On regular configurations

An Ann Cook entry

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Abstract

In this entry we briefly discuss the regular configurations on cycle graphs and an application to find disjoint cycles in shift graphs as well as the corresponding disjoint sequences in extremal sequence theory.

Configurations

Some models for configuration $\mathrm{Config}(a, b)$

A polygon of $a+b$ sides with a sides of length α and b sides of length β.

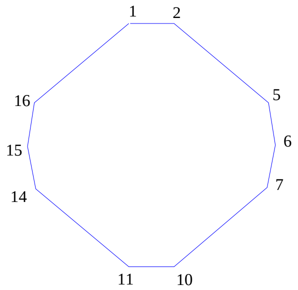


Figure 1: Polygon Model

A way of putting a red balls and b black balls in a round.

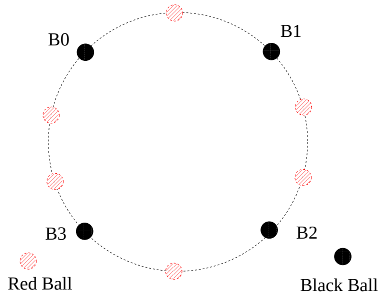


Figure 2: Ball Model

A zig-zag line from $(0, 0)$ to (a, b) in integer lattice $\mathbb{Z}^2$.[1]

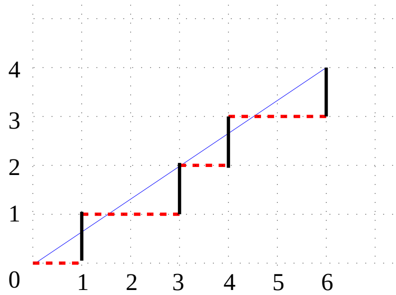


Figure 3: Line Model

A map ϕ from $V(C_{a+b})$ to $\{+, -\}$ with $|\phi^{-1}(+)| = a$. [2]

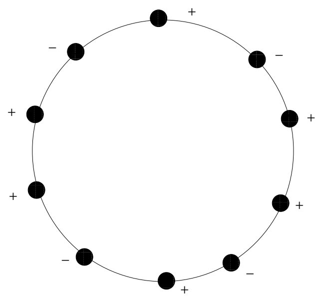


Figure 4: Ising Model

For these models, we have the 1-1 corresponding as follows:
A side of length $\alpha \leftrightarrow$ *a red ball* $\leftrightarrow$ *one horizonal step* $\leftrightarrow$ *a vertex mapped to* $+$.

Configuration representation

Fixed a labelling on the black balls in the ball model, a configuration is uniquely given by a sequence $\{x_0, \cdots, x_{b-1}\}$ where x_i is the number of red balls between black balls with labelling B^i and B^{i+1}. Two sequences $\{x_0, \cdots, x_{b-1}\}$ and $\{x'_0, \cdots, x'_{b-1}\}$ represent the same configuration if and only if $\exists\ 0 \leq \sigma < b$ such that $x_i = x'_{i+\sigma}$ for all i, where the index of the sequence should be understood as an element in $\mathbb{Z}_b$. The sequence for the above example (Fig 2) is $\{1, 2, 1, 2\}$.

[1]One step from (x, y) is either $(x + 1, y)$ or $(x, y + 1)$. This model is widely used in computer graphics to draw a line between two points on the screen.

[2]This is Kawasaki-Ising Model in statistical mechanics on cycle graphs C_{a+b}. Here we

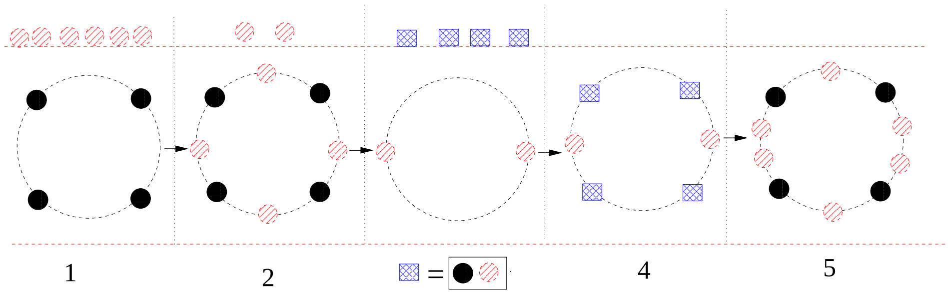
Regular configurations

Definition A configuration of $\mathrm{Config}(a, b)$ is *regular* [3] if it satisfies the following condition:

$$\frac{a}{b}k - 1 < x_i + x_{i+1} + \cdots + x_{i+k-1} < \frac{a}{b}k + 1$$

for $0 \leqslant i \leqslant b - 1, 1 \leqslant k \leqslant 1 + \lfloor \frac{b}{2} \rfloor$.

An example[4]



Construction

Algorithm 1: FindRegular (a, $\boxed{\text{R}}$;b, $\boxed{\text{B}}$)

- if $(a < b)$, return FindRegular(b, $\boxed{\text{B}}$;a, $\boxed{\text{R}}$);
- else do:
  - compute t, k such that $a = bt + k$ for $1 \leq t,\ 0 \leq k < b$.
  - build box $\boxed{\text{B,R,...,R}}$ with t $\boxed{\text{R}}$ inside.
    - if $(k \neq 0)$ return FindRegular(b, $\boxed{\text{B,R,...,R}}$; k, $\boxed{\text{R}}$);
    - else return $\underbrace{\boxed{\text{B,R,...,R}}, \cdots, \boxed{\text{B,R,...,R}}}_{b}$

Theorem

$\boxed{\text{The regular configuration is unique, and is given by Algorithm 1.}}$ [5]

are allowed to exchange two neighboring spins rather than flip a spin.

[3]The regular configuration is the one which is closest to the random one as the expectation for each x_i is a/b. In other words, it has the biggest entropy.

[4]Step 1: put 4 black balls in a round; Step 2: insert one red ball in each pair of them; Step 3: build 4 boxes; Step 4: configure these boxes and two remaining red balls; Step 5: unpack the boxes to get the regular configuration of $\mathrm{Config}(6, 4)$.

[5]In the line model, Brisenham Algorithm is used to find the regular configuration, which is more efficient while the one presented here is better for analysis.

Application: disjoint cycles in shift graphs.

Shift Graph

Shift(n,m): also circulant graph or cyclic graph.

- a directed graph for $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$.
- vertex set: [n-1]=$\{0, 1, 2, \cdots, n-1\}$
- edge set: $(i, i+1) (\bmod n)$ and $(i, i+m)\ (\bmod n)$
- the Cayley graph of $\mathbb{Z}_n$ with generators $\{1, m\}$.

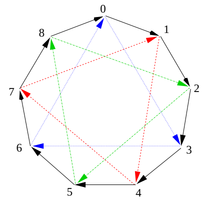


Figure 5: Shift (9,3)

Cycles and sequences:

The vertices of d-cycle with natural order is a *d-sequence*: $A = \{v_0, v_1, \cdots, v_{d-1}\}$ and the polygon under the d-cycles is the *difference sequence*: $\Delta(A) = \{v_1 - v_0, \cdots, v_{d-1} - v_{d-2}, v_0 - v_{d-1}\}\ (\bmod n)$, which is denoted as $\delta(A)$ when viewed as a multiset. By shifting $\{v_0, v_1, \cdots, v_{d-1}\}$ with distance σ we get another cycle $\{v_{0+\sigma}, v_{1+\sigma}, \cdots, v_{d-1+\sigma}\}$. [6] For instance, $\{0, 3, 6\}$ is a cycle in Shift$(9, 3)$ and we can shift it by distance 1 and 2 to get other two disjoint cycles.

An algorithm to find the disjoint cycles

Algorithm 2: FindCycles (n,m)

- Compute a, b s.t $n = am + b$ where $1 \leq a, 0 \leq b < a$, and let $k = \frac{n}{a+b}$.
- Find a **regular polygon** of $a + b$ sides with a side of length m and b side of length 1 via **Algorithm 1**.
- Get a cycle from this polygon which starts from vertex 0
- Shift this cycle with distance $(m-1), \cdots, (k-1)(m-1)$ to get other $k-1$ disjoint cycles.

Theorem

$\boxed{\text{There are } k \text{ vertex disjoint cycles in Shift}(n, m).}$ [7]

$$\boxed{\begin{array}{l}\text{Given } n\text{, there exist } k \text{ disjoint } d\text{-sequences } A_1, \cdots, A_k \\ \text{such that } \delta(A_i) = \{\underbrace{1, \cdots, 1}_{b}, \underbrace{m, \cdots, m}_{a}\} \text{ for } 1 \leqslant i \leqslant k.\end{array}}$$ [8]

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[6] The index of sequences should be understood as an element in $\mathbb{Z}_d$.

[7] In some cases k is not the best possible and the set of such cycles is not unique though Algorithm 2 always produces one.

[8] k is best possible here and the corresponding theorem in set theory is trivial as $[n]$ obviously contains k disjoint sets of size d.

Other topics

Dual configurations

The dual configuration of Config(a, b) is Config(b, a), or is obtained by switching the color of balls in the ball model. (See Fig. 6 for the dual configuration of Config$(6, 4)$). *A configuration is regular if and only if its dual configuration is regular.* This lemma is crucial to prove the uniqueness of the regular configuration.

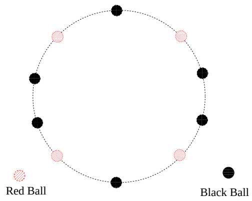

Figure 6: Dual configuration

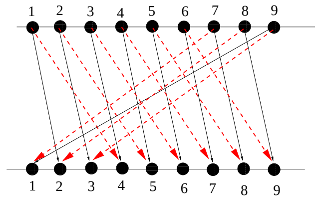

Figure 7: Network (9,3)

Motivation

By identifying the senders (top vertices) and receivers (bottom bertices) in the network (n, m) we can get Shift(n, m). (See Fig 7, which produces Shift$(9, 3)$.) Riis defines a parameter, "guessing number", in such networks [3] and our results give a lower bound for it.

Further direction

A configuration is symmetric if its automorphism group has maximal size for all possible configurations (in the polygon model). The symmetric configuration is not unique, (see Fig 1 and Fig 9 for two different symmetric configurations for Config$(6, 4)$), therefore symmetry does not lead to regularity. But we conjecture the converse is true: regularity implies symmetry. This is our next step to understand regularity. On the other hand, it would also be interesting to understand regularity in other graphs and find more applications in other fields.

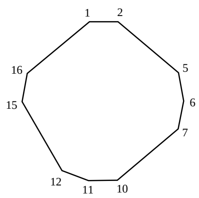

Figure 8: Not symmetry

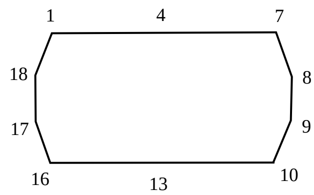

Figure 9: Symmetry but not regular

Reference

1 : Wu,T., *Disjoint cycles in shift graphs.* Draft.

2 : Riis,S., and Wu,T., *On the guessing number of shift graphs.* In preparation.

3 : Riis, S., *Information flows, graphs and their guessing numbers.* Submitted.