CHAPTER 14

WILL ELEMENTARY ALGEBRA DISAPPEAR WITH THE USE OF NEW GRAPHING CALCULATORS?

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We begin with an historical look at the development of algebra by Diophantus, Al-Khowarizmi, Lucas Pacioli, Tartaglia, Descartes, etc. Then the relationship is established between the algebraic content and procedures necessary for students in secondary schools, which today can be solved very easily with a graphing calculator. A few examples will be given with the TI-92. Finally, we will insist that even if algebraic procedures will be obsolete in the near future, what will not become obsolete are the algebraic thinking strategies and the reasoning processes that permit us to model as equations situations that are given as verbal descriptions. These not only will not lose their importance with the appearance of graphing calculators; they should, much to the contrary, be the main objective of secondary teaching. Therefore, it appears that a profound revision of the algebra curriculum must be completed in order to adapt it to the future.

A LOOK TO THE PAST

What do we understand elementary algebra to be? Elementary algebra is the language with which we communicate the majority of mathematics. Thanks to algebra we can work with concepts at an abstract level and then apply them.

Elementary algebra begins as a generalization of arithmetic and then focuses on its own structure and greater logical coherence. From there comes the importance of the various uses of algebraic symbols. When we write A + B, we can be indicating the sum of two natural numbers, the sum of two algebraic expressions, or even the sum of two matrices. Thus there is, at first, representations and symbolism, and later the development of algorithms and procedures to work formally with algebraic expressions. But what we today understand to be algebra has been the fruit of the efforts of many generations that have been contributing their grains of sand in constructing this magnificent building.

It seems that the Egyptians already knew methods for solving first degree equations. In the *Papyrus of Ahmes* (1650 BC) one finds: "Calculate

1. Translated from the Spanish original by Rick Scott.

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the value of an amount if that amount and a seventh part of that amount equals 19". But to solve such problems, they used arithmetic methods, such as, for example, the rule of false position. The Babylonians solved some simple systems of linear equations. Also, according to a discovery by Neugebauer in 1930, the Babylonians handled quadratic equations with great skill.

By the 6th Century BC the deductive method appeared in Greek mathematics, and by the 4th Century BC what can be called *geometric algebra* with such exercises as: "Given the sum and the product of the sides of a rectangle, find those sides". Some of that geometric algebra was treated by Euclid in his *Elements*.

But the most important of the Greek algebraists was Diophantus of Alexandria. Little is known of his life, even though on his tomb there is an inscription that translates to a linear equation that gives us some information about him. Diophantus can be considered to be the father of ancient algebra. His most important work was *Arithmetic*, but of its thirteen books only the first six have survived. It is not an algebra text, but a collection of problems about the application of algebra. The influence of Diophantus has been greater that is often believed. Even Pierre de Fermat (1601-1665) arrived at his famous last theorem when he tried to generalize a problem that he had seen in Diophantus' Arithmetic: "decompose a square given the sum of two other square".

The Indian mathematicians Brahmagupta (598-?) and Bhaskara (1114-1185) contributed general solutions of quadratic equations to the development of algebra, including those of two roots with one of them negative. One of the most distinguished members of the House of Wisdom in Baghdad was the mathematician and astronomer, Al-Khowarizmi (around 780-850). In his work, entitled *Algebra*, he studied in detail the six types of linear and quadratic equations that have positive roots. His way of solving them was essentially geometric, connected thus to the Greek algebra of Euclid.

But classic algebra, as we know it today, was developed more completely during the Renaissance, which is when the first break between ancient and classic algebra occurred. In 1494, the Italian Lucas Pacioli (1445-1514) published *Summa Arithmetica* in which he included the solution of first and second degree equations. In that work, a rudimentary symbolic algebra began to be used. Pacioli was very pessimistic about the possibility of solving cubic equations, and thought that cubic, not to mention quartic, equations were beyond the reach of algebra. Scipione Ferro (1465-1526) accepted the challenge of Pacioli and arrived at a formula for cubic equations of the form $ax^3 + cx + d = 0$. He revealed his discovery to his disciple, Antonio de Fiore (1506-?). Niccolo Fontana (Tartaglia) (1500-1557) claimed to be able to solve cubic equations of the form $x^3 + mx^2 = n$. However, he apparently did not know how to solve those of the form

$ax^3 + cx + d = 0$. Challenged by Fiore, he managed to find solutions to these equations. Ludovico Ferrari (1522-1565) was able to find a procedure for solving fourth degree algebraic equations.

Girolamo Cardan (1501-1576) published *Ars Magna* in which he included the method for solving some kinds of cubic equations that Tartaglia had shown him, even though he promised not to make them public. It is today known as "solution by radicals". But if negative numbers were considered to be suspect in the Sixteenth Century, their square roots were deemed to be totally absurd. It was, however, the beginning of the appearance of what are now called complex numbers.

In 1572 Rafael Bombelli (1526-1573) published his treatise, *Algebra*, in which he gave one more step in the solution of cubic equations, expressing solutions in a form such as $2 + \sqrt{-1}$. We should thank Bombelli for discovering that imaginary numbers play an important role in the development of algebra.

The French mathematician François Vieta (1540-1603) proposed a new focus for the solution of cubic equations. He also began the study of the relationship between the roots and the coefficients of an equation. That work was completed by the Flemish mathematician Albert Girard (1590-1633) with the publication of *Invention nouvelle en l'algèbre* in 1629. Rene Descartes (1596-1650) in his *La Géométrie: Livre Premier* included a detailed system of instructions for solving quadratic equations, but with geometric methods like those used by the ancient Greeks.

Neither Cardan nor Ferrari were able to find solutions to fifth degree equations. It was not until 1824 that the young Norwegian mathematician Neils Abel (1802-1829) shook the entire scientific community by proving that it was impossible to solve fifth or higher degree equations by radicals. This did not mean that fifth or higher degree equations did not have solutions, it just meant that there did not exist an algebraic method that permitted obtaining the roots of an equation as a function of its coefficients. The discovery of Abel links to the pessimism of Lucas Pacioli and shows the failure of algebra when you go beyond the fourth degree.

Girard, facing the problem of taking square roots of negative numbers, was, in 1629, the first to dare to conjecture the following: "An equation of degree n has exactly n roots, as long as you count the impossible ones". Later, Euler (1707-1783), Lagrange (1736-1813) and D'Alembert (1717-1783), studied the same theorem, but without thinking that the roots could be complex numbers. It is to Gauss (1777-1855) that we owe the name of this fact, *The Fundamental Theorem of Algebra*. In his doctoral dissertation, finished in 1799, he critiqued the work of Euler, Lagrange and D'Alembert, and gave a proof of the theorem, based on geometric considerations, which was not completely convincing. In 1816 he published two new proofs and five years before his death he published the fourth proof attempting to find purely algebraic procedures.

At the beginning of the Seventeenth Century, the English mathematician Roger Cotes (1682-1716) and the French immigrant to England, Abraham de Moivre (1667-1754), reduced the solution of the equation $z^n - 1 = 0$, to the division of the circumference of a circle in n equal parts, thus showing a good understanding of complex numbers. But algebra continued to develop and in the Eighteenth and Nineteenth Centuries another rupture occurred, this time between classic algebra and modern algebra. This was due to the contributions, among others, of George Peacock (1791-1858). He initiated axiomatic thinking, that later would be developed by Augustus de Morgan (1806-1871), Sir William Rowan Hamilton (1805-1865), and George Boole (1815-1864). We should not forget the two most prolific algebraists of the time, the Englishmen Arthur Cayley (1821-1895) with his study of matrices and J.J. Sylvester (1814-1897) with the theory of invariants. Finally, Evariste Galois (1811-1832) with the creation group made that abstract concept the central idea of the theory of algebraic equations, signaling an end to the era of classic algebra.

ALGEBRAIC CONTENT NECESSARY FOR PRE–UNIVERSITY STUDENTS

In Spain, pre-university students receive Obligatory Secondary Education from approximately age 12 to 16, and high school from age 16 to 18. Therefore, in general, pre-university teaching is directed to students who are under 19 years old.

What is the algebraic content necessary for those students?

Taking into consideration the current curriculum, and without trying to be exhaustive, the following algebra topics are being studied by students of that age:

- Polynomials: operations
- Algebraic expressions: operations
- Solution of first degree equations
- Study of first degree inequalities
- Solution of second degree equations
- Study of second degree inequalities
- Solution of cubic equations with integer roots
- Solution of biquadratic equations
- Solution of algebraic equations of degree n with n-2 integer roots

- Solution of irrational equations
- Solution of transcendental equations (logarithmic, exponential, trigonometric, etc.)
- Matrices: operations
- Determinant of a square matrix. Inverse matrix.
- Systems of linear equations: Methods of Gauss, Cramer, Rouche

APPLICATIONS OF ALGEBRAIC SYMBOLIC MANIPULATION WITH THE TI-92

With the appearance of graphing calculators, and their ability to construct tables, it is possible to solve any kind of equation, whether it be algebraic or transcendental, by numeric or graphical methods. More recently, with the arrival of the TI-92, which has an incorporated symbolic computer algebra system, it is possible to do any kind of algebraic manipulation. Thus for example:

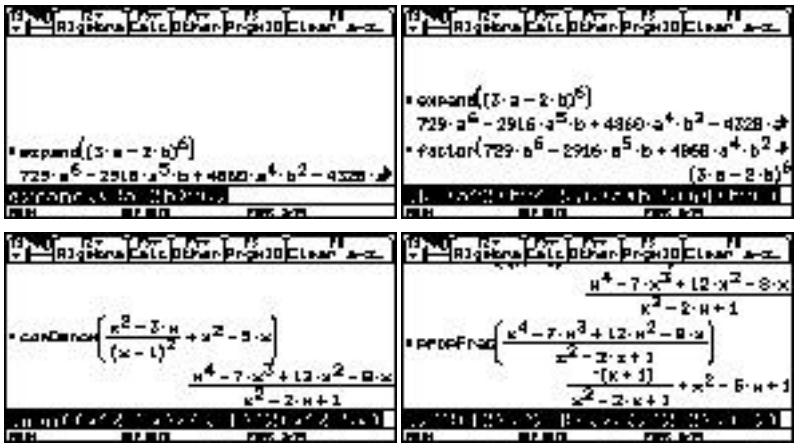

Figure 1.

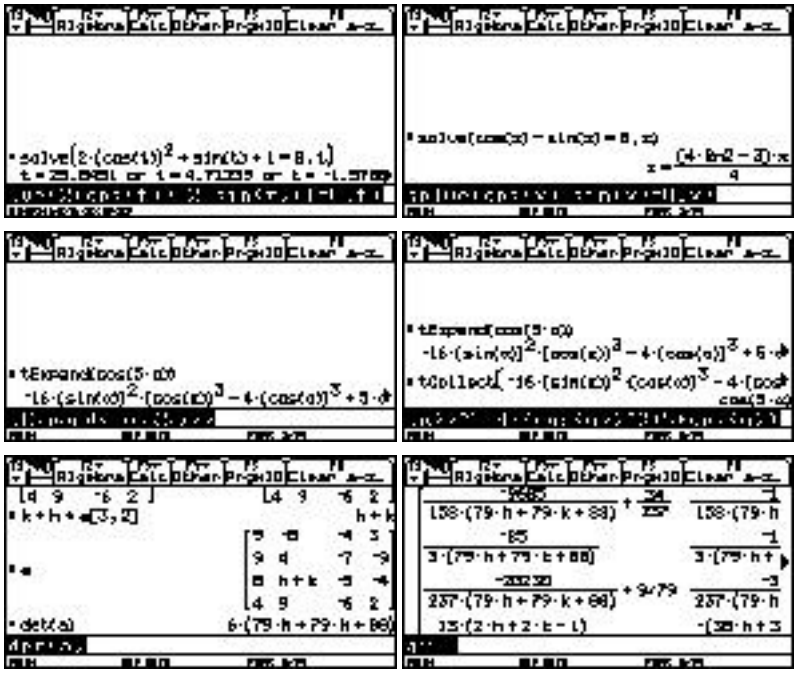

Figure 1.

So we can conclude that the algebraic procedures with the symbolic computer algebra system do not present any difficulties.

A LOOK TO THE FUTURE

From what we have seen up until now it can be deduced that the effort realized by humanity to find algorithms and procedures that permit the development of algebraic skills such as solving equations have been laudable, and important during their time, but today they are totally obsolete. Today, what sense does it make to solve a cubic equation by radicals, when we have just seen it done so simply? Keeping in mind that calculators can solve equations so easily, does it mean that students at this level will not have to study algebra?

Obviously our students should continue to study algebra, but the algebra curriculum that can be proposed for the future does not have to place its emphasis on manipulative agility. Instead, it can give more importance to the very conceptual structure of algebra as a means of representation and to algebraic methods as tools for solving problems. But this should lead to a change in the amount of teaching time that is given to each of the topics.

Up until recently each time we taught a topic, for example matrix operations, a lot of time was spent calculating expressions with matrix operations. But these kind of calculations are difficult and laborious. As a result, when we are ready to begin studying the applications of matrix calculations, which is what is really important, there is no time left. And, on the other hand, it is humanly impossible to calculate the tenth power of matrix with paper and pencil in order to study Markov's problem.

However, in the future we propose to dedicate more time to the concepts and the structure of algebraic thought, and less time to algebraic manipulation, such as in the solution of equations, given that they can now be solved in an instant and always with the same procedure. As part of this change we can dedicate much more time to applying algebra to various situations, for which we can use the existing technology.

Let's simplify things a bit and show just one example. Let's suppose that 14 year old students are being taught the solution of linear equations for the first time. Traditionally this takes a lot of time, approximately four weeks solving equations that are presented out of context, making it repetitive and boring. Then, usually very little time, often only a week, is dedicated to solving first degree equations that come from more interesting word problems. These problems are obviously much more motivating than simply using a series of rules and recipes that the students must repeat in great detail, but often without really knowing what they are doing, nor why they are doing it.

Given this situation, we propose a substantial change in which we begin with concrete situations that are sufficiently motivating that they justify the need for algebraic language so we can, using symbolism and representations, write the corresponding linear equation. Later, and then only, with sufficiently simple equations, should we try to use paper and pencil to solve them with traditional algebraic procedures. But this manipulative stage for algebraic procedures can be shortened drastically, taking into account that the students can use appropriate technology that permits them to solve equations very easily, by analytic, as well as by numerical and graphical, methods. And therefore, it appears to be much more interesting to dedicate more time to setting up equations that are deduced from reading various applied problem situations from different aspects of daily life that are much closer to students' experience, than to solve those equations by hand, which can be done so much easier with a calculator.

CONCLUSIONS

Obviously the main objective in teaching secondary mathematics in all countries is to prepare students to confront the challenges that life can present them in the future. So how can we best help our students: by teach-

ing them to apply, like robots, rules and recipes to do things that can be done easily with technology, or by teaching them to think, something that neither computers nor calculators can yet do? Even considering future university students, those who obviously should make a more frequent use of algebraic skills, an important objective will have to be getting an adequate level of those kinds of skills and procedures. Nevertheless, those students also are going to find technology present and there is no need to mention that they will be required to have different kinds of skills.

In summary, and by way of example, it is necessary to put more emphasis in setting up equations as a way of developing structures of algebraic thought than in the mere solution of the equation, which today is not so important. In fact, we should keep in mind that there still is no calculator or computer that can read a problem and automatically set up the equation. However, today there are various computer programs and calculators where you can enter an equation and just by tapping the Enter key you obtain the solution, in just a few seconds. By way of example, let's take another look at some famous classic problems:

The life of Diophantus

> This tomb holds Diophantus. Ah, what a marvel.
>
> And the tomb tells scientifically the measure of his life.
>
> God vouchsafed that he should be a boy for the sixth part of his life;
>
> when a twelfth was added, his cheeks acquired a beard;
>
> He kindled for him the light of marriage after a seventh,
>
> and in the fifth year after his marriage He granted him a son.
>
> Alas! late-begotten and miserable child, when he reached the measure of half his father's life, the chill grave took him.
>
> After consoling his grief by this science of numbers for four years, he reached the end of his life.

From the Lilavati of Brahmin Bahskara (1114-1185), Indian Mathematician

In a swarm of bees, 1/5 of the bees are going toward the lotus blossom, and 1/34 toward a banana. A number equal to three times the difference between the two preceding numbers (Oh my beauty with eyes of a gazelle) flew toward the Codaga tree (with bitter bark like quinine). Another one, the last, flew back and forth, aided by the breezes, attracted for a time by the delicious perfume of the jasmine and pandanus. Tell me, my beloved, how many bees were there?

A problem from Euler (1707-1783)

A father left an inheritance of 8600 pounds sterling to his four sons. According to the will, the oldest should get 100 pounds less than twice the second son. That second share, should be 200 pounds less than triple the third son. And the third share should be 300 pounds less than quadruple to share of the youngest. What share should each receive?

The series of Rachinski (XIX Century)

Find a series of five consecutive integers such that the sum of the squares of the first three is equal to the sum of the squares of the other two.

Where is the difficulty: in coming up with the equations, or, once you have them, solving them?

REFERENCES

Boyer, C. B. (1987). *Historia de la Matemática*. Madrid: Alianza Universidad Textos.

Diez Barrabes, M. (1994). Dificultades en el uso y significación de las letras en álgebra. Propuesta de una nueva ingeniería didáctica. *Comunicación presentada al II CIBEM*. Blumenau. Brasil.

Dunham, W. (1992). *Viaje a través de los genios*. Madrid: Editorial Pirámide.

NCTM (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston: NCTM.

Paradis, J., et al. (1989). *El álgebra en el período renacentista. La recuperación de los clásicos griegos*. Barcelona: PPU.

Perelman, Y. (1965). *Algebra recreativa*. Moscú: Editorial Mir.

Radice, L (1983). *La matemática de Pitágoras a Newton*. Barcelona: Laia.

Wussing, H., Arnold, W. (1989). *Biografías de grandes matemáticos.* Zaragoza: U de Zaragoza.

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