channel simultaneously if either $i + R \ge j$ or $(j+R) \bmod N \ge i$. For example, in the topology of Fig. 4, cell 9 can use the channel simultaneously with all cells except cells 7, 8, 10, and 0. Besides being a natural generalization of a linear array, this topology is a basic step in understanding the channel allocation problem for two-dimensional hexagonal topologies which contain circles as special cases.

When N is a multiple of R, the policy for the linear array described in Section II-A can be applied directly to cells arranged on a circle. When N is not a multiple of R, the policy can still be applied with minor modifications (see Section III-A, Case 2). However, the following example shows that the

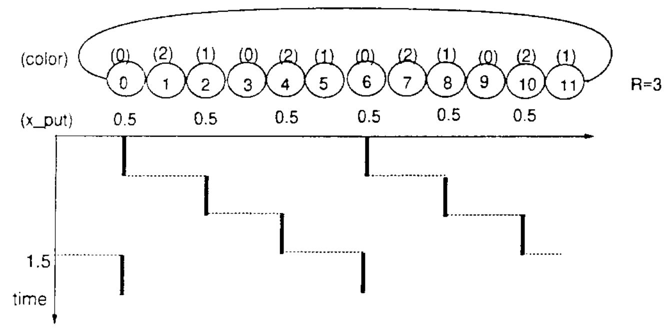

Fig. 5. Counterexample for circular array.

policy may not achieve maximal throughput. Let $N = 12$, $R = 3$, and the required throughput of cell i be defined by

$$\hat{\beta}_i = \begin{cases} 0.5, & \text{for } i = 0, 2, 4, 6, 8, 10 \\ 0, & \text{else.} \end{cases}$$

There exists a policy that provides the required throughputs, namely, the one that repeats the following schedule: first cells 0, 4, 8 for $T/2$ time, then cells 2, 6, 10 for $T/2$ time. Since cell i transmits for $T/2$ time in every interval of duration T, it is guaranteed its required throughput. The same, however, is not true for the policy described in Section II-A. As shown in Fig. 5, that policy only provides throughput $0.5/1.5 = 1/3 < 1/2$ to the even-numbered cells.

Interestingly, for the special case $R = 2$, the policy described in Section II (with a minor modification when N is odd) achieves maximum throughput. This is the main subject of the next section.

B. Maximum Throughput for $R = 2$

The policies described here are similar to that in Section II. For simplicity of exposition, we only consider the case in which every cell j transmits for $S_j(n) = S_j = \beta_j T$ time in *every* transmission cycle. To simplify the notation, we will use the notation $T_i(n)$, instead of $\tilde{T}_i(n)$, to denote the cycle start times in this case. The extensions to the general case $S_j(n) \le \beta_j T$ can be handled in a manner similar to Section II-A. For the rest of this section, we will use the convention that all negative and positive subscripts are modulo N, i.e., $T_i = T_{i \bmod N}$, $S_i = S_{i \bmod N}$, and so on.

Case 1: First consider the case when N is even. The policy described in Section II-A is applicable in this case without any modifications. Since any two adjacent cells may not transmit simultaneously, the achievable cell throughputs under any policy must satisfy the conditions

$$\hat{\beta} := \max_{0\le i\le N-1} (\hat{\beta}_i + \hat{\beta}_{i+1}) \le 1. \tag{24}$$

Note that (24) includes a condition $\hat{\beta}_{N-1} + \hat{\beta}_0 \le 1$ that did not arise for a linear array.

We show that the policy guarantees throughput β_i to cell i whenever the policy parameters $\{\beta_i\}$ satisfy (24). Let $\beta = \max_{0\le i\le N-1}(\beta_i + \beta_{i+1})$. We show the following stronger version of Lemma 2:

$$T_j(n+1) - T_j(n) \le \beta T, \qquad n \ge 0,\ 0 \le j \le N-1. \tag{25}$$

The result is stronger since it holds for $n \geq 0$ as opposed to $n \geq \lceil j/R \rceil$ in Lemma 2. This immediately implies that cell j is guaranteed throughput β_j when $\beta \leq 1$.

It is easy to check that as in (8), $T_{2i}(n+1)$ can be written as [recall the assumption $S_i(n) = S_i = \beta_i T$]

$$\max_{l \leq 2} \left\{ T_{2(i+1-l)}(n) + \max_{\mathbf{k} \in \mathcal{K}_l} \sum_{m=1}^{2} S_{2\left(i+1-\sum_{j=m}^{2} k_j\right)-(m-1)} \right\}. \tag{26}$$

The situation is similar to that in Section II, except that for this special case,

$$\begin{aligned} \sum_{m=1}^{2} & S_{2\left(i+1-\sum_{j=m}^{2} k_j\right)-(m-1)} \\ &= T \sum_{m=1}^{2} \beta_{2\left(i+1-\sum_{j=m}^{2} k_j\right)-(m-1)} \\ &\leq T \max_{0 \leq i \leq N-1} (\beta_i + \beta_{i+1}) \leq \beta T. \end{aligned} \tag{27}$$

Using (26), (27), and Lemma 8, the desired result (25) can be obtained. The details would be repetitive, and hence are omitted.

Case 2: Now, consider the more interesting case when N is odd. First, note that since at most $\lfloor N/2 \rfloor$ cells may be transmitting simultaneously, the achievable cell throughputs must satisfy, in addition to (24), the following condition.

$$\sum_{i=0}^{N-1} \hat{\beta}_i \leq \tilde{N} := \lfloor N/2 \rfloor. \tag{28}$$

If N is even, (28) is redundant since it is implied by (24). [To see this, simply add all the N constraints implied by (24) and divide both sides by 2.] This is not the case however, for N odd.

Consider the applicability of the policy for the linear array in this case. Since both the cells 0 and $N-1$ have color 0 but are within the reuse distance of each other, they may not transmit simultaneously. We therefore modify the policy as follows. Cell 0 is given higher priority than cell $N-1$ for every transmission cycle; however, cell 0 may not start its $n+1$th cycle before cell $N-1$ has completed its nth. For an illustration, see Fig. 6. We show next that this policy guarantees throughput β_i to cell i whenever the policy parameters $\{\beta_i\}$ satisfy (24) and (28).

It is not difficult to construct examples to show that (25) does not hold in this case. However, a similar result holds when we consider $\tilde{N}$ (recall that $\tilde{N} = \lfloor N/2 \rfloor$) cycles together. Our main result in this direction is the following.

Theorem 2: For every even cell j

$$m\hat{N} - \tilde{N}\beta T \leq T_j(m\tilde{N}) \leq m\hat{N} + \beta T, \qquad m = 1, 2, \cdots \tag{29}$$

where

$$\hat{N} = T \max \left\{ \sum_{k=0}^{N-1} \beta_k, \, \tilde{N}\beta \right\}.$$

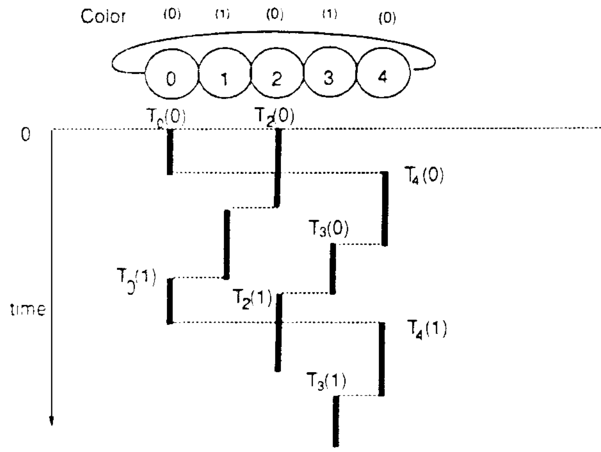

Fig. 6. Policy operation for circular array.

Remark: As in Section II, Theorem 2 implies that the policy can provide guarantees for the throughput and the queue size of any cell j as long as the policy parameters $\{\beta_i\}$ satisfy the necessary conditions (24) and (28). By following an argument identical to that in the proof of Corollary 1, it becomes evident that the right-hand-side inequality in (29) implies a guaranteed throughput β_j to node j. On the other hand, using both inequalities in (29) and recalling the definition of $n_j(t_0)$ from Theorem 1, we have that, for any $t_0 \geq 0$,

$$T_j[n_j(t_0) + m\tilde{N}] - t_0 \leq m\hat{N} + C$$

where C does not depend on m, which implies the boundedness of the queue size at node j along the lines of Corollary 2.

We begin with a result similar to Proposition 1 that expresses the start time of the $n+1$st cycle of color 0 cells in terms of the start times of the nth cycle of an appropriate set of color 0 cells. The equations are identical to those in Proposition 1 (specialized to $R = 2$), except for the boundary cells 0 and $N-1$, and just this difference leads to substantial complications. The proof is essentially identical to that of Proposition 1, and therefore is omitted.

Lemma 5: Subject to the initial conditions

$$T_{2i}(0) = \begin{cases} S_0, & \text{if } 2i = N-1 \\ 0, & \text{otherwise} \end{cases}$$

the following relations hold. For $1 \leq i < \tilde{N}$, $T_{2i}(n+1)$ is given by

$$\max_{0 \leq l \leq 2} \left\{ T_{2(i+1-l)}(n) + \max_{\mathbf{k} \in \mathcal{K}_l} \sum_{m=1}^{2} S_{2\left(i+1-\sum_{j=m}^{2} k_j\right)-(m-1)} \right\} \tag{30}$$

while the two special end nodes follow the relations

$$\begin{aligned} T_0&(n+1) \\ &= \max\{T_2(n) + S_2 + S_1, \, T_0(n) + S_0 + S_1, \\ &\quad T_{N-1}(n) + S_{N-1}\} \end{aligned} \tag{31}$$

$$\begin{aligned} T_{N-1}&(n+1) \\ &= \max\{T_{N-1}(n) + S_{N-1} + S_{N-2}, \\ &\quad T_{N-3}(n) + S_{N-3} + S_{N-2}, \, T_0(n+1) + S_0\}. \end{aligned} \tag{32}$$

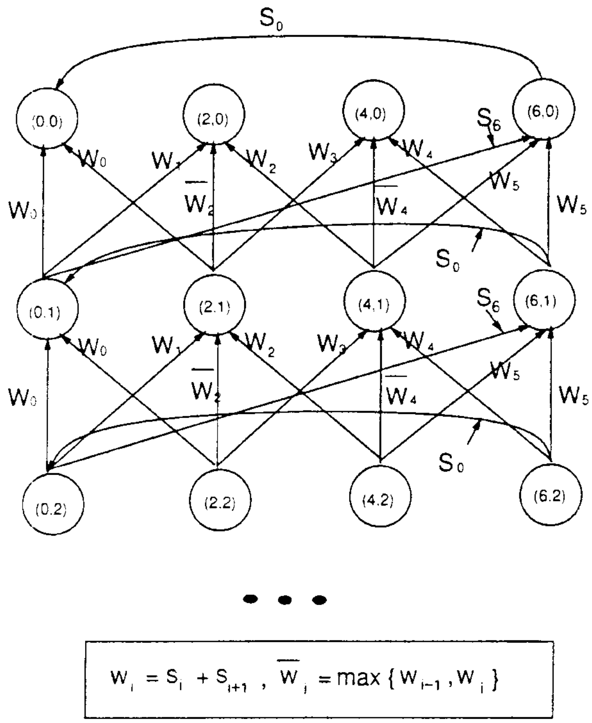

Fig. 7. Graph representation of system evolution.

The proof of Theorem 2 needs arguments of a different nature than those given so far. Induction arguments for proving (29) become cumbersome since an induction step has to cover $\tilde{N}$ steps (instead of one step as in Section II-C). Our arguments are based on the following directed infinite graph G that represents the interdependence of $\{T_{2i}(n)\}$ as given by Lemma 5. Each node of G corresponds to some $T_{2i}(n)$, and is represented as $(2i, n)$, $i = 0, 1, \cdots, \tilde{N}$, $n = 0, 1, 2, \cdots$. There is a directed edge from $(2j, m)$ to $(2i, n)$, denoted by $[(2j, m), (2i, n)]$, of weight w in G if there is a factor $\tilde{T}_{2i}(n) + w$ inside the maximum function in the recursive expression for $T_{2j}(m)$ as given by (30)–(32). As an illustration, the graph G along with the edge weights for seven nodes is shown in Fig. 7. Using the notation of the figure, we have from Lemma 5 that

$$T_2(1) = \max\{T_0(0) + W_0, T_2(0) + \overline{W}_2, T_4(0) + W_3\}.$$

Therefore, there are edges [(2, 1), (0, 0)], [(2, 1), (2, 0)], and [(2, 1), (4, 0)] of weights W_0, $\overline{W}_2$, and W_3, respectively, in G. Other edges are explained similarly.

A directed path from node v_1 to node v_2 in G will be denoted as $P[v_1 \rightsquigarrow v_2]$ and the set of all such paths as $\mathcal{P}[v_1 \rightsquigarrow v_2]$. The cost of a directed path P in G is defined to be the sum of edge weights along the path, and is denoted by $C(P)$. From the construction of G, it is easily seen by an inductive argument that $2j \neq N - 1$,

$$T_{2i}(n) := \max\{C(P)\text{: path } P \in \mathcal{P}[(2i, n) \rightsquigarrow (2j, 0)], \\ 2j \neq N - 1\}. \tag{33}$$

Proof of Theorem 2: We first prove the upper bound in (29). In view of (33), we have to show that the cost of every path from $(2i, m\tilde{N})$ to $(2j, 0)$, $2j \neq N - 1$, is at most $m\hat{N} + \beta T$.

Fix a positive integer m and a path $P = P[(2i, m\tilde{N}) \rightsquigarrow (2j, 0)]$ for some $2j \neq N - 1$. Let K_1, (respectively, K_2) denote the number of times the path P enters (resp., leaves) the set of nodes

$$\mathcal{G} := \{(N-1, n)\text{: } 0 \le n \le m\tilde{N} - 1\}$$

by an edge of the form $[(0, n), (N-1, n-1)]$, (respectively, $[(N-1, n), (0, n)]$). Note that $S_i + S_{i+1} = (\beta_i + \beta_{i+1})T \le \beta T$, and therefore, from (30)–(32), it follows that βT is an upper bound on the weight of each edge in G. The path P has $m\tilde{N} + K_2$ edges. If $K_2 = 0$, then P has $m\tilde{N}$ edges, each of weight less than βT, and the desired bound clearly holds. Consider next the case $K_2 \ge 1$. The proof will be given by considering the following two cases.

Case 1: $K_1 \ge K_2 \ge 1$. In this case, the sum of the weights of the K_2 edges of the form $[(N-1, n), (0, n)]$ of P (each having weight $\beta_0 T$), and K_2 of the K_1 edges of the form $[(0, n), (N-1, n-1)]$ (each having weight $\beta_{N-1}T$) have combined weight at most $K_2\beta T$. This follows since, by the definition of β, $\beta_0 + \beta_{N-1} \le \beta$. Since the number of remaining edges of P is $m\tilde{N} + K_2 - 2K_2 = m\tilde{N} - K_2$, each with weight at most βT, the desired bound $C(P) \le K_2\beta T + (m\tilde{N} - K_2)\beta T = m\tilde{N}\beta T \le m\hat{N}$ follows.

Case 2: $K_2 = K_1 + l$, $l \ge 1$. We split the sum of the edge weights of P into three disjoint groups.

1) The first group contains the weights of K_1 edges of the form $[(N-1, n), (0, n)]$ and K_1 edges of the form $[(0, n), (N-1, n-1)]$. As in Case 1, this sum is at most $K_1\beta T$.
2) Consider a path P^*
   - that starts from a node of the form $(0, k_1)$
   - whose last edge is of the form $[(N-1, k_2), (0, k_2)]$, $k_2 < k_1$
   - that contains no edge of the form $[(0, n), (N-1, n-1)]$.

   The path P^* must contain $\tilde{N} + 1$ edges of the form $[(0, n_0), (2, n_0 - 1)]$, $[(2, n_1), (4, n_1 - 1)]$, $\cdots$, $[(N-3, n_{\tilde{N}-1}), (N-1, n_{\tilde{N}-1} - 1)]$, $[(N-1, n_{\tilde{N}}), (0, n_{\tilde{N}})]$ for some $k_1 \le n_0 < n_1 < \cdots < n_{\tilde{N}} \le k_2$. The sum of the weights of these $\tilde{N}+1$ edges is $\sum_{k=0}^{N-1} S_k$. Consider next a path P_1^* such that
   - its origin node is of the form $(0, k_a)$
   - its last edge is of the form $[(N-1, k_b), (0, k_b)]$, $k_b < k_a$
   - it contains no other edge $[(N-1, k), (0, k)]$.

   Note that path P contains $K_2 - I$ nonoverlapping subpaths of the form P_1^*, where $I = 0$ if the origin node of P is of the form $(0, m\tilde{N})$ and $I = 1$ otherwise. Observe next that, of these $K_2 - I$ paths, at most K_1 contain an edge of the form $[(0, n), (N-1, n-1)]$ while the rest are of the form P^*. Since $K_2 = K_1 + l$, the path P must contain at least $l - 1$ subpaths of the form P^*; the second group in the summation contains the weights of $\tilde{N}+1$ edges, with weight sum $\sum_{k=0}^{N-1} S_k$, from each of $l-1$ subpaths of P that are of the form P^*. The total weight of this group is at most $(l-1)\sum_{k=0}^{N-1} S_k$.

3) The last group contains the remaining $m\tilde{N}+K_2-2K_1-(l-1)(\tilde{N}+1)=[m-(l-1)]\tilde{N}-K_1+1$ edges, each of weight at most βT.

Combining the contributions of the three groups and noting that $[m-(l-1)]\tilde{N}\geq 0$, we have that $C(P)$ is bounded from above by

$$\begin{aligned} &K_1\beta T+(l-1)\sum_{k=0}^{N-1} S_k+\{[m-(l-1)]\tilde{N}-K_1+1\}\beta T \\ &\quad=[m-(l-1)]\tilde{N}\beta T+(l-1)\sum_{k=0}^{N-1} S_k+\beta T \\ &\quad\leq [m-(l-1)]\hat{N}+(l-1)\hat{N}+\beta T \\ &\quad= m\hat{N}+\beta T. \end{aligned}$$

To derive the lower bound in (29), we argue as follows. Let $2i_0$ be a node such that $S_{2i_0}+S_{2i_0+1}=\beta T$. Consider the path P that starts from the node $(2i, m\tilde{N})$, enters the node $(2i_0, m\tilde{N}-k)$, $k\leq\tilde{N}$, and then follows the path $(2i_0, m\tilde{N}-k)$, $(2i_0, m\tilde{N}-k-1), \cdots, (2i_0, 0)$. Since the weight of an edge of the form $[(2i_0, n), (2i_0, n-1)]$ is βT, it is easy to see that $C(P)\geq(m\tilde{N}-\tilde{N})\beta T$. Consider next the path P_1 along the nodes $(2i, m\tilde{N})$, $(2i+2, m\tilde{N}-1), \cdots, [\tilde{N}-(\tilde{N}-i)]$, $[0, m\tilde{N}-(\tilde{N}-i)]$, $[2, m\tilde{N}-(\tilde{N}-i)-1], \cdots, [2i, (m-1)\tilde{N}], \cdots, (2i, 0)$. The cost of this path is $C(P_1)=m\sum_{k=0}^{N} S_k$. It follows from (33) that

$$T_{2i}(m\tilde{N})\geq\max\{C(P), C(P_1)\}\geq m\hat{N}-\tilde{N}\beta T.$$

□

B. Maximum Throughput for $R\geq 3$

The problem of achieving maximal cell throughputs becomes very difficult when the reuse distance is at least 3. This is because the achievable cell throughputs must now satisfy a very large set of constraints and, moreover, this set of constraints is not completely known. To illustrate this, let us try to enumerate some of the constraints that the throughputs must satisfy.

Associate a graph G with a given circular array as follows. The nodes in the graph correspond to the cells in the array, and there is an edge between two nodes in G if the corresponding cells in the circular array are within the reuse distance R of each other. The graph G is also known as the interference graph of the circular array. A set of cells S in the circular array can transmit simultaneously if S is an independent set of G, i.e., no two nodes in S are connected by an edge. Hence, the problem of characterizing constraints for achievable cell throughputs is the same as characterizing the independent set polytope [10] for G. For general graphs, this is a well-known [10] hard problem. But the situation simplifies for specific graphs, e.g., interference graphs of linear arrays and circular arrays with $R=2$. For interference graphs of circular arrays with $R\geq 3$, however, the problem does not simplify much, as we illustrate below.

First, the achievable cell throughputs must satisfy the constraints of the type encountered in the previous sections:

$$\max_{0\leq i\leq N-1}\sum_{j=i}^{i+R-1}\hat{\beta}_j\leq 1. \tag{34}$$

Since the cells $\{i, i+1, \cdots, i+R-1\}$ form a clique in G, constraints of the form (34) are called clique constraints. Next, the throughputs must satisfy the hole constraints. A hole H is a sequence of nodes $(i_1, i_2, \cdots, i_k)$ in G satisfying the following properties: 1) k is odd, 2) two nodes of H are adjacent only if they appear in consecutive positions in H. Therefore, there are edges between i_j and i_{j+1} $1\leq j\leq k-1$ and i_k and i_1 in G, but no edges between i_j and i_k for $|j-k|\geq 2$. For example, for a circular ring with $N=9$ and $R=3$, (0, 2, 4, 6, 8), and (0, 2, 3, 5, 7) are holes, while (0, 1, 2) is not a hole. Letting $|H|$ denote the number of nodes in H, observe that at most $(|H|-1)/2$ cells in a hole can transmit at a time. Hence, the achievable throughputs must also satisfy the hole constraints:

$$\sum_{i\in H}\hat{\beta}_i\leq(|H|-1)/2,\qquad\text{for every hole } H. \tag{35}$$

A more general set of constraints has been discovered by Padberg [11]. These constraints have the form

$$\sum_{i\in H}\hat{\beta}_i+\sum_{i\in H'}c_i\hat{\beta}_i\leq(|H|-1)/2,\ \text{for every hole } H \tag{36}$$

where H' is the set of nodes in G that are adjacent to some node in H and $\{c_i\}$ are computed from an optimization problem described in [11]. Consider, for example, $N=12$, $R=4$, and the hole $H=(0, 3, 5, 8, 11)$. From [11, Theorem 3.3], it follows that the following inequality must hold:

$$\sum_{i\in H}\hat{\beta}_i+\sum_{i=1,2,6}\hat{\beta}_i\leq 2.$$

It turns out that, in general, there are even more constraints on the achievable throughputs than those described above, and the complete constraint set is not even known. It seems therefore, highly unlikely that a simple policy can be designed that guarantees cell throughputs under the weakest possible constraints. In what follows, we present a simple policy that achieves maximal throughputs asymptotically as $R/N\to 0$, e.g., for large circular arrays with fixed reuse distance. Intuitively, this is plausible since a circular array with small R/N is very close to a linear array.

The policy works with parameters β_i and T. We present a centralized policy first. Let $\mathcal{L}_p$, $p=0, \cdots, N-1$, denote the linear array formed by removing the nodes $p, p+1, \cdots, p+R-1$ from the circular array. From the results in Section II, we can determine a schedule with parameters $\{\beta_i\}$ and T that organizes the transmission of the cells of a linear array so that cell i transmits for $\beta_i T$ time, and additionally, if the parameters $\{\beta_i\}$ satisfy (2), then the length of the schedule is at most T. Accordingly, let π_p denote the schedule with parameters $\{N\beta_i/(N-R+1)\}$ and T for the linear array $\mathcal{L}_p$. The schedule for the circular array is then as follows. In the interval $[kT, (k+1)T)$, the schedule π_p is applied, where $p = k \bmod N$.

Our main result is the following.

Theorem 3: If the policy parameters $\{\beta_i\}$ satisfy

$$\max_{0 \le i \le N-1} \sum_{j=i}^{i+R-1} \beta_i \le 1 - \frac{R-1}{N}, \qquad (37)$$

then the policy guarantees throughput β_i to cell i.

Proof: It suffices to show that in any interval $[lT, (l+N)T)$, cell i receives $\beta_i NT$ units of service. Since

$$\max_{0 \le i \le N-1} \sum_{j=i}^{i+R-1} \frac{N\beta_i}{N-R+1} \le 1,$$

any cell i that is allowed to transmit during schedule π_p, $p = k \bmod N$, receives $N\beta_i T/(N-R+1)$ units of service in the interval $[kT, (k+1)T)$. Since cell i is allowed to transmit in $N-R+1$ out of N consecutive intervals of the form $[kT, (k+1)T)$, we conclude that cell i receives $N\beta_i T$ service during any interval $[lT, (l+N)T)$. □

Comparing (34) and (37), we conclude that as $R/N \to 0$, the proposed policy can guarantee cell throughputs under conditions that are very close to necessary. It is easy to provide a distributed version of the policy described above, along the lines of Section II. However, the analysis of such a policy becomes now very complicated. While we believe that the distributed policy can also provide throughput guarantees under (34), we could not prove this assertion rigorously.

IV. Extensions to Planar Hexagonal Cells

In this section, we adapt the policy to planar hexagonal cell arrays when there is one frequency for sharing. We assume that the frequency can be reused in two hexagonal cells if they do not touch along a face. This is equivalent to the statement that if the frequency is used in a cell, then it cannot be used in the ring of six hexagonal cells around that cell. This assumption is mainly for notational convenience; other periodic reuse patterns can be treated similarly. Two cells are said to be interfering if they cannot use the frequency simultaneously.

We use the coordinate system shown in Fig. 8. It is well known that one can assign three colors (say RED, YELLOW, and GREEN) to the cells in such a way that two interfering cells have different colors. The coloring can be done as shown in Fig. 8. Formally, in the given coordinate system, we color the cells as follows.

1) Color (0, 0), (1, 0), (0, 1) RED, YELLOW, and GREEN, respectively.
2) Iteratively do the following for every cell until all of the cells are colored: if (x, y) is colored C (C = RED, YELLOW, GREEN), color cells $(x+1, y+1)$, $(x+2, y-1)$, $(x+1, y-2)$, $(x-1, y-1)$, $(x-2, y+1)$, and $(x-1, y+2)$ also with color C.

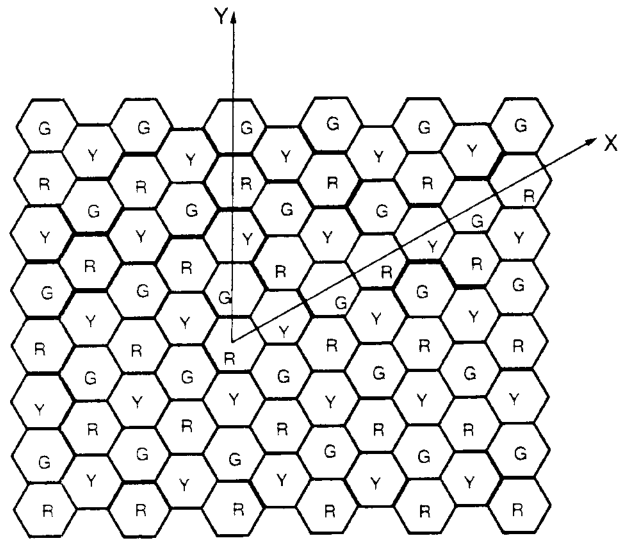

Fig. 8. Coloring and coordinate system of hexagonal array.

The policy described in Section II-A can be adapted to this case as follows. The policy is parameterized by $\{\beta_{xy}\}_{x,y}$ and a global constant T. The parameter β_{xy} is associated with cell (x, y). The neighbors of a cell (x, y) are defined as those that are within its reuse distance, namely, the six cells that touch (x, y). Whenever a cell (x, y) gets an opportunity to transmit, it may transmit for up to $\beta_{xy}T$ time. The RED cells transmit first, and send a Done$(\cdot)$ message to their neighboring YELLOW cells. A YELLOW, (respectively, GREEN, RED) cell transmits only after it has received a Done$(\cdot)$ message from its neighboring RED (resp., YELLOW, GREEN) cells since its last transmission, and after completing transmission, sends a Done$(\cdot)$ message to its neighboring GREEN (resp., RED, YELLOW) cells.

We first show that the policy may not achieve maximal throughput. Consider the system in Fig. 9. All empty cells have zero throughput requirements. The requirements of the rest of the cells are shown in the figure. It is easy to see that when the policy is applied, the nth cycle of the RED cells starts at time $1.2nT$, which implies that all of the cells requiring nonzero throughput get less than what they require, e.g., cell (0, 0) gets throughput $0.5/1.2 = 0.417 < 0.5$. However, there exists the following policy that provides the required throughputs: cells (0, −3), (2, −3), (4, −3), (6, −3), (4, −1), (2, 1), (0, 3), (0, 1), (0, −1) transmit first for $0.5T$ time; then cells (3, −3), (3, 0), (0, 0) transmit from time $0.5T$ up to time T; in parallel, cells (1, 3), (5, −2), (0, 2) transmit from time $0.5T$ up to time $0.7T$, and cells (0, −2), (5, −3), and (1, 2) transmit from time $0.7T$ up to time $0.9T$. After time T, the same schedule is repeated.

Next, we will identify regions of cell throughputs achievable by the proposed policy. Let $S_{xy}(n)$ denote the time for which cell (x, y) transmits during its nth transmission cycle which begins at time $T_{xy}(n)$. In the spirit of the discussion in Sections II and III, we first develop a recursive expression relating successive cycle start times of RED cells. Let $\mathcal{A}$ denote the set $\{(0, 1), (1, -1), (-1, 0)\}$. Observe that the set of GREEN, (respectively, YELLOW, RED) neighbors of

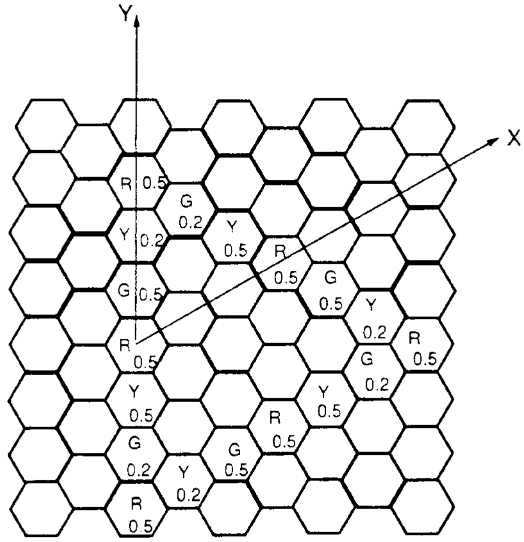

Fig. 9. Counterexample.

a RED (resp., GREEN, YELLOW) cell (x, y) is given by $\{(x+i, y+j)\colon (i, j) \in \mathcal{A}\}$. From the operation of the policy, it is immediate that for a RED cell (x, y),

$$T_{xy}(n+1) = \max_{(i,j)\in\mathcal{A}} \{T_{x+i, y+j}(n) + S_{x+i, y+j}(n)\}$$

while for GREEN and YELLOW cells,

$$T_{xy}(n) = \max_{(i,j)\in\mathcal{A}} \{T_{x+i,y+j}(n) + S_{x+i,y+j}(n)\}.$$

Combining these equations, we obtain the following.

Lemma 6: For a RED cell (x, y), $T_{xy}(n+1)$ is given by

$$\max_{\substack{(i_m,j_m)\in\mathcal{A}\\ m=1,2,3}} \left\{ T_{x+\sum_{m=1}^3 i_m,\, y+\sum_{m=1}^3 j_m}(n) + \sum_{m=1}^{3} S_{x+\sum_{k=1}^m i_k,\, y+\sum_{k=1}^m j_k}(n) \right\}. \tag{38}$$

Using Lemma 6, we provide a set of sufficient conditions on the policy parameters $\{\beta_{xy}\}$ under which throughputs can be guaranteed.

Lemma 7: If the policy parameters $\{\beta_{xy}\}$ satisfy

$$\beta_{xy} + \beta_{x+i,y+j} \le 2/3, \qquad (i, j) \in \mathcal{A} \tag{39}$$

then the policy guarantees throughput β_{xy} to every cell (x, y).

Proof: Let $\{[i_m^{(1)}, j_m^{(1)}]\}_{m=1}^3$, [respectively, $\{[i_m^{(2)}, j_m^{(2)}]\}_{m=1}^3$] attain the maximum in the expression for $T_{xy}(n+1)$ [resp., $T_{x+\sum_{m=1}^3 i_m^{(1)},\, y+\sum_{m=1}^3 j_m^{(1)}}(n)$] in (38). After applying (38) twice and using $S_{xy}(n) \le \beta_{xy}T$, we have the following upper bound on $T_{xy}(n+1)$:

$$T_{x+\sum_{m=1}^3 i_m^{(1)}+\sum_{m=1}^3 i_m^{(2)},\, y+\sum_{m=1}^3 j_m^{(1)}+\sum_{m=1}^3 j_m^{(2)}}(n-1) + T\Delta$$

where

$$\Delta = \sum_{m=1}^{3} \beta_{x+\sum_{k=1}^m i_k^{(1)},\, y+\sum_{k=1}^m j_k^{(1)}} + \sum_{m=1}^{3} \beta_{x+\sum_{k=1}^3 i_k^{(1)}+\sum_{k=1}^m i_k^{(2)},\, y+\sum_{k=1}^3 j_k^{(1)}+\sum_{k=1}^m j_k^{(2)}}. \tag{40}$$

We show that $\Delta \le 2$. Indeed, using (39), successive terms in (40) can be paired so that the contribution of each of the three pairs is at most 2/3. More specifically, the $m = 1$ and $m = 2$ terms of the first sum are paired together, the $m = 3$ term of the first sum is paired with the $m = 1$ term of the second sum, and the remaining two terms are paired together. It follows now, by using standard arguments, that

$$T_{xy}(2n) \le 2nT, \qquad \text{every } (x, y). \tag{41}$$

It can be easily seen that inequality (41) implies that cell (x, y) gets its required throughput. □

Note that the cell throughputs achievable by any policy must satisfy the condition

$$\beta_{xy} + \beta_{x+i,y} + \beta_{x,y+i} \le 1, \qquad i = \pm 1$$

since only one cell out of three adjacent cells of different colors can transmit at a time. From the discussion in Section III-B, we see that, in fact, the achievable cell throughputs must satisfy many more constraints, and furthermore, not all of them are known. In view of this, (39) seem fairly weak.

V. Conclusions

We have proposed and analyzed a class of distributed channel allocation policies for PCN. By appropriately choosing the policy parameters β_i, the policy can guarantee throughput β_i to every cell i, independently of the traffic behavior in the rest of the cells. These guarantees can be provided under certain conditions on achievable cell throughputs $\{\beta_i\}$, which are the weakest possible in the cases of a linear array of cells and a circular array with reuse distance equal to 2. For a general circular array, the conditions are weakest asymptotically as the ratio of reuse distance to the number of cells tends to zero. For a planar hexagonal array of cells and reuse distance 2, the conditions are fairly weak, although not the weakest possible.

A cell can use the throughput guarantees to support connections that require certain transmission rates (not necessarily identical), e.g., voice and video. Alternatively, these guarantees can be viewed as a means of providing a server of a given minimum speed to a cell. As such, they can be used to control delays for bursty traffic, provided that the arrival rate of packets of this type of traffic remains smaller than a percentage of the guaranteed server speed. Again, in this case, the delay performance of the packets belonging to bursty traffic will not be significantly affected by variations of traffic in neighboring cells. A study of this interesting possibility is the subject of further investigation.

APPENDIX

In this Appendix, we provide the proofs of a number of results stated in the main body of the paper.

Proof of Lemma 1:

a) Since $S_i(n) \le \beta_i T$, we have

$$\sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}(n) \le T \sum_{m=1}^{R} \beta_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}. \quad (42)$$

Letting $\gamma_m = (i+1-\sum_{j=m}^{R} k_j)R-(m-1)$, observe that $|\gamma_{m+1}-\gamma_m| \le R-1$. From the definition of β, it follows that the sum of any two consecutive terms on the right in (42) is at most βT. Since the sum contains R terms, the desired bound follows.

b) Observe that from the recursive expression for $T_{iR}(n)$ given by Proposition 1 and the bound (15), we have for all $n \ge 0$,

$$T_{iR}(n) \le \max_{0\le l\le R}\{T_{(i+1-l)R}(n)\} + \beta T\lceil R/2\rceil.$$

The desired bound for $T_{iR}(n)$ follows by induction, using the last inequality. Now, fix $j \ne iR$, and observe that since cell $\lceil j/R\rceil R$ can only begin its $(n+1)$st cycle after cell j has started its nth, we have

$$\tilde{T}_j(n) \le \tilde{T}_{\lceil j/R\rceil R}(n+1), \qquad n \ge 0. \quad (43)$$

Now, using the already derived bound for $\tilde{T}_{\lceil j/R\rceil R}(n+1)$ in (43), the desired bound for $T_j(n)$ follows.

c) Note that both $T_j(n)$ and $\tilde{T}_j(n)$ satisfy recursion (8). For cells of the form $j = iR$, the result follows from (8) by an induction on n and using the fact $S_j(n) \le \beta_j T$ in the induction step. For other cells of the form $iR-c$, a standard induction argument on c using (11) can be given. The details are straightforward. □

We now proceed to the proofs of Lemmas 2 and 3. The following preliminary result is needed. Recall the definition $S_i = \beta_i T$.

Lemma 8: For every $i, n \ge 0$,

$$\tilde{T}_{(i+1)R}(n) - \tilde{T}_{iR}(n) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \le \beta T. \quad (44)$$

Proof: For $n = 0$, the result follows simply from the initial conditions $T_{iR}(0) = 0$ and the definition of β; therefore, assume $n \ge 1$. From Proposition 1, it follows that for appropriate $\{k_j\}_{j=1}^{R}$,

$$\tilde{T}_{(i+1)R}(n) = \tilde{T}_{\left(i+2-\sum_{j=1}^{R} k_j\right)R}(n-1) \cdot \sum_{m=1}^{R} S_{\left(i+2-\sum_{j=m}^{R} k_j\right)R-(m-1)}. \quad (45)$$

Consider the following three cases.

Case 1: $k_R = 1$. Define $\{\tilde{k}_j\}$ as follows:

$$\tilde{k}_j = \begin{cases} k_j, & \text{for } j \le R-1 \\ 0, & \text{for } j = R \end{cases}$$

so that

$$\sum_{j=m}^{R} k_j = 1 + \sum_{j=m}^{R} \tilde{k}_j, \qquad m = 1, 2, \cdots, R. \quad (46)$$

Using first Proposition 1 and then (45) and (46), we obtain

$$\begin{aligned}\tilde{T}_{iR}(n) \ge\, & \tilde{T}_{\left(i+1-\sum_{j=1}^{R} \tilde{k}_j\right)R}(n-1) + \sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} \tilde{k}_j\right)R-(m-1)} \\ =\, & \tilde{T}_{\left(i+2-\sum_{j=1}^{R} k_j\right)R}(n-1) + \sum_{m=1}^{R} S_{\left(i+2-\sum_{j=m}^{R} k_j\right)R-(m-1)} \\ =\, & \tilde{T}_{(i+1)R}(n). \end{aligned} \quad (47)$$

From (47) and the definition of β, we obtain

$$\tilde{T}_{(i+1)R}(n) - \tilde{T}_{iR}(n) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \le \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \le \beta T.$$

Case 2: $k_R = 0$, but $k_j = 1$ for some j. With $p := \max\{j \le R\colon k_j = 1\}$, define

$$\tilde{k}_j = \begin{cases} k_j, & \text{for } j \le p-1 \\ 0, & \text{for } p \le j \le R. \end{cases}$$

It follows that

$$\sum_{j=m}^{R} k_j = 1 + \sum_{j=m}^{R} \tilde{k}_j, \qquad m = 1, 2, \cdots, p \quad (48)$$

$$\sum_{j=m}^{R} k_j = \sum_{j=m}^{R} \tilde{k}_j = 0, \qquad m = p+1, \cdots, R. \quad (49)$$

Using first Proposition 1 and then (48) and (49), we write

$$\begin{aligned}\tilde{T}_{iR}(n) \ge\, & \tilde{T}_{\left(i+1-\sum_{j=1}^{R} \tilde{k}_j\right)R}(n-1) + \sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} \tilde{k}_j\right)R-(m-1)} \\ =\, & \tilde{T}_{\left(i+2-\sum_{j=1}^{R} k_j\right)R}(n-1) + \sum_{m=1}^{p} S_{\left(i+2-\sum_{j=m}^{R} k_j\right)R-(m-1)} \\ & + \sum_{m=p+1}^{R} S_{(i+1)R-(m-1)}. \end{aligned} \quad (50)$$

Now, (45) and (50) can be combined to show that

$$\begin{aligned}\tilde{T}_{(i+1)R}(n) - \tilde{T}_{iR}(n) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} &\le \sum_{m=p+1}^{R} S_{(i+2)R-(m-1)} - \sum_{m=p+1}^{R} S_{(i+1)R-(m-1)} \\ &\quad + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \\ &= \sum_{m=p+1}^{R} S_{(i+2)R-(m-1)} + \sum_{m=1}^{p} S_{(i+1)R-(m-1)} \\ &\le \beta T\end{aligned}$$

where the final inequality follows from the definition of β.

Case 3: $k_j = 0$ for every j. In this case, from (45), we obtain

$$\tilde{T}_{(i+1)R}(n) = \tilde{T}_{(i+2)R}(n-1) + \sum_{m=1}^{R} S_{(i+2)R-(m-1)} \quad (51)$$

while Proposition 1 yields

$$\tilde{T}_{iR}(n) \ge \tilde{T}_{(i+1)R}(n-1) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)}. \quad (52)$$

Combining (51) and (52),

$$\begin{aligned}&\tilde{T}_{(i+1)R}(n) - \tilde{T}_{iR}(n) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \\ &\quad \le \tilde{T}_{(i+2)R}(n-1) - \tilde{T}_{(i+1)R}(n-1) + \sum_{m=1}^{R} S_{(i+2)R-(m-1)}.\end{aligned}$$

We continue in this manner until we reach Case 1 or Case 2 or the boundaries $n-l=0$ or $(i+l)R > N$ for some l. The boundaries are easy to handle since $T_{iR}(0) = 0$ for all i and $\sum_{m=1}^{R} \beta_{(i+2)R-(m-1)} \le \beta$. □

Proof of Lemma 2: We first show that for all $n \ge i$, $i = 0, 1, 2, \cdots$

$$\tilde{T}_{iR}(n+1) - \tilde{T}_{iR}(n) \le \beta T. \quad (53)$$

The proof is by induction. From Proposition 1, the fact $\tilde{T}_{iR}(0) = 0$, and the definition of β, we obtain

$$\tilde{T}_0(1) = \max\left\{\sum_{m=1}^{R} S_{R-(m-1)}, \sum_{m=1}^{R} S_{R-m}\right\} \le \beta T \quad (54)$$

which shows (53) for $i = n = 0$. Now, assume that (53) is true for n and all i, $0 \le i \le n$. It suffices to show that for $0 \le i \le n$,

$$\tilde{T}_{iR}(n+2) - \tilde{T}_{iR}(n+1) \le \beta T \quad (55)$$
$$\tilde{T}_{(i+1)R}(n+2) - \tilde{T}_{(i+1)R}(n+1) \le \beta T. \quad (56)$$

Again, the recursive expression for $\tilde{T}_{iR}(n+1)$ as given in Proposition 1 will be the key.

Proof of (55): From Proposition 1, it follows that for appropriate $\{k_j\}_{j=1}^{R}$,

$$\begin{aligned}\tilde{T}_{iR}(n+2) &= \tilde{T}_{\left(i+1-\sum_{j=1}^{R} k_j\right)R}(n+1) \\ &\quad + \sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}.\end{aligned} \quad (57)$$

Consider two cases.

Case 1: $k_j = 0$ for all j. In this case, (57) reduces to

$$\tilde{T}_{iR}(n+2) = \tilde{T}_{(i+1)R}(n+1) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)} \quad (58)$$

and using Lemma 8, we obtain that $\tilde{T}_{iR}(n+2) - \tilde{T}_{iR}(n+1)$, which can be rewritten as

$$\tilde{T}_{(i+1)R}(n+1) - \tilde{T}_{iR}(n+1) + \sum_{m=1}^{R} S_{(i+1)R-(m-1)}$$

is at most βT.

Case 2: $k_j = 1$ for some j. In this case, using Proposition 1, we note that $\tilde{T}_{iR}(n+1)$ is at least

$$\tilde{T}_{\left(i+1-\sum_{j=1}^{R} k_j\right)R}(n) + \sum_{m=1}^{R} S_{\left(i+1-\sum_{j=m}^{R} k_j\right)R-(m-1)}$$

so that $\tilde{T}_{iR}(n+2) - \tilde{T}_{iR}(n+1)$ is at most

$$\tilde{T}_{\left(i+1-\sum_{j=1}^{R} k_j\right)R}(n+1) - \tilde{T}_{\left(i+1-\sum_{j=1}^{R} k_j\right)R}(n)$$

and the desired conclusion (55) follows from the induction hypothesis since $\sum_{j=1}^{R} k_j \ge 1$.

Proof of (56): The proof is similar to that for (55). Note that the proof in Case 1 does not depend on the induction hypothesis, and that in Case 2 breaks down only for the case $\sum_{j=1}^{R} k_j = 1$ since it is not known whether $\tilde{T}_{(i+1)R}(n+1) - \tilde{T}_{(i+1)R}(n) \le \beta T$. So we only have to show (56) for the following case ($\sum_{j=1}^{R} k_j = 1$ in (8)):

$$\begin{aligned}&\tilde{T}_{(i+1)R}(n+2) \\ &\quad = \tilde{T}_{(i+1)R}(n+1) + \max_{1 \le c \le R}\left[\sum_{m=1}^{c} S_{(i+1)R-(m-1)}\right. \\ &\qquad \left. + \sum_{m=c+1}^{R} S_{(i+2)R-(m-1)}\right]\end{aligned} \quad (59)$$

But from the definition of β, the expression inside $[\cdots]$ in (59) is smaller than βT, and (56) follows. This completes the proof of (53).

To extend (53) to other cells, we show by an induction on c that

$$\tilde{T}_{iR-c}(n+1) - \tilde{T}_{iR-c}(n) \le \beta T, \qquad 0 \le c \le R-1,\ n \ge i \quad (60)$$

holds. Observe first from (53) that (60) holds for $c = 0$. Now, suppose (60) holds for $c \ge 0$, and that in the expression for $\tilde{T}_{iR-(c+1)}(n+1)$ as given by (53), $\tilde{T}_{iR-c}(n+1) + S_{iR-c} <$

$\tilde{T}_{(i-1)R-c}(n+1)+S_{(i-1)R-c}$ (the other case is similar). Then from the induction hypothesis,

$$\begin{aligned}\tilde{T}_{iR-c-1}(n+1) - \tilde{T}_{iR-c-1}(n) &= \tilde{T}_{(i-1)R-c}(n+1) + S_{iR-c-1} \\ &\quad - \max_{k\in\{0,1\}}\{\tilde{T}_{(i-k)R-c}(n) + S_{(i-k)R-c}\} \\ &\leq \tilde{T}_{(i-1)R-c}(n+1) - \tilde{T}_{(i-1)R-c}(n) \leq \beta T. \qquad \square\end{aligned}$$

Proof of Lemma 3: Consider first the case $n+1 \leq N/R$. From Lemma 1b) and this bound on $n+1$, we obtain that $T_{iR}(n+1) - T_{jR}(n)$ is upper bounded by

$$T_{iR}(n+1) \leq (N/R)\beta T(R+1)/2 = (\beta NT/2)(1+1/R).$$

Now, consider the case $n \geq N/R$. Fix two color 0 cells iR and jR. Observe that cell jR can start its $n+1$st cycle only after cells $(j-1)R$ and $(j+1)R$ have both completed their nth cycles. Therefore, when cell jR begins its nth cycle, every cell sR, $s = 0, 1, \cdots, N/R$, must have completed its $(n-N/R)$th cycle, that is,

$$T_{sR}(n - N/R) \leq T_{jR}(n). \tag{61}$$

Next, let l_1, $\{k_j^{(1)}\}_{j=1}^R$ correspond to the term that achieves the maximum in the expression for $T_{iR}(n+1)$ in (8). With the notation $j_1 := i+1-l_1$ and using Lemma 1a), we obtain

$$T_{iR}(n+1) - T_{j_1R}(n) \leq \beta T\lceil R/2\rceil. \tag{62}$$

If $T_{j_1R}(n) \leq T_{jR}(n)$, then using (62), we derive that $T_{iR}(n+1) - T_{jR}(n)$ is upper bounded by

$$T_{iR}(n+1) - T_{j_1R}(n) \leq \beta T\lceil R/2\rceil \leq (\beta NT/2)(1+1/R) \tag{63}$$

and (17) follows; else we continue. Let l_2, $\{k_j^{(2)}\}_{j=1}^R$, $j_2 := j_1 + 1 - l_2$ correspond to the term that achieves the maximum in the expression for $T_{j_1R}(n)$ in (8). Another application of Lemma 1 yields

$$T_{j_1R}(n) - T_{j_2R}(n-1) \leq \beta T\lceil R/2\rceil. \tag{64}$$

Combining (62) and (64) yields

$$T_{iR}(n+1) - T_{j_2R}(n-1) \leq 2\beta T\lceil R/2\rceil. \tag{65}$$

If $T_{j_2R}(n-1) \leq T_{jR}(n)$, then using (65), we again derive

$$\begin{aligned}T_{iR}(n+1) - T_{jR}(n) &\leq T_{iR}(n+1) - T_{j_2R}(n-1) \\ &\leq 2\beta T\lceil R/2\rceil \\ &\leq (\beta NT/2)(1+1/R).\end{aligned}$$

Else, we continue similarly with $T_{j_2R}(n-1)$. Note that, because of (61), the procedure will stop after at most N/R steps, thereby yielding (17). $\square$

The Appendix concludes with a proof of Lemma 4.

Proof of Lemma 4: We show by an induction on c that for $0 \leq i \leq N/R$, $0 \leq c \leq R-1$,

$$T_{iR-c}^{\Delta}(n) \leq \tilde{T}_{iR-c}(n) + \Delta. \tag{66}$$

Consider first the case $c = 0$. Let $S_j^{\Delta}(n)$ denote the time for which cell j transmits during its nth cycle; clearly, this depends on Δ because of the arrivals during [0, Δ]. Using the fact

$$\begin{aligned}&\max_{\mathbf{k}\in\mathcal{K}_l}\left(\sum_{m=1}^{R} S^{\Delta}_{\left(i+1-\sum_{j=m}^{R}k_j\right)R-(m-1)}(n)\right) \\ &\quad\leq T\max_{\mathbf{k}\in\mathcal{K}_l}\left(\sum_{m=1}^{R}\beta_{\left(i+1-\sum_{j=m}^{R}k_j\right)R-(m-1)}\right) =: X_i\end{aligned}$$

in Proposition 1, we have

$$T_{iR}^{\Delta}(n+1) \leq \max_{0\leq l\leq R}\{T_{(i+1-l)R}^{\Delta}(n) + X_i\} \tag{67}$$

$$\tilde{T}_{iR}(n+1) = \max_{0\leq l\leq R}\{\tilde{T}_{(i+1-l)R}(n) + X_i\}. \tag{68}$$

The desired inequality (66) for $c = 0$ follows from (67) and (68) by an induction on n.

For the induction step, we assume (66) for c, and show it for $c+1$. This can be done by expressing $T_{iR-(c+1)}^{\Delta}(n)$ and $\tilde{T}_{iR-(c+1)}(n)$, respectively, in terms of $T_{iR-c}^{\Delta}(n)$ and $\tilde{T}_{iR-c}(n)$ using (11), and then using the induction hypothesis. The details are straightforward. $\square$

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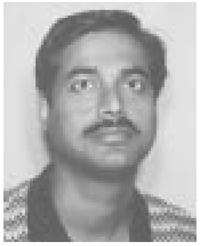
REFERENCES

[1] H. Ahmadi, Ed., *IEEE Personal Commun. Mag.*, vol. 1, 1Q 1994.
[2] D. F. Bantz and F. J. Bauchot, "Wireless LAN design alternatives," *IEEE Network Mag.*, vol. 8, pp. 43–53, Mar./Apr. 1994.
[3] R. Braden, L. Zhang, S. Berson, S. Herzog, and S. Jamin, "Resource ReSerVation Protocol: Version 1, Functional Spec.," in ftp://ietf.org/internet-drafts/draft-ietf-rsvp-spec-13.txt, 1996.
[4] L. J. Cimini, G. F. Foschini, and C.-L. I, "Call blocking performance of distributed algorithms for dynamic channel allocation in microcells," in *Proc. IEEE ICC*, Chicago, IL, June 1992, pp. 1327–1332.
[5] R. L. Cruz, "A calculus for network delay, part I: Network analysis," *IEEE Trans. Inform. Theory*, vol. 37, pp. 114–131, Jan. 1991.
[6] M. Frodigh, "Optimum dynamic channel allocation in certain street microcellular radio systems," in *Proc. IEEE Veh. Technol. Soc. 42nd VTS Conf.*, Denver, CO, May 1992, pp. 658–661.
[7] D. J. Goodman and S. X. Wei, "Efficiency of packet reservation multiple access," *IEEE Trans. Veh. Technol.*, vol. 1, pp. 170–176, Feb. 1991.
[8] S. S. Kuek and W. C. Wong, "Ordered dynamic channel assignment scheme with reassignment in highway microcells," *IEEE Trans. Veh. Technol.*, vol. 41, pp. 271–277, Aug. 1992.
[9] R. O. LaMaire, A. Krishna, and H. Ahmadi, "Analysis of a wireless MAC protocol with client-server traffic," in *Proc. IEEE INFOCOM'93* San Francisco, CA, Mar. 1993, IEEE Computer Society Press.
[10] L. Lovasz, M. Grotschel, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization.* Berlin: Springer-Verlag, 1988.
[11] M. W. Padberg, "On the facial structure of set packing polyhedra," *Math. Programming*, vol. 5, pp. 199–215, 1973.
[12] P. Raymond, "Performance analysis of cellular networks," *IEEE Trans. Commun.*, vol. 39, pp. 1787–1793, Dec. 1991.
[13] M. Zhang and T.-S. P. Yum, "Comparisons of channel assignment strategies in cellular mobile telephone systems," *IEEE Trans. Veh. Technol.*, Nov. 1989.



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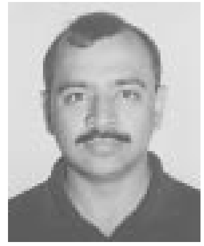

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