Discrete- and Continuous-Time Local Cosine Bases with Multiple Overlapping

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***Abstract*— Cosine-modulated filter banks (CMFB's) are filter banks whose impulse responses are obtained by modulating a window with cosines. Among their applications are video and audio compression and multitone modulation. Their continuous-time counterpart is known as local cosine bases. Even though there is an extended literature on the discrete-time case both for single and multiple overlapping, the continuous-time case has received less attention, and only the single overlapping case has been solved. This work gives a solution to the problem of continuous-time local cosine bases with multiple overlapping via a general theory that emphasizes the deep connection between discrete and continuous time. A sampling theorem for local cosine basis and an efficient algorithm to compute the expansion of a signal are also given.**

***Index Terms*—Lapped orthogonal transform, local cosine bases, smooth localized trigonometric transforms.**

I. Introduction

Cosine-modulated filter banks (CMFB) are filter banks whose impulse responses are obtained by modulating a window with harmonic trigonometric functions [1]–[3]. Among their advantages are easy design and fast computation with an FFT-like algorithm. The fact that they can be interpreted as a "smooth DCT" make them interesting for compression purposes [4], [5]. Recently, they have also found application in multitone modulation systems [6].

In discrete time, the first perfect reconstruction (PR) version of the CMFB has been introduced by Princen and Bradley [7]. In such a construction, the filter length L is twice the sampling period M, giving rise to *single overlapping* CMFB. The first results on the multiple overlapping case, more precisely for $L = 2kM$, are due to Malvar [8] and Koilpillai and Vaidyanathan [9]. In [10], Poize *et al.* show that it is not necessary to use cosines as modulating functions, as long as the modulating functions enjoy some type of symmetry and periodicity. All the cited works use an algebraic approach, relying on popular signal processing tools like the z-transform and polyphase components [2].

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The continuous-time case has received less attention in the signal processing literature. The continuous-time counterpart of the CMFB is known as local cosine bases (LCB), and it has been introduced by Coifman and Meyer [11]. Such a device has been used by Auscher *et al.* [12] to construct the Lemarié and Meyer wavelet [13]. Recently, Matviyenko [14] introduced biorthogonal LCB, showing that the dual is still an LCB but with a different window. All the cited works consider only the single overlapping case. The only result known to the authors for multiple overlapping in continuous time is due to Malvar, which, in [15], shows that by modulating a raised cosine, we get an orthonormal basis for $L^2(\mathbb{R})$.

Bernardini and Kovačević in [16] explore both continuous and discrete time. Inspired by [12], they approach the problem with a vector space point of view, interpreting PR as a decomposition of $\ell^2(\mathbb{Z})$ [or $L^2(\mathbb{R})$] into a direct sum of subspaces of compactly supported signals. The theory presented in [16] works both in continuous and discrete time and, like [10], relies only on symmetries, but it is usable only in the single overlapping case.

The goal of this paper is twofold: A first immediate one is to give a solution to the problem of continuous-time LCB with multiple overlapping; a second result is to present a general theory of LCB that emphasizes the deep connection between discrete and continuous time. The approach is similar to the one used in [16]; we will study LCB via the orthogonality of some subspaces of $L^2(\mathbb{R})$ or $\ell^2(\mathbb{Z})$. The theory relies on the idea of *folding operator*[1] that has an intuitive interpretation. Using this concept, we can deduce the constraints that a window must satisfy in order to have PR. The idea of folding operator can be readily extended to the discrete-time case by simple "sampling." For reasons of space, we will develop in detail only the continuous-time case by simply pointing out how the theory should be modified in discrete time.

The outline is as follows. In Section II, we present the notation and give the problem statement. In Section III, we introduce the framework that will be used in this paper. In Section IV, we revisit the continuous-time, single overlapping case using the techniques introduced in Section III. In Section V, we attack the case of multiple overlapping. In Section VI, we discuss the main differences between continuous and discrete time and present a sampling theorem and a Mallat-like algorithm for LCB. In Section VII, we show how to design a continuous-time window with arbitrary smoothness. Section VIII gives the conclusions.

[1] Matviyenko [14] also introduces a folding operator, but it is slightly different from ours.

II. Notation and Problem Statement

A. Notation

The scalar product between two vectors f, g of vector space V will be denoted as $\langle f, g\rangle_V$ or $\langle f, g\rangle$ when no confusion about the vector space can arise. For complex signals, we will suppose the scalar product linear with respect to the second argument, that is, $\langle f, \alpha g\rangle = \alpha\langle f, g\rangle$ for every $\alpha \in \mathbb{C}$. With notation $f(\cdot - j)$, we will denote the signal obtained by translating f of j. Such a notation is convenient in expressions like $\langle f(\cdot - j)w, g\rangle$, where $f(t-j)$ could be interpreted as the scalar value assumed by f in $t-j$. Continuous and discrete-time signals will be differentiated by writing their argument between parenthesis or square brackets, respectively (e.g., $f(x)$ or $w[n]$).

B. Problem Statement

In continuous time, a (time-invariant) local cosine basis (LCB) is made of functions

$$g_{j,k}(t) \triangleq \sqrt{2}w(t-j)\cos_k(t-j), \qquad j \in \mathbb{Z}, k \in \mathbb{N} \tag{1}$$

where $w(t)$ is a function having as support an interval $[a, b]$, and $\cos_k(t)$ is defined as

$$\cos_k(t) \triangleq \cos[\pi(k+\tfrac{1}{2})(t-\tfrac{1}{2})]. \tag{2}$$

We assumed, without loss of generality, an elementary shift step of 1. Other steps can be obtained by scaling.

In a time-invariant discrete-time cosine-modulated filter bank (CMFB), the generic basis function has the form

$$g_{j,k}[n] \triangleq \sqrt{2}w[n-Nj]\cos_k(n/N - j + \phi_N), \quad j, n \in \mathbb{Z}, \quad k = 0, 1, \cdots, N-1 \tag{3}$$

with $\phi_N \triangleq 1/2 + 1/2N$. Note that in the discrete-time case, we cannot normalize the elementary step N by scaling.

If the window length $b-a$ is less or equal to twice the elementary step, the support of $w(t)$ (or $w[n]$) overlaps only the support of adjacent windows. This is the *single overlapping case*. If $b-a$ is greater than twice the elementary step, the support of $w(t)$ also intersects the support of nonadjacent windows, and we have the *multiple overlapping case*.

The main objective in our study of continuous-time LCB is to find conditions on $w(t)$ that lead to functions in (1) to form an orthonormal basis for $L^2(\mathbb{R})$. Similarly, in the study of CMFB, we search for conditions on $w[n]$ such that the functions in (3) form an orthonormal basis of $\ell^2(\mathbb{Z})$.

III. The Framework

A. Vector Spaces Characterization: Continuous Time

As a first step, it is instrumental to "collect" together the functions $g_{j,k}(t)$, $k = 0, 1, \cdots$ relative to the same translation j. Let V_j be the subspace of $L^2(\mathbb{R})$ generated by their linear combinations, that is

$$V_j \triangleq \text{span}\{g_{j,k}\}_{k\in\mathbb{N}}. \tag{4}$$

With this definition of V_j, orthonormality of functions $g_{j,k}$ can be split into two types of orthogonality:

- an "external" orthogonality between vector spaces V_j and V_m, $j \neq m$;
- an "internal" orthogonality between two functions $g_{j,k}$ and $g_{j,\ell}$, $k \neq \ell$ belonging to the same vector space V_j.

Such a separation will make the study of LCB easier. Completeness is also split in two parts:

- "external" completeness: $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} V_j$;
- "internal" completeness: functions $\{g_{j,k}\}_{k\in\mathbb{N}}$ form a basis for V_j.

Actually, we just need to check the external completeness since the internal one is automatically granted by definition (4).

As a second step, let us give a characterization of the functions belonging to V_j. Note that vector space V_j is just a translated version of V_0; more precisely, $f(t) \in V_0 \Leftrightarrow f(t-j) \in V_j$. Because of this, we can limit ourselves to the study of V_0.

If a function $f \in V_0$, then there exists a real, square summable sequence α_k, $k \in \mathbb{N}$ such that

$$f(t) = \sum_{k\in\mathbb{N}} \alpha_k g_{0,k}(t) = \sum_{k\in\mathbb{N}} \alpha_k w(t)\cos_k(t) = w(t)\sum_{k\in\mathbb{N}} \alpha_k \cos_k(t) = w(t)s(t) \tag{5}$$

that is, if $f(t) \in V_0$, then f can be written as the product of the window $w(t)$ with a function $s(t) \triangleq \Sigma_{k\in\mathbb{N}} \alpha_k \cos_k(t)$ belonging to the space $\mathcal{C}_0 \triangleq \text{span}\{\cos_k\}_{k\in\mathbb{N}}$. The translated version of $\mathcal{C}_0$ will be called $\mathcal{C}_j$. Functions in $\mathcal{C}_0$ are not in $L^2(\mathbb{R})$; however, it is easy to show that space $\mathcal{C}_0$ is a Hilbert space when endowed with the scalar product

$$\langle s_1, s_2\rangle_{\mathcal{C}_0} \triangleq \int_{-1/2}^{1/2} s_1(t)s_2(t)\,dt. \tag{6}$$

The reason for limiting the integral in (6) between −1/2 and 1/2 stems from the fact that every $\cos_k(t)$ is symmetric around −1/2, antisymmetric around 1/2, and *skew periodic* with period 2, that is

$$\cos_k(-1-t) = \cos_k(t) \quad \cos_k(1-t) = -\cos_k(t)$$
$$\cos_k(t+2) = -\cos_k(t). \tag{7}$$

Because of the symmetries in (7), every function belonging to $\mathcal{C}_0$ is uniquely determined by the values assumed on [−1/2, 1/2]. Definition (6) follows from such a fact. We can characterize $\mathcal{C}_0$ as follows.

Property 1: The vector space $\mathcal{C}_0$ is the space of the functions that are square summable on [−1/2, 1/2] and enjoy symmetries (7).

Proof: We give just a sketch of the proof. By definition, every function $s(t) \in \mathcal{C}_0$ can be written as $\Sigma_k \alpha_k \cos_k(t)$ with α_k, $k = 0, 1, \cdots$ a square summable sequence; therefore, $s(t)$ is square summable on [−1/2, 1/2]. Moreover, $s(t)$ inherits symmetries (7) from cosines.

Now, let $f(t)$ be a function square summable on [−1/2,1/2] and enjoying symmetries (7). Since $f(t) = f(t+4)$, $f(t)$ can be expressed with a Fourier series. Skew periodicity

$f(t) = -f(t+2)$ implies that only odd harmonics are used, and symmetry around −1/2 implies that the series contains only cosines.

Since $V_0 = w\mathcal{C}_0$, that is, V_0 is a "windowed" version of $\mathcal{C}_0$, we obtain the following characterization of V_0.

Corollary 1: The vector space V_0 is the space of the functions that can be written as $f(t) = w(t)s(t)$ with $s(t)$ square summable on [−1/2, 1/2] and enjoying symmetries (7).

We will need some other vector spaces similar to $\mathcal{C}_0$ but differing in the symmetry signs. More exactly, we will consider vector spaces

$$\mathcal{C}_0^- \triangleq \{s(t) \in L^2([-1/2, 1/2]): s(-1-t) = -s(t), s(1-t) = +s(t)\} \tag{8a}$$

$$\mathcal{S}_0 \triangleq \{s(t) \in L^1([-1/2, 1/2]): s(-1-t) = +s(t), s(1-t) = +s(t)\} \tag{8b}$$

$$\mathcal{S}_0^- \triangleq \{s(t) \in L^1([-1/2, 1/2]): s(-1-t) = -s(t), s(1-t) = -s(t)\}. \tag{8c}$$

It is possible to prove that functions in $\mathcal{C}_0^-$ are skew periodic with period 2, whereas functions in $\mathcal{S}_0$ and $\mathcal{S}_0^-$ are periodic with period 2.

We report here some useful properties enjoyed by vector spaces $\mathcal{C}_0, \mathcal{C}_0^-, \mathcal{S}_0$, and $\mathcal{S}_0^-$. The proofs can be found in Proofs A.1–A.3 of Appendix A.

Property 2: For every $j \in \mathbb{Z}$, $\mathcal{C}_{2j} = \mathcal{C}_0$, and $\mathcal{C}_{2j+1} = \mathcal{C}_0^-$.

Property 3: Every function $s(t) \in \mathcal{C}_0$ is antisymmetric around $2\ell + 1/2$ and symmetric around $(2\ell - 1) + 1/2, \ell \in \mathbb{Z}$. Dually, every function $s(t) \in \mathcal{C}_0^-$ is symmetric around $2\ell + 1/2$ and antisymmetric around $(2\ell - 1) + 1/2$, $\ell \in \mathbb{Z}$.

Property 4: If $s_1, s_2 \in \mathcal{C}_0$ or $s_1, s_2 \in \mathcal{C}_0^-$, then $s_1 s_2 \in \mathcal{S}_0$; if $s_1 \in \mathcal{C}_0$, $s_2 \in \mathcal{C}_0^-$, then $s_1 s_2 \in \mathcal{S}_0^-$.

B. Vector Spaces Characterization: Discrete Time

The theory presented in this work does not use any particular characteristic of continuous time, and everything could be repeated also in discrete time, with just a change of language. For sake of convenience, let us just summarize the characterization of V_0 in discrete time because we will need it in Section VI-B.

Property 5: The vector space V_0 contains the functions $f[n]$ that can be written as $f[n] = w[n]s[n]$, with $s[n]$ satisfying the symmetries

$$s[N-1-n] = -s[n] \qquad s[-N-1-n] = s[n]. \tag{9}$$

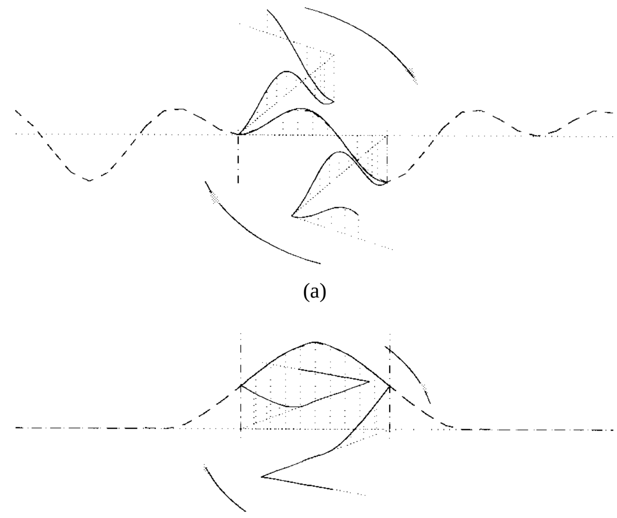
C. The Folding Operators

The following operators acting on compactly supported functions of $L^2(\mathbb{R})$ will prove useful.

$$\mathcal{Q}_0^+ f(t) \triangleq \sum_{\ell \in \mathbb{Z}} f(t+2\ell) + f(1-t+2\ell)$$

$$\mathcal{Q}_0^- f(t) \triangleq \sum_{\ell \in \mathbb{Z}} f(t+2\ell) - f(1-t+2\ell)$$

$$\mathcal{Q}_0^\pm f(t) \triangleq \sum_{\ell \in \mathbb{Z}} (-1)^\ell [f(t+2\ell) - f(1-t+2\ell)]. \tag{10}$$



(a)

(b)

Fig. 1. Action of the folding operator when computing $\int_{\mathbb{R}} r(t)s(t)\,dt$, with $s(t) \in \mathcal{S}_0$. (a) Folding of function $s(t)$ around the symmetry point $1/2 + \mathbb{Z}$ causes the overlay of the single pieces of the function. (b) Folding on $r(t)$ induced by the folding in (a).

Operators (10) will be called *folding operators*. The one with the most intuitive action is $\mathcal{Q}_0^+$, whose action can be described as "folding" $f(t)$ around the symmetry points $1/2 + \ell, \ell \in \mathbb{Z}$ (see Fig. 1). Operators $\mathcal{Q}_0^-$ and $\mathcal{Q}_0^\pm$ act like $\mathcal{Q}_0^+$, but they weight each term in the sum differently. It is easy to prove that the result of operators (10) belongs to $\mathcal{S}_0$, $\mathcal{S}_0^-$, and $\mathcal{C}_0$, respectively. Note that the weights that each operator (10) assigns to the symmetry points $1/2 + \ell$ match with the weights of the corresponding vector space. The folding operators (10) will be used to simplify scalar products, according to the following property.

Property 6: Let $s(t) \in \mathcal{S}_0$, $s_-(t) \in \mathcal{S}_0^-$, and $s_\pm(t) \in \mathcal{C}_0$, and let $r(t)$ be a function with compact support; then

$$\int_{\mathbb{R}} r(t)s(t)\,dt = \int_{-1/2}^{1/2} \mathcal{Q}_0^+ r(t)s(t)\,dt \tag{11a}$$

$$\int_{\mathbb{R}} r(t)s_-(t)\,dt = \int_{-1/2}^{1/2} \mathcal{Q}_0^- r(t)s_-(t)\,dt \tag{11b}$$

$$\int_{\mathbb{R}} r(t)s_\pm(t)\,dt = \int_{-1/2}^{1/2} \mathcal{Q}_0^\pm r(t)s_\pm(t)\,dt. \tag{11c}$$

Proof: We just give a graphical sketch of the proof in Fig. 2. A more formal proof can be easily obtained by using Fig. 2 as a guide. Fig. 2 shows that one can "fold" the left-hand integral of (11) around the symmetry point 1/2 without changing the value. Such a folding can be repeated for every symmetry point of $s(t)$ to obtain the right side of (11). □

D. Internal Orthogonality

Let $z(t)$ be the indicator function of the interval [−1/2, 1/2]. We will prove internal orthogonality via the following lemma.

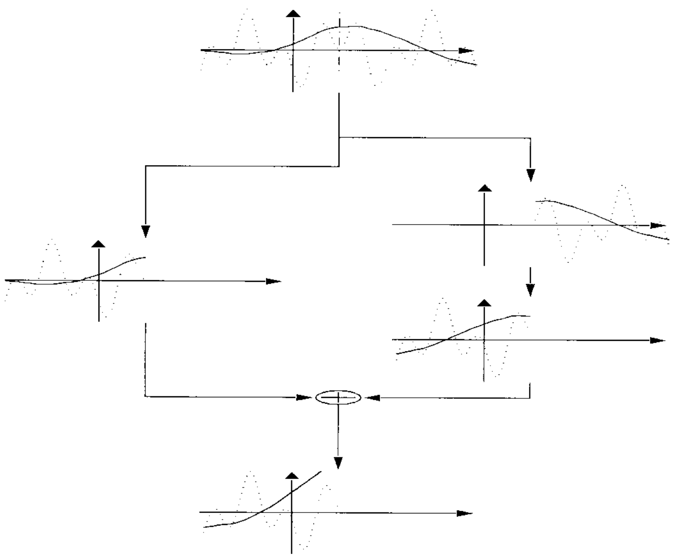


Fig. 2. Proof of folding operator properties. Every plot symbolizes a scalar product in $L^2(\mathbb{R})$ and should be read as "the integral of the product of the two plotted functions." 1) Continuous line shows a compact support function $r(t)$; dashed line shows a function $s(t)$ of $\mathcal{S}_0$; dash-and-dot line shows the position of 1/2. 2) and 3) The integral on $\mathbb{R}$ is split into two integrals for $x < 1/2$ and $x > 1/2$, respectively. 4) The integral for $x > 1/2$ is flipped around 1/2. This does not change the value of the integral. 5) The integrals relative to 2) and 4) have the same support, and they can be summed together. Since the dashed plots are equal, the resulting plot corresponds to $\int_{-\infty}^{1/2} (r_1(t)+r_2(t))s_1(t)\,dt$, that is, the two continuous lines are added, whereas the dashed line remains the same.

Lemma 1: If the window $w(t)$ verifies the *power-complementarity* conditions

$$\mathcal{Q}_0^+(w^2) = \sum_{\ell\in\mathbb{Z}} w^2(t-2\ell) + w^2(1-t-2\ell) = 1, \quad \forall t \in [-1/2, 1/2] \tag{12}$$

then linear mapping

$$\eta\colon w(t)s(t) \to z(t)s(t). \tag{13}$$

maps V_0 into $L^2([-1/2, 1/2])$ preserving the scalar product, that is

$$\langle w(t)s_0(t), w(t)s_1(t)\rangle = \int_{-1/2}^{1/2} s_0(t)s_1(t)\,dt, \quad \forall s_0, s_1 \in \mathcal{C}_0. \tag{14}$$

The action of η can be graphically described as in Fig. 3. Function $w(t)s(t)$ [Fig. 3(a)] is "unwindowed," and the resulting function $s(t) \in \mathcal{C}_0$ [Fig. 3(b)] is forced to zero outside the interval [−1/2, 1/2] [Fig. 3(c)]. It is clear from the same figure that η can be inverted by extending by symmetry the function in Fig. 3(c) to $\mathbb{R}$ and multiplying the result by $w(t)$.

Since the functions $\{\cos_k(t)\}_{k\in\mathbb{N}}$ form an orthonormal basis of $\mathcal{C}_0$, internal orthogonality follows from (14) and (6) with $s_0(t) = \cos_k(t), s_1(t) = \cos_j(t)$.

Proof of Lemma 1: To prove that (14) follows from (12), write explicitly the scalar product $\langle w(t)s_0(t), w(t)s_1(t)\rangle$

$$\langle w(t)s_0(t), w(t)s_1(t)\rangle = \int_{\mathbb{R}} w^2(t)s_0(t)s_1(t)\,dt. \tag{15}$$

Since the functions $s_0, s_1 \in \mathcal{C}_0$, their product belongs to $\mathcal{S}_0$ (Property 4), and we can apply Property 6 to rewrite (15) as

$$\langle w(t)s_0(t), w(t)s_1(t)\rangle = \int_{-1/2}^{1/2} \mathcal{Q}_0^+ w^2(t)s_0(t)s_1(t)\,dt. \tag{16}$$

Expression (16) is equal to (14) for every s_0, s_1 if and only if (12) is true.

In the single overlapping case, since the window support is [−1, 1], (12) assumes the more usual form

$$\begin{aligned} w^2(t) + w^2(1-t) &= 1 \qquad 0 \le t \le 1/2 \\ w^2(t) + w^2(-1-t) &= 1 \qquad -1/2 \le t \le 0. \end{aligned} \tag{17}$$

E. Projection

We will search for an expression for the projection on V_0 that does not depend on the chosen basis. It is worth spending a few words to explain why this could be interesting. Let us start from a simpler case: a discrete-time modulated filter bank (cosine modulated or DFT filter bank). Call N the sampling interval and w the prototype filter, and let c_k, $k = 1, \cdots, N$ be the modulating functions (cosines or complex exponentials). The basis associated with such a filter bank is

$$g_{j,k}[n] = w[n - Nj]c_k[n - Nj] \quad j \in \mathbb{Z}, k = 1, 2, \cdots, N. \tag{18}$$

Note that in discrete time, we have just a finite number of modulating functions. From a linear space point of view, to compute the filter bank output at time j corresponds to compute the scalar products

$$\langle f, g_{j,k}\rangle = \langle f, w[\cdot - Nj]c_k[\cdot - Nj]\rangle, \quad k = 1, 2, \cdots, N. \tag{19}$$

For the sake of simplicity, in the following, we will concentrate on the case $j = 0$. With the usual scalar product of $\ell^2(\mathbb{Z})$, we can move the window w in (19) to the same side of f to obtain

$$\langle f, g_{0,k}\rangle = \langle wf, c_k\rangle, \qquad k = 1, 2, \cdots, N. \tag{20}$$

Equation (20) can be interpreted as saying that the filter bank output can be obtained by windowing the input signal f with the prototype w and by computing the scalar product of the result with the modulating functions.

If the window is rectangular (that is, the filter bank implements a DCT or a DFT), the product of the input signal with the window is the projection on V_0 and (20) can be interpreted as two-step procedure: First, f is projected on V_0, and then, the result is projected on basis vectors c_k. The first projection is an "external" projection, and the second one is an "internal" one. If the window is not rectangular, the product with w is not a projection since it is not idempotent. We will see that to obtain a projection one must take one more step (folding).

To find the projection on V_0, remember that because of internal orthogonality, the set $\{g_{0,k}\}$ forms an orthonormal

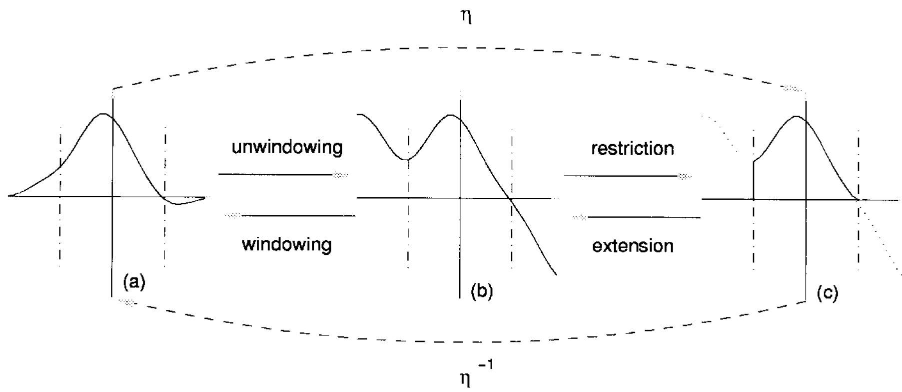


Fig. 3. Graphical description of the unitary transformation η between V_0 and $L^2([-1/2, 1/2])$. (a) Function $w(t)s(t)$ belonging to V_0. (b) Function $S(t)$. (c) Restriction of function $s(t)$ to $[-1/2,\ 1/2]$.

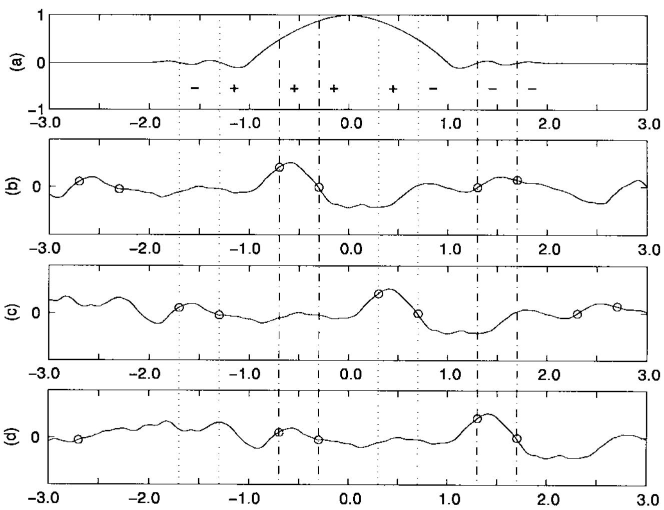


Fig. 4. Computation of the projection of $q_t[n]$ on V_0 for $n = 0, 1, 2$. (a) Window $w(t)$. (b) Signal $f(t)$. (c) and (d) Translated versions $f(t-1)$ and $f(t-2)$. To compute $q_t[n]$, we have to multiply the values of $f(t-n)$ marked with little circles by the corresponding window values, with a possible change of sign, according to the little signs reported next to the window.

basis of V_0. Therefore, the projection on V_0 can be written as

$$\begin{aligned} P_0 f(t) &= \sum_{k\in\mathbb{N}} g_{0,k}(t)\langle f, g_{0,k}\rangle \\ &= w(t)\sum_{k\in\mathbb{N}} \cos_k(t)\langle f, w\cos_k\rangle \\ &= w(t)\sum_{k\in\mathbb{N}} \cos_k(t)\int_{\mathbb{R}} f(x)w(x)\cos_k(x)\,dx. \end{aligned} \tag{21}$$

Since $\cos_k(t) \in \mathcal{C}_0$, we can apply Property 6 to rewrite (21) as

$$P_0 f(t) = w(t)\sum_{k\in\mathbb{N}} \cos_k(t)\int_{-1/2}^{1/2} \mathcal{Q}_0^{\pm}(wf)(x)\cos_k(x)\,dx. \tag{22}$$

Remember that $\mathcal{Q}_0^{\pm}(wf) \in \mathcal{C}_0$. Since cosines $\{\cos_k(t)\}_{k\in\mathbb{N}}$ are an orthonormal basis for $\mathcal{C}_0$, the sum in (22) is equal to $\mathcal{Q}_0^{\pm}(wf)$, and (22) can be rewritten as

$$P_0 f(t) = w(t)\mathcal{Q}_0^{\pm}(wf). \tag{23}$$

If our goal is to compute the scalar products $\langle f, g_{0,k}\rangle$, we can exploit the isomorphism η between V_0 and $L^2([-1/2, 1/2])$ by simply expressing $z(t)\mathcal{Q}_0^{\pm}(wf)$ as a linear combination of $\{\cos_k(t)\}$. In discrete time, this is just a DCT. This is how the fast algorithm for discrete-time CMFB's works. Indeed, such an algorithm can be described as follows.

- The input signal is multiplied by the window (N products, with N the window length) and folded ($N/2$ sums). This corresponds to the external projection.
- The DCT (for which fast algorithms of complexity $N \log N$ exist) of the resulting signal is computed. This corresponds to the internal projection.

F. Completeness

We will prove that completeness of the LCB follows from power complementarity. More formally, we have the following.

Property 7: If $w(t)$ satisfies the power complementarity conditions (12) and $f(t) \in L^2(\mathbb{R})$ is orthogonal to every V_j, $j \in \mathbb{Z}$, then $f(t) = 0$, that is

$$(\forall j \in \mathbb{Z}, f(t) \perp V_j) \Rightarrow f(t) = 0 \tag{24}$$

or, equivalently

$$(\forall j \in \mathbb{Z}, f(t-j) \perp V_0) \Rightarrow f(t) = 0. \tag{25}$$

The proof of Property 7 is reported in Sections IV-B and V-B, for the single and multiple overlapping case, respectively.

In the proofs, for technical convenience, $f(t)$ is supposed continuous. Since the subset of continuous functions is dense in $L^2(\mathbb{R})$, completeness for continuous functions implies completeness for $L^2(\mathbb{R})$. The proofs will use the auxiliary signal

$$q_t[2n+m] \triangleq \begin{cases} \mathcal{Q}_0^{\pm}[w(\cdot)f(\cdot - 2n)](t), & \text{if } m = 0 \\ \mathcal{Q}_0^{\pm}[w(\cdot)f(\cdot - (2n+1))](t+1), & \text{if } m = 1. \end{cases} \tag{26}$$

Since $w(t+m)q_t[2n+m]$ is the projection of $f(t-(2n+m))$ evaluated in $t+m$, signal f is orthogonal to every V_n if and only if $q_t[n] = 0$ for every $t \in [-1, 0], n \in \mathbb{Z}$. We will prove that in such a case, $f = 0$. To have an intuitive reason for the

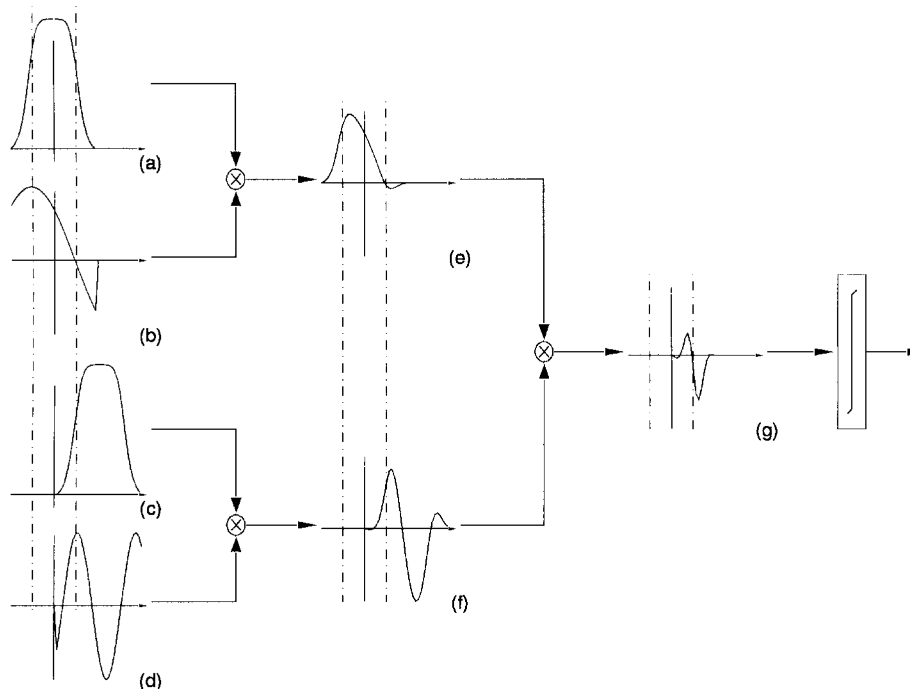


Fig. 5. Computation of the scalar product between a function of V_0 and a function of V_1. Symbol $\otimes$ denotes the product of two signals; symbol $\int$ inside a box means that the input signal is integrated over $\mathbb{R}$. Dash-and-dot lines denote the position of $-1/2$ and 1/2. (a) Window $w(t)$ relative to space V_0. (b) Function s_0 belonging to space $\mathcal{C}_0$. (c) Translated window $w(t-1)$. (d) Function $s_1(t)$ belonging to space $\mathcal{C}_1$. (e) Function $f_0(t) = w(t)s_0(t) \in V_0$. (f) Function $f_1(t) = w(t-1)s_1(t) \in V_1$. (g) Product $f_0(t)f_1(t)$.

choice of the signal (26), write explicitly the expression for $\mathcal{Q}_0^{\pm}$ in (26) to obtain, after some algebra

$$q_t[2n] = \sum_{\ell\in\mathbb{Z}} (-1)^\ell [f(t+2\ell-2n)w(t+2\ell) - f(1-t+2\ell-2n)w(1-t+2\ell)] \tag{27a}$$

$$q_t[2n+1] = \sum_{\ell\in\mathbb{Z}} (-1)^\ell [f(t+2\ell-2n)w(1+t+2\ell) + f(1-t+2\ell-2n)w(2-t+2\ell)]. \tag{27b}$$

Interpreting (27) with the help of Fig. 4, we can see that the computation of $q_t[n]$ always requires the same set of values of $\{f(t+2i), f(t+2i+1)\}_{i\in\mathbb{Z}}$ for every n. Such a fact will be exploited in Sections IV-B and V-B to write (27) as a PR filter bank and prove the completeness.

IV. Single Overlapping Revisited

In this section, we briefly revisit the single overlapping case [11] to show how the framework presented in Section III can be used.

A. External Orthogonality

We need to check $V_j \perp V_k$, $j \neq k$, $j, k \in \mathbb{Z}$. Because of the support restriction, only $V_j \perp V_{j+1}$ needs to be checked, which, by translation invariance, reduces to $V_0 \perp V_1$. We will prove that external orthogonality follows from window symmetry $w(t) = w(-t)$. A simple consequence of window symmetry that will be exploited in the following is that $w(t-1) = w(1-t)$, that is, translating $w(t)$ is equivalent to taking its symmetric around 1/2.

Fig. 5 shows the computation of the scalar product of a function $f_0(t) \in V_0$ with a function $f_1(t) \in V_1$. The proof of the orthogonality between V_0 and V_1 becomes immediate from symmetry considerations by rearranging Fig. 5 as in Fig. 6. It is worth summarizing how the scheme of Fig. 6 works because the same reasoning holds also for multiple overlapping and odd shifts.

1) Window $w(t)$ is even; this implies that $w(t-1)$ can be obtained via a symmetry around 1/2 and that the product $w(t)w(t-1)$ is symmetric around 1/2.
2) The function $s_1(t) \in \mathcal{C}_1 = \mathcal{C}_0^-$ (Property 2), whereas $s_0(t) \in \mathcal{C}_0$. Therefore, they have different symmetries around 1/2, and their product is antisymmetric.
3) The overall product $w(t)w(t-1)s_0(t)s_1(t)$ is antisymmetric, and its area is zero.

B. Completeness

As anticipated, we will prove completeness by showing that if the auxiliary signal $q_t[n]$ introduced in (26) is identically zero, then $f = 0$. We present such a proof because it introduces, in a simpler context, the technique that will be used in the multiple overlapping case.

Proof of Property 7 (Single Overlapping): By using in (27) the fact that the window support is $[-1, 1]$ and the window symmetry, we get

$$q_t[2n+m] = \begin{cases} f(t-2n)w(t) + f(-1-t-2n)w(-1-t) \\ \quad \text{if } m=0 \\ f(t-2n)w(-1-t) - f(-1-t-2n)w(t) \\ \quad \text{if } m=1. \end{cases} \tag{28}$$

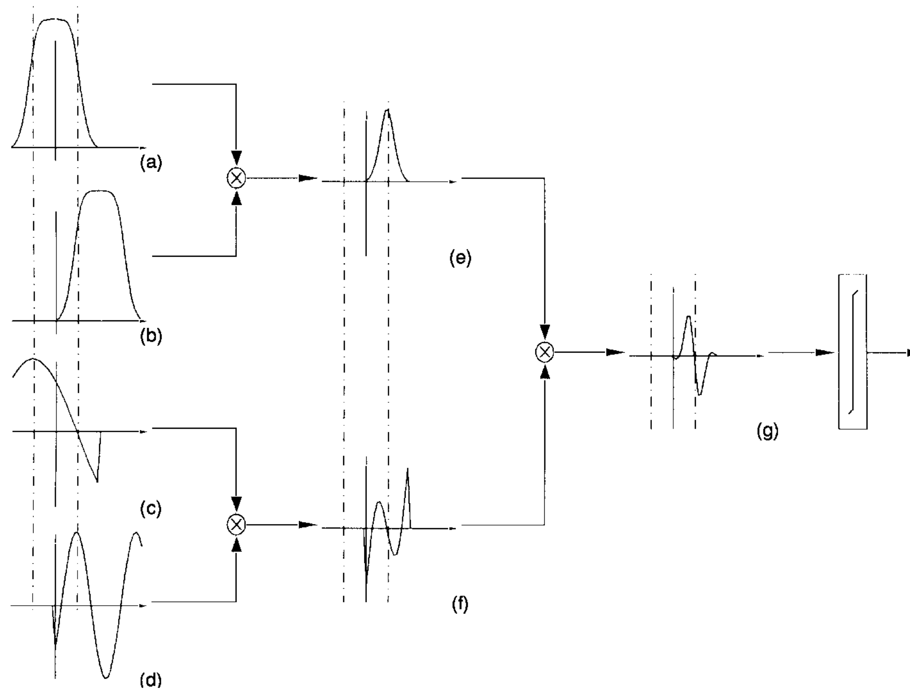


Fig. 6. Rearrangement of the signals of Fig. 5. (a) Window $w(t)$ symmetric with respect to 0, that is, $w(t) = w(-t)$. (b) Window $w(t-1) = w(1-t)$. (c) Signal $s_0(t) \in \mathcal{C}_0$ symmetric around $-1/2$ and antisymmetric around 1/2. (d) Signal $s_1(t) \in \mathcal{C}_1$ symmetric around 1/2. (e) Product $w(t)w(t-1) = w(t)w(1-t)$ symmetric around 1/2. (f) Product $s_0(t)s_1(t)$ antisymmetric around 1/2 because of the two different types of symmetry of $s_0(t)$ and $s_1(t)$. (g) Overall product $w(t)w(t-1)s_0(t)s_1(t)$ antisymmetric because product of the symmetric function $w(t)w(t-1)$ with the antisymmetric one $s_0(t)s_1(t)$. Its integral is zero.

Rewriting (28) in matrix form gives

$$\begin{bmatrix} q_t[2n] \\ q_t[2n+1] \end{bmatrix} = \begin{bmatrix} w(-1-t) & w(t) \\ -w(t) & w(-1-t) \end{bmatrix} \cdot \begin{bmatrix} f(-1-t-2n) \\ f(t-2n) \end{bmatrix}. \tag{29}$$

Because of the power complementarity conditions, the matrix in (29) is invertible $\forall t \in [-1, 0]$. This implies that if $q_t[n] \equiv 0$, then

$$\begin{bmatrix} f(-1-t-2n) \\ f(t-2n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \forall t \in [-1, 0], n \in \mathbb{Z}. \tag{30}$$

Since for every $t_0 \in \mathbb{R}$ there exist $t \in [-1, 0]$ and $n \in \mathbb{Z}$ such that $t_0 = t - 2n$ or $t_0 = -1 - t - 2n$, (30) implies $f(t) = 0$, $\forall t \in \mathbb{R}$.

V. Multiple Overlapping in Continuous Time

Now, we attack an original construction, namely, the case of LCB with multiple overlapping.

A. External Orthogonality

In the case of multiple overlapping, window symmetry still leads to external orthogonality but only for odd translations.

- If the window is even, we can write $w(t - (2j+1)) = w((2j+1) - t)$, that is, a translation of $2j+1$ gives the same result of a symmetry around $(2j+1)/2$. Therefore, the product $w(t)w(t - (2j+1))$ is symmetric around $(2j+1)/2 = j + 1/2$.
- The product $s_0(t)s_1(t)$ is antisymmetric around $(2j+1)/2$ since $s_0(t) \in \mathcal{C}_0$ and $s_1(t) \in \mathcal{C}_{2j+1} = \mathcal{C}_0^-$.
- The approach of the single overlapping case still works: $w(t)w(t-(2j+1))$ is symmetric around $j+1/2$, whereas $s_0(t)s_1(t)$ is antisymmetric. Therefore, the overall product is antisymmetric, and the two spaces are orthogonal.

We can summarize such a fact in a property.

Property 8: If $w(t) = w(-t)$, then V_0 is orthogonal to V_{2j+1} for every $j \in \mathbb{Z}$.

What happens for a translation of $2j$? For even translations, both $s_0(t)$ and $s_1(t)$ belong to $\mathcal{C}_0$; therefore, $s_0 s_1 \in \mathcal{S}_0$ is symmetric around $\ell + 1/2$, $\ell \in \mathbb{Z}$, and the approach of the single overlapping case cannot be applied anymore.

How can we obtain orthogonality? The answer is contained in the following property.

Property 9: Vector spaces V_0 and V_{2j}, $j \in \mathbb{Z}$, $j \neq 0$ are orthogonal if and only if the window $w(t)$ is *self-orthogonal* in the sense that

$$\begin{aligned} \mathcal{Q}_0^+(w(\cdot)w(\cdot - 2j)) = \sum_{\ell \in \mathbb{Z}} & w(t + 2\ell)w(t + 2\ell - 2j) \\ & + w(1 - t + 2\ell)w(1 - t + 2\ell - 2j) \\ = 0, & \quad \forall t \in \mathbb{R}. \end{aligned} \tag{31}$$

Proof: If V_0 has to be orthogonal to V_{2j}

$$\int_{\mathbb{R}} w(t)w(t-2j)s_1(t)s_2(t)\, dt = 0, \quad \forall s_1(t) \in \mathcal{C}_0, s_2(t) \in \mathcal{C}_{2j} = \mathcal{C}_0 \tag{32}$$

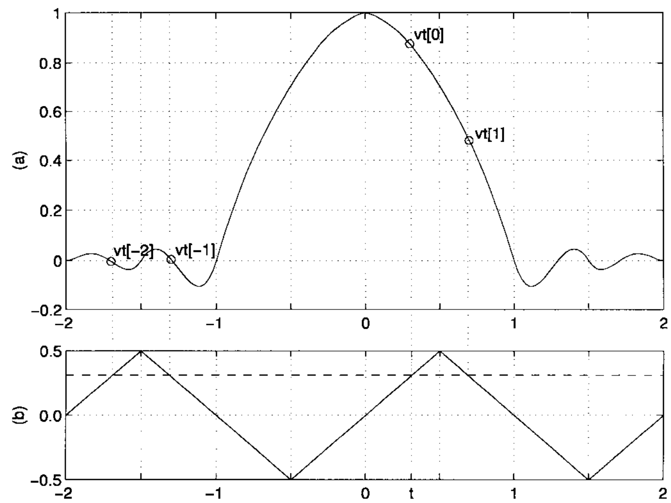


Fig. 7. (a) Example of construction of discrete-time signal $v_t[n]$. (b) Triangular sinus giving the position of the samples of $v_t[n]$.

must hold. In (32), $s_1(t)s_2(t) \in \mathcal{S}_0$ (Property 4) and from Property 6, we get

$$\int_{-1/2}^{1/2} [\mathcal{Q}_0^+(w(\cdot)w(\cdot - 2j))](t)s_1(t)s_2(t)\, dt = 0, \quad \forall s_1(t), s_2(t) \in \mathcal{C}_0. \tag{33}$$

Equation (33) is verified if and only if (31) is true.

Equation (31) is not a continuous-time condition but a continuum of discrete-time conditions (a condition for every $t \in [-1/2, 1/2]$). By using such a fact, we can prove the following property.

Property 10: The window $w(t)$ enjoys power complementarity and self-orthogonality if and only if, for each t, $v_t[\cdot]$ is a branch of a two-channel PR filter bank.

Property 10 is interesting because two-channel PR filter banks have a nice parameterization. Such a fact will be exploited in the section relative to window design.

Proof: Define the following discrete-time signal

$$v_t[2n+m] \triangleq \begin{cases} w(t+2n), & \text{if } m=0 \\ w(1-t+2n), & \text{if } m=1. \end{cases} \tag{34}$$

Fig. 7(a) shows the construction of signal $v_t[n]$ for $t = 0.4$. The values of $w(t)$ used to construct $v_t[n]$ are marked with little circles, and the corresponding samples of $v_t[n]$ are written next to them. By using (34) in (31) and recalling the power complementarity condition $\langle v_t[\cdot], v_t[\cdot] = 1\rangle$, we get

$$\sum_{\ell \in \mathbb{Z}} v_t[2\ell]v_t[2\ell - 2n] + v_t[2\ell+1]v_t[2\ell+1-2n] = \langle v_t[\cdot], v_t[\cdot - 2n]\rangle = \delta(n), \qquad n \neq 0 \tag{35}$$

that is, v_t is orthogonal to its even translations or, equivalently, it is a branch of a two-channel PR filter bank. □

It is worth to pointing out two properties of $v_t[n]$ that come directly from its definition (the first equality requires window symmetry) and that will be exploited in the following.

$$v_{-t}[-n] = v_t[n] \tag{36a}$$

$$v_{1-t}[2n+m] = v_t[2n+(1-m)]. \tag{36b}$$

1) Other Uses of the Folding Operator: In this section, we show some other applications of the folding operator.

Window symmetry is a sufficient, but not necessary, condition for external orthogonality for odd translations. To obtain a necessary condition, we can prove, with a reasoning similar to the one used in the case of even translations, that external orthogonality for odd translations is equivalent to $\mathcal{Q}_0^- w(t)w(t-(2j+1)) = 0$ that can be rewritten as

$$\sum_{\ell \in \mathbb{Z}} v_t[2\ell]v_{-t}[2j+2\ell-1] - \sum_{\ell \in \mathbb{Z}} v_t[2\ell+1]v_{-t}[2j+2\ell] = \sum_{\ell \in \mathbb{Z}} (-1)^\ell v_t[\ell]v_{-t}[\ell+2j-1] = 0, \quad \forall j \in \mathbb{Z}. \tag{37}$$

Since every $v_t[n]$ is a branch of a PR filter bank, (37) implies that $u[n] \triangleq (-1)^n v_{-t}[n-1]$ must be the conjugate (in a PR sense) of $v_t[n]$. Window symmetry clearly fulfills such a condition.

If LCB's are used for multitone modulation [6], the window at the receiver is a distorted version $\tilde{w}(t)$ of the window at the transmitter $w(t)$. Because of this, we can loose external orthogonality, and this causes intersymbol interference. To measure the deviation from orthogonality we could use the norm of the folded product $\|\mathcal{Q}_0^+[w(t)\tilde{w}(t-2j)]\|$. More generally, by using Property 6 and the Cauchy–Schwartz inequality, we can prove the following property.

Property 11: Let $s_1(t) \in \mathcal{C}_0$ and $s_2(t) \in \mathcal{C}_j$, $j \in \mathbb{Z}$, and let $w_1(t)$, $w_2(t)$ be two windows; then

$$\max_{\|s_1 s_2\|=1} |\langle w_1 s_1, w_2 s_2\rangle| = \begin{cases} \|\mathcal{Q}_0^+(w_1 w_2)\|, & \text{if } j \text{ even} \\ \|\mathcal{Q}_0^-(w_1 w_2)\|, & \text{if } j \text{ odd.} \end{cases} \tag{38}$$

As an example of application of Property 11, we can consider Matviyenko's biorthogonal local trigonometric bases [14]. In [14], the two dual bases are obtained by modulating two different windows $w(t)$ and $\tilde{w}(t)$ with the same cosines. Using Property 11 with $w_1(t) = w(t)$ and $w_2(t) = \tilde{w}(t)$, we can write the biorthogonality conditions as $\mathcal{Q}_0^+(w(\cdot)\tilde{w}(\cdot - 2j)) = \delta(j)$ and $\mathcal{Q}_0^-(w(\cdot)\tilde{w}(\cdot - (2j+1))) = 0$.

B. Completeness

Let $f(t) \in L^2(\mathbb{R})$ be a continuous function, and define $q_t[n]$ as in (26). We want to prove that $q_t[n] = 0$ for all t and n implies $f = 0$.

Proof of Property 7 (Multiple Overlapping): Observe Fig. 4. When $f(t)$ is translated, values $f(t+2\ell-2n)$ and $f(1-t+2\ell-2n)$ fall alternatively under the influence of two different sets of window values. This suggests that $q_t[n]$ can be written as a convolution. To such an end, define

$$g[2n+m] \triangleq \begin{cases} f(t+2n), & \text{if } m=0 \\ f(1-t+2n), & \text{if } m=1 \end{cases} \tag{39}$$

$$u_0[2n+m] \triangleq \begin{cases} (-1)^n w(t+2n), & \text{if } m=0 \\ (-1)^{n+1} w(1-t+2n), & \text{if } m=1 \end{cases} \tag{40}$$

$$u_1[2n+m] \triangleq \begin{cases} (-1)^n w(1+t+2n), & \text{if } m=0 \\ (-1)^n w(-t+2n+2), & \text{if } m=1. \end{cases} \tag{41}$$

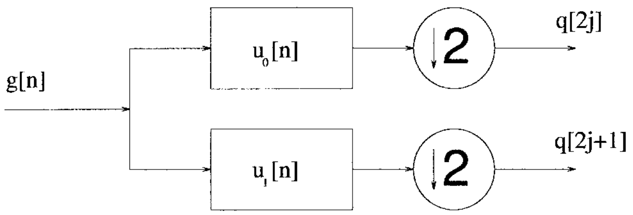


Fig. 8. Two-channel perfect reconstruction filter bank induced by the projection on V_0.

By using such signals, we can rewrite (27) as

$$\begin{aligned} q_t[2n] &= \sum_{\ell\in\mathbb{Z}} g[2\ell-2n]u_0[2\ell] \\ &\quad + g[2\ell+1-2n]u_0[2\ell+1] \\ &= \sum_{\ell\in\mathbb{Z}} g[\ell-2n]u_0[\ell] \quad (42a) \\ q_t[2n+1] &= \sum_{\ell\in\mathbb{Z}} g[2\ell-2n]u_1[2\ell] + g[2\ell \\ &\quad +1-2n]u_1[2\ell+1] \\ &= \sum_{\ell\in\mathbb{Z}} g[\ell-2n]u_1[\ell]. \quad (42b) \end{aligned}$$

A possible interpretation of (42) is presented in Fig. 8; values $q_t[n]$ can be obtained by filtering $g[n]$ with a two-channel filter bank having $u_0[n]$ and $u_1[n]$ as impulse responses. The even samples exit from u_0, whereas the odd ones exit from u_1. Such a filter bank structure will be even more interesting after we will have proved a fundamental relationship between $u_0[n]$ and $u_1[n]$. As a first step, rewrite $u_0[n]$ and $u_1[n]$ in terms of $v_t[n]$ and $v_{-t}[n]$ as

$$\begin{aligned} u_0[2n+m] &= (-1)^{n+m} v_t[2n+m] \\ u_1[2n+m] &= (-1)^n v_t[2n+m+1]. \end{aligned} \quad (43)$$

From (43), it is easy to prove that

$$\begin{aligned} \langle u_0[\cdot], u_0[\cdot{+}2n]\rangle &= (-1)^n \langle v_t[\cdot], v_t[\cdot+2n]\rangle = \delta(n) \\ \langle u_1[\cdot], u_1[\cdot{+}2n]\rangle &= (-1)^n \langle v_{-t}[\cdot], v_{-t}[\cdot+2n]\rangle = \delta(n) \end{aligned} \quad (44)$$

that is, $u_0[n]$ and $u_1[n]$ are orthogonal to their even translation. Moreover, by using (43) in (37), we can see that $u_0[n]$ and $u_1[n]$ are conjugate quadrature filters, and the scheme of Fig. 8 is a PR filter bank! Therefore, if $q_t[n] = 0$ for every $n \in \mathbb{Z}$, then $g[n] = 0$ because of the PR property. If such a fact is verified for every t, then $f(t) = 0$.

It is interesting to observe from the proof that a continuous-time LCB can be interpreted as a continuum of discrete-time two channel filter banks.

It is worth summarizing what we found so far.

Theorem 1: Let $w(t)$ be a continuous window satisfying symmetry, power complementarity, and self orthogonality constraints

$$w(t) = w(-t) \quad (45a)$$

$$\sum_{\ell\in\mathbb{Z}} w^2(t-2\ell) + w^2(1-t-2\ell) = 1 \quad (45b)$$

and

$$\begin{aligned} &\sum_{\ell\in\mathbb{Z}} w(t+2\ell)w(t+2\ell-2j) \\ &\quad + w(1-t+2\ell)w(1-t+2\ell-2j) = 0. \end{aligned} \quad (45c)$$

Define V_0 as the subspace of $L^2(\mathbb{R})$ spanned by $\{w(t)\cos_k(t)\}_{k\in\mathbb{Z}}$ with $\cos_k(t)$ defined as in (2). Let V_j be the translation of V_0 in the sense that $f(t) \in V_0 \Leftrightarrow f(t-j) \in V_j$. Then, $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} V_j$, and functions $g_{j,k}(t) \triangleq w(t-j)\cos_k(t-j)$, $j,k \in \mathbb{Z}$ make an orthonormal basis for V_j.

VI. Relation Between Discrete-Time and Continuous-Time Cases, a Sampling Theorem, and a Mallat-Like Algorithm for LCB

A. Discrete-Time Case

The continuous-time theory can be easily rephrased in discrete time. This is suggested from the fact that the conditions of Theorem 1 have a "pointwise" nature. For example, self-orthogonality (45c) is a pointwise condition, and it does not require t to belong to a continuous set. It is worth emphasizing the major differences between the two cases.

- The elementary step cannot be normalized, and we have one more parameter: the step size N. The cosine symmetries are not around $\pm 1/2$ but are around $\pm N/2$.
- The scalar product is computed via sums and not integrals, and properties like Property 6 should be suitably rewritten. The folding operator remains the same but with $t \in \mathbb{Z}$.
- The vector spaces V_j have finite dimension, whereas in continuous time, their dimension is infinite. Since we never used the dimension of V_j, the difference is of no consequence.
- The proof of the fact that η is a unitary mapping still works. This time, the role of $L^2([-1/2, 1/2])$ is played by the space of sequences with support $\{0, \cdots, N-1\}$.
- It is possible that the symmetry points do not belong to $\mathbb{Z}$. This is not a problem since the proofs rely on the property that an antisymmetric function has zero mean, and this holds independently from the fact that the symmetry point belongs, or does not belong, to $\mathbb{Z}$.
- Self-orthogonality condition (31) gives rise to a finite set of constraints and not a continuum.

An important relationship between the continuous and discrete-time cases is that by sampling a continuous-time LCB with a lattice symmetric with respect to $\pm 1/2$ (in this way, the symmetry characteristics of LCB still make sense), we get a discrete-time CMFB. It is convenient to define some new symbols for the sampled versions of the generic signal $f(t)$, the window $w(t)$, the cosines $\cos_k(t)$, and the basis functions $g_{j,k}(t)$. They will be denoted as

$$\begin{aligned} w_N[n] &\triangleq w(1/2 + 1/2N + n/N) \\ f_N[n] &\triangleq f(1/2 + 1/2N + n/N) \quad (46a) \\ c_k[n] &\triangleq \cos_k(1/2 + 1/2N + n/N) \\ h_{j,k}^{(N)}[n] &\triangleq g_{j,k}(1/2 + 1/2N + n/N) \\ &= w_N[n]c_k[n]. \quad (46b) \end{aligned}$$

"Phase" $\phi_N \triangleq 1/2 + 1/2N$ in (46) is necessary in order to make the sampling lattice symmetric with respect to ±1/2. Now, we can state the announced property.

Property 12: Consider a continuous-time LCB with window $w(t)$. Let $N \in \mathbb{N}, N > 0$. Define $V_0^{(N)}$ as the space obtained by sampling V_0; more precisely

$$V_0^{(N)} \triangleq \{r[n]: r[n] = f(1/2 + 1/2N + n/N) \quad n \in \mathbb{Z}, f(t) \in V_0\}. \tag{47}$$

The vector space $V_0^{(N)}$ is a discrete-time local cosine space of dimension N relative to the window $w_N[n]$ and having functions $h_{j,k}^{(N)}[n], k = 0, \cdots, N-1$ as an orthonormal basis.

It is clear that $V_0^{(N)}$ has dimension N since there are N independent samples (the ones inside [−1/2, 1/2]). The proof of Property 12 requires the following "aliasing lemma."

Lemma 2: The following "aliasing relationships" hold:

$$\begin{aligned} c_{2N-1-k}[n] &= -c_k[n] \\ c_{k+2\ell N}[n] &= (-1)^\ell c_k[n], \quad \forall \ell \in \mathbb{Z} \end{aligned} \tag{48a}$$

$$\begin{aligned} h_{2N-1-k,n}^{(N)} &= -h_{k,n}^{(N)}; \\ h_{k+2\ell N,n}^{(N)} &= (-1)^\ell h_{k,n}^{(N)}, \quad \forall \ell \in \mathbb{Z}. \end{aligned} \tag{48b}$$

Equations (48) have an intriguing interpretation. If we consider $c_k[n]$ as a "function valued signal" in k, the equations in (48) claim that $c_k[n]$ is antisymmetric around $N - 1/2$ and skew periodic with period $2N$, that is, the same type of symmetries enjoyed by the cosine functions in time!

Proof: Equations (48) follow immediately from cosine properties and the definition of $h_{k,n}^{(N)}$. □

Sketch of the Proof of Property 12: Functions $h_{0,k}^{(N)}[n], k \in \mathbb{Z}$ generate $V_0^{(N)}$. Because of Lemma 2, we can restrict our attention to functions $h_{0,k}^{(N)}[n], k < N$. The window $w_N[n]$ inherits from $w(t)$ symmetry, power complementarity, and self-orthogonality. Functions $c_k[n]$ are the cosines used in CMFB's [compare with (3)]. Therefore, $h_{0,k}^{(N)}[n] = w_N[n]c_k[n], k < N$ are the impulse responses of a CMFB, and the thesis follows.

B. A Sampling Theorem for LCB's

The sampling theorem is an important result of Fourier analysis. Since LCB's, like the Fourier transform, are a frequency analysis tool, we expect a result similar to the sampling theorem based on LCB. In this section, we are going to give such a result. It will be seen that the class of perfectly reconstructible signals in the LCB sense is different from the class of perfectly reconstructible signals in the Fourier sense (bandlimited signals).

To emphasize the connection with the Fourier transform, we will change, in this section only, index k in $\cos_k$ and $g_{j,k}$ with ω. Moreover, we will use notations $F(j,\omega) \triangleq \langle f, g_{j,\omega}\rangle$ and ${}_N F(j,\omega) \triangleq \langle f_N, h_{j,\omega}^{(N)}\rangle$ for continuous- and discrete-time functions, respectively. Before stating the main result, let us define the concept of "bandlimited signal" in an LCB sense.

Definition 1: Define B_N as the "locally bandlimited" subspace of $L^2(\mathbb{R})$, that is

$$\begin{aligned} B_N &\triangleq \{f: f \in L^2(\mathbb{R}), F(j,\omega) = 0, \forall \omega \geq N, j \in \mathbb{Z}\} \\ &= \text{span}\{g_{j\omega}\}_{\omega < N, j \in \mathbb{Z}}. \end{aligned} \tag{49}$$

Theorem 2: If $f(t) \in B_N$, then $f(t)$ can be recovered from its sampled version $f_N[n]$ via

$$f(t) = \sum_{\ell \in \mathbb{Z}} f_N[\ell] r(t, \phi_N + \ell/N) \tag{50}$$

where $r(t,x)$ is defined as

$$r(t,x) \triangleq \sum_{j \in \mathbb{Z}} \sum_{\omega=0}^{N-1} g_{j,\omega}(t) g_{j,\omega}(x). \tag{51}$$

If $f(t) \notin B_N$, the quadratic norm of the reconstruction error is minimized by sampling

$$f_2(t) = \int_{\mathbb{R}} f(x) r(x,t)\, dx. \tag{52}$$

Equation (52) is the LCB correspondent of the lowpass filtering used in the usual sampling. Note that a signal can be locally bandlimited without being bandlimited in the usual sense. Therefore, according to Theorem 2, we can find signals with infinite bandwidth that can be perfectly reconstructed in an LCB sense (and vice versa). To prove Theorem 2, we need the following lemma that describes the relationship between the local cosine transforms of $f(t)$ and its sampled version f_N.

Lemma 3: Let $F(j,\omega)$ and ${}_N F(j,\omega)$ be the local cosine transforms of $f(t) \in L^2(\mathbb{R})$ and its sampled version f_N, respectively; then

$$\begin{aligned} {}_N F(j,\omega) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell [&F(j, 2\ell N + \omega) \\ &- F(j, 2\ell N + (2N - 1 - \omega))]. \end{aligned} \tag{53}$$

Lemma 3 descends directly from Lemma 2. Note that instead of the usual aliasing of the classical sampling theorem, here we have a folding! Lemma 3 has an immediate corollary.

Corollary 2: If $f(t) \in B_N$, then $F(j,\omega) = {}_N F(j,\omega)$, $\omega = 0, 1, \cdots, N-1$, $j \in \mathbb{Z}$.

Lemma 4: Expression (52) is the projection of $f(t)$ on B_N.

Proof: Write the projection R_N on B_N in terms of $g_{j,\omega}$, $j \in \mathbb{N}$, $\omega = 0, \cdots, N-1$

$$\begin{aligned} (R_N f)(t) &= \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \langle f, g_{j,k}\rangle g_{j,k}(t) \\ &= \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \int_{\mathbb{R}} f(x) g_{j,k}(x) g_{j,k}(t)\, dx \\ &= \int_{\mathbb{R}} f(x) \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} g_{j,k}(x) g_{j,k}(t)\, dx \\ &= \int_{\mathbb{R}} f(x) r(x,t)\, dx. \end{aligned} \tag{54}$$

The compact support of the functions $\{g_{j,k}\}$ allowed us to bring the sums inside the integral. □

Now, we can proof Theorem 2.

Proof of Theorem 2: Let $f(t) \in B_N$. From Corollary 2, it follows that for each $j \in \mathbb{Z}$ and $k = 0, \cdots, N-1$, $\langle f, g_{j,k}\rangle = F(j,k) = {}_N F(j,k) = \langle f_N, h_{j,k}^{(N)}\rangle$. Therefore, we can write

$$\begin{aligned} f(t) &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \langle f_N, g_{j,k}\rangle g_{j,k}(t) \\ &= \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} \langle f_N, h_{j,k}^{(N)}\rangle g_{j,k}(t). \end{aligned} \quad (55)$$

By replacing the scalar product in (55) with its expression, we get

$$\begin{aligned} f(t) &= \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} \left(\sum_{\ell\in\mathbb{Z}} f_N[\ell] h_{j,k}^{(N)}[\ell]\right) g_{j,k}(t) \\ &= \sum_{\ell\in\mathbb{Z}} f_N[\ell] \left(\sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} g_{j,k}(t) g_{j,k}[\phi_N + \ell/N]\right) \\ &= \sum_{\ell\in\mathbb{Z}} f_N[\ell] r(t, \ell/N + \phi_N). \end{aligned} \quad (56)$$

The second part of the theorem is clear since the reconstruction formula gives a function of B_N, and $f_2(t)$ in (52) is the function of B_N having minimum distance from $f(t)$. □

Note the necessity of projecting $f(t)$ on B_N before sampling it. This is similar to what happens with the wavelet expansion using Mallat's algorithm; we must first calculate a projection onto the space spanned by the scaling functions at a chosen scale. Another example is the case of classical sampling; if the signal is not bandlimited, we must use a lowpass filter (that is, the projection on the space of bandlimited signals).

C. A Mallat-Like Algorithm for Continuous-Time Local Bases

Let us exploit the results of the previous section to develop an algorithm to compute $\langle f, g_{j,k}\rangle$.

If $f(t) \in B_N$, it is clear that because of Corollary 2, we can sample $f(t)$ and compute the scalar products in discrete time. This can be efficiently done with a CMFB. If $f(t)$ is not bandlimited, we have, according to Theorem 2, to project it on B_N. By evaluating $R_N f$ at instants $\phi_N + n/N$, $n \in \mathbb{Z}$, we get

$$(R_N f)(\phi_N + n/N) = \langle f, r(\cdot, \phi_N + n/N)\rangle. \quad (57)$$

The resulting analysis algorithm is shown in Fig. 9(a). Fig. 9(b) shows the corresponding synthesis scheme.

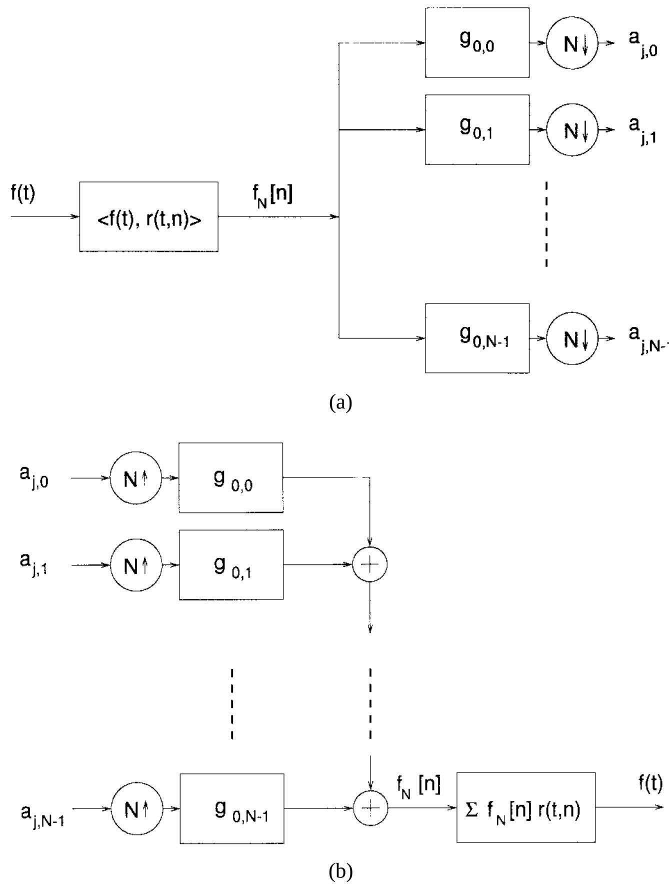


Fig. 9. Mallat-like algorithm for local cosine bases. (a) Analysis—For each $n \in \mathbb{Z}$, the scalar product of the input signal $f(t)$ with the kernel $r(t,n)$ is computed. The resulting discrete-time sequence $f_N[n]$ is processed with a CMFB whose output are components $a_{j,k}$. (b) Synthesis—The N input signals $a_{j,0}, \cdots, a_{j,N-1}$ are sent into a synthesis filter bank whose output is the original sequence $f_N[n]$ because of the perfect reconstruction property. The discrete-time sequence is sent into an interpolator that reconstructs the original signal $f(t)$ (or its projection on B_N if originally $f(t)$ was not bandlimited).

VII. Window Design

In this section, we show how to design a window for a continuous-time, multiple overlapping LCB. Let us state explicitly our objective.

Problem 1: Let $D \in \mathbb{N}$. Design a D-time differentiable window satisfying the constraints of power complementarity, symmetry, and self-orthogonality.

Recall the definition (34) of the signal $v_t[n]$. Remember that $v_t[n]$ is a branch of a PR filter bank. Let us, in this section, change the notation $v_t[n]$ into $v(t;n)$. We need to know how to obtain $w(t)$ from the filters $v(t;n)$.

Property 13: For every $t \in \mathbb{R}$, there exist $q(t) \in \mathbb{Z}, \tau(t) \in [0, 1/2]$ such that $w(t) = v(\tau(t); q(t))$. Moreover, for every $t_0 \in \mathbb{R} - (\mathbb{Z}/2)$, the function $\tau(t)$ is arbitrarily differentiable in t_0, and there exists a neighborhood of t_0 such that $q(t)$ is constant.

Proof: Call $\text{tsin}(t)$ (as *triangular sinus*) the function from $\mathbb{R}$ to $[-1/2, 1/2]$, which is shown in Fig. 7(b). It is clear from the figure that $\forall t \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $w(t) = v(\text{tsin}(t); n)$. If $\text{tsin}(t) \geq 0$, we are done; otherwise, exploit window symmetry to obtain $w(t) = w(-t) = v(\text{tsin}(-t); -n)$, with $\text{tsin}(-t) = -\text{tsin}(t) > 0$ since $\text{tsin}(t)$ is odd. □

The integer $q(t)$ in Property 13 is not necessarily unique; indeed, for every $t \in \mathbb{Z}/2$, there exist two integers satisfying Property 13. We will avoid such an ambiguity by imposing left continuity to $q(t)$. By exploiting Property 13, we can restate Problem 1 as follows.

Problem 2: Find a family of filters $v(t;n)$, parameterized by $t \in [0, 1/2]$, such that the corresponding window is D-time differentiable.

Every two channel filter bank can be expressed via the lattice factorization as a sequence of rotations and unit delays [1], [2]. Every rotation is identified by an angle a_i, and a $2N$-length filter bank requires N rotations. Collect all the angles in a vector $\boldsymbol{a} \triangleq [a_1, \cdots, a_N]$. Let $L_n(\boldsymbol{a})$ be the function giving the nth sample of the filter relative to angles $\boldsymbol{a}$. To determine a family of filters, make every angle a_i a function of $t \in [0, 1/2]$, and define $v(t;n) = L_n(\boldsymbol{a})(t)$. Note that functions $a_i(t)$ have no constraint. With such a parameterization Problem 2 becomes the following.

Problem 3: Find a set of functions $a_i(t), t \in [0, 1/2]$, such that the window $w(t) = v(\tau(t); q(t)) = L_{q(t)}(\boldsymbol{a}(\tau(t)))$ is D-time differentiable.

Now, we have to map the requirement of window smoothness into constraints on $a_i(t)$. Since $L_n(\boldsymbol{a})$ and $\tau(t)$ are arbitrarily smooth if $t \in \mathbb{R} - (\mathbb{Z}/2)$, it is clear that as long as $q(t)$ does not change, $w(t)$ is as smooth as $\boldsymbol{a}(t)$. Since for every $t \in \mathbb{R} - (\mathbb{Z}/2)$ there is a neighborhood of t in which $q(t)$ is constant, we have the following property.

Property 14: If every function $a_i(t)$ is D-times differentiable in $t_0 \in \mathbb{R} - (\mathbb{Z}/2)$, then $w(t)$ is D-times differentiable in t_0.

Therefore, to achieve smoothness in $\mathbb{R} - (\mathbb{Z}/2)$, we can, for example, use for every $a_i(t)$ a polynomial in t. Instead, if $t_0 \in \mathbb{Z}/2$, function $q(t)$ assumes two different values in every neighborhood of t_0, and we have continuity if

$$\lim_{\epsilon \downarrow 0} L_{q(t_0+\epsilon)}(\boldsymbol{a}(\tau(t_0+\epsilon))) = \lim_{\epsilon \downarrow 0} L_{q(t_0-\epsilon)}(\boldsymbol{a}(\tau(t_0-\epsilon))) \tag{58}$$

where $\lim_{\epsilon \downarrow 0}$ denotes the limit for ϵ going to zero from the right. Call n_1 and n_2 the two constant values $q(t_0+\epsilon)$, $q(t_0-\epsilon)$ in (58). Since, for fixed n, $L_n(\boldsymbol{a}(\tau(t)))$ is a continuous function of t, we can rewrite (58) as

$$L_{n_1}(\boldsymbol{a}(\tau(t_0))) = L_{n_2}(\boldsymbol{a}(\tau(t_0))). \tag{59}$$

Since $\tau(t_0) = |\text{tsin}(t_0)|$ can only assume values 0, 1/2 if $t_0 \in \mathbb{Z}/2$ [see Fig. 7(b)], (59) can be rewritten as a set of boundary conditions on functions $a_i(t)$ for $t = 0, 1/2$. Actually, it is shown in Appendix B that the window must satisfy the following constraints on $\mathbb{Z}/2$:

$$\begin{aligned} w(0) &= 1; \quad w(1/2) = w(-1/2) = 1/\sqrt{2} \\ w(n/2) &= 0, \qquad n \in \mathbb{Z}, |n| > 1. \end{aligned} \tag{60}$$

Constraints (60) map themselves into constraints for $a_i(t)$. It is immediate to see that if $a_i(t)$ are continuous and satisfy constraints (60), the limits in (58) are necessarily equal.

The reasoning used to obtain (59) can be repeated for every order of differentiability, giving rise to the following boundary conditions on the derivatives of $L_n(\boldsymbol{a}(t))$ with respect to t

$$L_{n_1}^{(D)}(\boldsymbol{a}(\tau(t_0))) = (-1)^D L_{n_2}^{(D)}(\boldsymbol{a}(\tau(t_0))). \tag{61}$$

The term $(-1)^D$ in (61) comes from the fact that the derivative of $\tau(t)$ is $+1$ or -1, depending on the direction t approaches t_0. Since every $L_n(\boldsymbol{a})$ is a linear combination of products of sines and cosines of $a_i(t)$, it is possible to show that (61) becomes a linear system in unknowns $\boldsymbol{a}^{(D)}(0)$ and $\boldsymbol{a}^{(D)}(1/2)$ that can be solved with usual techniques. The obtained boundary conditions can be easily matched by using for each $a_i(t)$ a polynomial in t.

Although everything can be carried out in closed form, because of the involved form of the functions $L_n(\boldsymbol{a}(t))$ and their derivatives, a program for symbolic mathematics can prove useful.

A. Continuous-Time Design with Discrete-Time Techniques and Vice Versa

Since the self-orthogonality conditions for discrete time are just the sampled version of the continuous time ones, we can get a good discrete-time window by sampling a continuous-time one. Such a fact can be exploited in two ways.

1) Design a discrete-time window (with some known technique, e.g., [1]), and use its samples as "anchor points." The continuous-time window is obtained by (nonlinear) interpolation.

   More precisely, suppose the discrete-time window has N samples between 0 and 1/2.[2] Window samples $w(\phi_{2N} + \ell/2N), \ell \in \mathbb{Z}$ can be mapped into angle values $a_i(n/2N), n = 0, 1, \cdots, N-1$ that can be interpolated with smooth functions.

2) Design a good continuous-time window; then, obtain a good discrete-time one by sampling. This approach could be used, for example, to transform a window for N_1 channels with a given overlapping into a window for $N_2 > N_1$ channels with the same overlapping. To do that, just interpolate the original window to continuous time, and sample the result.

B. Design Example

Fig. 10(a) shows the plots of the sinc function (dashed line) and of a twice differentiable window for quadruple overlapping (continuous line). The window frequency response is shown with continuous line in Fig. 10(b) together the frequency response of the single overlapping window shown in Fig. 10(c) (dotted line). Samples of the window of Fig. 10(a) can be found in Table I.

Fig. 10(e) and (f) show time and frequency domain views of a window obtained by interpolating, with the technique of Section VII-A, the discrete-time window reported in [1] for $M = 16$, $K = 4$, and $\omega_s = \pi/M$. It is interesting to observe that the resulting window does not satisfy the boundary conditions in $\mathbb{Z}/2$ and, therefore, is not continuous (the discontinuities are evident in $t = \pm 2.5, \pm 3, \pm 3.5$). However, allowing such discontinuities gives more freedom to the window, and the resulting frequency response has a better stopband attenuation, although it decreases more slowly for high frequencies. It is worth observing that the cosine window for double overlapping presented by Malvar in [15] is discontinuous as well.

[2] For simplicity, the discrete-time window domain is supposed to be $(1/2N)\mathbb{Z}$ instead of the more common $\mathbb{Z}$.

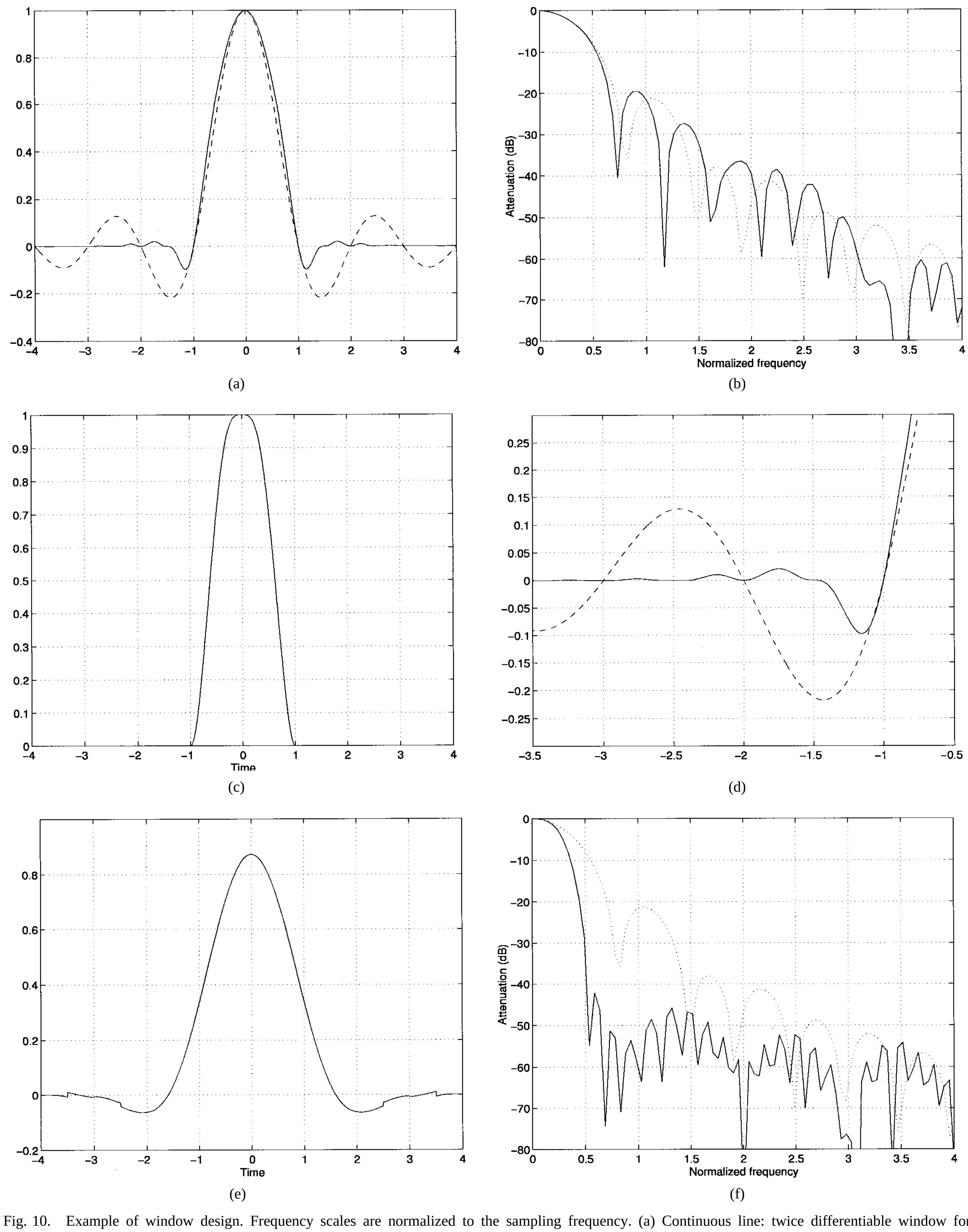


Fig. 10. Example of window design. Frequency scales are normalized to the sampling frequency. (a) Continuous line: twice differentiable window for quadruple overlapping. Dashed line: sinc function for the same sampling frequency. (b) Continuous line: frequency response of the window in (a). Dotted line: frequency response of the window in (c). (c) Single overlapping window used for comparison. (d) "Zoom" on a tail of (a). This closer view explains why the two tails are doomed to be different; the first minimum of $\mathrm{sinc}(t)$ is around $t = -1.5$, where $w(t)$ must be zero. (e) and (f) Like (a) and (b), but for a window obtained by interpolating a discrete-time one. Note that the window has some discontinuities at $t = \pm 2.5, \pm 3, \pm 3.5$. This is because the discrete-time window did not satisfy the boundary conditions in $\mathbb{Z}/2$. Despite that, the frequency response of the window (d) is better than the frequency response of the window (a).

TABLE I
SAMPLES OF THE WINDOW FOR QUADRUPLE OVERLAPPING SHOWN IN FIG. 10 FOR $t = 1/50 + n/25, n = 0, 1, \cdots, 99$. THE TABLE MUST BE READ COLUMNWISE, TOP TO BOTTOM, LEFT TO RIGHT

9.9996401e-01	-5.9381407e-03	1.8085379e-05	4.7595983e-10
9.9714078e-01	-4.7960558e-02	1.4166675e-03	4.2825537e-06
9.9016782e-01	-7.7841672e-02	4.4108398e-03	5.6628850e-05
9.7956648e-01	-9.4160549e-02	7.6151584e-03	2.2891095e-04
9.6554735e-01	-9.7509738e-02	9.6795605e-03	5.1447048e-04
9.4793139e-01	-8.9927145e-02	9.8507279e-03	7.7856613e-04
9.2635105e-01	-7.4470997e-02	8.2055305e-03	8.5059323e-04
9.0057747e-01	-5.4862763e-02	5.5148390e-03	6.7567248e-04
8.7077650e-01	-3.5013478e-02	2.8145831e-03	3.6940426e-04
8.3751552e-01	-1.8345793e-02	9.0811603e-04	1.1469735e-04
8.0150050e-01	-7.0357110e-03	4.7636882e-05	5.0346091e-06
7.6323524e-01	-1.4464086e-03	-7.2658345e-05	-5.0535093e-06
7.2284268e-01	-2.9816804e-05	-3.7408560e-06	-7.7098491e-08
6.8006913e-01	1.5497145e-04	-1.3421552e-05	4.5354349e-07
6.3439941e-01	2.0748571e-03	-6.6962791e-05	5.3529318e-06
5.8531216e-01	6.4556683e-03	1.2499595e-04	-1.4042451e-05
5.3240283e-01	1.2107395e-02	7.9167361e-04	-1.0209609e-04
4.7561137e-01	1.7174185e-02	1.7640640e-03	-2.2973221e-04
4.1540335e-01	2.0085753e-02	2.5779724e-03	-3.0572754e-04
3.5272035e-01	2.0108382e-02	2.8111595e-03	-2.7511986e-04
2.8874577e-01	1.7453573e-02	2.3720158e-03	-1.7200151e-04
2.2468847e-01	1.3032317e-02	1.5345587e-03	-7.2085165e-05
1.6174465e-01	8.0605310e-03	7.1833684e-04	-1.8040071e-05
1.0122505e-01	3.7047671e-03	2.0664859e-04	-1.9987838e-06
4.4704868e-02	8.4915981e-04	2.0116661e-05	-3.4454387e-08

VIII. CONCLUSIONS

A theory for local cosine basis with multiple overlapping has been presented. Although only the continuous-time case has been studied in detail, the theory also works in discrete time. Such a fact allowed us to obtain a sampling theorem for local cosine bases and an efficient Mallat-like analysis algorithm. The problem of window design has also been analyzed, and some example of windows for multiple overlapping have been given.

APPENDIX A
PROOFS

Proof A.1: We will prove that $\mathcal{C}_2 = \mathcal{C}_0$ and $\mathcal{C}_1 = \mathcal{C}_0^-$. The property will follow by induction.

Let $s(t) \in \mathcal{C}_0$, and call $s_1(t) = s(t-1)$. Since symmetries (7) are not independent one another, but every two of them imply the third one, we just need to prove that $s_1(t)$ is symmetric around 1/2 because skew periodicity is not affected by translations.

$$\begin{aligned} s_1(1-t) &= s(1-t-1) = s(-t) = s(-1-(-t)) \\ &= s_1(t). \end{aligned} \tag{62}$$

Therefore, $s_1(t) \in \mathcal{C}_0^-$. To prove that $\mathcal{C}_2 = \mathcal{C}_0$, just observe that $s(t-2) = -s(t) \in \mathcal{C}_0$. □

Proof A.2: We will carry out the proof only for $\mathcal{C}_0$ since the proof for $\mathcal{C}_0^-$ differs only in the signs. Function $s(t)$ enjoys the following symmetries and skew periodicity

$$s(t) = s(-1-t) \quad s(t) = -s(1-t) \quad s(t) = (-1)^n s(t-2n). \tag{63}$$

From (63), it follows that

$$\begin{aligned} s(4\ell+1-t) &= (-1)^{-2\ell} s(1-t) = -s(t); \\ s(4\ell-1-t) &= (-1)^{-2\ell} s(-1-t) = s(t). \end{aligned} \tag{64}$$

□

Proof A.3: We only give the proof for $s_1, s_2 \in \mathcal{C}_0$. The proof for the other cases differs only in the signs. Since s_1 and s_2 are both square summable on [−1/2, 1/2], then $s_1 s_2 \in L^1([-1/2, 1/2])$. We just need to prove that $s_1 s_2$ enjoys the symmetries characteristic of $\mathcal{S}_0$

$$\begin{aligned} s_1(1-t)s_2(1-t) &= (-s_1(t))(-s_2(t)) = s_1(t)s_2(t); \\ s_1(-1-t)s_2(-1-t) &= s_1(t)s_2(t). \end{aligned} \tag{65}$$

□

APPENDIX B
BOUNDARY CONDITIONS ON $v_t[n]$ FOR $t = 0, 1/2$

Equation (36a) for $t = 0$ implies $v_0[n] = v_0[-n]$. Since $\langle v_0[\cdot], v_0[\cdot - 2j]\rangle = \delta(j)$, $j \in \mathbb{Z}$, then $v_0[n] = \delta(n)$.

B.1. Constraints for $t = 1/2$

Equation (36b) for $t = 1/2$ gives $v_{1/2}[2n] = v_{1/2}[2n+1]$ $\forall n$. Therefore, (35) becomes

$$\begin{aligned} &\langle v_{1/2}[\cdot], v_{1/2}[\cdot + 2n]\rangle \\ &\quad = 2\sum_{\ell \in \mathbb{Z}} v_{1/2}[2\ell] v_{1/2}[2(\ell+n)] = \delta[n]. \end{aligned} \tag{66}$$

Equation (66) claims that $u[n] \stackrel{\Delta}{=} v_{1/2}[2n]$ is a non-null finite length signal orthogonal to *all* its translations. This implies $u[n] = 1/\sqrt{2}\delta[n]$, which in turn implies

$$v_{1/2}[2n] = v_{1/2}[2n+1] = w(1/2 - 2n) = \frac{1}{\sqrt{2}}\delta[n]. \tag{67}$$

By window symmetry, we deduce from (67) the values assumed by the window on $2\mathbb{Z} - 1/2$.

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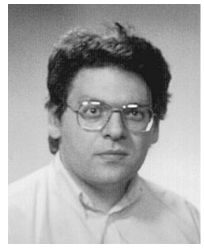

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Martin Vetterli (F'95), for a photograph and biography, see p. 1255 of the May 1998 issue of this TRANSACTIONS.