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CUT POINTS IN ČECH-STONE REMAINDERS

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ABSTRACT. We investigate cut points of subcontinua of $\beta\mathbb{R}\setminus\mathbb{R}$. We find, under CH, the topologically smallest type of subset of $\mathbb{R}$ that can support such a cut point. On the other hand we answer Question 66 of Hart and van Mill's *Open problems on* $\beta\omega$ [Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 97–125] by showing that it is consistent that all cut points are trivial (in a sense to be made precise in the paper).

INTRODUCTION

In this paper we study some types of points of $\mathbb{H}^*$, where $\mathbb{H}$ is the half line $[0,\infty)$. It is well known that $\mathbb{H}^*$ is an indecomposable continuum. As such it does not have any cut points, but it does have *sub cut points*, i.e., points that are cut points of some subcontinuum.

To find sub cut points one only has to look in ω^*: for each n let $\mathbb{I}_n$ be the interval $[n-1/4,n+1/4]$ and for $u\in\omega^*$ put $\mathbb{I}_u=\bigcap_{U\in u}\operatorname{cl}\bigcup_{n\in U}\mathbb{I}_n$. It is readily verified that $\mathbb{I}_u$ is a continuum and that u is a cut point of it. This argument shows that every point that is in the closure of a closed and discrete subset of $\mathbb{H}$ (a so-called *near point*) is in fact a sub cut point.

On the other hand, $\mathbb{H}^*$ also has points that are not sub cut points; such points were found by van Douwen in [2] and van Mill and Mills in [7]. This gives a clear cut reason why the space $\mathbb{H}^*$ is not homogeneous: there are points with visibly different topological behaviour.

It can also be shown that if x is a sub cut point of $\mathbb{H}^*$ then there are a discrete sequence $\langle\mathbb{I}_n:n\in\omega\rangle$ of closed intervals in $\mathbb{H}$ and a $u\in\omega^*$ such that x is a cut point of $\mathbb{I}_u$.

These results prompted further investigation of the structure of the continua $\mathbb{I}_u$. One question — mentioned as Question 66 (or Problem 265) in Hart and van Mill [5]—was whether every cut point of $\mathbb{I}_u$ is of the form $u\text{-}\lim x_n$, where $\langle x_n:n\in\omega\rangle$ is a sequence such that $x_n\in\mathbb{I}_n$ for all n; for clearly every such point is a cut point of $\mathbb{I}_u$. Let us call such points *trivial cut points.* We shall abbreviate $u\text{-}\lim x_n$ by x_u (the u-th term of the sequence).

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Nontrivial cut points were constructed by Baldwin and Smith in [1] and by Zhu in [9] using Martin's Axiom for Countable posets and CH, respectively.

The main result of our paper shows that in Laver's model for the Borel Conjecture (Laver [6]) no $\mathbb{I}_u$ has a nontrivial cut point. This confirms a conjecture mentioned in the paper of Hart and van Mill.

The paper is organized as follows: In Section 1 we summarize some known facts about cut points of $\mathbb{I}_u$, needed in Section 2. We also prove, under CH, a result that shows that every $\mathbb{I}_u$ may have a nontrivial cut point and that also identifies, in some sense, the topologically smallest type of set that can support a nontrivial cut point (Proposition 1.4 and Theorem 1.5 will give a meaning to the phrase 'topologically smallest'). In Section 2 we prove our main result and interpret it in terms of $\mathbb{H}^*$.

The survey in [4] will provide proofs for statements not proven here. In that paper a sub cut point is called a weak cut point. The term sub cut point seems more appropriate.

1. Various kinds of cut points

For convenience we consider the space $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I} = [0,1]$. We write $\mathbb{I}_n = \{n\} \times \mathbb{I}$ and we interpret $\mathbb{I}_u$ accordingly.

As we shall have no reason to take a union of a sequence of intervals in $\mathbb{I}$ itself we shall relieve our notational burden somewhat by agreeing that $\bigcup_n [a_n, b_n]$ means $\bigcup_n \{n\} \times [a_n, b_n]$, whenever $\langle [a_n, b_n] : n \in \omega \rangle$ is a sequence of intervals in $\mathbb{I}$.

We begin by giving the most convenient—for us—characterizations of cut points in the continua $\mathbb{I}_u$. Let $x \in \mathbb{I}_u$. We define two subsets A_x and B_x of $\mathbb{I}^\omega$ as follows:

$$A_x = \Bigl\{ a \in \mathbb{I}^\omega : x \in \operatorname{cl} \bigcup_{n\in\omega} \bigl[a(n), 1\bigr] \Bigr\}$$

and

$$B_x = \Bigl\{ b \in \mathbb{I}^\omega : x \in \operatorname{cl} \bigcup_{n\in\omega} \bigl[0, b(n)\bigr] \Bigr\},$$

where the closures are taken in $\beta\mathbb{M}$ of course.

Now if x is of the form x_u for some sequence $\langle x_n : n \in \omega \rangle$ in $\mathbb{I}^\omega$ then $A_x \cap B_x$ consists exactly of those sequences $\langle y_n : n \in \omega \rangle$ for which there is $U \in u$ such that $x(n) = y(n)$ for all $n \in U$. Otherwise the intersection of A_x and B_x is empty.

Thus the trivial cut points of $\mathbb{I}_u$ are characterized by the fact that $A_x \cap B_x \neq \varnothing$. The following proposition characterizes the nontrivial cut points in terms of A_x and B_x. If U is an open subset of $\mathbb{M}$ then $\operatorname{Ex} U$ denotes the largest open subset of $\beta\mathbb{M}$ whose intersection with $\mathbb{M}$ is U. Let us call a sequence $\langle r_n : n \in \omega \rangle$ that consists of positive reals and converges to 0 a *null sequence*.

Proposition 1.1. *Let x be a point of $\mathbb{I}_u$ for which $A_x \cap B_x = \varnothing$. Then the following are equivalent:*

1. *The point x is a cut point of $\mathbb{I}_u$.*
2. *The sets of the form $\operatorname{Ex} \bigcup_{n\in U} \bigl(a(n), b(n)\bigr)$, where $U \in u$, $a \in A_x$, and $b \in B_x$, form a local base at x.*
3. *For every null sequence $\langle r_n : n \in \omega \rangle$ there are $a \in A_x$ and $b \in B_x$ such that the set $U = \bigl\{ n : b(n) - a(n) < r_n \bigr\}$ belongs to u.*

Proofs can be found in Hart [4] and Zhu [10].

Before we continue we would like to insert a remark that may take away the confusion that tends to be caused by condition (2) in the previous proposition.

Remark 1.2. Consider the following two conditions that one may impose on a point x of $\mathbb{I}_u$ for which $A_x \cap B_x = \varnothing$:

(α) If F is closed in $\mathbb{M}$ and $x \notin \operatorname{cl} F$ then there are $a \in A_x$, $b \in B_x$, and $U \in u$ such that $F(a,b,U) = \bigcup_{n\in U} \bigl[a(n), b(n)\bigr]$ is disjoint from F.

(β) If F is closed in $\mathbb{M}$ and $x \in \operatorname{cl} F$ then there are $a \in A_x$, $b \in B_x$, and $U \in u$ such that $F(a,b,U)$ is contained in F.

Note that condition (α) is equivalent to condition (2) in Proposition 1.1 and hence characterizes nontrivial cut points. Condition (β) on the other hand is stronger than (α) but not equivalent to it; in [9, 10] Zhu calls points that satisfy (β) *simple* cut points.

In Section 2 we shall need the following variation of condition (3):

Lemma 1.3. *A point x of $\mathbb{I}_u$ with $A_x \cap B_x = \varnothing$ is a cut point if and only if for every $f \in \omega^\omega$ with $f(n) > 0$ for all n there is $g \in \omega^\omega$ with $g(n) < f(n)$ for all n and such that the closed set $\bigcup_n \bigl[g(n)/f(n), \bigl(g(n)+1\bigr)/f(n)\bigr]$ belongs to x.*

Proof. The condition of the lemma clearly implies condition (3) of Proposition 1.1.

For the other direction fix f and pick $a \in A_x$ and $b \in B_x$ such that $b(n) - a(n) < 1/f(n)$ for all n. Now choose g such that $g(n) \leqslant a(n) \cdot f(n) < g(n) + 1$ for all n. Then either g or $g + 1$ will do. □

We now investigate how close to trivial a nontrivial cut point can be.

A consequence of condition (3) in Proposition 1.1 is that a nontrivial cut point is not in the closure of any closed discrete subset of $\mathbb{M}$, i.e., every nontrivial cut point is a *far point.* (A far point is a point that is not near.)

Indeed, if D is closed and discrete in $\mathbb{M}$ then $D_n = D \cap \mathbb{I}_n$ is finite for every n. For each n let r_n be the minimum distance between two adjacent points of D_n ($r_n = 1$ if $D_n = \varnothing$). Now let x be a nontrivial cut point of $\mathbb{I}_u$ and take $a \in A_x$ and $b \in B_x$ such that $b(n) - a(n) < r_n$ for all n. Then $\bigcup_n \bigl(a(n), b(n)\bigr)$ picks at most one point from each D_n; but this gives us a set that, by definition, x has to avoid.

On the other hand, under CH or weaker one can find nontrivial cut points that are in the closure of nowhere dense subsets of $\mathbb{M}$ (see Hart [4] or Zhu [9]; Theorem 1.5 provides another example).

To see what kind of sets can still support nontrivial cut points we consider scattered sets, as these are arguably the topologically smallest type of subsets of $\mathbb{R}$.

We assume the reader is familiar with the notion of a scattered space. We use $X^{(\alpha)}$ to denote the α-th derived set of X. If X is compact then the last α for which $X^{(\alpha)} \neq \varnothing$ is the *scattered height* of X, denoted $\operatorname{ht}(X)$. Furthermore, if $x \in X$ then the last α for which $x \in X^{(\alpha)}$ is called the *scattered rank* of x.

Consider now a scattered closed subset D of $\mathbb{M}$. We define its scattered height as $\operatorname{ht}(D) = \sup_n \operatorname{ht}(D \cap \mathbb{I}_n)$. We have seen that D cannot support a nontrivial cut point if D is discrete or, equivalently, if its scattered height is 0. The same can be said if its scattered height is finite; this follows from the following proposition.

Proposition 1.4. *If x is a far point and D is a closed subset of $\mathbb{M}$ of finite scattered height then $x \notin \operatorname{cl} D$.*

Proof. The argument is by induction on height: If the height of D is k then $D^{(k)}$ is closed and discrete, hence x is not in its closure and there is a neighborhood, X, of it that is disjoint from $D^{(k)}$. The scattered height of $X \cap D$ is then at most $k-1$, so by our inductive assumption x is not in the closure of $X \cap D$. It follows that x is not in the closure of D. □

The next result shows that this is best possible; under CH there is a closed set of scattered height ω that supports a nontrivial cut point for every $\mathbb{I}_u$. This theorem also solves, negatively, Question 13.2 from Hart [4], which asks for an $\mathbb{I}_u$ without nontrivial cut points.

Theorem 1.5 (CH)**.** *Take, for each n, a copy K_n of the ordinal space ω^n+1 in $\mathbb{I}_n$. Then for every $u \in \omega^*$ there is a nontrivial cut point of $\mathbb{I}_u$ that is in the closure of $\bigcup_n K_n$.*

Proof. We shall construct two sequences $\langle a_\alpha : \alpha \in \omega_1 \rangle$ and $\langle b_\alpha : \alpha \in \omega_1 \rangle$ in $\mathbb{I}^\omega$ such that:

(1) If $\alpha \in \beta \in \omega_1$ then $a_\alpha <^* a_\beta <^* b_\beta <^* b_\alpha$.
(2) For every null sequence $\langle r_n \rangle_n$ there is $\alpha \in \omega_1$ such that $b_\alpha(n) - a_\alpha(n) < r_n$ for all but finitely many n.
(3) For every $x \in \mathbb{I}^\omega$ there is $\alpha \in \omega_1$ such that $x(n) \notin \bigl[a_\alpha(n), b_\alpha(n)\bigr]$ for all but finitely many n.

To avoid cumbersome notation we shall use $I(\alpha, n)$ to denote both $\bigl[a_\alpha(n), b_\alpha(n)\bigr]$ and $\{n\} \times \bigl[a_\alpha(n), b_\alpha(n)\bigr]$; the context should always dictate which meaning we use.

Consider now the closed set

$$C = \mathbb{M}^* \cap \bigcap_{\alpha \in \omega_1} \operatorname{cl} \bigcup_{n \in \omega} I(\alpha, n).$$

Condition (1) implies that C meets every $\mathbb{I}_u$, condition (2) implies that $C \cap \mathbb{I}_u$ consists of exactly one point for every u, and condition (3) implies that every such point is a nontrivial cut point (see Proposition 1.1). To make sure that C is a subset of the closure of $\bigcup_n K_n$ we are forced, by Proposition 1.4, to add the following condition to our list.

(4) For every α we have $\lim_{n\to\infty} \operatorname{ht}\bigl(I(\alpha, n) \cap K_n\bigr) = \infty$.

Now let $\{x_\alpha : \alpha \in \omega_1\}$ list $\mathbb{I}^\omega$ and let $\{c_\alpha : \alpha \in \omega_1\}$ list the set of all null sequences. During the construction we make sure that for every α the set

$$\bigl\{n : x_\alpha(n) \in I(\alpha, n) \text{ and } b_\alpha(n) - a_\alpha(n) \geqslant c_\alpha(n)\bigr\}$$

is finite.

We start by putting $a_{-1}(n) = 0$ and $b_{-1}(n) = 1$ for all n.

Now let $\alpha \in \omega_1$ and assume everything has been taken care of below α. If α is a successor or 0 put $a^\alpha = a_{\alpha-1}$ and $b^\alpha = b_{\alpha-1}$. If α is a limit first choose an increasing cofinal sequence $\langle \gamma_i : i \in \omega \rangle$ in α. Then choose an increasing sequence $\langle n_i : i \in \omega \rangle$ in ω such that for every i: if $n \geqslant n_i$ then $\operatorname{ht}\bigl(I(\gamma_i, n) \cap K_n\bigr) \geqslant i$ and if $j < i$ then $a_{\gamma_j}(n) < a_{\gamma_i}(n)$ and $b_{\gamma_j}(n) > b_{\gamma_i}(n)$.

Now define a^α and b^α by: if $n < n_0$ then $a^\alpha(n) = 0$ and $b^\alpha(n) = 1$, and: if $n_i \leqslant n < n_{i+1}$ then $a^\alpha(n) = a_{\gamma_i}(n)$ and $b^\alpha(n) = b_{\gamma_i}(n)$. In either case $\lim_{n\to\infty} \operatorname{ht}\Bigl(\bigl[a^\alpha(n), b^\alpha(n)\bigr] \cap K_n\Bigr) = \infty$.

Choose, for every n a point y_n in $\bigl[a^\alpha(n), b^\alpha(n)\bigr] \cap K_n$ of maximum scattered rank, say k_n, and an interval J_n around y_n of length at most $c_\alpha(n)$ and contained in $\bigl[a^\alpha(n), b^\alpha(n)\bigr]$. Now pick, if $k_n \geqslant 1$, a point z_n in J_n of rank $k_n - 1$ but not equal to $x_\alpha(n)$. Finally then choose a_α and b_α in such a way that $I(\alpha, n)$ has z_n in its interior but does not contain $x_\alpha(n)$ whenever $k_n \geqslant 1$, in the finitely many cases when $k_n = 0$ the choice of $a_\alpha(n)$ and $b_\alpha(n)$ is immaterial. This concludes the proof. □

Remark 1.6. We note that a suitable modification of the arguments of Baldwin and Smith from [1] can be used to show that under Martin's Axiom for Countable posets there is some $u \in \omega^*$ for which $\mathbb{I}_u$ has a nontrivial cut point that is in the closure of a scattered set of height ω.

Our stronger assumption of CH yields a stronger conclusion: *every* $\mathbb{I}_u$ has such a nontrivial cut point.

Let us summarize what kind of nontrivial cut points we can have.

To begin, every nontrivial cut point is a far point and hence not in the closure of a set of finite scattered height. On the other hand, we just constructed a non-trivial cut point in the closure of a set of scattered height ω.

The point constructed by Baldwin and Smith is *remote*, which means that it is not in the closure of any closed nowhere dense subset of $\mathbb{M}$. Finally, the point constructed by Zhu is not remote but still quite far—it has a base of perfect sets. We remark that a cut point is remote iff it is simple, i.e., satisfies condition (β) of Remark 1.2.

In [10] Zhu showed that in Laver's model there are no remote cut points; in the next section we show that in this model there are no nontrivial cut points at all, by showing that no far point is a cut point.

2. A model without nontrivial cut points

In this section we prove that there are no nontrivial cut points in Laver's model for the Borel conjecture. We need to describe Laver's poset of course.

To begin, a *Laver tree* is a subtree T of ${}^{<\omega}\omega$ of the following form: There is a node s_T of T such that for every $t \in T$ either $s_T \subseteq t$ or $t \subseteq s_T$; we call s_T the root node of T. Furthermore, if $t \in T$ extends s_T then the set of $i \in \omega$ for which $t ^\frown i \in T$ is infinite. The set of branches through T is denoted by $[T]$.

The Laver poset $\mathbb{L}$ is the set of Laver trees, ordered by inclusion. If G is a generic filter on $\mathbb{L}$ then it determines a new real, a Laver real, as follows: In $V[G]$ the intersection $\bigcap [T] : T \in G\bigr\}$ consists of one point f; this point f is the Laver real and it determines G because $G = \bigl\{T : f \in [T]\bigr\}$. The function f goes much faster to infinity than any function from V and this is what we shall use to destroy nontrivial cut points.

A crucial property of Laver forcing is the following.

Lemma 2.1. *If $T \in \mathbb{L}$, F is a finite set in V and $T \Vdash \dot{x} \in F$ then there are $S \leqslant T$* with the same root as T *and $a \in F$ such that $S \Vdash \dot{x} = a$.*

Laver's model is obtained from a model of CH, using a countable-support iterated forcing construction of length ω_2 where each time one forces with the poset $\mathbb{L}$.

Theorem 2.2. *In Laver's model there are no nontrivial cut points.*

Proof. Let $\langle \mathbb{P}_\alpha : \alpha \leqslant \omega_2 \rangle$ be a countable support iteration, where at every stage $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{L}}$. We let $\mathbb{P} = \mathbb{P}_{\omega_2}$ and we let G be a generic filter on $\mathbb{P}$.

Finally we assume that, in $V[G]$, x is a far point of $\mathbb{M}$; we must show that x is not a cut point of the $\mathbb{I}_u$ that contains it.

An easy reflection argument will produce an ordinal $\alpha < \omega_2$ such that, in the model $V[G \restriction \alpha]$, the point $x \restriction \alpha$ is a far point of $\mathbb{M}$. We show that, in $V[G]$, there is no nontrivial cut point extending the filter $x \restriction \alpha$ (in particular, x is not a cut point).

For this we consider the Laver real f, added at the next stage. It induces partitions of the intervals $\mathbb{I}_n$:

$$\mathbb{I}_n = \bigcup \Big\{ \big[i/f(n), (i+1)/f(n)\big] : i < f(n) \Big\}.$$

In what follows we shall denote the interval $\big[i/f(n), (i+1)/f(n)\big]$ by $I(n,i)$.

Our task will be complete once we show that for every $g : \omega \to \omega$ from $V[G]$ satisfying $(\forall n \in \omega)\big(g(n) < f(n)\big)$ there is an $X \in x \restriction \alpha$ disjoint from the set $\bigcup_n I\big(n, g(n)\big)$.

The intuition behind this is that f grows so fast that the intervals $I\big(n, g(n)\big)$ become very thin, as thin as points. As x is a far point, we must then be able to avoid those intervals.

For convenience we drop (as we may) all references to α and simply assume that we have a far point x in V and show that it has the property mentioned in the previous paragraph.

So assume that g, from $V[G]$, is a function below the first Laver real f. A straightforward application of Lemma 6 from Laver [6] gives us, in $V[f]$, a function F such that $F(n)$ is an n-element subset of $f(n)$ and $g(n) \in F(n)$ for all n.

In V this gives us a $T \in \mathbb{L}$ that forces all this:

$$T \Vdash_{\mathbb{L}} (\forall n \in \omega)\Big(\dot{F}(n) \subseteq \dot{f}(n) \wedge \big|\dot{F}(n)\big| = n\Big),$$

and there is a condition $p \in \mathbb{P}$, with first coordinate T, such that

$$p \Vdash_{\mathbb{P}} (\forall n \in \omega)\big(\dot{g}(n) \in \dot{F}(n)\big).$$

We will be done once we show that the set of $S \in \mathbb{L}$ for which there is $X \in x$ such that

$$S \Vdash_{\mathbb{L}} X \cap \bigcup \big\{ I(n,i) : i \in \dot{F}(n), n \in \omega \big\} = \varnothing$$

is dense below T. It suffices to find such an S below T (the same argument works below any other element below T).

We begin by observing that if $t \in T$ and $t \frown i \in T$ then

$$T \restriction (t \frown i) \Vdash \dot{f}\big(|t|\big) = i,$$

and hence

$$T \restriction (t \frown i) \Vdash \dot{F}\big(|t|\big) \subseteq i.$$

Therefore we may, using Lemma 2.1, assume that we have a partial function $H : T \times \omega \to [\omega]^{<\omega}$ such that

$$T \restriction (t \frown i) \Vdash \dot{F}\big(|t|\big) = H(t,i).$$

We enumerate $H(t,i)$ in increasing order as $\big\{h(t,i,j) : j < |t|\big\}$ and we denote the interval $\big[h(t,i,j)/i, \big(h(t,i,j)+1\big)/i\big]$ by $I(t,i,j)$. It should be clear that all we need is a Laver tree S below T and an element X of x such that

$$X \cap \big(\{n\} \times I(t,i,j)\big) = \varnothing \qquad (*)$$

whenever n, t, i, and j are such that $t^\frown i \in S$, $|t| = n$, and $j < n$. This then is the goal of the rest of the proof.

Let n_0 be the length of the root node of T and denote the root node by t_{n_0}. We shall thin out T by induction on $n \geqslant n_0$. At each step, when we have a tree T_{n-1}, we choose a node t_n of T_{n-1} and we make sure that the nodes $t_0, \ldots, t_n$ will be in the tree T_n. Furthermore the choice will be made in such a way that in the end the tree $T_\infty = \{t_n : n \in \omega\}$ is a Laver tree with root t_{n_0} (for $n < n_0$ we let $t_n = t_{n_0} \restriction n$). This amounts to what is generally called a 'standard fusion argument'; the t_n can be chosen in a very canonical way, described in detail in [6, p. 156].

At step n_0 we choose an infinite subset of $\{i : t_{n_0}{}^\frown i \in T\}$ such that the sequence of cubes $\prod_{j<n_0} I(t_{n_0}, i, j)$ converges to a point $r_{n_0}(t_{n_0})$ of $\mathbb{I}^{n_0}$. After throwing away the other i's we end up with the tree T_{n_0}.

Now let $n > n_0$. Consider the tree T_{n-1} and choose t_n in T_{n-1} as an immediate successor of some t_k with $n_0 \leqslant k < n$. Using the nodes t_k we split T_{n-1} into subtrees: if $n_0 \leqslant k \leqslant n$ then $T_{n,k}$ is the union of all $T \restriction t$ where t is an immediate successor of t_k, but none of the t_j for $k < j \leqslant n$ (note that t_k is the root node of $T_{n,k}$).

Fix k between n_0 and n. For every $t \in \operatorname{Lev}(T_{n,k}, n)$ we choose an infinite subset of $\{i : t^\frown i \in T_{n,k}\}$ such that the cubes $\prod_{j<n} I(t, i, j)$ converge to a point $r_n(t)$ of $\mathbb{I}^n$ and we throw away the parts of $T_{n,k}$ above the other successors of t. Note that such a choice is possible because for every fixed t the diameters of the cubes $\prod_{j<n} I(t, i, j)$ converge to 0. We continue down to the level of t_k, all the time thinning out $T_{n,k}$ further to a tree $T'_{n,k}$ with the property that for every t with $|t_k| \leqslant |t| < n$ the sequence

$$\left\langle r_n(t^\frown i) : t^\frown i \in T'_{n,k} \right\rangle$$

converges to a point $r_n(t) \in \mathbb{I}^n$.

When this is done for every k we piece the trees $T'_{n,k}$ together to form the tree T_n.

In the end we get the tree $T_\infty = \bigcap_n T_n = \{t_n : n \in \omega\}$. It gives us, for $n \geqslant n_0$, the following picture in $\mathbb{I}^n$ (it may be better to think of this as taking place in $\mathbb{I}^n_n$): for every $k \in [n_0, n]$ a point $r_n(t_k)$ with a sequence $\left\langle r_n(t^\frown i) : t^\frown i \in T_\infty \right\rangle$ converging to it. Each of the terms of this sequence has a sequence converging to it and so on until we reach the points above t_k that are on level n, then we get sequences of cubes converging to the corresponding $r_n(t)$.

In $\mathbb{I}_n$ we get the finite set of coordinates of the points $r_n(t_k)$:

$$F_n = \Big\{ \big\langle n, r_n(t_k, j) \big\rangle : n_0 \leqslant k \leqslant n, j < n \Big\},$$

together with sequences converging to them:

$$G_n = \Big\{ \big\langle n, r_n(t_k{}^\frown i, j) \big\rangle : n_0 \leqslant k \leqslant n, t_k{}^\frown i \in T_\infty, j < n \Big\}.$$

Because x is a far point of $\mathbb{M}$ we may find an element X of x that is disjoint from $\bigcup_n (F_n \cup G_n)$ (each $F_n \cup G_n$ has scattered height 1). Our intuition did not let us down, we will manage to confine g near the set $\bigcup_n (F_n \cup G_n)$.

Now we are ready for the final recursive trim.

The points $\big\langle n_0, r_{n_0}(t_{n_0}, j) \big\rangle$ are not in X, so for all but finitely many i with $t_{n_0}{}^\frown i \in T_\infty$ every interval $\{n_0\} \times I(t_{n_0}, i, j)$ is disjoint from X. Discard those finitely many i.

Now let t_m be one of the surviving direct successors of t_{n_0}. The property that allows the trimming to continue is that $\langle n, r_n(t_m, j)\rangle \notin X$ for all $n \geqslant |t_m|$ (for $n \geqslant m$ use the fact that $\langle n, r_n(t_m, j)\rangle \in F_n$ and for $n < m$ use the fact that $\langle n, r_n(t_m, j)\rangle \in G_n$; it's in the sequence converging to $\langle n, r_n(t_{n_0}, j)\rangle$).

For every n from $|t_m| + 1$ through m we have to discard those finitely many successors t of t_m for which some $\langle n, r_n(t, j)\rangle$ is in X and any of the finitely many more $t = t_m \frown i$ for which $\{|t_m|\} \times I(t_m, i, j)$ meets X. For $n > m$ we have $\langle n, r_n(t_m, j)\rangle \in F_n$, hence for all direct successors t of t_m we have $\langle n, r_n(t, j)\rangle \in G_n$ and so no more trimming of successors of t_m is required.

The same strategy applies to nodes higher up in T_∞: whenever a node t_k has survived it will lose finitely many direct successors. These successors must be discarded because of possible nonempty intersections with X in $\mathbb{I}_n$ for $|t_k| \leqslant n \leqslant k$. In the end we get our tree S satisfying $(*)$. □

To end this paper we show how Theorem 2.2 may be applied in the theory of far points of $\mathbb{H} = [0, \infty)$. The theorem implies that in Laver's model a point of $\mathbb{H}^*$ is a near point if and only if it is a sub cut point and hence that the set of near points is topologically invariant in $\mathbb{H}^*$.

This is a partial answer to a question of van Douwen from [3] whether the set of remote points of $\mathbb{H}$ is topologically invariant. Under CH it is not: in [8] Yu showed that if u is a P-point then any two cut points of $\mathbb{I}_u$ can be mapped to each other by an autohomeomorphism of $\mathbb{M}^*$ that leaves every $\mathbb{I}_v$ invariant; so, for example, a remote point can become a near point. This can then be modified to produce an autohomeomorphism of $\mathbb{H}^*$ with the same effect.

This suggests the obvious question whether the set of remote points of $\mathbb{H}^*$ is topologically invariant in Laver's model.

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