KLEINIAN GROUPS IN HILBERT SPACES

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The theory of discrete groups acting on finite dimensional Euclidean open balls by hyperbolic isometries was borne around the end of 19th century within the works of Fuchs, Klein and Poincaré. We develop the theory of discrete groups acting by hyperbolic isometries on the open unit ball of an infinite dimensional separable Hilbert space. We present our investigations on the geometry of limit sets at the sphere at infinity with an attempt to highlight the differences between the finite and infinite dimensional theories. We discuss the existence of fixed points of isometries and the classification of isometries. Various notions of discreteness that were equivalent in finite dimensions, no longer turn out to be in our setting. In this regard, the robust notion of strong discreteness is introduced and we study limit sets for properly discontinuous actions. We go on to prove a generalization of the Bishop-Jones formula for strongly discrete groups, equating the Hausdorff dimension of the radial limit set with the Poincaré exponent of the group. We end with a short discussion on conformal measures and their relation with Hausdorff and packing measures on the limit set.

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I am very lucky indeed to have such good vibrations showered upon me.

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CHAPTER 1

INTRODUCTION

The theory of discrete groups acting on (finite-dimensional) Euclidean open balls by hyperbolic isometries was born around the end of 19th century within the works of Fuchs, Klein and Poincaré. The modern theory of these groups and their associated geometry has been largely shaped by Mostow, Margulis, Thurston, Sullivan and Gromov among others.

The earliest instance we could find that mentioned infinite-dimensional hyperbolic geometry modeled on Hilbert space was in [49] where J. B. Wilker encouraged their study in the closing remarks of his notes on inversive geometry. We also mention that spectral theory in such a setting was studied since the early '40s by Pontryagin, Iohvidov, Krein and then pushed further by Ky Fan in the '60s - more on this can be found in section on fixed points. Dennis Sullivan, in his Seminar on Conformal and Hyperbolic Geometry [42] indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable infinite-dimensional Hilbert space. This was the earliest reference we could find, however we are confident that such an idea must have been discussed prior to that. Later in the early '90s, Misha Gromov lamented the paucity of results regarding such actions in his lectures "Asymptotic Invariants of Infinite Groups" [13] where he encouraged their investigation: "The spaces like this ... look as cute and sexy to me as their finite dimensional siblings but they have been for years shamefully neglected by geometers and algebraists alike".

We develop the theory of discrete groups acting by hyperbolic isometries on the open unit ball of an infinite-dimensional separable Hilbert space. We present our investigations on the geometry of limit sets at the sphere at infinity and build appropriate forms of thermodynamic formalism, ergodic theory, and geometric measure theory to aid us in this regard. There will be an attempt to highlight the differences between the finite- and infinite-dimensional theories to provide a perspective on what aspects of the classical finite-dimensional theory are dimension-free.

The existence of fixed-points of isometries and their structure will be discussed. Note that classical fixed point theorems do not apply. The concept of *strongly discrete* groups is introduced and we study limit sets for properly discontinuous actions. We define *convex-cobounded* groups and characterize them in terms of radial points in the limit set and go on to characterize the groups whose limit sets are compact. We go on to prove a generalization of the Bishop-Jones formula, $\mathrm{HD}(L_r(G)) = \delta_G$. Although the original proof of Bishop and Jones crucially uses the compactness of the sphere at infinity, our proof avoids such a dependence. We emphasize that this is proved without any knowledge of conformal measures, in fact, without mentioning them at all.

Conformal measures, introduced for Fuchsian groups by S. J. Patterson and extended to any finite dimension by D. P. Sullivan, are indispensable for a finer analysis of the fractal structure of the limit set. In the finite-dimensional case these measures always exist. In the Hilbertian case they cannot even be defined if the Poincaré exponent is equal to $+\infty$. Even if this exponent is finite, the fact that the boundary of hyperbolic space is no longer compact forces us to tread with care.

CHAPTER 2

SOME INFINITE-DIMENSIONAL HYPERBOLIC GEOMETRY

2.1. Infinite-Dimensional Models of Hyperbolic Geometry

DEFINITION 2.1. Let us define

$$\mathcal{H} \;:=\; \ell_2 \;:=\; \{x = (x_i)_{i=1}^{\infty} |\; x_i \in \mathbb{R} \; and \; \sum_{i=1}^{\infty} {x_i}^2 < \infty\}.$$

The space $\mathcal{H}$ comes with a standard inner product defined by

$$\langle x, y\rangle := \langle x, y\rangle_{\mathcal{H}} := \sum_i x_i y_i \ ,$$

which in turn induces a norm

$$\|x\| := \|x\|_{\mathcal{H}} := \langle x, x\rangle^{1/2} = \Big(\sum_i x_i^2\Big)^{1/2}$$

via which we can define a metric

$$d_{\mathcal{H}}(x, y) := \|x - y\|.$$

We note that the standard inner product is what allows us to do geometry on Hilbert space as it allows us to define the notions of length, angle, (infinitesimal) length element and so on.

DEFINITION 2.2. We denote the orthogonal group of $\mathcal{H}$ by

$$\begin{aligned}\mathscr{O}(\mathcal{H}) :=& \{T : \mathcal{H} \to \mathcal{H} |\; T \text{ is a bijective linear isometry}\} \\ =& \{T \in L(\mathcal{H}) |\; T^*T = \mathrm{id} = TT^*\}.\end{aligned}$$

Definition 2.3. Let

$$\mathbb{L}_\infty \ := \ \{(a,x) \in \mathbb{R} \times \ell_2 : a > 0 \ , \ -a^2 + \|x\|^2 = -1\}$$

and

$$\mathbb{B}_\infty \ := \ \{x \in \ell_2 : \|x\| < 1\}$$

and

$$\mathbb{H}_\infty \ := \ \{x \in \ell_2 : x_1 > 0\}.$$

On $\mathbb{L}_\infty$ define

$$\langle (a,u),(b,v)\rangle_{x,\mathbb{L}} \ := \ -ab + \langle u,v\rangle \ .$$

On $\mathbb{B}_\infty$ define

$$\langle u,v\rangle_{x,\mathbb{B}} \ := \ \frac{4\langle u,v\rangle}{(1-\langle x,x\rangle)^2} \ .$$

On $\mathbb{H}_\infty$ define

$$\langle u,v\rangle_{x,\mathbb{H}} \ := \ \frac{\langle u,v\rangle}{x_1^2} \ .$$

When there is no need to distinguish between the models they will be denoted simply by $\mathbb{K}_\infty$ for the rest of this section. However, in the sequel we will use $\mathbb{K}_\infty$ for either the ball model $\mathbb{B}_\infty$ or the half-space model $\mathbb{H}_\infty$ and distinguish the Lorentzian model by using its original notation, viz. $\mathbb{L}_\infty$.

Given the Riemannian inner product (metric) we define a field of norms

$$\|u\|_{x,\mathbb{K}} := \langle u,u\rangle_{x,\mathbb{K}}^{1/2} \ .$$

Let us denote the length element by

$$ds_{\mathbb{K}} := \|dx\|_{x,\mathbb{K}} \ .$$

Now for every piecewise C^1-path $\gamma : [a,b] \to \mathbb{K}_\infty$ we define the length of γ to be

$$l_{\mathbb{K}}(\gamma) = \int_\gamma ds_{\mathbb{K}} = \int_a^b \|\gamma'(t)\|_{\gamma(t),\mathbb{K}} dt \ .$$

Now define the distance function as usual as follows: for any two points $x, y \in \mathbb{K}_\infty$, let

$$d_{\mathbb{K}}(x,y) := \inf_{\gamma : x \rightsquigarrow y} l_{\mathbb{K}}(\gamma) \cdot$$

We have the following formulae for the associated length elements:

$$\begin{aligned} ds^2_{\mathbb{L}} &= -da^2 + \|dx\|^2 \ , \\ ds^2_{\mathbb{B}} &= \frac{4\|dx\|^2}{(1-\|x\|^2)^2} \quad \text{and} \\ ds^2_{\mathbb{H}} &= \frac{\|dx\|^2}{x_1^2} \ . \end{aligned}$$

Here $\|\cdot\|$ refers to the standard norm in Hilbert space, i.e. $\|dx\|^2 = \sum_i dx_i^2$.

NOTATION 2.4. For emphasis we may decorate the notation for the standard Hilbertian inner product, norm and metrics by $\langle \cdot \ , \ \cdot \rangle_{\mathcal{H}}$, $\| \cdot \|_{\mathcal{H}}$ and $d_{\mathcal{H}}(\cdot \ , \ \cdot)$. Similarly for $\mathbb{L}_\infty$, $\mathbb{B}_\infty$ and $\mathbb{H}_\infty$ we have $\langle \cdot \ , \ \cdot \rangle_{\mathbb{L}}$, $\| \cdot \|_{\mathbb{L}}$, $d_{\mathbb{L}}(\cdot \ , \ \cdot)$ and $\langle \cdot \ , \ \cdot \rangle_{\mathbb{B}}$, $\| \cdot \|_{\mathbb{B}}$, $d_{\mathbb{B}}(\cdot \ , \ \cdot)$ and $\langle \cdot \ , \ \cdot \rangle_{\mathbb{H}}$, $\| \cdot \|_{\mathbb{H}}$, $d_{\mathbb{H}}(\cdot \ , \ \cdot)$ respectively. We will also use $\langle \cdot \ , \ \cdot \rangle_{\mathbb{K}}$, $\| \cdot \|_{\mathbb{K}}$ and $d_{\mathbb{K}}(\cdot \ , \ \cdot)$ in case we do not want to specify a particular model of hyperbolic Hilbert space. For example $B_{\mathbb{K}}(0,r)$ would mean the ball centered at the origin of hyperbolic radius r, without specifying the model. Here, we implicitly assume that the origin for $\mathbb{L}_\infty$ is $(1,0,0,\dots) \in \mathbb{R} \times \ell_2$, that the origin for $\mathbb{H}_\infty$ is the point $\uparrow := (1,0,0,\dots) \in \ell_2$ while it is the usual $(0,0,0,\dots) \in \ell_2$ for $\mathbb{B}_\infty$. Similarly, $S_{\mathbb{H}}(a,\alpha)$ would be the sphere in the $\mathbb{H}_\infty$ model with center $a \in \mathbb{H}_\infty$ and with hyperbolic radius α. Finally note that if we do not put a subscript we mean the associated notion in the standard Hilbert space with its usual Euclidean structure.

2.2. Conformal Maps and Liouville's Theorem

We define an infinite-dimensional Riemannian manifold to be Hilbert manifold M equipped with a Riemannian structure, viz. a field a inner products $\big(\langle \cdot \ , \cdot \rangle_x\big)_{x \in M}$ such that:

- For every $x \in M$, $\big(\langle \cdot \ , \cdot \rangle_x\big)$ is an inner product on the tangent space $T_x M$.

- If $U \subseteq M, V \subseteq \mathcal{H}$ and if $\phi : U \leftrightarrow V$ is a coordinate chart, then for every $x \in U$, there exists a C_x such that for all $u \in T_x M$

$$\frac{1}{C_x}\|\phi'[u]\| \leq \|u\|_x \leq C_x\|\phi'[u]\|$$

where, $\|u\|_x := \sqrt{\langle u, u\rangle_x}$.

As a result of our transition maps being diffeomorphisms, for each $x \in M$ it suffices to check the second condition for a particular chart from which it follows that it holds for every other chart, with different values for C_x.

We define conformality for maps between Hilbert manifolds M and N, where $T_x M$ denotes the tangent space at x to M, and give a concrete description of an important class of conformal maps, viz. inversions in spheres.

Definition 2.5. Let $f : M \to N$ be a diffeomorphism. Then f is called *conformal* when there exists a differentiable positive function $\alpha : M \to \mathbb{R}$ such that for all $x \in M$ and for all $v, w \in T_x M$

$$\langle d_x f(v), d_x f(w)\rangle_{f(x)} = \alpha^2(x)\langle v, w\rangle_x \cdot$$

For each x, we call the number $\alpha(x)$ the *scaling constant* of the map $d_x f$. Note that $\alpha(x)$ is equal to the operator norm $\|d_x f\|$.

Definition 2.6. Let $x \in \mathcal{H}$ and $\alpha > 0$. Then the *inversion with respect to the sphere* $S(x, \alpha)$ is the map

$$i_{x,\alpha} : z \mapsto \alpha^2 \frac{z - x}{\|z - x\|^2} + x$$

We make the convention that $i_x := i_{x,1}$. Note that as defined $i_{x,\alpha} : \mathcal{H} \setminus \{x\} \to \mathcal{H} \setminus \{x\}$; however, we are more interested in its extension to $\widehat{\mathcal{H}} := \mathcal{H} \cup \{\infty\}$, $i_{x,\alpha} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ by defining $x \mapsto \infty$ and $\infty \mapsto x$.

We define the topology on $\mathcal{H}$ as follows: a subset $U \subseteq \widehat{\mathcal{H}}$ is open if and only if $U \cap \mathcal{H}$ is open and $\mathcal{H} \setminus U$ is bounded if $\infty \in U$. We remark that the word *bounded* cannot be replaced by the word *compact* in this definition as one may recall from the familiar definition for the

one-point compactification of a locally compact space. Notice that the map $i_{x,\alpha} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ is a homeomorphism.

It may be verified that $i_{x,\alpha} : \mathcal{H} \setminus \{x\} \to \mathcal{H} \setminus \{x\}$ is a conformal map and that the following formulae for the scaling constant and "direction" of $i'_{x,\alpha}(z)$ analogous to the finite-dimensional case are still valid:

$$\|i'_{x,\alpha}(z)\| = \frac{\alpha^2}{\|z-x\|^2} \text{ and}$$

$$\frac{i'_{x,\alpha}(z)}{\|i'_{x,\alpha}(z)\|} = \mathrm{id} - 2P_{z-x} \ ,$$

where, $P_{z-x} : \mathcal{H} \to \mathcal{H}$ is orthogonal projection onto the linear subspace spanned by $z - x$. Note here that $\|i'_{x,\alpha}(z)\|$ refers to the operator norm, i.e. the scaling constant in this case, of the operator $i'_{x,\alpha}(z) : \mathcal{H} \to \mathcal{H}$.

The following theorem generalizes the classical result known as Liouville's theorem, that tells us that conformal transformations from finite-dimensional Euclidean spaces (or their respective unit spheres) to themselves in dimensions greater than 3 are restrictions of Möbius transformations.

THEOREM 2.7 (Liouville). *Let $U \in \mathcal{H}$ be a non-empty open connected set and let $\phi : U \to \mathcal{H}$ be a conformal map, then one of the following cases holds:*

- *there exist $\lambda > 0$, $x, y \in \mathcal{H}$ and $M \in \mathscr{O}(\mathcal{H})$ such that*

$$\phi(z) = \lambda M[i_x(z)] + y,$$

- *there exist $\lambda > 0$, $y \in \mathcal{H}$ and $M \in \mathscr{O}(\mathcal{H})$ such that*

$$\phi(z) = \lambda M[z] + y.$$

Note that in either case the map ϕ extends uniquely to a conformal map $\hat{\phi} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. As in the finite dimensional case, the map $\hat{\phi}$ is called a Möbius transform and we denote the class of such maps by $\mathcal{M}ob(\widehat{\mathcal{H}})$. To avoid clutter and when it is clear we avoid decorating the extension.

REMARK 2.8. It follows from the formula for the derivative of an inversion given above that for any given conformal map ϕ of the form $\phi(z) = \lambda M(i_x(z)) + y$ we have that

$$\|\phi'(z)\| = \frac{\lambda}{\|z - x\|^2}$$

for every $z \in \mathcal{H}$. Note that if ϕ is of the form $\phi(z) = \lambda M(z) + y$ we have that $\|\phi'(z)\| = \lambda$.

REMARK 2.9. Nevanlinna gave a proof which was valid if g is C^4 [22] and it has been noted that Nevanlinna's proof extends to infinite-dimensional Hilbert spaces as well [14], [51]. It seems that many authors have repeated this version of Nevanlinna's proof (e.g. [6], [14], [2]) without noticing a small omission towards the end of the proof, at the place where it breaks into a few cases. This is probably because in the case of Riemannian geometry (finite or infinite-dimensional), the case which Nevanlinna ignored is relatively easy to dispose of, completing the proof of the theorem. However if one wanted to generalize to the Lorentzian setting, there is work to be done.

2.3. The Equivalence of Models

Consider $\mathbb{B}_\infty, \mathbb{H}_\infty \subseteq \widehat{\mathcal{H}}$. Recall that the topology on $\mathcal{H}$ was defined as follows: a subset $U \subseteq \widehat{\mathcal{H}}$ is open if and only if $U \cap \mathcal{H}$ is open and $\mathcal{H} \setminus U$ is bounded if $\infty \in U$. In what follows, $\partial\mathbb{B}_\infty$, $\overline{\mathbb{B}}_\infty$, $\partial\mathbb{H}_\infty$ and $\overline{\mathbb{H}}_\infty$ refer to the boundary and closure with respect to the topology on $\widehat{\mathcal{H}}$. We then have the following descriptions:

$$\partial\mathbb{B}_\infty = \{x \in \mathcal{H} : \|x\| = 1\},$$

$$\overline{\mathbb{B}}_\infty = \{x \in \mathcal{H} : \|x\| \leq 1\},$$

$$\partial\mathbb{H}_\infty = \{x \in \mathcal{H} : x_1 = 0\} \cup \{\infty\} \text{ and}$$

$$\overline{\mathbb{H}}_\infty = \{x \in \mathcal{H} : x_1 \geq 0\} \cup \{\infty\}.$$

REMARK 2.10. We mention here the beautiful exposition of models and their equivalence in Cannon et al [7] so that the interested reader may extend their computations to the Hilbertian setting. For example, just as in the finite-dimensional situation, the conformal map $i_{-e_1,\sqrt{2}} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ is a homeomorphism between $\overline{\mathbb{H}}_\infty$ and $\overline{\mathbb{B}}_\infty$ and an isometry between

$\mathbb{H}_\infty$ and $\mathbb{B}_\infty$ with their respective metrics. For the purposes of this work, we do not talk about the boundary in the Lorentzian setting.

To give some flavor for these computations let us make the following observations. Note that x, u, e_1 will refer to vectors in Hilbert space; though when we write $(x)_1$ we refer to the first coordinate of the vector x, e.g. $(e_2)_2 = 1$.

OBSERVATION 2.11. The map $e_{\mathbb{L},\mathbb{B}} : \mathbb{L}_\infty \to \mathbb{B}_\infty$ defined by

$$e_{\mathbb{L},\mathbb{B}}(a, x) = \frac{x}{1+a}$$

is an isometry between $\mathbb{L}_\infty$ and $\mathbb{B}_\infty$.

PROOF. Note that the derivative of $e_{\mathbb{L},\mathbb{B}} : \mathbb{L}_\infty \to \mathbb{B}_\infty$ is given by

$$e'_{\mathbb{L},\mathbb{B}}(a, x)[(b, u)] = \frac{u}{1+a} - \frac{bx}{(1+a)^2}.$$

Now we compute

$$\begin{aligned}
\|e'_{\mathbb{L},\mathbb{B}}(a, x)[(b, u)]\|^2_{e_{\mathbb{L},\mathbb{B}}(a,x),\mathbb{B}} &= \left\| \frac{u}{1+a} - \frac{bx}{(1+a)^2} \right\|^2_{\frac{x}{1+a},\mathbb{B}} \\
&= 4 \frac{\left\| \frac{u}{1+a} - \frac{bx}{(1+a)^2} \right\|^2}{\left[1 - \left\| \frac{x}{(1+a)^2} \right\|^2\right]^2} \\
&= \frac{\left[\frac{\|u\|^2}{(1+a)^2} - \frac{2ab^2}{(1+a)^3} + \frac{b^2(a^2-1)}{(1+a)^4} \right]}{\left[1 - \frac{a^2+1}{(1+a)^2}\right]^2} \\
&= 4 \frac{\left[\frac{\|u\|^2}{(1+a)^2} + \frac{b^2(-2a+a-1)}{(1+a)^3} \right]}{\left[1 - \frac{a-1}{a+1}\right]^2} \\
&= 4 \frac{\left[\frac{\|u\|^2}{(1+a)^2} - \frac{b^2(1+a)}{(1+a)^3} \right]}{\left[\frac{2}{1+a}\right]^2} \\
&= \|u\|^2 - b^2 \\
&= \|(b, u)\|^2_{(a,x),\mathbb{L}} \cdot
\end{aligned}$$

□

OBSERVATION 2.12. The map $e_{\mathbb{B},\mathbb{H}} : \mathbb{B}_\infty \to \mathbb{H}_\infty$ defined by

$$e_{\mathbb{B},\mathbb{H}}(x) = -e_1 + 2\frac{x+e_1}{\|x+e_1\|^2}$$

is an isometry between $\mathbb{B}_\infty$ and $\mathbb{H}_\infty$.

PROOF. Note that the derivative of $e_{\mathbb{B},\mathbb{H}} : \mathbb{B}_\infty \to \mathbb{H}_\infty$ is given by

$$e'_{\mathbb{B},\mathbb{H}}(x)[u] = 2\frac{u}{\|x+e_1\|^2} - 4\frac{\langle x+e_1, u\rangle}{\|x+e_1\|^4}(x+e_1)\cdot$$

As before we now compute

$$\begin{aligned}
\|e'_{\mathbb{B},\mathbb{H}}(x)[u]\|^2_{e_{\mathbb{B},\mathbb{H}}(x),\mathbb{H}} &= \frac{\left\|2\frac{u}{\|x+e_1\|^2} - 4\frac{\langle x+e_1,u\rangle(x+e_1)}{\|x+e_1\|^4}\right\|^2}{\left(-1+2\frac{x_1+1}{\|x+e_1\|^2}\right)^2} \\
&= \frac{\left\|2u - 4\frac{\langle x+e_1,u\rangle(x+e_1)}{\|x+e_1\|^2}\right\|^2}{\left(2(x_1+1) - \|x+e_1\|^2\right)^2} \\
&= \frac{4\|u\|^2 - \frac{16\langle x+e_1,u\rangle^2}{\|x+e_1\|^2} + \frac{16\langle x+e_1,u\rangle^2}{\|x+e_1\|^2}}{\left(2x_1+2-[\|x\|^2+2x_1+1]\right)^2} \\
&= \frac{4\|u\|^2}{\left(1-\|x\|^2\right)^2} \\
&= \|u\|^2_{x,\mathbb{B}}
\end{aligned}$$

□

2.4. Isometries of Hyperbolic Space

DEFINITION 2.13. Denote the set of Möbius maps that preserve $\mathbb{K}_\infty$ by

$$\mathcal{M}_{\mathbb{K}} := \{g \in \mathcal{M}ob(\widehat{\mathcal{H}}) : g(\mathbb{K}_\infty) = \mathbb{K}_\infty\}$$

and let

$$\mathcal{I}_{\mathbb{K}} := \{g : \mathbb{K}_\infty \to \mathbb{K}_\infty : \|g_*[u]\|_{g(x),\mathbb{K}} = \|u\|_{x,\mathbb{K}} \ \ \forall x \in \mathbb{K}_\infty, u \in T_x\mathbb{K}_\infty\}$$

be the group of diffeomorphisms of $\mathbb{K}_\infty$ that preserve the infinitesimal metric.

DEFINITION 2.14 (Poincaré Extension). Given a Hilbert space $\mathcal{H}$, define $\mathrm{Sim}(\mathcal{H})$ to be the set of all affine maps of the form $x \mapsto \lambda Mx+y$ with $\lambda > 0$, $y \in \mathcal{H}$ and $M \in \mathscr{O}(\mathcal{H})$. Isometries correspond to $\lambda = 1$. Notice that $\mathcal{E} := \partial\mathbb{H} \setminus \{\infty\}$ is a Hilbert space. For $g \in \mathrm{Sim}(\mathcal{E})$, the Poincaré extension of g is defined as

$$\widehat{g}(x) := \lambda\widehat{M}x + y$$

where $g(x) = \lambda Mx + y$ and $\widehat{M}x = \widehat{M}(x_1, x_2, \ldots) = \big(x_1, M\big((x_2, x_3, \ldots)\big)\big)$.

THEOREM 2.15. *Let $\mathbb{K}_\infty$ be either $\mathbb{B}_\infty$ or $\mathbb{H}_\infty$ and let g be a diffeomorphism of $\mathbb{K}_\infty$. Then the following are equivalent:*

(1) *g is a conformal isomorphism of $\mathbb{K}_\infty$.*
(2) *g extends to a Möbius transform $\widehat{g} \in \mathcal{M}_\mathbb{K}$.*
(3) *$g \in \mathcal{I}_\mathbb{K}$.*
(4) *g is an isometry, i.e. $d_\mathbb{K}(gx, gy) = d_\mathbb{K}(x, y)$ for every $x, y \in \mathbb{K}_\infty$.*

REMARK 2.16. The equivalence (2) $\Leftrightarrow$ (3) shows that $\mathcal{M}_\mathbb{K}$ is isomorphic to $\mathcal{I}_\mathbb{K}$, which in turn ($\Leftrightarrow$ (4)) equals the *group* of isometries of $\mathbb{K}_\infty$. We remark that $\mathcal{M}_\mathbb{K}$ makes sense for $\mathbb{B}_\infty$ and $\mathbb{H}_\infty$ whereas $\mathcal{I}_\mathbb{K}$ makes sense for $\mathbb{L}_\infty$, $\mathbb{B}_\infty$ and $\mathbb{H}_\infty$.

PROOF. We provide an outline of the argument. First notice the equivalence (1) $\Leftrightarrow$ (2) follows from Liouville's Theorem 2.7 and that the implication (2) $\Rightarrow$ (3) is a straightforward but somewhat tedious computation. We next explain why (3) $\Rightarrow$ (1): for $g \in \mathcal{M}_\mathbb{K}$,

$$\frac{\|g'(x)[u]\|}{f_\mathbb{K}(g(x))} = \|g'(x)[u]\|_{g(x),\mathbb{K}} = \|u\|_{x,\mathbb{K}} = \frac{\|u\|}{f_\mathbb{K}(x)},$$

where

$$f_\mathbb{K}(x) = \begin{cases} \frac{1-\|x\|^2}{2} & \text{when } \mathbb{K} = \mathbb{B} \\ x_1 & \text{when } \mathbb{K} = \mathbb{H} \end{cases}.$$

Therefore

$$\|g'(x)[u]\| = \frac{f_\mathbb{K}(g(x))}{f_\mathbb{K}(x)}\|u\|$$

and so g is conformal. Of the remaining directions (3) $\Rightarrow$ (4) is straightfoward and we will prove the final non-trivial direction, viz. (4) $\Rightarrow$ (3). Note that it is enough to show

$$\|u\|_{x,\mathbb{K}} = \frac{\partial}{\partial t}\Big[d_{\mathbb{K}}(x, x+tu)\Big]_{t=0} := \lim_{t\searrow 0} \frac{1}{t} d_{\mathbb{K}}(x, x+tu).$$

The $\geq$ direction is easier. Indeed

$$\begin{aligned}
\frac{1}{t} d_{\mathbb{K}}(x, x+tu) &\leq \frac{1}{t}\int_{s=0}^{1} \|\frac{\partial}{\partial s}[x+stu]\|_{x+stu,\mathbb{K}} ds \\
&= \int_{s=0}^{1} \|u\|_{x+stu,\mathbb{K}} ds \\
&\xrightarrow{t\to 0} \int_{s=0}^{1} \|u\|_{x,\mathbb{K}} ds \ .
\end{aligned}$$

For the $\leq$ direction first fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all h with $\|h\| < \delta$ and for all u we have

$$\|u\|_{x+h,\mathbb{K}} \geq (1-\varepsilon)\|u\|_{x,\mathbb{K}} \ .$$

Now fix u, t and let $\gamma : x \rightsquigarrow x+tu$. Then

- Case 1. $[\gamma \subseteq B(x,\delta)]$

  Then $l(\gamma) \geq \int_{s=0}^{1}(1-\varepsilon)\|\gamma'(s)\|_{x,\mathbb{K}} ds \geq (1-\varepsilon)\|tu\|_{x,\mathbb{K}}$ and therefore

  $$\frac{1}{t} l(\gamma) \geq (1-\varepsilon)\|u\|_{x,\mathbb{K}} \ .$$

- Case 2. $[\gamma \not\subseteq B(x,\delta)]$

  Let s_0 be minimal so that $\gamma(s_0) \notin B(x,\delta)$. Then

  $$\begin{aligned}
  l(\gamma) &\geq \int_{s=0}^{s_0} (1-\varepsilon)\|\gamma'(s)\|_{x,\mathbb{K}} ds \\
  &\geq (1-\varepsilon)\|\gamma(s_0) - x\|_{x,\mathbb{K}} \\
  &\geq \frac{1}{C_x}(1-\varepsilon)\delta \ .
  \end{aligned}$$

  Now taking the infimum over γ we get

  $$\frac{1}{t} d(x, x+tu) \geq (1-\varepsilon)\min\big(\|u\|_{x,\mathbb{K}}, \frac{\delta}{tC_x}\big) \ .$$

  Therefore

  $$\liminf_{t\searrow 0} \frac{1}{t} d(x, x+tu) \geq (1-\varepsilon)\|u\|_{x,\mathbb{K}}$$

since the second term in the minimum goes to infinity. We had chosen $\varepsilon > 0$ to be arbitrary and thus are done with the proof.

□

For $g \in \mathcal{I}_{\mathbb{K}}$ we will also denote by g its unique extension $g : \overline{\mathbb{K}}_\infty \to \overline{\mathbb{K}}_\infty$.

Proposition 2.17.

(1) $Stab_{\mathcal{M}_{\mathbb{H}}}(\infty) = \{\lambda M + y : \lambda > 0, y \in \mathcal{H}, M \in \mathscr{O}(\mathcal{H}),$
$\langle y, e_1\rangle = 0, M[e_1] = e_1\}$.

(2) $Stab_{\mathcal{M}_{\mathbb{B}}}(0) = \mathscr{O}(\mathcal{H})$.

Note that it follows from (1) that

$$\mathrm{Stab}_{\mathcal{M}_{\mathbb{H}}}(\infty) = \widehat{\mathrm{Sim}}(\mathcal{E}) := \{\widehat{g} : g \in \mathrm{Sim}(\mathcal{E})\}.$$

Proof. Recall that

$$\mathrm{Stab}_{\mathcal{M}_{\mathbb{H}}}(\infty) := \{g \in \mathcal{M}_{\mathbb{H}} : g(\infty) = \infty\}$$

and that

$$\mathrm{Stab}_{\mathcal{M}_{\mathbb{B}}}(0) := \{g \in \mathcal{M}_{\mathbb{B}} : g(0) = 0\}.$$

(1) [Proof of $\supseteq$] First notice that g is clearly Möbius and that $g(\mathbb{H}_\infty) = \mathbb{H}_\infty$ since $g(x) \in \mathbb{H}_\infty \Leftrightarrow (g(x))_1 > 0 \Leftrightarrow \lambda(Mx)_1 + y_1 > 0 \Leftrightarrow \lambda x_1 > 0 \Leftrightarrow x_1 > 0 \Leftrightarrow x \in \mathbb{H}_\infty$.
[Proof of $\subseteq$] Note that $g \in \mathcal{M}ob(\widehat{\mathcal{H}})$ come in two flavors and that g cannot be of the first type, i.e. $g(z) = \lambda M \circ i_{p,r}(z) + y$, since these maps don't preserve the point $\infty \in \widehat{\mathcal{H}}$. Therefore g must be of the form $g(x) = \lambda M x + y$ with $\lambda > 0$. Left to check $\langle y, e_1\rangle = 0$ and $M[e_1] = e_1$. The first is true since $y = g(0) \in \partial\mathbb{H}_\infty$ and so $\langle y, e_1\rangle = 0$. For the second fact, notice that $g(\partial\mathbb{H}) = \partial\mathbb{H}$ and thus $g(e_1^\perp) = e_1^\perp$, where $^\perp$ denotes the orthogonal complement. Therefore $M \in \mathscr{O}(\mathcal{H})$ and so $M[e_1] = \pm e_1$. But g preserves $\mathbb{H}_\infty$ and so $M[e_1] = e_1$.

(2) [Proof of $\supseteq$] This is clear.
[Proof of $\subseteq$] Let $g \in \mathrm{Stab}_{\mathcal{M}_{\mathbb{B}}}(0)$ and let $h := i \circ g \circ i$ where $i := i_{0,1}$ denotes reflection in the unit sphere. Then $h \in \mathcal{M}ob(\widehat{\mathcal{H}})$ and $h(\infty) = \infty$. As in the argument above,

h must be of the form $h(x) = \lambda Mx + y$ with $\lambda > 0$. Now $h(\mathbb{B}_\infty) = \mathbb{B}_\infty$ and so $\lambda = 1$ and $y = 0$ and therefore $h(x) = Mx$. Therefore $g(x) = Mx \in \mathscr{O}(\mathcal{H})$ and we are done.

□

PROPOSITION 2.18. *The group $\mathcal{M}_\mathbb{K}$ acts transitively on $\mathbb{K}_\infty$ and triply transitively on $\partial\mathbb{K}_\infty$.*

PROOF. The proof splits into three parts. First we show that $\mathcal{M}_\mathbb{H}$ acts transitively on $\mathbb{H}_\infty$, then by equivalence of models we get the same result for $\mathcal{M}_\mathbb{B}$. Next we show that $\mathcal{M}_\mathbb{B}$ acts transitively on $\partial\mathbb{B}_\infty$ and again by equivalence of models we get the same result for $\mathcal{M}_\mathbb{H}$ acting on $\partial\mathbb{H}_\infty$. Finally, we show that $\mathcal{M}_\mathbb{H}$ acts triply transitively on $\partial\mathbb{H}_\infty$ and once more by equivalence of models we get the same result for $\mathcal{M}_\mathbb{B}$.

(1) [$\mathcal{M}_\mathbb{H}$ acts transitively on $\mathbb{H}_\infty$] For any $p \in \mathbb{H}_\infty$, let $\lambda = p_1$, $y = (0, p_2, p_3, \ldots)$ and $M = \mathrm{id}$, the identity map. Now define $g(x) := \lambda Mx + y$ and notice that $g \in \mathrm{Stab}_{\mathcal{M}_\mathbb{H}}(\infty)$ (and therefore $g \in \mathcal{M}_\mathbb{H}$) and that $g(e_1) = p$. Therefore the action is transitive on $\mathbb{H}_\infty$.

(2) [$\mathcal{M}_\mathbb{B}$ acts transitively on $\partial\mathbb{B}_\infty$] Take $\xi \in \partial\mathbb{B}_\infty$ and extend to an orthonormal basis $\mathcal{B} := \{\xi = \xi_1, \xi_2, \ldots\}$. This is possible since every maximal orthonormal set is an orthonormal basis. Now define M by $M[e_i] = \xi_i$. Then $M \in \mathscr{O}(\mathcal{H})$ and therefore $M \in \mathcal{M}_\mathbb{B}$ since $\mathrm{Stab}_{\mathcal{M}_\mathbb{B}}(0) = \mathscr{O}(\mathcal{H})$ and we also have $M[e_1] = \xi$. Therefore the action is transitive on $\partial\mathbb{B}_\infty$.

(3) [$\mathcal{M}_\mathbb{H}$ acts triply transitively on $\partial\mathbb{H}_\infty$] Fix $\xi_1, \xi_2, \xi_3 \in \partial\mathbb{H}$ distinct. Then by transitivity of $\mathcal{M}_\mathbb{H}$ there exists $g \in \mathcal{M}_\mathbb{H}$ such that $g(\infty) = \xi_1$. Let $\eta_i := g^{-1}(\xi_i)$, $\lambda := \|\eta_3 - \eta_2\|$ and $y := \eta_2$ and choose $M \in \mathcal{O}(\mathcal{E})$ such that $M[e_2] = \frac{\eta_3 - \eta_2}{\|\eta_3 - \eta_2\|}$ (again this possible by arguments similar to those above, viz. that the orthogonal group acts transitively on the unit sphere). Now define $\widehat{M} : \mathcal{H} \to \mathcal{H}$ by $(x_1, \ldots) \mapsto (x_1, M[(x_2, \ldots)])$ (i.e. via the Poincaré extension) and let $h(x) := \lambda\widehat{M}[x] + y$. Then $h(\infty) = \infty$, $h(0) = \eta_2$ and $h(e_2) = \eta_3$ and therefore $gh(\infty) = \xi_1$, $gh(0) = \xi_2$ and $gh(e_2) = \xi_3$. Thus $\mathcal{M}_\mathbb{H}$ acts triply transitively on $\partial\mathbb{H}_\infty$.

□

We end this subsection with a few concluding remarks to highlight a few similarities and differences with the classical finite-dimensional theory.

REMARK 2.19. If $\phi \in \mathcal{M}_{\mathbb{B}}$, then for all $x, y \in \mathbb{B}_\infty$ the following formulae hold

$$\frac{\|\phi(x) - \phi(y)\|^2}{(1 - \|\phi(x)\|^2)(1 - \|\phi(y)\|^2)} = \frac{\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}$$

and

$$\|\phi(x) - \phi(y)\|^2 = \|\phi'(x)\| \cdot \|\phi'(y)\| \cdot \|x - y\|^2.$$

The second has been referred to as the *geometric mean value theorem* for Möbius transforms in finite dimensions.

REMARK 2.20. Just as in finite dimensions, every $g \in \mathcal{M}_{\mathbb{B}}$ may be factored as follows:

$$g = \gamma \circ i_{v, \sqrt{\|v\|^2 - 1}} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$$

for some $v \in \widehat{\mathcal{H}}$ and $\gamma \in \mathscr{O}(\mathcal{H})$. Recall that $i_{x,\alpha}$ was reflection in the sphere centered at x of radius α and that the sphere $S(v, \sqrt{\|v\|^2 - 1})$ is orthogonal to $\partial\mathbb{B}_\infty$.

REMARK 2.21. If we restrict to the subclass of $\mathcal{M}_{\mathbb{K}}$ defined by

$$\mathcal{M}^*_{\mathbb{K}} := \{\ g \in \mathcal{M}_{\mathbb{K}} \mid g \textit{ is the composition of finitely many inversions}\},$$

then we have that

$$\mathcal{M}^*_{\mathbb{K}} \subsetneq \mathcal{M}_{\mathbb{K}}.$$

Let $g(z) = \lambda M \circ i_x(z) + y$ for some (λ, x, y, M) as in Liouville's Theorem. Then g can be written as a finite composition of inversions if and only if $\mathrm{Fix}(M)$ has finite codimension. For example, the shift map on $\ell_2(\mathbb{Z})$ cannot be written as a finite composition of inversions. Say $\mathrm{Fix}(M)$ has finite codimension, then one can find a finite dimensional subspace V such that the entire map is the Poincaré extension of its restriction V. This reduces us to the finite dimensional statement [28] that every Möbius map is a composition of finitely many

inversions which may then be re-extended. On the other hand if one computes the composition of two inversions it can be shown that $\mathrm{Fix}(M)$ has codimension 1 and so composing finitely many inversions only adds one finitely many times to the codimension.

REMARK 2.22. One cannot make sense of *orientation-preserving* transformations in infinite dimensions as one cannot define a meaningful notion of orientation. If one wanted to define orientation-preserving via the kernel of a continuous homomorphism $\mathcal{O} : \mathscr{O}(\mathcal{H}) \to \mathbb{Z}_2$ one would easily fall into a trap. For example, any reflection in a hyperplane on $\ell_2(\mathbb{Z})$ would be orientation-preserving. To see why this is true, take the commutator of the shift map squared and the map that switches consecutive pairs of nonnegative coordinates, i.e. 0 and 1, 2 and 3, etc.

2.5. Bounded Distortion

We now proceed to establish *bounded distortion* for conformal maps in our setting. We remark that this allows us access to the tools and methods developed in [21].

Define $s_{\mathbb{B}} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ and $s_{\mathbb{H}} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ to be $s_{\mathbb{B}} := i_0$ and $s_{\mathbb{H}}(x) = (-x_1, x_2, x_3, \ldots)$ respectively. Then note that

$$e_{\mathbb{B},\mathbb{H}} \circ s_{\mathbb{B}} = s_{\mathbb{H}} \circ e_{\mathbb{B},\mathbb{H}}.$$

OBSERVATION 2.23. For every $g \in \mathcal{M}_{\mathbb{K}}$, g commutes with $s_{\mathbb{K}}$.

PROOF. It is enough to prove it in the upper-half space model $\mathbb{H}_\infty$. Let the commutator be denoted by $k := [gsg^{-1}s^{-1}]$. Notice that $k : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ is conformal and k restricted to $\partial\mathbb{H}_\infty$ is the identity. In particular $k(\infty) = \infty$ and therefore by Proposition 2.17 we have that k is affine, i.e. $k(x) = \lambda M[x] + y$ where $\lambda > 0$, $M \in \mathscr{O}(\mathcal{H})$ and $v \in \mathcal{H}$. Next we can see that $k(x) = \lambda M[x]$ (i.e. $y = 0$) since $k(0) = 0$ and further that $k(x) = M[x]$ (i.e. λ=1) since k restricted to $\partial\mathbb{H}_\infty$ is the identity. Finally notice that we have a linear isometry M that is the identity when restricted to $\partial\mathbb{H}_\infty$ and that k and therefore M must preserve $\mathbb{H}_\infty$ and so $M[e_1] = e_1$. But then

$$\ker(M - \mathrm{id}) \supseteq \mathbb{R}\text{-span}\{e_1, \partial\mathbb{H}_\infty\} = \mathbb{H}_\infty.$$

Therefore k is the identity and we are done. $\square$

THEOREM 2.24. *Fix $\sigma > 0$ and let $g \in \mathcal{M}^*_{\mathbb{B}}$. Then there exists a region $R := R_{\sigma,g}$ and a constant $K := K_\sigma \asymp e^{2\sigma}$ such that*

$$\frac{|g'(y)|}{|g'(x)|} \leq K,$$

for all $x, y \in R$.

PROOF. As above, let s denote inversion in the unit sphere $\partial\mathbb{B}_\infty$. Then s commutes with g and therefore we have that

$$gg^{-1}(0) = 0 \implies g(sg^{-1}(0)) = \infty.$$

We call $a := sg^{-1}(0)$ and so $g(a) = \infty$. Now let us define

$$R := R_{\sigma,g} := \{z \in \mathcal{H} : \langle z, v_a\rangle_{euc} \leq \tanh(\sigma/2)\} \cap \overline{\mathbb{B}}_\infty,$$

where v_a is the unit vector in the direction of a. Then we have that

$$\operatorname{dist}_e(a, R) \geq \sup\{d_e(x, y) : x \in R, y \in \partial\mathbb{B}_\infty\} =: \widetilde{K}_\sigma := \widetilde{K}.$$

One way to see this is to observe that $g(a) = \infty = s(0) \implies s(a) = g^{-1}(0)$, again using that s and g commute. Note also that $\widetilde{K}_\sigma \asymp e^{-\sigma}$ as can be seen from elementary arguments in plane hyperbolic geometry. Note that $K \to 1$ as $\sigma \to \infty$. Next let us derive an analog of Theorem 4.1.3 in [21].Now $\operatorname{diam}(R) \leq 2$ and so

$$\begin{aligned}\sup_{x,y\in R} \frac{\|y-a\|}{\|x-a\|} &\leq \sup_{x,R} \frac{\|x-a\|+2}{\|x-a\|} \\ &= 1 + \frac{2}{\inf_{x\in R}\|x-a\|} \\ &\leq 1 + \frac{2}{\widetilde{K}_\sigma}.\end{aligned}$$

Recall that we have that

$$\frac{|g'(y)|}{|g'(x)|} = \frac{\|x-a\|^2}{\|y-a\|^2}$$

and therefore

$$\frac{|g'(y)|}{|g'(x)|} \leq \left(1 + \frac{2}{\widetilde{K}_\sigma}\right)^2 \asymp (1 + 2e^\sigma)^2 \asymp e^{2\sigma}$$

for all $x, y \in R$. Let $K := K_\sigma = (1 + 2/\widetilde{K}_\sigma)^2$ and notice we are done. $\square$

2.6. Fixed Points of Isometries

We will employ the seminal notion of a Gromov product in our scenario - one which may be defined as follows:

DEFINITION 2.25. Fix $z \in \mathbb{K}_\infty$. Let $\langle \cdot\ ,\ \cdot \rangle_z : \mathbb{K}_\infty \times \mathbb{K}_\infty \to [0, \infty)$ be defined as:

$$\langle x, y \rangle_z := \frac{1}{2}[d_{\mathbb{K}}(x, z) + d_{\mathbb{K}}(y, z) - d_{\mathbb{K}}(x, y)].$$

The Gromov product is a measure of the "defect" in the triangle inequality. Fix x and y as two points on this page. Then as you (based at z) zoom further back, away from the page - the "angle" you subtend, $\angle(x, z, y)$, becomes smaller and smaller, while the Gromov product $\langle x, y \rangle_z$ gets larger and larger. The idea here is "angle seen from z" $\sim e^{-\text{gromov product}}$.

The main reason why this product is useful for the purposes of this section is that it extends continuously to the boundary $\partial\mathbb{K}_\infty$. In fact we have the following

PROPOSITION 2.26. *Fix $z \in \mathbb{K}_\infty$. Then the function $f_z : \mathbb{K}_\infty \times \mathbb{K}_\infty \to [0, \infty)$ given by $(x, y) \mapsto \langle x, y \rangle_z$ extends uniquely to a continuous function $\bar{f}_z : \overline{\mathbb{K}}_\infty \times \overline{\mathbb{K}}_\infty \to [0, \infty]$ such that*

$$\bar{f}_z(x, y) = \infty \Leftrightarrow x = y \text{ and } x \in \partial\mathbb{K}_\infty. \tag{1}$$

PROOF. Without a loss in generality, we let $\mathbb{K}_\infty = \mathbb{B}_\infty$ and $z = 0$. A short computation gives us that f_z may be written as

$$\langle x, y \rangle_0 = \log \left[\frac{(1 + \|x\|)(1 + \|y\|)}{\sqrt{\|x\|^2\|y\|^2 - 2\langle x, y \rangle + 1} + \|x - y\|} \right].$$

Now the proof follows from considering the terms describing this formula, in particular to characterize when the denominator vanishes. $\square$

We therefore make the convention that for $x, y \in \overline{\mathbb{K}}_\infty$ and $z \in \mathbb{K}_\infty$ we have

$$\langle x, y \rangle_z = \bar{f}_z(x, y).$$

COROLLARY 2.27. *The following statements are true:*

(i) For $\mathbb{K}_\infty$: If $\langle z_m, z_n\rangle_0 \to \infty$ as $m, n \to \infty$, then there exists $z \in \partial\mathbb{K}_\infty$ such that $z_n \to z$ as $n \to \infty$.

(ii) For $\mathbb{B}_\infty$: If $\langle z_m, w_m\rangle_0 \to \infty$ as $m \to \infty$, then $\|z_m - w_m\| \to 0$ and $\|z_m\| \to 1$ both as $m \to \infty$.

PROOF. Note that

$$e^{\langle x,y\rangle_0} = \frac{(1+\|x\|)(1+\|y\|)}{\sqrt{\|x\|^2\|y\|^2 - 2\langle x,y\rangle + 1} + \|x-y\|} \leq \frac{4}{\|z+w\| + (1-\|z\|\|w\|)}.$$

We use the Cauchy-Schwartz inequality in the last step and the proof of the corollary follows by considering the terms on the right-hand-side of this inequality, especially what is necessitated by the vanishing of the denominator. □

THEOREM 2.28 (Pontryagin-Iohvidov-Krein-Fan-Karlsson-Simmons). *For every $g \in \mathcal{M}_\mathbb{K}$, there exists a $x \in \overline{\mathbb{K}}_\infty$ such that $gx = x$.*

REMARK 2.29. We remark that we first learnt a proof of this theorem in 2010 from David Simmons [unpublished] who provided such by employing the standard spectral theory of bounded linear operators acting on Hilbert spaces. Much later in 2011 we discovered a beautiful proof by Anders Karlsson [17] in the general setting of (not necessarily proper) Gromov hyperbolic spaces that used Gromov products to great effect. In 2012 we discovered a far-reaching generalization of this theorem by Ky Fan [12] from the '60s that in turn generalized previous work of Pontryagin-Iohvidov-Krein from the '40s. We are led to believe [29] that these studies may have been inspired by Dirac's radical '41 Bakerian lecture [9] titled *The Physical Interpretation of Quantum Mechanics* where through considering "indefinite bilinear forms as defining the square of the lengths of vectors in the Hilbert space [of states, one is lead to] the consequence that *operators with only positive eigenvalues can have negative expectation values*. One can express it also by saying that one introduces negative probabilities that certain positive eigenvalues of an observable are realized", see [26].

For completeness we provide a proof employing ideas from [17] - this to our taste is the most elementary proof and the technique deserves to be better known. The proof

divides into two cases - one in which we prove the existence of a fixed point in the interior (essentially an idea that goes back to E. Cartan, repopularized by Bruhat-Tits: see e.g. [5]) and the other in which we prove the existence of a fixed point on the boundary. In the first case, the orbit of any point under g is bounded as $n \to \infty$, while in the second the orbit is unbounded and clusters at $\partial\mathbb{K}_\infty$ as $n \to \infty$.

Note however that unlike in the finite-dimensional setting this leads to interesting new phenomena: there exist examples in the second case where the orbit of a point may be unbounded but yet returns infinitely often to some bounded region. The earliest example we could find in the literature is due to M. Edelstein [10] from 1964 who was studying the iteration of non-expansive mappings in Banach spaces. However we wouldn't be surprised if further search brought up earlier instances of such objects. We recently found some beautifully written notes (unpublished) of lectures by Alain Valette [48] that discuss the Edelstein example in detail and include various other interesting leads to topics of current research interest.

Lastly for contrast, we note that the proof of this result in the finite-dimensional setting depends on the Brouwer fixed-point theorem which is not valid in the infinite-dimensional setting where our ambient space is not locally compact.

PROOF. Let $x \in \mathbb{K}_\infty$. Then the sequence $(d_{\mathbb{K}}(x, g^n(x)))$ is either unbounded or bounded.

Case 1 (Unbounded orbit):

For each $k \in \mathbb{N}$ let $n_k := \min\{n : d_{\mathbb{K}}(x, g^n(x)) > k\}$. Now fix $k, l \in \mathbb{N}$ with $k < l$. Then

$$\begin{aligned} d(g^{n_k}(x), g^{n_l}(x)) &= d_{\mathbb{K}}(x, g^{n_l - n_k}(x)) \\ &\leq l \\ &< d(x, g^{n_l}(x)). \end{aligned}$$

We then have that $\langle g^{n_k}(x), g^{n_l}(x)\rangle_x > \frac{1}{2} d_{\mathbb{K}}(x, g^{n_k}(x)) > \frac{k}{2}$ which in turn implies that $\langle g^{n_k}(x), g^{n_l}(x)\rangle_x \to \infty$ as $k, l \to \infty$. By Corollary 2.27 there exists a $y \in \partial\mathbb{K}_\infty$ such that $g^{n_k}(x) \to y$ as $k \to \infty$. Now we show that y is fixed.

Notice that

$$\begin{aligned}\langle g^{n_k}(x), g^{n_k+1}(x)\rangle_x &\geq \frac{1}{2}\big[d_{\mathbb{K}}(g^{n_k}(x), x) - d_{\mathbb{K}}(g^{n_k}(x), g^{n_k+1}(x))\big] \\ &= \frac{1}{2}\big[d_{\mathbb{K}}(g^{n_k}(x), x) - d_{\mathbb{K}}(x, g(x))\big] \\ &> \frac{1}{2}\big[k - d_{\mathbb{K}}(x, g(x))\big] \\ &\to \infty \text{ as } k \to \infty.\end{aligned}$$

Now by Proposition 2.26 we have that

$$\begin{aligned}\infty = \lim_{k\to\infty}\langle g^{n_k}(x), g^{n_k+1}(x)\rangle_x &= \left\langle \lim_{k\to\infty} g^{n_k}(x), g(\lim_{k\to\infty} g^{n_k}(x))\right\rangle_x \\ &= \langle y, g(y)\rangle_x\end{aligned}$$

and therefore that $y = g(y)$ is a fixed point of g.

Case 2 (Bounded orbit):

We only sketch the main idea, a complete proof of which may be found in [5]. The following observation is the key to the argument in this case. In our setting (and more generally in a Hadamard space) given a bounded subset A of $\mathbb{K}$ and some $x \in \mathbb{K}$, define $r_x(A) := \sup_{y\in A} d_{\mathbb{K}}(x, y)$ and $r(A) := \inf_{x\in\mathbb{K}} r_x(A)$. Then there exists a unique $z \in \mathbb{K}$, called the *circumcenter of* A, such that $r_z(A) = r(A)$, i.e. $A \subset \overline{B}(z, r(A))$. This observation when applied to the bounded orbit in question produces the required fixed point. □

2.7. Classification of Isometries

We now provide a classification of isometries that is analogous to the one carried out by Klein for those of two-dimensional hyperbolic space.

PROPOSITION 2.30. *Any isometry of hyperbolic space i.e. a $g \in \mathcal{M}_{\mathbb{K}}$, is conjugate to exactly one of the following:*

(1) *[Elliptic] A bijective linear isometry on $\mathbb{B}_\infty$, i.e. $T \restriction \mathbb{B}_\infty$ for some orthogonal map $T \in \mathscr{O}(\mathcal{H})$.*
(2) *[Parabolic] A bijective affine euclidean isometry on $\mathbb{H}_\infty$ with no fixed points in the interior.*

(3) *[Hyperbolic] A map of the form* $g = \lambda M : \mathbb{H}_\infty \to \mathbb{H}_\infty$*, where* $0 < \lambda < 1$ *and* M *is a bijective linear isometry of* $\mathbb{H}_\infty$*.*

PROOF. The proof splits into the following cases:

Case 1 [There exists $x \in \mathbb{K}$ with $g(x) = x$.]

In this case, conjugate x to 0 and $\mathbb{K}$ to $\mathbb{B}$. Then $g \in \mathrm{Stab}_{\mathcal{M}_\mathbb{B}}(0) = \mathscr{O}(\mathcal{H})$ by Proposition 2.17.

Case 2 [There exists $\xi \in \partial\mathbb{K}$ with $g(\xi) = \xi$.]

In this case, conjugate ξ to ∞ and $\mathbb{K}$ to $\mathbb{H}$. Then $g \in \mathrm{Stab}_{\mathcal{M}_\mathbb{H}}(\infty)$ and by Proposition 2.17 we have that $g(x) = \lambda\widehat{T}[x] + b$ with $\lambda > 0$, $T \in \mathscr{O}(\mathcal{E})$ and $b \in \mathcal{E}$.

Case 2A [$\lambda = 1$]

In this case g is a Euclidean isometry. If there were no fixed points in the interior we're in Case 1. If not, then we're in case (2) in the statement of the Proposition.

Case 2B [$\lambda < 1$]

The Banach contraction principle gives us that g has a fixed point in $\mathcal{E}$. Using Proposition 2.18 conjugate the fixed point to 0 while leaving ∞ fixed. Then without loss of generality we may assume that $b = 0$ and so $g(x) = \lambda\widehat{T}[x]$.

Case 2C [$\lambda > 1$]

The map g^{-1} is in Case 2B and therefore has a fixed point in $\mathcal{E}$. The same argument then shows that $g(x) = \lambda\widehat{T}[x]$.

□

DEFINITION 2.31. We call $g \in \mathcal{M}_\mathbb{K}$ *elliptic*, *parabolic* or *hyperbolic* according to which of the three classes it falls into via the Proposition above.

REMARK 2.32. The classification of an isometry g determines the behavior of the orbit of any point in $\mathbb{K}_\infty$. The *elliptic* case corresponds to the case where every orbit is bounded. In the *hyperbolic* case, there is a pair of fixed points at infinity: one is attracting and the other repelling. In this case, every orbit is unbounded and the forward orbit approaches

the attracting fixed point while the backward orbit approaches the repelling fixed point. Furthermore, there exists a unique geodesic connecting the two fixed points that is invariant under the action of g. In the *parabolic* case, every orbit is also unbounded; however there is only one fixed point (at infinity). However this case affords the Edelstein examples (see [10], [48]) referred to above and is therefore more subtle that the previous ones. Notice that these arguments show us that $\gamma \in G$ is *parabolic* if and only if γ has exactly one fixed point and similarly, *hyperbolic* if and only if γ has exactly two fixed points.

DEFINITION 2.33. We will call $\alpha \in \partial\mathbb{K}_\infty$ *parabolic* if it is the unique fixed point of a parabolic element in G. We will call $\alpha \in \partial\mathbb{K}_\infty$ *hyperbolic* if it is a fixed point of a hyperbolic element in G. For $\alpha \in \partial\mathbb{K}_\infty$ that is hyperbolic, there exists $\beta \neq \alpha$ in $\partial\mathbb{K}_\infty$ and there exists $\gamma_{\alpha,\beta} := \gamma \in G \setminus \{\mathrm{id}\}$ such that $\gamma(\alpha) = \alpha$ and $\gamma(\beta) = \beta$. The geodesic $l_{\alpha,\beta}$ joining α and β is called the *hyperbolic axis* of α and β.

We may summarize our discussion in the following table:

Type	$\#[\mathrm{Fix}(g) \cap \partial\mathbb{K}]$	$\#[\mathrm{Fix}(g) \cap \mathbb{K}]$	End behaviour of $(g^n(x))_n$
Parabolic	1	0	Unbounded orbits: $(g^n(x))_n$ either converges to or clusters at the fixed point
Hyperbolic	2	0	Unbounded orbits: $g^n(x) \to \alpha_+$ as $n \to +\infty$ $g^n(x) \to \alpha_-$ as $n \to -\infty$
Elliptic	0	1	Bounded orbits
	2	$\lvert\mathbb{R}\rvert$	
	$\lvert\mathbb{R}\rvert$	$\lvert\mathbb{R}\rvert$	

TABLE 2.1. Classification of an isometry $g \in \mathcal{M}_{\mathbb{K}}$.

REMARK 2.34. The classification of isometries into these three classes may also be given the following elegant geometric interpretation in terms of *minimal displacement*, see [5]. The following definitions can be made for a metric space (X, d) and a mapping $f : X \to X$. The *minimal displacement* of f is defined as

$$\lambda(f) := \inf_{x \in X} d(x, f(x)).$$

The *minimal set* of f is defined as

$$\mathrm{Min}(f) := \{x \in X | d(x, f(x)) = \lambda(f)\}.$$

Then given an isometry g of hyperbolic space, exactly one of the following holds:

(i) g is parabolic $\Leftrightarrow \mathrm{Min}(g) = \emptyset$.

(ii) g is elliptic $\Leftrightarrow \mathrm{Min}(g) \neq \emptyset$ and $\lambda(f) = 0$.

(iii) g is hyperbolic $\Leftrightarrow \mathrm{Min}(g) \neq \emptyset$ and $\lambda(f) > 0$.

2.8. A Digression on CAT(-1) and Gromov Hyperbolic Spaces

We introduce some notions from the theory of Gromov hyperbolic spaces and CAT(-1) spaces that can be found in [8] and in much greater detail in [47] and [5].

PROPOSITION 2.35. *The Gromov product satisfies the following identities and inequalities:*

$$
\begin{aligned}
&(a) \qquad \langle x|y\rangle_z = \langle y|x\rangle_z \geq 0 \\
&(b) \qquad \langle x|y\rangle_z = \operatorname{dist}(x,z) - \langle y|z\rangle_x \\
&(c) \qquad \langle x|y\rangle_z \leq \min(\operatorname{dist}(x,z), \operatorname{dist}(y,z)) \\
&(d) \qquad \langle x|y\rangle_z = \langle x|y\rangle_w + \langle x|w\rangle_z - \langle y|z\rangle_w \\
&\qquad\qquad\quad = \langle x|y\rangle_w + \operatorname{dist}(z,w) - \langle x|z\rangle_w - \langle y|z\rangle_w
\end{aligned}
$$

The proof of this proposition follows directly from the definitions. We will refer to (d) as the *change of viewpoint formula*. Proofs of (a), (b), and (c) may be found in [47]. In the sequel we will frequently use the symbols $\lesssim$, $\gtrsim$, and $\asymp$ to denote inequalities and equalities up to positive additive constants. For example, $A \lesssim B$ means that $A \leq B + C$ for some $C > 0$ and all A and B from the set of objects under consideration. A subscript of $\times$ will be used to indicate multiplicative relations. For example $A \gtrsim_\times B$ means that $A \geq BC$ for some $C > 0$ and all A and B from the set of objects under consideration.

DEFINITION 2.36. The metric space (X, dist) is called *hyperbolic* (or *Gromov hyperbolic*) if for every four points $x, y, z, w \in X$ we have

$$\langle x|z\rangle_w \gtrsim \min(\langle x|y\rangle_w, \langle y|z\rangle_w).$$

We will refer to this inequality as *Gromov's inequality*.

The following spaces are examples of hyperbolic spaces : Standard hyperbolic space $\mathbb{H}_n$ or $\mathbb{B}_n$ and complex hyperbolic spaces; all Riemannian manifolds with negative sectional curvature; infinite-dimensional hyperbolic space; $\mathbb{R}$-trees, including non-locally compact ones and word-hyperbolic groups. This last example is the only one of our examples that is not

CAT(-1). Any Cayley graph of a word-hyperbolic group gives a proper invariant hyperbolic metric on the group. A good reference for these examples is [5].

The following definitions are taken from [5] to which the reader is referred to for more details.

DEFINITION 2.37. X is a CAT(-1) space if it is a geodesic space all of whose geodesic triangles satisfy the CAT(-1) inequality. Recall that given a metric space X and Δ, a geodesic triangle in X we denote $\overline{\Delta} \subset \mathbb{H}^2$ to be a comparison triangle for Δ. Then Δ satisfies the CAT(-1) inequality if for every $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\overline{x}, \overline{y}).$$

Note that it follows from this definition that CAT(-1) spaces are uniquely geodesic.

DEFINITION 2.38. An $\mathbb{R}$-tree is a metric space T such that:

(i) there is a unique geodesic segment (denoted $[x, y]$) joining each pair of points $x, y \in T$

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

The boundary of a hyperbolic space (X, dist) is defined in a similar way to the completion of a metric space as the set of Cauchy sequences modulo their equivalence. We fix a distinguished point $0 \in X$. However, it will be clear that the definitions given are independent of which point is chosen.

DEFINITION 2.39. A sequence $(x_n)_1^\infty \subset X$, is called a *Gromov sequence* if

$$\lim_{m,n\to\infty} \langle x_n | x_m \rangle_0 = \infty.$$

Two Gromov sequences $(x_n)_n$, $(y_n)_n$ are called *equivalent* if

$$\lim_{m,n\to\infty} \langle x_n | y_n \rangle_0 = \infty.$$

It is readily verified using Gromov's inequality that equivalence of Gromov sequences is indeed an equivalence relation.

DEFINITION 2.40. The *Gromov boundary* of X is the set of Gromov sequences modulo equivalence. It is denoted ∂X. The *Gromov closure* of X is the set $\overline{X} := X \cup \partial X$.

Without going to too many details, let us mention the following key observation:

OBSERVATION 2.41. The formulas of Proposition 2.35 together with Gromov's inequality hold for points on the boundary as well, if the equations there are replaced by ones with additive constants. In fact, (a) and (c) of Proposition 2.35 hold in $\overline{X}$ in the usual sense, i.e. as exact formulas without additive constants.

DEFINITION 2.42. For point $\xi \in \partial X$ we say that

$$\lim_{n\to\infty} x_n = \xi \quad \text{if} \quad \lim_{n\to\infty} \langle x_n | \xi \rangle_0 = \infty. \tag{2}$$

We put a metric, and therefore a topology, on the space ∂X. In fact, the following proposition (see [47] Propositions 5.16 and 5.31) demonstrates that the spherical metric on the boundary of a standard (either finite or infinite dimensional) hyperbolic space can be generalized to the setting of hyperbolic metric spaces.

PROPOSITION 2.43. *For each $a > 1$ sufficiently small, there exists a complete metric* $\mathrm{Dist} = \mathrm{Dist}_a$ *on ∂X satisfying the following multiplicative equation:*

$$\mathrm{Dist}(\xi, \eta) = \mathrm{Dist}_a(\xi, \eta) \asymp_\times a^{-\langle \xi | \eta \rangle_0}. \tag{3}$$

We will refer to Dist as the "visual metric from the point 0" respective to the parameter a.

REMARK 2.44. For CAT(-1) spaces, and in particular for the standard model of hyperbolic geometry, Proposition 2.43 holds for any $1 < a \leq e$.

CHAPTER 3

DISCRETE GROUPS

3.1. Discrete Groups of Isometries

Let $X = \mathbb{K}_\infty$ and let G be a subgroup of the isometry group $\mathrm{Iso}(X)$. In finite dimensions, i.e. when $X = \mathbb{K}^n$ the following definitions are equivalent:

(1) For every bounded $B \subset X$, $\#[g \in G : gB \cap B \neq \emptyset] < \infty$.

(2) For every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow gx = x.$$

(3) G is a discrete subset of $\mathrm{Iso}(X)$ w.r.t. compact-open topology.

They all may be taken as a definition of a *discrete* group of isometries. However in infinite dimensions one must tread more carefully. For starters, notice that although (1) $\Rightarrow$ (2) even in infinite dimensions, there exist natural examples of groups that show us (2) $\not\Rightarrow$ (1).

Example 3.1. Consider $\mathbb{H}_\infty$ and let

$$V := \bigcup_{m \geq 2} \{(0, n_2, n_3, \ldots, n_m, 0, 0, \ldots) : n_i \in \mathbb{Z} \ \ \forall i = 2, \ldots, m\} \subseteq \mathcal{E}$$

Then v is a $\mathbb{Z}$-vector space and the group

$$G := \langle x \mapsto x + v : v \in V \rangle$$

is an example of one that satisfies (2) but not (1).

It may be somewhat harder to imagine the right infinite-dimensional analogue of (3). Let us start with the following definitions:

Definition 3.2. The group G is *strongly discrete (SD)* if (1) holds, i.e. for every bounded $B \subset X$,

$$\#[g \in G : gB \cap B \neq \emptyset] < \infty.$$

The group G is *weakly discrete (WD)* if (2) holds, i.e. for every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow gx = x.$$

DEFINITION 3.3. An group G acts *properly discontinuously* if for every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow g = \mathrm{id}.$$

Equivalently, if there exists $r > 0$ such that

$$B(0,r) \ \cap \bigcup_{g \in G \setminus \{\mathrm{id}\}} g\left(B(0,r)\right) = \emptyset.$$

We remark that unlike strong discreteness which turns out to be a rather robust notion in infinite dimensions, the notion of being properly discontinuous is far more fragile.

Recall that a group is *torsion-free (TF)* if every element of finite order is the identity and that a group action is *free (F)* if $\mathrm{Fix}(g) \neq \emptyset \Rightarrow g = \mathrm{id}$. Now we summarize the connections between our various notions in the following

OBSERVATION 3.4. Let (X, d) be a metric space and let G be a subgroup of the isometry group $\mathrm{Iso}(X)$. Then the following hold:

1. SD $\Rightarrow$ WD.
2. SD + TF $\Rightarrow$ F.
3. Properly discontinuous $\Leftrightarrow$ WD + F.
4. SD + TF $\Rightarrow$ PD.
5. If X is a CAT(0) space, then F $\Rightarrow$ TF.

We make a couple of remarks before proving the observation.

REMARK 3.5. Strongly discrete torsion-free groups are always properly discontinuous. In the reverse direction

- In finite dimensions, or when X is proper, properly discontinuous groups are strongly discrete via 3., since WD $\Rightarrow$ SD in such a situation.

- On the other hand 5. tells that that in CAT(0) setting properly discontinuous groups are torsion-free.

We also remark that in a CAT(0) setting, if your group has torsion elements then it has no chance of being properly discontinuous. However for properly discontinuous groups:

(i) The fixed points of elements $g \in G$ do not occur in the interiors of our models of hyperbolic Hilbert space.

(ii) If $g \in G$ has three fixed points, then $g = \mathrm{id}$.

Proof. [Of Observation 3.4]

- Part 1. [SD $\Rightarrow$ WD]

  Fix $x \in X$. Then $\#[g : d(x, gx)] < \infty$ by strong discreteness. Therefore $\varepsilon := \min\{d(x, gx) : gx \neq x\} > 0$. So set $U := B(x, \varepsilon/2)$. Then we have that for every g with $gx \neq x$ we have $d(x, gx) \geq \varepsilon$ which implies $gU \cap U = \emptyset$.

- Part 2. [SD + TF $\Rightarrow$ F]

  By way of contradiction if G were not free then $gx = x$ for some $g \neq \mathrm{id}$. Then for every n we have $d(g^n x, x) = 0$. Now strong discreteness implies that $\#[g^n : n \in \mathbb{Z}] < \infty$ which in turn produces an N for which $g^N = \mathrm{id}$. This contradicts the assumption that G was torsion-free.

- Part 3. [Properly discontinuous $\Leftrightarrow$ WD + F]

  This follows straight from the definitions.

- Part 4. [SD + TF $\Rightarrow$ PD]

  Follows from Parts 1., 2. and 3.

- Part 5. [If X is a CAT(0) space, then F $\Rightarrow$ TF.]

  Suppose not, then $g^n = \mathrm{id}$ for some $g \neq \mathrm{id}$. Then $\#[H] < \infty$ where $H := \langle g \rangle$. Therefore the orbit $H(0)$ is bounded and by the Cartan's lemma there exists an $x \in \mathrm{Fix}(H) = \mathrm{Fix}(g)$. But we had assumed that G was free.

□

We make the simplifying assumption that all subgroups of $\mathcal{M}_{\mathbb{K}}$ are without torsion-elements.

REMARK 3.6. A discrete group G is strongly discrete if and only if for every sequence $(g_n)_1^\infty$ of distinct elements of G,

$$\limsup_{n\to\infty} d_{\mathcal{H}}(g_n(0), \partial\mathbb{B}_\infty) = 0.$$

This may observed from the following equivalences $\limsup_{n\to\infty} d_{\mathcal{H}}(g_n(0), \partial\mathbb{B}_\infty) = 0 \Leftrightarrow \liminf_{n\to\infty} d_{\mathcal{H}}(g_n(0), 0) = 1 \Leftrightarrow \liminf_{n\to\infty} d_{\mathbb{B}}(g_n(0), 0) = +\infty$.

REMARK 3.7. We remark that properly discontinuous actions are also known as *covering space actions* and in this language we could also have said a group G acts properly discontinuously on a topological space X if and only if

$$\Pi : X \to X/G \text{ is a covering map.}$$

Therefore in our case $\mathbb{K}_\infty/G$ is a hyperbolic Hilbertian manifold, and we let $q_G : \mathbb{K}_\infty \to \mathbb{K}_\infty/G$ denote the corresponding projection map. Since G acts as a group of hyperbolic isometries, the manifold $\mathbb{K}_\infty/G$ is uniquely endowed with an infinite-dimensional Riemannian metric projected down from $\mathbb{K}_\infty$.

3.2. Poincaré Series and Its Exponent of Convergence

DEFINITION 3.8. Fix $x \in \mathbb{K}_\infty$ and $s > 0$. The *Poincaré series* of the group G is defined by

$$\Sigma_{x,y}(s) = \sum_{g\in G} e^{-s d_{\mathbb{K}}(x, g(y))}. \tag{4}$$

With this notation, the *Poincaré exponent* of G is given by

$$\delta_G = \inf\{s > 0 : \Sigma_{x,y}(s) < +\infty\}. \tag{5}$$

It may be shown by using the triangle inequality that the definition is independent of our choice in x and y.

OBSERVATION 3.9. In the finite-dimensional case the Poincaré exponent of a discrete group is always finite. However in the Hilbertian case one may easily construct a strongly discrete (Schottky) group G such that $\delta_G = +\infty$. Construct a sequence of positive reals (a_n) with $a_n \to 0$ and $\sum_{n\geq 1} a_n^s = \infty$ for every s. Then with some care one may choose infinitely

many generators $(g_n)_n$ such that for every n we have that $e^{-d(g_n(0),0)} \asymp a_n$ and so that each generator has a distinct coordinate axis as its hyperbolic axis.

However the following fact remains true even in infinite dimensions.

OBSERVATION 3.10. If G is a subgroup of $\mathcal{M}_{\mathbb{K}}$ and $\delta_G < +\infty$, then the group G is strongly discrete.

PROOF. Suppose that G is not strongly discrete. Therefore that there exists some bounded set W such that $\#W \cap \overline{G(0)} = \infty$. Now the fact that W is bounded implies there exists some $B_{\mathbb{K}}(0, K) \supseteq W$ and therefore for every $g(0) \in W$, we have that

$$\sum_{g(0)\in W} e^{-td(0,g(0))} \geq \sum_{g(0)\in W} e^{-tK} = +\infty.$$

Since t was arbitrary we are done. □

3.3. Fundamental Domains

DEFINITION 3.11. We define a Dirichlet domain for G by

$$D := \{x : d(0, x) < d(x, g(0)) \ \forall g \neq \mathrm{id}\} \ .$$

OBSERVATION 3.12.

(i) $g(D) \cap D = \emptyset$ for all $g \neq \mathrm{id}$.

(ii) If G is strongly discrete, then $G\big(\overline{D}\big) = \mathbb{K}_\infty$.

PROOF.

(i) Suppose we had a $x \in D \cap g(D)$. Then we would have

$$d(g(x), 0) < d(g(x), g(0)) = d(x, 0) < d(x, g^{-1}(0)) = d(g(x), 0)$$

which is a contradiction.

(ii) Fix $x \in \mathbb{K}_\infty$. Since G is strongly discrete, we have that $\min_{g\in G}\{d(x, g(0))\}$ exists and therefore $d(x, g(0)) \leq d(x, h(0))$ for every h. In particular $d(x, g(0)) \leq d(x, gh(0))$ which is the same as $d(g^{-1}(x), 0) \leq d(g^{-1}(x), h(0))$. This means that $g^{-1}(x) \in \overline{D}$ and therefore we have proved that $G\big(\overline{D}\big) = \mathbb{K}_\infty$.

□

REMARK 3.13. In general however, tiling Hilbert spaces turns out to be a very subtle problem and has been studied (among others) by Klee, Fonf-Lindenstrauss and most recently by Preiss [27]. In brief, the acts of tiling by the group and of taking the closure of D do not commute.

3.4. Limit Sets and Their Basic Properties

We now will define the limit sets of our group actions and then move on to describe basic properties of such sets. When not specified we may assume that G is a subgroup of $\mathcal{M}_{\mathbb{K}}$ that acts properly discontinuously on $\mathbb{K}_\infty$ In most cases though this is more than we need and it suffices to assume that G is weakly discrete.

DEFINITION 3.14. For $x \in \mathbb{B}_\infty$, let

$$L_x(G) := \{\alpha \in \overline{\mathbb{B}}_\infty \mid \exists (g_n)_n \ \forall n \neq m, \ \ g_n \neq g_m \ \ \textit{and} \ \ \lim_{n\to\infty} g_n(x) = \alpha\}.$$

PROPOSITION 3.15.

(i) $L_x(G) \subseteq \partial\mathbb{B}_\infty$.

(ii) $L_x(G)$ *is independent of the choice of* x, *i.e.* $\forall x, y \in \mathbb{B}_\infty, L_x(G) = L_y(G)$.

PROOF. Let us start with

(i) Suppose that $\alpha \in L_x(G) \cap \mathbb{B}_\infty$. Then there exists a neighbourhood U of α such that for every $g \neq \mathrm{id}$, $g(U) \cap U = \emptyset$. But then $g_n(x) \in U$ for at most one value of n. But then $\alpha \notin L_x(G)$.

(ii) Let $g_n(x) \to \alpha \in L_x(G) \subseteq \partial\mathbb{B}_\infty$ as $n \to \infty$. Since the g_n are isometries, we have that $d_{\mathbb{B}}(g_n x, g_n y) = d_{\mathbb{B}}(x, y)$ for all $n \in \mathbb{N}$. It then follows (for example from Corollary 2.27) that $\|g_n x - g_n y\| \to 0$ as $n \to \infty$. Now by the triangle inequality,

$$\|g_n y - \alpha\| \leq \|g_n x - g_n y\| + \|g_n y - \alpha\|$$

and so $g_n y \to \alpha$ as $n \to \infty$. Therefore, $\alpha \in L_y(G)$ and we are done.

□

Now since $L_x(G)$ is independent of the choice of x, we may make the following

DEFINITION 3.16. We define the *limit set* of G to be

$$L(G) := L_x(G)$$

for some $x \in \mathbb{B}_\infty$.

REMARK 3.17. Note that any non-trivial properly discontinuous group has a non-empty limit set $L(G)$. Indeed, let $g \in G \setminus \{\mathrm{id}\}$. Then g must either be parabolic or hyperbolic since we have a discontinuous action. Now since it is clear that $\mathrm{Fix}(g) \subseteq L(G)$, the remark follows, see Table 2.1. We remark that the proof of this fact depends crucially on the existence of fixed points for non-trivial isometries of hyperbolic space; which followed from standard fixed point theory in finite dimensions, but required a separate development in infinite dimensions. See Section 2.6 for details.

OBSERVATION 3.18.

(i) $L(G)$ is closed.

(ii) $L(G)$ is G-invariant, i.e. for all $\gamma \in G$, $\gamma(L(G)) = L(G)$.

PROOF. These are fairly easy to see - the first follows from the definition of being a set of accumulation points. The second may be seen as follows: For $\gamma \in G, \alpha \in L(G)$ there exists a sequence of distinct elements from G, viz. (g_n) such that $g_n(0) \to \alpha$ as $n \to \infty$. Then $\gamma g_n(0) \to \gamma(\alpha)$ as $n \to \infty$. Therefore $\gamma(\alpha) \in L(G)$ and we have proved $\gamma(L(G)) \subseteq L(G)$. Now the other direction follows since $L(G) = \gamma^{-1}(\gamma(L(G))) \subseteq \gamma(L(G))$. □

DEFINITION 3.19. A group G is called *elementary* whenever $\#L(G) \in \{0, 1, 2\}$.

THEOREM 3.20 (No Global Fixed Points). *Any non-elementary weakly discrete group G has no global fixed points.*

PROOF. Assume by way of contradiction that we have a global fixed point and then conjugate to send it to ∞. Then by Liouville's theorem g must be of the form

$$g(x) = \lambda_g T[x] + b.$$

The proof now splits into two cases, viz.

Case 1: [$\lambda_g = 1$ for every $g \in G$.]

Then each hyperplane $\{x : x_0 = \alpha\}$ is fixed by G for every $\alpha > 0$ and therefore $L(G) = \{\infty\}$. This leads to a contradiction since we assumed that G was non-elementary.

Case 2: [There exists a $g \in G$ with $\lambda_g < 1$.]

Without loss of generality we assume that $g(0) = 0$ since g is hyperbolic and so $l_{0,\infty}$ is the invariant axis. Then g is of the form

$$g(x) = g_1(x) = \lambda_1 T_1[x]$$

with $\lambda_1 < 1$. We claim that there exists a $g_2(x) := \lambda_2 T_2[x] + b$ with $\lambda_2 < 1$ and $b \neq 0$. Indeed, since G is non-elementary, there exists a third limit point say ξ that is neither 0 nor ∞. Now fix a point on the invariant axis at distance greater than one from the boundary $\partial\mathbb{H}_\infty$. Then there exists an element of the group that brings it arbitrarily close to ξ and this element is the one we were after. Note that since $\xi \notin \{0, \infty\}$ its translation vector is non-zero as claimed.

Now let us calculate the commutator of g_1 and g_2 and call it $g_3 := [g_2, g_1]$. Let

$$y = g_3(x) = g_2^{-1} \circ g_1^{-1} \circ g_2 \circ g_1(x).$$

One may compute the following form

$$y = T_2^{-1} \circ T_1^{-1} \circ T_2 \circ T_1[x] + T_2^{-1} \circ T_1^{-1} \frac{I - \lambda_1 T_1}{\lambda_1 \lambda_2}[b]$$

which we rewrite as $g_3(x) = \hat{T}[x] + \hat{b}$. Note that $\hat{b} \neq 0$ [suppose not, then $T_2^{-1} \circ T_1^{-1}(I - \lambda_1 T_1)[b] = 0 \Leftrightarrow \lambda_1 T_1[b] = b$ which leads to a contradiction since $\lambda_1 < 1$ and $b \neq 0$]. Recall that $\uparrow = e_1$. Now consider

$$\begin{aligned} g_1^n g_3 g_1^{-n}(\uparrow) &= g_1^n g_3(\lambda_1^{-n} T_1^{-n} \uparrow) \\ &= g_1^n \big(\lambda_1^{-n} \hat{T}[T_1^{-n} \uparrow] + \hat{b}\big) \\ &= T_1^n \hat{T}[T_1^{-n} \uparrow] + \lambda_1^n T_1^n[\hat{b}] \\ &= \uparrow + \lambda_1^n T_1^n[\hat{b}]. \end{aligned}$$

Note that the last term goes to zero in norm as $n \to \infty$ since $\lambda_1 < 1$. But this contradicts the fact that G is weakly discrete.

□

Let us single out a useful observation from the proof above

OBSERVATION 3.21. Given a weakly discrete group G, let $g, h \in G$ be such that $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$. If g is hyperbolic then so is h and $\mathrm{Fix}(g) = \mathrm{Fix}(h)$.

PROOF. By way of contradiction suppose that there was a common fixed point and let the other two be distinct. Then conjugate the common fixed point to ∞ and have the one of other two go to $0 \in \mathcal{E}$. Then $l_{0,\infty}$ is the invariant axis. Then our maps are of the form $g_1(x) = \lambda_1 T_1[x]$ with $\lambda_1 < 1$ and $g_2(x) = \lambda_2 T_2[x] + b$ with $\lambda_2 < 1$ and $b \neq 0$. Now the remainder of the proof works in same way as that Theorem 3.20 □

We have the following characterization of elementary groups.

THEOREM 3.22. *For every G that is properly discontinuous the following are equivalent:*

(1) $\#[L(G)] < \infty$.

(2) $\exists F \subseteq \overline{\mathbb{K}}_\infty$ *non-empty, finite and G-invariant.*

(3) Either

- $G = \langle \mathrm{id} \rangle$.
- $\exists! \xi \in \partial \mathbb{K}_\infty$ *parabolic with* $G(\xi) = \xi$ *and G consists entirely of parabolics.*
- $G = \langle g \rangle$, *with g hyperbolic.*

(4) $\nexists\ g, h \in G$ *hyperbolic with* $Fix(g) \cap Fix(h) = \varnothing$.

PROOF. We start by noticing that (3) implies the remaining conditions. The remaining three steps are placed below:

Step 1 [(1) ⇒ (2)]

If $G = \langle \mathrm{id} \rangle$ then we're done. So assume $G \neq \langle \mathrm{id} \rangle$ and choose $g \neq \mathrm{id}$. Then g is either parabolic or hyperbolic. In either case $L(G) \neq \emptyset$ which in turn produces our finite G-invariant set.

Step 2 [(4) $\Rightarrow$ (2)]

This splits into two subcases

- *Case 2a.* [There exists a $g \in G$ which is hyperbolic].

  Let $F := \mathrm{Fix}(g)$. Then

  $$F = \bigcup_{\substack{h \in G \\ h \text{ hyperbolic}}} \mathrm{Fix}(h) \ .$$

  Now Observation 3.21 and (4) together imply that $\mathrm{Fix}(h) = \mathrm{Fix}(g)$ for all hyperbolic $h \in G$. All that's left to check is that $G(F) = F$ which follows from the fact that $g\mathrm{Fix}(h) = \mathrm{Fix}(ghg^{-1})$. Therefore (2) holds.

- *Case 2b.* [None of the $g \in G$ are hyperbolic].

  If $G = \langle \mathrm{id} \rangle$ then we're done. So assume $G \neq \langle \mathrm{id} \rangle$ and let g be parabolic with $F := \mathrm{Fix}(g)$. Again we are left to show $G(F) = F$. Suppose not, then $h(p) \neq p$ for $h \in G$. Conjugate p to ∞ so that g is a translation. Then there exists some ball B intersecting $\partial\mathbb{H}_\infty$ orthogonally with $h(B) \cap B = \emptyset$. Now since g is parabolic $\overline{g^n(h(B))} \subseteq B$ for some large n [Note that we cannot make the statement for sufficiently large n as g may have recurrent behaviour unlike in finite dimensions]. But then the infinite dimensional version of the Schwarz lemma implies that $g^n h$ is hyperbolic which leads us into contradiction. Let us say a few words about this last step. To be precise, the Schwarz lemma tells us that any conformal isomorphism $g : \mathbb{B}_\infty$ is semi-contracting in the hyperbolic metric on $\mathbb{B}_\infty$. We may then use this and the Banach contraction principle to prove the following claim. Suppose $g : \mathbb{H}_\infty \to \mathbb{H}_\infty$ and $\overline{g(B)} \subseteq B$ for some ball B intersecting $\partial\mathbb{H}_\infty$ orthogonally. Then g is hyperbolic and we have that both $\#[\mathrm{Fix}(g) \cap B] = 1$ as well as $\#[\mathrm{Fix}(g) \cap \mathbb{H} \setminus B] = 1$.

Step 3 [(2) $\Rightarrow$ (3)]

Suppose $G \neq \langle \mathrm{id} \rangle$ and let $g \in G \setminus \{\mathrm{id}\}$. Let $S := F \setminus \mathrm{Fix}(g)$ and notice that $g(S) = S$. Also notice that S is finite and in fact $S = \emptyset$. [Suppose not, then let $x \in S$. we would therefore have $g^n(x)$ clustering at $\mathrm{Fix}(g)$ which would imply that

$\mathrm{Fix}(g) \cap \overline{S} \neq \emptyset$. This leads to a contradiction via the definition of S]. Therefore $F \subseteq \mathrm{Fix}(g)$ which implies that F is fixed point-wise by G, i.e. $G(p) = p$ for $p \in F$. At this point the proof spilts into two more pieces. In the first case, if G were to consist entirely of parabolics then we'd be done by the second option in (3). If not, let $g \in G$ be hyperbolic. We will show that every element of G must be hyperbolic and in fact G itself is a cyclic group, i.e. $G = \langle g \rangle$.

For every $h \in G$, $p \in \mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$ and therefore h must also be hyperbolic and $\mathrm{Fix}(g) = \mathrm{Fix}(h)$ by Observation 3.21. It is left to show that G is cyclic. Without loss of generality let $\mathrm{Fix}(g) = \{0, \infty\} \subseteq \overline{\mathbb{H}}_\infty$. Then for every $h \in G$ we have that $h(x) = \lambda_h T[x]$ with $\lambda_h > 0$. The map $\lambda : G \to \mathbb{R}_+^*$ defined by $h \mapsto \lambda_h$ is a homomorphism from G to the multiplicative group of positive reals. Then it is enough to show that $\lambda(G)$ is a discrete subgroup of $\mathbb{R}_+^*$ since any such subgroup must be cyclic. Suppose it is not discrete, then there exists a sequence of elements with $\lambda(g_n) \to 1$ as $n \to \infty$. But then $g_n(\uparrow) = \lambda(g_n)(\uparrow)$ which tends to $\uparrow$ as $n \to \infty$. This contradicts the discreteness of G and we are done.

□

Remark 3.23. We used the fact that any discrete subgroup of $\mathbb{R}_+^*$ must be cyclic. Here is a short proof: let G be the subgroup in question and define

$$\lambda_0 := \min\{\lambda > 1 : \lambda \in G\}.$$

We have a minimum instead of an infinimum since G is discrete and we claim that $G = \langle \lambda_0 \rangle$. Given a $\lambda \in G$ let $\lambda = \tilde{\lambda}\lambda_0^n$ for some $1 \leq \tilde{\lambda} < \lambda_0$. Since $\tilde{\lambda} \in G$ we have that $\tilde{\lambda} = 1$ by the definition of λ_0 and we are done.

The proof of the following proposition is standard and will be omitted.

Proposition 3.24 (Perfection). *For every non-elementary group G, $L(G)$ is perfect, i.e. it contains no isolated points. Moreover, for every non-trivial group G, $\#L(G) \in \{1, 2, \#\mathbb{R}\}$.*

THEOREM 3.25 (Minimality). *For every non-elementary group G, $L(G)$ is the smallest non-empty closed G-invariant subset of $\partial\mathbb{K}_\infty$.*

PROOF. By Theorem 3.20 every non-elementary group has no global fixed points and so we may assume that E has at least two elements. Call them $\alpha_1, \alpha_2 \in \partial\mathbb{K}$. For an arbitrary $\xi \in L(G)$, we'd like to show that $\xi \in E$. There exists a sequence g_n such that $g_n(0) \to \xi$.

OBSERVATION 3.26. $g_n(x) \to \xi$ on a subseq, for $x \in \{\alpha_1, \alpha_2\}$.

Notice that $0 \sim \langle \alpha_1 | \alpha_2 \rangle_0 \gtrsim \min_{i=1,2} \langle g_n^{-1}(0) | \alpha_i \rangle_0$. [We use the fact that our space is a hyperbolic space in the sense of Gromov, see Section 2.8, Definition 2.36.] This implies that $\exists\ (i_n)_n$ with $\langle g_n^{-1}(0) | \alpha_i \rangle_0 \sim 0$ which means geometrically that one of the α_i's will be in a shadow as seen from $g_n^{-1}(0)$. So we push it forward by g_n to get $\exists\ (i_n)_n$ with $\langle 0 | g_n(\alpha_i) \rangle_{g_n(0)} \sim 0$. Again using the Gromov inequality we have

$$\langle g_n(\alpha_{i_n}) | \xi \rangle_0 \gtrsim \min_{i=1,2} \Big(\langle g_n(0) | g_n(\alpha_{i_n}) \rangle_0, \langle g_n(0) | \xi \rangle_0 \Big)$$

The first term

$$\langle g_n(0) | g_n(\alpha_{i_n}) \rangle_0 \sim d(0, g_n(0))$$

and the second term

$$\langle g_n(0) | \xi \rangle_0$$

both go to ∞ as $n \to \infty$ since we know that $g_n(0) \to \xi$. Therefore

$$\langle g_n(\alpha_{i_n}) | \xi \rangle_0 \underset{n}{\to} \infty, \quad \text{i.e.} \quad g_n(\alpha_{i_n}) \underset{n}{\to} \xi.$$

We're done with proving the observation and can now wrap up the proof. Since E is G-invariant this implies that $g_n(\alpha_{i_n}) \in E$ for every $n \in \mathbb{N}$ and since E is closed, $g_n(\alpha_{i_n}) \to \xi$ implies that $\xi \in E$. Therefore, $L(G) \subseteq E$ and we're done. There cannot be a smaller closed G-invariant subset of $L(G)$. $\square$

DEFINITION 3.27. A properly discontinuous group G is said to be of *compact type* when $L(G)$ is compact.

THEOREM 3.28. *For a properly discontinuous group G acting on $\mathbb{B}_\infty$, the following are equivalent:*

(1) G is of compact type.

(2) Every infinite subset of $G(0)$ contains an accumulation point.

(3) Each sequence $(g_n(0))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\|g_n(0)\| = 1$ has a converging subsequence, which necessarily accumulates at an element in $L(G)$.

(4) Every infinite subset of $G(0)$ contains a sequence $(z_n)_n$ such that

$$\langle z_n|z_m\rangle_0 \to \infty \text{ as } n, m \to \infty.$$

REMARK 3.29. In CAT(-1) spaces, any group of compact type acting properly discontinuously is strongly discrete.

PROOF. (of Theorem 3.28) Notice that (2) $\implies$ (3) is immediate. Let's start by proving (3) $\implies$ (1). Suppose that $(\xi_n)_n \subseteq L(G)$. Then for every n, there exists a m_n such that $\|g_{m_n}(0) - \xi_n\| \leq 1/n$. Since we have a discontinuous action, orbits must accumulate on the boundary, i.e. $\|g_{m_n}(0)\| \to 1$ and $n \to \infty$. By hypothesis, there exists a subsequence $(n_k)_k$ and $\xi \in L(G)$ such that $g_{m_{n_k}}(0) \to \xi$ as $k \to \infty$. We now have that

$$\|\xi_{n_k} - \xi\| \leq \|g_{m_{n_k}}(0) - \xi_n\| + \|g_{m_{n_k}}(0) - \xi\|,$$

where the first term is bounded by $1/n_k$ and therefore it and the second term both vanish as $k \to \infty$.

Next we we show that (1) $\implies$ (2). Let $(g_n(0))_n$ be our infinite subset of $G(0)$. Then for every n, $g_n \in G$. Now pick two points $\alpha, \beta \in L(G)$ and let $l_{\alpha,\beta}$ be the geodesic between them. Now fix some $z \in l_{\alpha,\beta}$ and pick some common subsequence such that $g_n(\alpha) \to \gamma$ and $g_n(\beta) \to \delta$ as $n \to \infty$. Now one may verify via calculation (or geometric intuition) that for every $\epsilon > 0$ there exists N_ϵ such that for every $n \geq N_\epsilon$, $g_n(l_{\alpha,\beta}) \subseteq B_{\mathcal{H}}(l_{\gamma,\delta}, \epsilon)$. One may check that the Hausdorff distance between the $g_n(l_{\alpha,\beta})$ is dependent on the distances $\|\gamma - g_n(\alpha)\|$ and $\|\delta - g_n(\beta)\|$ and vanishes as they decrease to zero. Therefore there exists a subsequence such that $g_n(l_{\alpha,\beta}) \subseteq B_{\mathcal{H}}(l_{\alpha,\beta}, 1/n)$ for every n. It now follows that for every n

there exists $w_n \in l_{\gamma,\delta}$ such that $d_{\mathcal{H}}(g_n(z), w_n) \leq 1/n$. Now there exists a subsequence $(w_n)_n$ (by compactness of $\lambda_{\gamma,\delta}$) such that $w_n \to w \in l_{\gamma,\delta}$ as $n \to \infty$. Finally one can see that $g_n(z) \to w$ as $n \to \infty$ since

$$\|g_n(z) - w\| \leq \|g_n(z) - w_n\| + \|w_n - w\|.$$

Finally notice that the equivalence with condition (4) follows from Corollary 2.27.

□

We remark that condition (4) is intrinsic to the interior of $\mathbb{B}_\infty$ and is not in terms of the boundary.

3.5. Finer Structures within the Limit Set

Fix $\alpha \in \partial\mathbb{K}_\infty$. Let S_α refer to the *hyperbolic ray* from 0, the origin, to α. Recall that the Busemann function may be extended to $\partial\mathbb{K}_\infty \times \mathbb{K}_\infty \times \mathbb{K}_\infty$ via

$$\lim_{z\to\alpha} \mathcal{B}_z(x, y) =: \mathcal{B}_\alpha(x, y),$$

where $\mathcal{B}_z(x, y) := d(x, z) - d(y, z)$.

Definition 3.30. Let us define the following distinguished subsets of $L(G)$:

- $\alpha \in L(G)$ is a *horospherical limit point* of G if there exists a sequence of orbit points $g(0)$ with
$$\mathcal{B}_\alpha(0, g(0)) \to -\infty.$$
- $\alpha \in L(G)$ is a *radial limit point* of G if there exists a constant $c = c(\alpha) > 0$ such that
$$S_\alpha \cap B(g(0), c) \neq \varnothing$$
for infinitely many $g(0)$ for every $g \in G$.
- $\alpha \in L(G)$ is a *uniformly radial limit point* of G if there exists a positive constant $c = c(\alpha) > 0$ such that
$$S_\alpha \subseteq \cup_{g\in G} B(g(0), c).$$

In other words, we may write

$$L_{ur}(G) = \cup_{\sigma>0} L_{ur,\sigma}(G)$$

and

$$L_r(G) = \cup_{\sigma>0} L_{r,\sigma}(G)$$

where $\alpha \in L_{ur,\sigma}(G)$ when the geodesic from 0 to α, S_α, is covered by hyperbolic balls $B(g(0), \sigma)$ over the G-orbit of the origin and similarly where $\alpha \in L_{r,\sigma}(G)$ when S_α intersects infinitely many hyperbolic balls $B(g_n(0), \sigma)$ for some subsequence of the G-orbit of the origin.

Definition 3.31. We define the projection map $\Pi : X \setminus \{0\} \to \partial X$ to be the unique map so that for all $x \in X \setminus \{0\}$, x is on the geodesic joining 0 and $\Pi(x)$. For $x \in X$ and $\sigma > 0$, it is useful to consider the set $\Pi(B(x, \sigma))$, which is called the "shadow" of the ball $B(x, \sigma)$, denoted by $\mathrm{Shad}(x, \sigma)$. It helps to if we imagine shining a light from the point 0 onto the boundary. In general there is nothing distinguished about the origin 0 and in fact we can similarly define Π_z and $\mathrm{Shad}_z(x, \sigma)$ when we are shining a light from $z \in \mathbb{K}_\infty$. Without the subscript we assume that the light is based at the origin.

Let us make a short digression on shadows in hyperbolic spaces which do not necessarily have geodesics. The reader may find more details including proofs of the next few facts in [8]. Here is a definition which uses only the Gromov product:

Definition 3.32. For each $\sigma > 0$ and $x \in X$, let

$$\mathrm{Shad}(x, \sigma) = \{\eta \in \partial X : \langle 0|\eta\rangle_x \leq \sigma\}.$$

Let us establish up front some geometric properties of shadows.

Observation 3.33. If $\eta \in \mathrm{Shad}(x, \sigma)$, then

$$\langle \eta|x\rangle_0 \leq d(x, 0) \asymp \langle \eta|x\rangle_0 + \langle \eta|0\rangle_x \lesssim \langle \eta|x\rangle_0 + \sigma$$

LEMMA 3.34. *For each $\sigma > 0$, there exists $\tau > 0$ such that for all $x, y \in X$ with* $\operatorname{dist}(0, y) \geq \operatorname{dist}(0, x)$ *and* $\operatorname{Shad}(x, \sigma) \cap \operatorname{Shad}(y, \sigma) \neq \emptyset$*, the following two formulas hold.*

$$\operatorname{Shad}(y, \sigma) \subseteq \operatorname{Shad}(x, \tau) \tag{6}$$

$$\operatorname{dist}(x, y) \asymp \mathcal{B}_0(y, x). \tag{7}$$

The key relation between $\Pi(B(x, \sigma))$ and $\operatorname{Shad}(x, \sigma)$ is made explicit in the following

COROLLARY 3.35. *Let X be standard hyperbolic space, take $X = \mathbb{K}_\infty$. For any $x \in X$ and $\sigma > 0$ we have*

$$\Pi(B(x, \sigma)) \subseteq \operatorname{Shad}(x, \sigma) \subseteq \Pi(B(x, \tau)),$$

where τ is as in Lemma 3.34.

REMARK 3.36. In the Corollary above, we may take X to be any hyperbolic space in which geodesics exist between points in the interior and points on the boundary, and in which geodesics between two points in the interior can be uniquely extended.

Let us redefine our subclasses of $L(G)$

OBSERVATION 3.37. Fix a sequence $(x_n)_n$ in X. Suppose that $\lim_{n\to\infty} \operatorname{dist}(0, x_n) = \infty$, and that for some $\sigma > 0$ the shadows $\operatorname{Shad}(x_n, \sigma)$ have a common intersection point η. Then $\lim_{n\to\infty} x_n = \eta$.

DEFINITION 3.38. In the situation above, we say that the sequence $(x_n)_1^\infty$ *converges radially to η*. We say that $(x_n)_1^\infty$ *converges uniformly radially to η* if it converges radially and if the sequence of distances $(\operatorname{dist}(x_n, x_{n+1}))_1^\infty$ is bounded.

The concept of radial convergence can be reformulated analytically as follows:

OBSERVATION 3.39. Suppose that $\lim_{n\to\infty} \operatorname{dist}(0, x_n) = \infty$. Then the sequence $(x_n)_1^\infty$ converges radially to a point $\eta \in \partial X$ if and only if either of the following equivalent relations

holds:

$$\langle 0|\eta\rangle_{x_n} \asymp 0$$

$$\langle x_n|\eta\rangle_0 \asymp \mathrm{dist}(0, x_n).$$

Definition 3.40. Let G be a Kleinian group. The sets

$$L(G) := \{\eta \in \partial X : \exists\ g_n(0) \underset{n}{\to} \eta\} \tag{8}$$

$$L_{\mathrm{h}}(G) := \{\eta \in \partial X : \exists\ g_n(0) \underset{n}{\to} \eta \text{ with } B_\alpha(0, g_n(0)) \underset{n}{\to} -\infty\} \tag{9}$$

$$\Lambda_{\mathrm{r}}(G) := \{\eta \in \partial X : \exists\ g_n(0) \underset{n}{\to} \eta \text{ radially}\} \tag{10}$$

$$\Lambda_{\mathrm{ur}}(G) := \{\eta \in \partial X : \exists\ g_n(0) \underset{n}{\to} \eta \text{ uniformly radially}\} \tag{11}$$

denote the *limit set*, *horospherical limit set*, *radial limit set*, and *uniformly radial limit set*, respectively.

Notice that we have the following inclusions:

$$\Lambda_{\mathrm{ur}} \subset \Lambda_{\mathrm{r}} \subset L_{\mathrm{h}} \subset L.$$

In the literature the complement of the horospherical set in the limit set has been sometimes referred to as the *Jørgensen limit set* and L_r was classically referred to as the *conical limit set*. Notice that though our limit sets coincide, to connect our definitions on the level of the $L_{ur,\sigma}$'s one may show that there exists κ such that for every σ, we have $L_{ur,\sigma} \subseteq L_{ur,\sigma+\kappa}$ and vice versa.

For $w, z \in \overline{\mathbb{K}}_\infty$, let $\gamma_{w,z}$ be the unique geodesic joining w and z. Let

$$C_{\mathbb{B}}(G) := \bigcup_{\xi_1,\xi_2 \in L(G)} \gamma^\circ_{\xi_1,\xi_2}\ .$$

Then notice that $C_{\mathbb{B}}(G)$ is G-invariant, i.e. for any $g \in G$, $g(C_{\mathbb{B}}(G)) = C_{\mathbb{B}}(G)$.

Definition 3.41. A properly discontinuous group G is *convex cobounded* if there exists a ball about the origin $B(0, R)$ such that $C_{\mathbb{B}}(G) \subset B(G(0), R) = G(B(0, R))$.

Theorem 3.42. *The following are equivalent for a group G of compact type:*

(i) The group is convex-cobounded.

(ii) $L_{ur,\sigma} = L(G)$ *for some* σ

(iii) $L_{ur}(G) = L(G)$.

(iv) $L_r(G) = L(G)$.

(v) $L_h(G) = L(G)$.

PROOF. For lack of time, we refer to [8] for a complete proof. □

CHAPTER 4

THE THEOREM OF BISHOP AND JONES

Throughout this section let (X, ρ) be a complete metric space.

DEFINITION 4.1. Given two sets $C, D \subseteq X$ and some $\kappa > 0$ we say that

$$C \subseteq_\kappa D \Leftrightarrow B(C, \kappa \text{diam}(C)) \subseteq D,$$

and read it as the *κ-thickening of C is contained in D*. We denote $B(C, \kappa \text{diam}(C)) := C_\kappa$ to be the κ-thickening of C.

We prove the following mass-redistribution result.

PROPOSITION 4.2 (Mass-redistribution). *Let (X, ρ) be a complete metric space and fix $t \geq 0$ and $\kappa \in (0, 1)$. For every $n \geq 1$ let E_n be a finite set. Set $E_i^j := E_i \times \ldots \times E_j$ for $0 \leq i \leq j$ and to avoid clutter we write E^n for E_1^n. Suppose that $\Sigma \subseteq E^* := \bigcup_{n \geq 1} E^n$, has the property that*

$$\widehat{\Sigma} := \{\omega|_{|\omega|-1} \in E^{|\omega|-1} : \omega \in \Sigma\} \subseteq \Sigma \cup \{\varnothing\}.$$

We denote $\Sigma^n := \Sigma \cap E^n$. Suppose further that for every $\omega \in \Sigma$, there exists a closed subset $A(\omega) \subseteq X$ with the following properties:

a) *For every $\omega \in \Sigma$, $A(\omega) \subseteq_\kappa A(\omega|_{|\omega|-1})$;*

b) *For every $\omega \in \Sigma$, $\text{diam}\big(A(\omega|_{|\omega|-1})\big) \leq \kappa^{-1} \text{diam}(A(\omega))$;*

c) *For $\omega, \tau \in E^n \cap \Sigma$, $\omega \neq \tau$, $A(\omega) \cap A(\tau) = \emptyset$;*

d) *For every $\omega \in \Sigma$*

$$\sum_{\substack{e \in E_{|\omega|+1} \\ \omega e \in \Sigma}} \text{diam}^t\big(A(\omega e)\big) \geq \text{diam}^t\big(A(\omega)\big); \text{ and}$$

e) $\lim_{n \to \infty} \max\{\text{diam}(A(\omega)) : \omega \in \Sigma^n\} = 0$.

Then it follows that

$$HD\left(\bigcap_{n=1}^{\infty} \bigcup_{\omega \in \Sigma^n} A(\omega)\right) \geq t.$$

REMARK 4.3. We make a few small remarks, before starting the proof of the Proposition.

- The sets E_i may be thought of as index sets or as alphabets if we think in terms of symbolic dynamics, or IFSs. Note that if we take all of them to be equal to the same set E then E^n is simply the n-fold product of the set E.
- It may help to explain what each of the conditions are saying
  (i) Condition a) says that the κ-thickenings are decreasing, while
  (ii) Condition b) tells us that they do not decrease too fast.
  (iii) Condition d) is the appropriate redistribution of mass that leads to the appropriate measure in the limit
  (iv) Condition c) is a natural disjointness condition which is necessary when building a measure.

  We note that we may prove the same proposition by specifying rates, i.e. different κ's for a) and b).
- In general metric spaces $\mathrm{diam}(A_\kappa) \leq (1+2\kappa)\mathrm{diam}(A)$. Whereas for Banach spaces, it turns out that we have equality. Therefore when in Hilbert spaces for instance, we have that (a) implies condition (e).

PROOF. Decreasing Σ if necessary we may assume without loss of generality that $A(\omega) \neq \emptyset$ for every $\omega \in \Sigma$. For every $n \geq 1$, let

$$J_n := \bigcup_{\omega \in E^n \cap \Sigma} A(\omega) \quad \text{and} \quad J := \bigcap_{n=1}^{\infty} J_n.$$

Note that for every $\omega \in \Sigma$, we have that $J \cap A(\omega) \neq \emptyset$ and so we fix a point $x_\omega \in J \cap A(\omega)$. Now define inductively the following sequence of Borel probability measures $(\mu_n)_{n \geq 0}$ on J as follows. Let μ_0 be an arbitrary Borel probability measure on J. For the inductive step, suppose that $n \geq 0$ and $\mu_0, \mu_1, \ldots, \mu_n$ (each a Borel probability measure on J) have already been defined. For every $\omega \in E^{n+1} \cap \Sigma$, set

$$\mu_{n+1} := \sum_{\omega \in E^{n+1} \cap \Sigma} \frac{\mathrm{diam}^t(A(\omega))}{\sum_{e \in E : \omega|_n e \in \Sigma} \mathrm{diam}^t(A(\omega|_n e))} \mu_n(A(\omega|_n)) \delta_{x_\omega}. \tag{12}$$

Now since μ_n is a probability measure and by c) all the sets $A(\tau), \tau \in E^n \cap \Sigma$ are pairwise disjoint, a straightforward direct computation shows that μ_{n+1} is a Borel probability measure. Now since $x_\omega \in J$ for every $\omega \in \Sigma$, we immediately get from (12) that

$$\mu_n(J) = 1 \text{ for every } n \geq 0. \tag{13}$$

Now suppose that $\tau \in E^n \cap \Sigma$. In view of (12) and c) we then get

$$\begin{aligned}
\mu_{n+1}(A(\tau)) &= \sum_{\omega \in E^{n+1} \cap \Sigma} \frac{\operatorname{diam}^t(A(\omega))}{\sum_{e \in E : \omega|_n e \in \Sigma} \operatorname{diam}^t(A(\omega|_n e))} \mu_n(A(\omega|_n)) \delta_{x_\omega}(A(\tau)) \\
&= \sum_{a \in E : \tau a \in \Sigma} \frac{\operatorname{diam}^t(A(\tau a))}{\sum_{e \in E : \tau e \in \Sigma} \operatorname{diam}^t(A(\tau e))} \mu_n(A(\tau)) \\
&= \mu_n(A(\tau)) \sum_{a \in E : \tau a \in \Sigma} \frac{\operatorname{diam}^t(A(\tau a))}{\sum_{e \in E : \tau e \in \Sigma} \operatorname{diam}^t(A(\tau e))} \qquad (14) \\
&= \mu_n(A(\tau)) \frac{\sum_{a \in E : \tau a \in \Sigma} \operatorname{diam}^t(A(\tau a))}{\sum_{e \in E : \tau e \in \Sigma} \operatorname{diam}^t(A(\tau e))} \\
&= \mu_n(A(\tau)).
\end{aligned}$$

Because of c) and since $x_\omega \in J \cap A(\omega)$ for every $\omega \in \Sigma$, we get that for every $n \geq 0$ and for all $0 \leq i \leq j$ that

$$\mu_{n+j}(A(\tau)) = \mu_{n+j}\left(\bigcup_{\omega \in E^{n+i}_{n+1} : \tau\omega \in \Sigma} A(\tau\omega) \right) = \sum_{\omega \in E^{n+i}_{n+1} : \tau\omega \in \Sigma} \mu_{n+j}(A(\tau\omega)). \tag{15}$$

Having this formula and treating (12) as the inductive step, we now prove the following

Observation 4.4. $\mu_{n+k}(A(\tau)) = \mu_n(A(\tau))$ for every $n, k \geq 0$ and all $\tau \in E^n \cap \Sigma$.

Proof. Fix $n \geq 0$. The proof now follows by induction on $k \geq 0$. For $k = 0$ it is a tautlogy. So suppose the claim is true for some $k \geq 0$ and all $\tau \in E^n \cap \Sigma$. Using (15) and (14) we then get

$$\begin{aligned}
\mu_{n+(k+1)}(A(\tau)) = \sum_{\omega \in E^k : \tau\omega \in \Sigma} \mu_{n+(k+1)}(A(\tau\omega)) &= \sum_{\omega \in E^k : \tau\omega \in \Sigma} \mu_{n+k}(A(\tau\omega)) \\
&= \mu_{n+k}(A(\tau)) \\
&= \mu_n(A(\tau)).
\end{aligned}$$

The inductive proof is complete. □

Now let $c := \left[\sum_{e\in E_1} \mathrm{diam}^t(A(e))\right]^{-1} < \infty$. Next we prove the following

OBSERVATION 4.5. For every $n \geq 1$ and for every $\omega \in E^n \cap \Sigma$, we have that $\mu_n(A(\omega)) \leq c\mathrm{diam}^t(A(\omega))$.

PROOF. We prove this by induction on $n \geq 1$. If $e \in E \cap \Sigma$, then it follows from (12) that

$$\begin{aligned}\mu_1(A(e)) &= \left[\sum_{a\in E_1} \mathrm{diam}^t(A(a))\right]^{-1} \mathrm{diam}^t(A(e)) \cdot \mu_0(X)\\ &= c \cdot \mathrm{diam}^t(A(e)).\end{aligned}$$

So suppose that the claim holds for some $n \geq 1$. For every $\omega \in E^{n+1} \cap \Sigma$, we then have from (14) and d) that

$$\begin{aligned}\mu_{n+1}(A(\omega)) &= \left[\sum_{e\in E_{n+1}:\omega|_n e\in\Sigma} \mathrm{diam}^t(A(\omega|_n e))\right]^{-1} \mathrm{diam}^t(A(\omega)) \cdot \mu_n(A(\omega|_n))\\ &\leq \left[\mathrm{diam}^t(A(\omega|_n))\right]^{-1} \mathrm{diam}^t(A(\omega)) \cdot \mu_n(A(\omega|_n))\\ &\leq c \cdot \mathrm{diam}^t(A(\omega)).\end{aligned}$$

We are done. □

Now the set $J = \bigcap_{n\geq 1} \bigcup_{\omega\in E^n\cap\Sigma} A(\omega) =: \bigcap_{n\geq 1} J_n$ is closed as it is the intersection of closed sets J_n. Recall e) gives us that

$$\lim_{n\to\infty} \max\{\mathrm{diam}(A(\omega)) : \omega \in E^n \cap \Sigma\} = 0$$

and now because of the finiteness of the index sets E_n, the set J is totally bounded. Thus J is compact since X is complete. Therefore by the Banach-Alaoglu theorem, the sequence $(\mu_n)_1^\infty$ of Borel probability measures on J contains a weakly convergent subsequence. Denote its weak limit by μ. Since $A(\omega) \cap J$ is a clopen subset of J (with respect to the topology relative to J) for every $\omega \in \Sigma$, it follows from Observation 4.5 that for every $\omega \in \Sigma$,

$$\mu(A(\omega)) \leq c \cdot \mathrm{diam}^t(A(\omega)). \tag{16}$$

Note that we have not yet used conditions $a)$ and $b)$ and we now do so in estimating from above the measures of balls centered at the points of J.

Let $z \in J = \cap_{n\geq 1} J_n$. Since for every $n \geq 1$, the sets $E_n \cap \Sigma$ are finite, it follows from König's Lemma that there exists $\omega \in E^{\mathbb{N}}$ such that $\omega|_n \in \Sigma$ for every $n \geq 1$ and $\{z\} = \bigcap_{n=1}^{\infty} A(\omega|_n)$. Because of $c)$ this $\omega \in E^{\mathbb{N}}$ is unique.

Now fix a radius $r \in (0, \kappa \min\{\mathrm{diam}(A(\omega)) : \omega \in E^2 \cap \Sigma\})$. Then there exists a largest $n = n(\omega, r) \geq 2$ such that

$$r \leq \kappa \mathrm{diam}(A(\omega|_n)). \tag{17}$$

Since $z \in A(\omega|_n)$ it follows from $a)$ that

$$B(z,r) \subseteq B\big(z, \kappa \mathrm{diam}(A(\omega|_n))\big) \subseteq B\big(A(\omega|_n), \kappa \mathrm{diam}(A(\omega|_n)))\big) \subseteq A(\omega|_{n-1}).$$

Now (16) implies that

$$\mu(B(z,r)) \leq c \cdot \mathrm{diam}^t(A(\omega|_{n-1})). \tag{18}$$

By the definition of n, we have that $\kappa \mathrm{diam}(A(\omega|_{n+1})) < r$ and then applying $b)$ twice we get that

$$\mathrm{diam}(A(\omega|_{n-1})) \leq \kappa^{-1} \mathrm{diam}(A(\omega|_n)) \leq \kappa^{-2} \mathrm{diam}(A(\omega|_{n+1})) \leq \kappa^{-3} r.$$

Inserting this into (18) we finally get

$$\mu(B(z,r)) \leq c\kappa^{-3t} r^t. \tag{19}$$

Note that $\mu(J) = 1$ (since its a probability measure on J) and thus by a direct application of Frostmann's Lemma, we have that $HD(J) \geq t$. We are done. □

Remark 4.6. In the case when (X, ρ) is a finite-dimensional euclidean space, condition $a)$ can be replaced by the requirement that all the sets $A(\omega)$ for $\omega \in \Sigma$ are uniformly undistorted balls and then a standard volume argument would work to get (19). Our Proposition 4.2 requires no extra structure on X and condition $a)$ will be proved to be satisfied in the course of our proof of the Bishop-Jones theorem for Hilbert spaces. We should note however that condition $a)$ is somewhat strong and for example it fails in the standard construction of

C, the middle-third Cantor set if $X = [0, 1]$. Note however if $X = C$, then a) is satisfied. Therefore one must take some care in the choice of X.

NOTATION 4.7. For $\xi \in \partial\mathbb{B}_\infty$, we denote $\widehat{B}_e(\xi, r)$ or equivalently $\lambda(\xi, r)$ to be the interior of the the hyperbolic half-plane containing ξ and chosen so that the projection of this half-plane onto $\partial\mathbb{B}_\infty$ gives the sphere centered at ξ of euclidean radius r.

Next we shall prove the following

LEMMA 4.8. *If G is a non-elementary strongly discrete group acting on $\mathbb{B}_\infty$, then for every $t < \delta_G$ there exists two distinct points $\xi_1, \xi_2 \in L(G)$ such that for all $r > 0$,*

$$\sum_{\gamma(0) \in \widehat{B}_e(\xi_i, r)} e^{-td(\gamma(0),0)} = +\infty \quad \textit{for} \quad i = 1, 2.$$

*We call such points ξ_i, t-*divergent points *of $L(G)$.*

PROOF. We assume that our group is non-elementary and so there exist at least two distinct hyperbolic elements. We prove the existence of one such point (as claimed in the Lemma) and the same argument will provide another point distinct from the first; as we have that these hyperbolic elements have distinct pairs of fixed points.

Suppose by way of contradiction, that there exist no t-divergent points in the limit set. Now pick a hyperbolic element g with axis l_g whose attracting and repelling endpoints are ξ_g^+ and ξ_g^- on $L(G)$. Let's look at ξ_g^- and refer to it as simply ξ.Then there exists an r_ξ such that the sum over the G-orbit of 0 within $A := \widehat{B}_e(\xi, r_\xi)$ is finite, i.e.

$$\sum_{\gamma(0) \in A} e^{-td(\gamma(0),0)} < +\infty \ .$$

Then we have that

$$\sum_{\gamma(0) \in \mathbb{B}_\infty \setminus A} e^{-td(\gamma(0),0)} = +\infty \ .$$

Now for an arbitrary $\varepsilon > 0$, there exists n large enough such that $g^n(\mathbb{B}_\infty \setminus A) \subseteq \widehat{B}_e(\xi_g^+, \varepsilon)$. It is enough to show that for such n, we have that

$$\sum_{g^n\gamma(0)\in g^n(\mathbb{B}_\infty\setminus A)} e^{-td(g^n\gamma(0),0)} = +\infty \ ,$$

since then we would have shown ξ_g^+ to be t-divergent and thus derived a contradiction. Notice that

$$\begin{aligned}
\sum_{g^n\gamma(0)\in g^n(\mathbb{B}_\infty\setminus A)} e^{-td(g^n\gamma(0),0)} &\geq \sum_{g^n\gamma(0)\in g^n(\mathbb{B}_\infty\setminus A)} e^{-td(0,g^n(0))} e^{-td(g^n(0),g^n\gamma(0))} \\
&\qquad \text{(by the triangle inequality)} \\
&= e^{-td(0,g^n(0))} \sum_{g^n\gamma(0)\in g^n(\mathbb{B}_\infty\setminus A)} e^{-td(g^n(0),g^n\gamma(0))} \\
&= e^{-td(0,g^n(0))} \sum_{\gamma(0)\in \mathbb{B}_\infty\setminus A} e^{-td(0,\gamma(0))} \\
&\quad \big[\text{since } g \text{ is an isometry and that} \\
&\qquad \gamma(0) \in \mathbb{B}_\infty \setminus A \iff g^n\gamma(0) \in g^n(\mathbb{B}_\infty \setminus A)\big] \\
&= +\infty \ .
\end{aligned}$$

We are done. □

Definition 4.9. Fix $\tau > 0$. For every integer $n \geq 0$ let $A_n(\tau) := \{z \in G(0) : \tau n \leq d(z,0) \leq \tau(n+1)\}$. The set $A_n(\tau)$ is called the *hyperbolic* (n, G)*-annulus* centered at 0 and of width τ.

We now prove the following lemma

Lemma 4.10. *Let G be a strongly discrete group. Fix $\tau > 0$ and $0 \leq s < t < \delta_G$. Let $\xi \in L(G)$ be a t-divergent point. Then for every $M > 0$ and for every $r > 0$ there exists*

$(n_j(\xi))_{j=1}^{\infty}$ *an increasing sequence of positive integers such that*

$$\sum_{g(0)\in\widehat{B}_e(\xi,r)\cap A_{n_j(\xi)}(\tau)} e^{-sd(g(0),0)} \geq M \quad \textit{for every} \quad j \geq 1.$$

PROOF. Suppose by way of contradiction that there exist $M, r > 0$ and an integer $q \geq 0$ such that

$$\sum_{z\in\widehat{B}_e(\xi,r)\cap A_n(\tau)} e^{-sd(z,0)} < M \quad \text{for every} \quad n \geq q+1.$$

Take $r_* \in (0, r]$ so small that $A_q(\tau) \cap \widehat{B}_e(\xi, r_*) = \emptyset$. Then $A_n(\tau) \cap \widehat{B}_e(\xi, r_*) = \emptyset$ for all $n = 0, 1, 2, \ldots, q$ and thus we get that

$$\sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-sd(z,0)} = 0 \leq M \quad \text{if} \quad n \leq q$$

and that

$$\sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-sd(z,0)} \leq \sum_{z\in\widehat{B}_e(\xi,r)\cap A_n(\tau)} e^{-sd(z,0)} \leq M \quad \text{if} \quad n \geq q+1.$$

Therefore we have that $\sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-sd(z,0)} \leq M$ for all $n \geq 0$. Hence

$$\begin{aligned}
\sum_{z\in G(0)\cap\widehat{B}_e(\xi,r_*)} e^{-td(z,0)} &= \sum_{n=0}^{\infty} \sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-(t-s)d(z,0)} e^{-sd(z,0)} \\
&\leq \sum_{n=0}^{\infty} \sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-(t-s)\tau n} e^{-sd(z,0)} \\
&= \sum_{n=0}^{\infty} e^{-(t-s)\tau n} \sum_{z\in\widehat{B}_e(\xi,r_*)\cap A_n(\tau)} e^{-sd(z,0)} \\
&\leq M \sum_{n=0}^{\infty} e^{-(t-s)\tau n} \\
&< +\infty.
\end{aligned}$$

The last inequality follows since s was chosen strictly smaller that t and so we have a geometric series that converges. But then this contradicts the hypothesis that ξ is a t-divergent point and finishes the proof. □

LEMMA 4.11. *For every $\sigma > 0$ there exists an $\alpha_\sigma \in (0,1)$ such that $\alpha_\sigma > \log 2$ and*

$$\Pi[B_h(z, \alpha_\sigma \sigma)] \subseteq B_e\Big(\Pi[z], \frac{1}{8}\mathrm{diam}_e \Pi[B_h(z,\sigma)]\Big),$$

for every z with $d(z,0) > \sigma$.

PROOF. One may prove this directly by elementary geometric arguments or use Corollary 3.35. □

For every $\sigma > 0$, let $r_\sigma > 0$ be chosen so small that

$$r_\sigma < \pi e^{-\alpha_\sigma \sigma}. \tag{20}$$

From this point on, for the remainder of the section, fix an arbitrary $\tau > 0$. The main ingredient, forming the inductive step in our proof of the Bishop-Jones Theorem, is the following

LEMMA 4.12. *Fix $0 < s < t < \delta_G$. Let $\xi_1, \xi_2 \in L(G)$ be two t-divergent points (see Lemma 4.8 for their existence). Then there exist $\sigma > 0$ and positive integers $l_1, l_2 \geq 1$ such that the following holds:*
For every $g \in G$ with $d(g(0),0) > \sigma$, there exists a set $\Gamma(g)$ contained in one of the sets

$$\Gamma_i := \{h \in G : h(0) \in \widehat{B}_e(\xi_i, r_\sigma) \cap A_{l_i(\xi_i)}(\tau)\} \quad \text{for } i = 1,2;$$

such that the following hold:-

a) *The family $\{\Pi[B(gh(0),\sigma)] : h \in \Gamma(g)\}$ consists of mutually disjoint balls.*

b) *For every $h \in \Gamma(g)$,*

$$\Pi[gh(0)] \in B_e\Big(\Pi[g(0)], \frac{1}{8}\mathrm{diam}_e \Pi[B(g(0),\sigma)]\Big).$$

c) *There exists a constant $\beta_\sigma \in (0,1)$ depending only on l_1, l_2 and σ (in particular independent of g) such that for every $h \in \Gamma(g)$,*

$$\frac{\beta_\sigma}{4}\mathrm{diam}\Big(\Pi[B(g(0),\sigma)]\Big) \leq \mathrm{diam}\Big(\Pi[B(gh(0),\sigma)]\Big) \leq \frac{1}{4}\mathrm{diam}\Big(\Pi[B(g(0),\sigma)]\Big).$$

d) *The following inequality holds*

$$\sum_{h\in\Gamma(g)} \operatorname{diam}^s\Big(\Pi[B(gh(0),\sigma)]\Big) \geq \operatorname{diam}^s\Big(\Pi[B(g(0),\sigma)]\Big).$$

PROOF. Take $\sigma > \log 2$ and as large as needed for Lemma 4.11 and so that

$$d_e(\xi_1,\xi_2) > 6\pi e^{-\alpha_\sigma \sigma}.$$

Now by the choice of r_σ, see (20), we have that

$$\inf\{d_e(x,y) : x \in \widehat{B}_e(\xi_1, r_\sigma), y \in \widehat{B}_e(\xi_2, r_\sigma)\} > 4\pi e^{-\alpha_\sigma\sigma}.$$

Since $\alpha_\sigma\sigma > \log 2$, it then follows from the Light Cone Lemma (Lemma 2.3 in [32]) that at least one of the balls $\widehat{B}_e(\xi_1, r_\sigma)$ or $\widehat{B}_e(\xi_2, r_\sigma)$ is contained in $g^{-1}\big(\Pi[B(g(0), \alpha_\sigma\sigma)]\big)$. Assume without loss of generality that $\widehat{B}_e(\xi_1, r_\sigma) \subseteq g^{-1}\big(\Pi[B(g(0), \alpha_\sigma\sigma)]\big)$. Consequently, if we fix two integers $l_1, l_2 \geq 1$ that we will specify later in the course of the proof, then

$$\Pi[h(0)] \in g^{-1}\big(\Pi[B(g(0), \alpha_\sigma\sigma)]\big)$$

for every $h \in \Gamma_1$. Therefore $g(\Pi[h(0)]) \in \Pi[B(g(0), \alpha_\sigma\sigma)]$ and g maps the geodesic from 0 to $\Pi[h(0)]$ with $h(0)$ on it to the geodesic from $g(0)$ to $g\Pi[h(0)]$ with $gh(0)$ on it. Therefore $gh(0)$ lies inside of the light cone generated by 0 and $B(g(0), \alpha_\sigma\sigma)$ and we have that

$$\Pi[gh(0)] \in \Pi[B(g(0), \alpha_\sigma\sigma)] \subseteq B_e\Big(\Pi[gh(0)], \frac{1}{8}\operatorname{diam}\Pi[B_h(g(0),\sigma)]\Big)$$

where the inclusion follows from Lemma 4.11. Note that condition b) of our Lemma has been established.

Assume now that $l_1 > \sigma/\tau$. Then $d(h(0), 0) > \sigma$ for all $h \in \Gamma_1$ and we can apply the Geometric Distortion Lemma (Lemma 2.2 in [32]) to get

$$d(g(0),0) + d(h(0),0) - 2\sigma \leq d(gh(0),0) \leq d(g(0),0) + d(h(0),0) \tag{21}$$

for every $h \in \Gamma_1$. Note that we also needed $g(\Pi[h(0)]) \in \Pi[B(g(0),\sigma)]$ to apply the Geometric Distortion Lemma, but we already have that $g(\Pi[h(0)]) \in \Pi[B(g(0), \alpha_\sigma\sigma)] \subseteq \Pi[B(g(0),\sigma)]$.

Now since $h(0) \in A_{l_1}(\tau)$, we have that $l_1\tau \leq d(h(0),0) < l_1\tau + \tau$ for every $h \in \Gamma_1$, we then get

$$d(g(0),0) + l_1\tau - 2\sigma \leq d(gh(0),0) \leq d(g(0),0) + l_1\tau + \tau \tag{22}$$

for every $h \in \Gamma_1$. Now define $\Gamma(g)$ to be a maximal (in the sense of inclusion) subset of Γ_1 such that

$$d(h_1(0), h_2(0)) > \tau + 6\sigma \tag{23}$$

for all $h_1, h_2 \in \Gamma_1$ with $h_1 \neq h_2$. Let us now prove item a). Suppose by way of contradiction that, $\Pi[B(gh_1(0),\sigma)] \cap \Pi[B(gh_2(0),\sigma)] \neq \emptyset$ for some $h_1, h_2 \in \Gamma_1$ with $h_1 \neq h_2$. This means that $B(gh_1(0),\sigma)$ intersects the light-cone viewed from 0 generated by $B(gh_2(0),\sigma)$. Let z_1 belong to this intersection and let $z_2 \in B(gh_2(0),\sigma)$ be chosen on the ray from 0 to z_1. Then the following estimates are true using (22):

$$d(g(0),0) + l_1\tau - 3\sigma < d(z_1,0) < d(g(0),0) + l_1\tau + \tau + \sigma$$

and

$$d(g(0),0) + l_1\tau - 3\sigma < d(z_2,0) < d(g(0),0) + l_1\tau + \tau + \sigma.$$

Therefore we have

$$\begin{aligned} z_1 &\in B(gh_1(0),\sigma) \\ &\subseteq \{w \in \mathbb{B}_\infty : d(g(0),0) + l_1\tau - 3\sigma < d(w,0) < d(g(0),0) + l_1\tau + \tau + \sigma\} \end{aligned}$$

and similarly that

$$\begin{aligned} z_2 &\in B(gh_2(0),\sigma) \\ &\subseteq \{w \in \mathbb{B}_\infty : d(g(0),0) + l_1\tau - 3\sigma < d(w,0) < d(g(0),0) + l_1\tau + \tau + \sigma\}. \end{aligned}$$

It then follows that

$$\begin{aligned} d(z_1,z_2) &= |d(0,z_1) - d(0,z_2)| \\ &< [d(g(0),0) + l_1\tau + \tau + \sigma] - [d(g(0),0) + l_1\tau - 3\sigma] \\ &= \tau + 4\sigma. \end{aligned}$$

Therefore we have that

$$\begin{aligned} d(h_1(0), h_2(0)) &= d(gh_1(0), gh_2(0)) \\ &\leq d(gh_1(0), z_1) + d(z_1, z_2) + d(z_2, gh_2(0)) \\ &< \sigma + [\tau + 4\sigma] + \sigma \\ &= \tau + 6\sigma \end{aligned}$$

which contradicts (23) and finishes the proof of item a). It follows directly from the left-hand side of (22) and Lemma 2.1 in [32] that with $l_1 \geq 1$ chosen large enough the right-hand side of c) holds. Similarly the left-hand side of c) follows form the right-hand side of (22) and Lemma 2.1 in [32].

We have only left to show item d) and this where the final specification of l_1 will be made. First for every $h \in \Gamma(g)$, define

$$Q_g(h) := \{f \in \Gamma(g) : d(f(0), h(0)) \leq \tau + 6\sigma\}.$$

By maximality of $\Gamma(g)$, see (23), we have that

$$\bigcup_{h \in \Gamma(g)} Q_g(h) = \Gamma_1.$$

Now since our group is strongly discrete, the number of elements in $Q_g(h)$ is bounded above by some constant, $C_1 = C_1(\tau, \sigma)$, depending only $(\tau + 6\sigma)$ and of course on G. Also because of (22) and Lemma 2.1 in [32], there exists another constant, $C_2 = C_2(\tau, \sigma)$, such that for every $g \in G$ with $d(g(0), 0) > \sigma$ and all $h_1, h_2 \in \Gamma_1$

$$C_2^{-1} \leq \frac{\operatorname{diam}\big(\Pi[B(gh_2(0), \sigma)]\big)}{\operatorname{diam}\big(\Pi[B(gh_1(0), \sigma)]\big)} \leq C_2.$$

Hence,

$$\begin{aligned}\sum_{h\in\Gamma_1}\mathrm{diam}^s(\Pi[B(gh(0),\sigma)]) &\leq \sum_{h\in\Gamma(g)}\sum_{f\in Q_g(h)}\mathrm{diam}^s(\Pi[B(gf(0),\sigma)])\\ &\leq \sum_{h\in\Gamma(g)} C_2(\tau,\sigma)^s \sum_{f\in Q_g(h)}\mathrm{diam}^s(\Pi[B(gh(0),\sigma)])\\ &= C_2(\tau,\sigma)^s \sum_{h\in\Gamma(g)}[\#Q_g(h)]\mathrm{diam}^s(\Pi[B(gh(0),\sigma)])\\ &\leq C_2(\tau,\sigma)^s C_1(\tau,\sigma)\sum_{h\in\Gamma(g)}\mathrm{diam}^s(\Pi[B(gh(0),\sigma)]).\end{aligned} \tag{24}$$

Now take $M = C_\sigma^s C_2(\tau,\sigma)^s C_1(\tau,\sigma)$ and choose $l_1 \geq 1$ to be one of the numbers $(n_j(\xi_1))_{j=1}^\infty$ appearing in Lemma 4.10 that is as large as required above, viz. that $l_1 > \sigma/\tau$. Now by Lemma 4.10, Lemma 2.1 in [32] and the right-hand-side of (21), we get that

$$\begin{aligned}\sum_{h\in\Gamma_1}\mathrm{diam}^s(\Pi[B(gh(0),\sigma)]) &\geq C_\sigma^{-s}\sum_{h\in\Gamma_1} e^{-sd(gh(0),0)}\\ &\geq C_\sigma^{-s}\sum_{h\in\Gamma_1} e^{-sd(g(0),0)}e^{-sd(h(0),0)}\\ &\geq C_\sigma^{-s}e^{-sd(g(0),0)}\sum_{h\in\Gamma_1} e^{-sd(h(0),0)}\\ &\geq MC_\sigma^{-s}e^{-sd(g(0),0)}\\ &\geq MC_\sigma^{-2s}\mathrm{diam}^s(\Pi[B(g(0),\sigma)])\\ &\geq C_2(\tau,\sigma)^s C_1(\tau,\sigma)\mathrm{diam}^s(\Pi[B(g(0),\sigma)]).\end{aligned}$$

Now inserting this into (24), we finally get

$$\sum_{h\in\Gamma(g)}\mathrm{diam}^s(\Pi[B(gh(0),\sigma)]) \geq \mathrm{diam}^s(\Pi[B(g(0),\sigma)])$$

which establishes d) and finished the proof of our Lemma. □

We are now in a position to prove the main result of this section, which is the extension of the Bishop-Jones result to the infinite-dimensional case.

THEOREM 4.13. *If G is a strongly discrete group acting on $\mathbb{B}_\infty$, then*

$$HD(L_r(G)) = HD(L_{ur}(G)) = \delta_G.$$

PROOF. As $L_{ur}(G) \subseteq L_r(G)$, we have that

$$HD(L_{ur}(G)) \leq HD(L_r(G)). \tag{25}$$

We shall first show that

$$HD(L_r(G)) \leq \delta_G.$$

If $\delta_G = +\infty$, then we are done and so let's assume that $\delta_G < +\infty$. Fix an arbitrary $s > \delta_G$. Write G as $(g_n)_{n=1}^{\infty}$. Fix $\sigma > \log 2$ and let

$$L_{r,\sigma}(G) := \bigcap_{n\geq 1} \bigcup_{k\geq n} \Pi\big[B(g_k(0), \sigma)\big].$$

Since $\sum_{n\geq 1} e^{-sd(g_n(0),0)} < +\infty$, we get that

$$\lim_{n\to\infty} \sum_{k\geq n} \operatorname{diam}^s(\Pi\big[B(g_k(0), \sigma)\big]) \asymp_\sigma \lim_{n\to\infty} \sum_{k\geq n} e^{-sd(g_k(0),0)} = 0.$$

Thus $HD(L_{r,\sigma}(G)) \leq s$ and consequently that $HD(L_{r,\sigma}(G)) \leq \delta_G$. Now $L_r(G) = \bigcup_{n\geq 3} L_{r,n}(G)$ and therefore by the σ-stability of Hausdorff dimension, we get

$$HD(L_r(G)) = \sup_{n\geq 3}\{HD(L_{r,n}(G))\} \leq \delta_G.$$

Along with (25) this gives $HD(L_{ur}(G)) \leq HD(L_r(G)) \leq \delta_G$, and we are left to show that

$$HD(L_{ur}(G)) \geq \delta_G. \tag{26}$$

By means of Lemma 4.12 we will perform a construction to which Proposition 4.2 will apply. In the setting of Lemma 4.12, for every $n \geq 1$ let

$$E_n := E := \Gamma_1 \cup \Gamma_2.$$

We define the set $\Sigma \subseteq E^*$ and the sets $A(\omega)$ for $\omega \in \Sigma$ by induction with respect to word length in Σ. For the base of your recursion, we take $E \cap \Sigma := E := \Gamma_1 \cup \Gamma_2$ and $A(h) := \Pi[B(h(0), \sigma)]$ for all $h \in E$. For the inductive step, suppose that the set $E^n \cap \Sigma$ has been defined and that all the sets $A(f)$, $f \in E^n \cap \Sigma$ have been defined as well. To define $E^{n+1} \cap \Sigma$ consider all the elements $g = f_1, \ldots, f_n \in E^n \cap \Sigma$ and declare that $f = f_1, \ldots, f_{n+1} \in E^{n+1} \cap \Sigma$ if $f_{n+1} \in E$ and $f_{n+1} \in \Gamma(f_1 \circ \ldots \circ f_n) = \Gamma(g)$. Then put $A(f) = \Pi[B(f_1 \circ \ldots \circ f_{n+1}(0), \sigma)]$.

Verifying now the hypotheses of Proposition 4.2, we see that $\widehat{\Sigma} \subseteq \Sigma$ directly by construction. Properties c), d), b) and a) of Proposition 4.2 follow respectively from property a) of Lemma 4.12; property d) of Lemma 4.12; the left-hand side of property c) of Lemma 4.12 and finally from both property b) and the right-hand-side of Property c) of Lemma 4.12 with $\kappa = 1/4$. Therefore all the properties of Proposition 4.2 have been verified and as a result of applying it we get that

$$HD\Big(\bigcap_{n\geq 1}\bigcup_{\omega\in E^n\cap\Sigma} A(\omega)\Big) \geq s.$$

It now follows from c) of Lemma 4.12, or more directly from (22), that $\bigcap_{n\geq 1}\bigcup_{\omega\in E^n\cap\Sigma} A(\omega) \subseteq L_{ur}(G)$ and so we have that $HD(L_{ur}(G)) \geq s$. Since was arbitrarily smaller than δ_G, we therefore get that $HD(L_{ur}(G)) \geq \delta_G$. This means that (26) has been established and we are done. $\square$

CHAPTER 5

CONFORMAL MEASURES

Conformal measures were introduced for Fuchsian groups by S. Patterson [25] and his construction was extended to Kleinian groups in any finite dimension by D. Sullivan [37]. These measures turn out to be indispensable for a finer analysis of the fractal structure of the limit set. Their construction which we will outline below is rather ingenious - they are formed as weak limits of purely atomic probability measures supported on an orbits of the group G in such a way that the mass is pushed out to infinity. In fact when they do exist, they are supported on the limit set $L(G)$. In the finite-dimensional case these measures always exist; however, in the Hilbertian case they cannot be defined if the Poincaré exponent is equal to positive infinity.

We assume that G is a strongly discrete subgroup of $\mathcal{M}_{\mathbb{B}}$. Note that by Theorem 3.10 it suffices to assume that the Poincaré exponent δ_G is finite.

DEFINITION 5.1. A finite Borel measure μ on $\overline{\mathbb{B}}_\infty$ is called a *conformal measure of dimension* s (with respect to G) if

$$\mu(gA) = \int_A |g'|^s d\mu$$

for every $g \in G$ and measurable $A \subseteq \overline{\mathbb{B}}_\infty$. Note that $|g'|$ refers to the operator norm, equivalently the scaling constant, of the derivative transformation. We will also call these *s-conformal* measures.

See equations (4) and (5) to recall the definition of the Poincaré series and the Poincaré exponent of the group. We simply write δ for δ_G. Then $\Sigma_{x,y}(s) = \sum_{g\in G} e^{-sd(x,g(y))}$ converges for $s > \delta$ and diverges for $s < \delta$. However at $s = \delta$ we may have have either case, leading to

DEFINITION 5.2. A group G is of *convergence type* if $\Sigma_{x,y}(\delta)$ converges and is of *divergence type* otherwise for some (equivalently, any) choice of x, y in $\mathbb{B}_\infty$.

We sketch the construction for groups of divergence-type first, where the main idea

is transparent. For each $s > \delta$ let us define

$$\mu_{x,s} := \frac{\sum_{g\in G} e^{-sd(x,g(0))}\delta_{g(0)}}{\Sigma_{x,0}(s)}$$

where δ_x is the Dirac point mass supported on x. Notice that the measures $\mu_{x,s}$ are probability measures. It is clear that for groups of compact-type, i.e. whose limits sets are compact, these probabilities are defined on the compact set $G(0) \cup L(G)$ and therefore by the Banach-Alaoglu theorem one has weak limits of such measures along certain subsequences $s_i \to \delta$. However, these limit measures depend upon the sequences that are chosen. We denote such a limiting measure by μ_x. For groups of convergence-type Patterson modified the Poincaré series via a slowly varying function ϕ such that the new series still converges for $s > \delta_G$ but now diverges for $s \leq \delta_G$.

Note that the divergence at the critical exponent is what pushes the mass out to $\partial\mathbb{B}_\infty$ in both cases. It turns out that for the divergence case, the limiting Patterson-Sullivan measures are δ-conformal. The art of Patterson's slowly varying function in the convergence case is that the modified limit measure retains its key geometric property of being δ-conformal. It turns out for non-elementary groups (see Proposition 5.10) that δ-conformality of a measure implies that its support is the whole limit set. One further obtains via Lemma 5.11 exactly how the measure scales shadows of balls in the interior seen from a fixed observation point, viz. they satisfy a power law with exponent δ on these shadows.

One constructs a continuous increasing function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ from the nonnegative reals to themselves such that:

$$\text{for every } \varepsilon > 0, \text{ there exists } r_\varepsilon \text{ such that } \phi(t+r) \leq e^{\varepsilon t}\phi(r) \text{ for all } r > r_\varepsilon, t > 0.$$

Then one defines just as in the divergence case

$$\Sigma^\phi_{x,0}(s) := \sum_{g\in G} e^{-sd(x,g(0))}\phi[d(x,g(0))]$$

and

$$\mu^\phi_{x,s} := \frac{\sum_{g\in G} e^{-sd(x,g(0))}\phi[d(x,g(0))]\delta_{g(0)}}{\Sigma^\phi_{x,0}(s)}.$$

Again as in the case of non-elementary strongly discrete groups of compact type one gets a weak limit μ_x^ϕ via Banach-Alaoglu's theorem.

The uniqueness of such a measure class is to be further investigated. Note that in finite dimensions, Ahlfors and Tukia have proved that this is the case for groups of divergence type using ideas of Thurston; for more on how this is generalized see [8].

We now describe the construction of Patterson's slowly varying function. Its usefulness will only become apparent when we check that the limit measure is δ-conformal.

PROPOSITION 5.3 (Slowly varying construction). *There exists a continuous increasing function $\phi : \mathbb{R}_{\geq)} \to \mathbb{R}_{>0}$ such that:*

(i) $\Sigma_{x,0}^\phi(\delta)$ diverges.

(ii) For every $\varepsilon > 0$, there exists t_ε large enough such that for all $s \geq 0$,

$$\phi(t+s) \leq e^{\varepsilon s}\phi(t).$$

PROOF. One starts by choosing a decreasing sequence of positive slopes $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$. We will use these to build a piece-wise defined function $t \mapsto \log \phi(t)$ from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$, where each piece of its domain $[t_n, t_{n+1})$ will form a sequence of successively longer intervals on which $\log \phi$ will be affine with slope ε_n.

For simplicity one can imagine the elements of G so enumerated so that $d(x, g_n(0))$ is an increasing sequence. Let $t_0 = 0$ and choose t_1 such that

$$\sum_{g \in G : d(x,g(0)) \leq t_1} e^{-\delta d(x,g(0))}\phi[d(x,g(0))] \geq 1.$$

We can find enough terms since by construction $\log \phi$ has slope ε_1, i.e. $\phi(t) = e^{\varepsilon_1(t-t_0)}\phi(t_0) = C_{t_0}e^{\varepsilon_0 t}$ for $t > t_0 = 0$. This contribution when added to the sum

$$\sum_{g \in G : d(x,g(0)) \geq t_0} e^{-\delta d(x,g(0))}\phi[d(x,g(0))] = \sum_{g \in G : d(x,g(0)) \geq t_0} e^{-\delta d(x,g(0))} C_{t_0} e^{\varepsilon_0 d(x,g(0))}$$

forces it to diverge, since we're summing below the critical exponent, i.e. at $\delta - \varepsilon_0$. At the second stage, we play the same game: affinely extend the graph of $\log \phi$ beyond t_1 to have

slope $\varepsilon_1 < e_0$ and find t_2 such that

$$\sum_{g\in G: t_1 < d(x,g(0)) \leq t_2} e^{-\delta d(x,g(0))}\phi[d(x,g(0))] \geq 1.$$

Again you will find enough terms since you have dipped below the critical exponent to force divergence of the full series, this time to $\delta - \varepsilon_1$. One carries on in such a manner and thereby forces divergence of $\Sigma^{\phi}_{x,0}(s)$. To check that (ii) is satisfied, notice that it says that for any ε there is some point after which the slope of $\log \phi$ will be less than ε, which is certainly true given that we chose our ε_n sequence to be decreasing to begin with. We are done. □

Let us continue with the geometric properties of the limiting Patterson-Sullivan measures. The following proposition includes the definition of Busemann functions, that were employed as a signed distance between horocycles.

Here is an alternate proof of the fact mentioned previously, namely that the Busemann can be continuously extended to the boundary, worked out in the setting of $\mathbb{B}_\infty$.

PROPOSITION 5.4. *The Busemann function* $\mathcal{B} : \mathbb{B}_\infty \times \mathbb{B}_\infty \times \mathbb{B}_\infty \to \mathbb{R}$ *defined by*

$$(w, z, z') \mapsto d(z, w) - d(z', w) =: \mathcal{B}_w(z, z')$$

has a continuous extension $\mathcal{B} : \overline{\mathbb{B}}_\infty \times \mathbb{B}_\infty \times \mathbb{B}_\infty \to \mathbb{R}$ *with*

$$(\xi, z, z') \mapsto \log \frac{P(z', \xi)}{P(z, \xi)} =: \mathcal{B}_\xi(z, z')$$

when $\xi \in \partial\mathbb{B}_\infty$.

Here $P : \mathbb{B}_\infty \times \partial\mathbb{B}_\infty \to \mathbb{R}$ defined as

$$P(x, \xi) := \frac{1 - |x|^2}{|x - \xi|^2}$$

denotes the Poisson kernel.

PROOF. We have the following formula from two dimensional hyperbolic geometry; see any classical text, e.g. those by Fenchel or Beardon. For $z, w \in \mathbb{B}_\infty$

$$4\sinh^2(d(z, w)/2) = \frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

This formula explains why we may continuously extend to the boundary and the fact that $\sinh(x/2) \asymp e^x$ for large x proves that

$$\lim_{w\to\xi} e^{\mathcal{B}_w(z,z')} = \lim_{w\to\xi} \frac{e^{-d(z',w)}}{e^{-d(z,w)}} = \frac{P(z',\xi)}{P(z,\xi)}.$$

This in turn may be written as

$$e^{-\mathcal{B}_\xi(z',z)} = \frac{P(z',\xi)}{P(z,\xi)}. \tag{27}$$

□

Now the geometric mean value theorem from Proposition 2.19 implies that:

$$|g'(g^{-1}(0))|.|g'(\xi)| = \frac{|0-g(\xi)|^2}{|g^{-1}(0)-\xi|^2} = \frac{1}{|g^{-1}(0)-\xi|^2}$$

and then applying the expression $|g'(z)| = \frac{1-|g(z)|^2}{1-|z|^2}$ allows us to relate the distortion and the Poisson kernel:

$$|g'(\xi)| = P(g^{-1}(0),\xi). \tag{28}$$

Observation 5.5. Let G be a strongly discrete group. Then for every $g \in G$ and for every measurable $A \subseteq \overline{\mathbb{B}}_\infty$ we have that

$$\mu_0(gA) = \mu_{g^{-1}(0)}(A).$$

Proof. Change of variables. □

Proposition 5.6 (Transformation Rule). *Given $z, z' \in \mathbb{B}_\infty$, we have that*

$$\frac{d\mu_z}{d\mu_{z'}}(\xi) = \left(\frac{P(z,\xi)}{P(z',\xi)}\right)^\delta = e^{-\delta\mathcal{B}_\xi(z,z')}\ .$$

Proof. This is the proof for the divergence case. Fix $s > \delta$. Then

$$\begin{aligned}
\mu_{z,s} &= \big(\Sigma_{z,0}(s)\big)^{-1} \sum_{g\in G} e^{-sd(z,g(0))}\delta_{g(0)} \\
&= \big(\Sigma_{z,0}(s)\big)^{-1} \sum_{g\in G} e^{-s[d(z,g(0))-d(z',g(0))]}e^{-sd(z',g(0))}\delta_{g(0)} \\
&= \big(\Sigma_{z,0}(s)\big)^{-1} \sum_{g\in G} e^{-s[\mathcal{B}_{g(0)}(z,z')]}e^{-sd(z',g(0))}\delta_{g(0)}
\end{aligned}$$

Now by Theorem 5.4 we have that $\mathcal{B}_{g(0)}(z,z') \to \mathcal{B}_\xi(z,z')$ when $g(0) \to \xi \in \partial\mathbb{B}_\infty$. Thus taking a limit $s_i \to \delta$ gives us

$$\frac{d\mu_z}{d\mu_{z'}}(\xi) = e^{-\delta\mathcal{B}_\xi(z,z')} \ .$$

The other form is via (27).

Now for the convergence case: fix $s > \delta$. Then

$$\begin{aligned}
\mu^\phi_{z,s} &= \left(\Sigma^\phi_{z,0}(s)\right)^{-1} \sum_{g\in G} e^{-sd(z,g(0))}\phi[d(x,g(0))]\delta_{g(0)} \\
&= \frac{1}{\Sigma^\phi_{z,0}(s)} \sum_{g\in G}\left[\frac{\phi[d(z,g(0))]}{\phi[d(z',g(0))]} e^{-s[d(z,g(0))-d(z',g(0))]}\right]\phi[d(z',g(0))]e^{-sd(z',g(0))}\delta_{g(0)} \\
&= \frac{1}{\Sigma^\phi_{z,0}(s)} \sum_{g\in G}\left[\frac{\phi[d(z,g(0))]}{\phi[d(z',g(0))]} e^{-s[\mathcal{B}_{g(0)}(z,z')]}\right]\phi[d(z',g(0))]e^{-sd(z',g(0))}\delta_{g(0)} \ .
\end{aligned}$$

As before, by Proposition 5.4 we have that $\mathcal{B}_{g(0)}(z,z') \to \mathcal{B}_\xi(z,z')$ when $g(0) \to \xi \in \partial\mathbb{B}_\infty$. So all we have to check is that $\frac{\phi[d(z,g(0))]}{\phi[d(z',g(0))]} \to 1$ as $g(0) \to \xi \in \partial\mathbb{B}_\infty$. Recall what Proposition 5.3 gives us, viz. for every $\varepsilon > 0$, there exists t_ε large enough such that for all $s \geq 0$,

$$\phi(t+s) \leq e^{\varepsilon s}\phi(t).$$

Now notice that in the triangle formed by $\Delta(z,z',g(0))$, both $d(z,g(0))$ and $d(z',g(0))$ can be taken as large as we like since $g(0)$ is marching off to the boundary as we run through the orbit and also notice that $d(z,z')$ is independent of g. Fix some ε; using (2), we have that

$$\frac{\phi[d(z,g(0))]}{\phi[d(z',g(0))]} \leq e^{\varepsilon d(z,z')}$$

and symmetrically that

$$e^{-\varepsilon d(z,z')} \leq \frac{\phi[d(z,g(0))]}{\phi[d(z',g(0))]}$$

for orbits outside of a sufficiently large ball around the origin. Now we take $\varepsilon \to 0$ and find the quotient squeezed to 1 as orbit points $g(0)$ tend to the boundary. Thus taking a limit $s_i \to \delta$ gives us

$$\frac{d\mu^\phi_z}{d\mu^\phi_{z'}}(\xi) = e^{-\delta\mathcal{B}_\xi(z,z')} \ .$$

Again the other form is via (27). We are done. □

We now glean the geometric relevance of this Proposition, viz.

PROPOSITION 5.7 (δ-harmonicity). *Every limit measure μ_0 satisfies*

$$\mu_0(gA) = \int_A P\big(g^{-1}(0), \xi\big)^{\delta} d\mu_0(\xi)$$

for every $g \in G$ and measurable $A \subseteq \overline{\mathbb{B}}_\infty$.

PROOF. Letting $z = g^{-1}(0)$ and $z' = 0$, the result follows from noticing $P(0, \cdot) = 1$ and the previous observation, viz. $\mu_{g^{-1}(0)}(A) = \mu_0(gA)$. □

It now follows immediately from equation (28) that

PROPOSITION 5.8 (δ-conformality). *Every limit measure μ_0 satisfies*

$$\mu_0(gA) = \int_A |g'(\xi)|^{\delta} d\mu_0(\xi)$$

for every $g \in G$ and measurable $A \subseteq \overline{\mathbb{B}}_\infty$.

REMARK 5.9. The terminology *δ-harmonic* is due to Stratmann. The usage of the word "harmonic" is inspired from the finite dimensional case, where it makes sense to do potential theory and study the connections with spectral properties of the hyperbolic Laplacian - there is an extensive development that we shall not touch upon, see [40], [37] and [38].

Let us notice that δ-conformality gives us the following

PROPOSITION 5.10. *The limit measure μ_0 is supported on the limit set, in fact* $\mathrm{supp}(\mu_0) = L(G)$. *More generally, given a non-elementary strongly discrete group G and any s-conformal measure ν with $s > 0$, we have that* $\mathrm{supp}(\nu) = L(G)$. *In particular, ν cannot be completely supported at a single point.*

PROOF. Notice that δ-conformality implies that $g_*\mu_0 := \mu_0 \circ g^{-1}$ is absolutely continuous to μ_0 and therefore we have that $\mathrm{supp}(\mu_0)$ is G-invariant.

Next note that $\mathrm{supp}(\mu_0)$ must contain at least 2 points since a conformal measure cannot charge all its mass on a single point. Suppose that it did, i.e. μ_0 is supported entirely on some $\{\xi\}$. Now since we have that any non-elementary group has no global fixed points

(by Theorem 3.20) we may choose $g \in G$ with $g(\xi) \neq \xi$. But this leads to a contradiction since

$$0 = \mu_0(\{g(\xi)\}) = \int_{\{\xi\}} |g'|^\delta d\mu_0 = |g'(\xi)|^\delta \cdot \mu_0(\{\xi\}) = |g'(\xi)|^\delta.$$

Since we have shown that $\mathrm{supp}(\mu_0)$ is G-invariant, has at least 2 elements and is by definition closed, therefore by the minimality of $L(G)$ from Theorem 3.25 we have that $\mathrm{supp}(\mu_0) = L(G)$. The same argument works in general for a s-conformal measure. □

We would now like to prove the following assertion:

THEOREM 5.11 (Sullivan's Shadow Lemma). *Given a strongly discrete group $G \subseteq \mathcal{M}_{\mathbb{B}}$, for sufficiently large radii σ and all $g \in G$ with $d(0, g(0)) > 2\sigma$, we have that*

$$\begin{aligned}\mu_0\big(\Pi[B(g(0),\sigma)]\big) &\asymp_\sigma \big(\mathrm{diam}_e \Pi[B(g(0),\sigma)]\big)^\delta \\ &\asymp_\sigma e^{-\delta d(0,g(0))} \\ &\asymp (1 - |g(0)|)^\delta.\end{aligned}$$

PROOF. First, some elementary geometric estimates given $B(z,r)$ and $\xi \in \Pi[B(z,r)]$, viz.

$$\mathrm{diam}_e\big(\Pi[B(z,r)]\big) \asymp_r 1 - |z| \asymp e^{-d(0,z)}$$

and

$$P(z,\xi) \asymp_r \frac{1}{1-|z|}.$$

Next we make the following claim:

$$\text{For } \sigma \text{ sufficiently large, } \mu_0\big(g^{-1}\Pi[B(g(0),\sigma)]\big) \asymp_\sigma 1.$$

PROOF OF CLAIM. It is enough to show that $\mu_0\big(g^{-1}\Pi[B(g(0),\sigma)]\big) \geq \varepsilon$ for some $\varepsilon > 0$ that is independent of g.

So far we have talked about shadows cast by a light source at the origin. For light sources z not at the origin, we denote their shadows by Π_z. Imagine the pullback of $\Pi[B(g(0),\sigma)]$ by g^{-1}, which is $\Pi_{g^{-1}(0)}[B(0,\sigma)]$. It follows from elementary plane geometry that the diameter of its complement $\mathrm{diam}_e\big(\partial\mathbb{B} - g^{-1}\Pi[B(g(0),\sigma)]\big) = \mathrm{diam}_e\big(\partial\mathbb{B} - \Pi_{g^{-1}(0)}[B(0,\sigma)]\big)$ goes to zero as we increase the radius σ. In fact it is bounded above

by the angle subtended from the origin to its extremities which one can calculate to be $4\csc^{-1}\cosh(\sigma)$: this clearly goes to zero as $\sigma \to \infty$.

Recall from Proposition 5.10 that a conformal measure cannot charge all its mass on a single point and so pick z_1, z_2 in its support. Let $\delta := \frac{1}{3}d(z_1, z_2)$ and $\varepsilon := \min_{i=1,2} \mu_0(\partial\mathbb{B}_\infty \setminus S)$. Now choose the radius large enough, say σ, so that the diameter of its complement $\operatorname{diam}_e\big(\partial\mathbb{B} - g^{-1}\Pi[B(g(0),\sigma)]\big) \leq \delta$. Then for any S with $\operatorname{diam}_e(S) \leq \delta$ one has that $\mu_0\big(\partial\mathbb{B}_\infty - S\big) \geq \varepsilon$. □

Returning to the proof of the theorem, notice that the result follows from applying the observation to

$$\begin{aligned} 1 \asymp_\sigma \mu_0\big(g^{-1}\Pi[B(g(0),\sigma)]\big) &= \int_{\Pi[B(g(0),\sigma)]} P(g(0),\xi)^\delta d\mu_0(\xi) \\ &\asymp_\sigma \int_{\Pi[B(g(0),\sigma)]} \operatorname{diam}_e\big(\Pi[B(g(0),\sigma)]\big)^{-\delta} d\mu_0(\xi) \\ &= \operatorname{diam}_e\big(\Pi[B(g(0),\sigma)]\big)^{-\delta} \mu_0\big(\Pi[B(g(0),\sigma)]\big). \end{aligned}$$

We are done with proving the theorem. □

Before we move to local geometric properties of conformal measures we introduce some

NOTATION 5.12. Recall from Defintion 3.30 that

$$L_{ur}(G) = \cup_{\sigma>0} L_{ur,\sigma}(G)$$

and

$$L_r(G) = \cup_{\sigma>0} L_{r,\sigma}(G)$$

where $\xi \in L_{ur,\sigma}(G)$ when the geodesic from 0 to ξ, S_ξ, is covered by hyperbolic balls $B(g(0),\sigma)$ over the G-orbit of the origin and similarly $\xi \in L_{r,\sigma}(G)$ when S_ξ intersects infinitely many hyperbolic balls $B(g_n(0),\sigma)$ for some subsequence of the G-orbit of the origin. Let $\mu := \mu_0$ a Patterson-Sullivan measure and let H^δ and P^δ denote the δ-dimensional Hausdorff and δ-dimensional packing measures respectively. Then the restrictions of the δ-dimensional Hausdorff measure to $L_{ur,\sigma}(G)$, $L_{ur}(G)$, $L_{r,\sigma}(G)$ and $L_r(G)$ will be denoted

as $H^{\delta}_{ur,\sigma}$, H^{δ}_{ur}, $H^{\delta}_{r,\sigma}$ and H^{δ}_{r} respectively. There will be similar subscript notation for the restrictions of μ and P^{δ} to these subsets of $L(G)$.

We write $\mu \lesssim_r \nu$ to mean $\mu \leq K\nu$ for some constant $K = K(r)$ that depends on r. Let $\mu \asymp_r \nu$ mean $\nu \lesssim_r \mu \lesssim_r \nu$, i.e. $\frac{1}{K}\nu \leq \mu \leq K\nu$ for some constant $K = K(r)$ that depends on r.

We write $\mu \ll \nu$ to mean μ is *absolutely continuous* with respect to ν, viz. that $[\nu(A) = 0] \implies [\mu(A) = 0]$ for all measurable sets A, and write $\mu \sim \nu$ to mean that they are *equivalent*, i.e. $\nu \ll \mu \ll \nu$.

The proofs of the first four items in Proposition 5.14 below follow from covering and differentiation theorems for Radon measures in arbitrary metric spaces. Most references including Federer, Evans-Gariepy or Heinonen use analogs of Besicovitch covering theorems in their proofs, thereby restricting their applicability to finite dimensions. However in [18], there are proofs in the setting of arbitrary metric spaces and we remind the reader of the following results which are taken from Section 8 of [18].

THEOREM 5.13 (Differentiation of measures). *Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous evenly varying function. Suppose (X, d) is a metric space and μ a Borel probability measure on X. Fix $A \subseteq X$ and $c \in (0, +\infty]$. Let us make the convention that $1/\infty = 0$. Then*

$$\frac{1}{c} \leq^{\circledast} \limsup_{r \to 0} \frac{\mu(B(x,r))}{\phi(r)} \leq c$$

$$\Downarrow$$

$$\frac{1}{c}\mu(E) \leq H_{\phi}(E) \leq^{\circledast} c\mu(E) \ .$$

If the lim sup *statement holds for all $x \in A$, then the corresponding statement about measures is true for all Borel $E \subseteq A$. Dually, we have*

$$\frac{1}{c} \leq \liminf_{r \to 0} \frac{\mu(B(x,r))}{\phi(r)} \leq^{\circledast} c$$

$$\Downarrow$$

$$\frac{1}{c}\mu(E) \leq^{\circledast} P_{\phi}(E) \leq c\mu(E) \ .$$

Again if the $\liminf$ *statement holds for all* $x \in A$*, then the corresponding statement about measures is true for all Borel* $E \subseteq A$*. The* $\circledast$ *decorations on the* $\limsup$ *and* $\liminf$ *inequalities indicate which of the corresponding inequalities about measures are implied.*

PROPOSITION 5.14. *Let* G *be non-elementary with finite exponent of convergence* $\delta := \delta_G$ *(and therefore strongly discrete). Assume that we have a* δ*-conformal measure* μ*. Then the following statements are true:*

(i)

$$H^{\delta}_{ur,\sigma} \asymp_{\sigma} \mu_{ur,\sigma} \asymp_{\sigma} P^{\delta}_{ur,\sigma}$$

(ii)

$$H^{\delta}_{r,\sigma} \lesssim_{\sigma} \mu_{r,\sigma} \lesssim_{\sigma} P^{\delta}_{r,\sigma}$$

(iii)

$$H^{\delta}_{ur} \sim \mu_{ur} \sim P^{\delta}_{ur}$$

(iv)

$$H^{\delta}_{r} \ll \mu_{r} \ll P^{\delta}_{r}$$

(v) μ_r *has no atoms.*

(vi) H^{δ}_{r} *is* σ*-finite.*

(vii) If G *is convex co-bounded then* $0 < H^{\delta} \restriction_{L(G)} < \infty$*.*

PROOF. Let us sketch the basic outline that gives the result.

(i) Fix $\xi \in L_{ur,\sigma}$ for some $\sigma > 0$ and choose g such that $S_{\xi} \cap B(g(0), \sigma) \neq \varnothing$. Recall that $B(\cdot\,,\cdot)$ refer to hyperbolic balls, while $B_e(\cdot\,,\cdot)$ refer to Euclidean balls on $\partial\mathbb{B}_{\infty}$. Then there exists a r_g such that

$$B_e(\xi, r_g) \subseteq \Pi[B(g(0), 2\sigma)] \subseteq B_e(\xi, 3r_g) \ .$$

Note that one may use Corollary 3.35 here. Therefore we have that

$$\mu\big(B_e(\xi, r_g)\big) \leq \mu\big(\Pi[B(g(0), 2\sigma)]\big) \leq \mu\big(B_e(\xi, 3r_g)\big)$$

and also that

$$r_g \leq \mathrm{diam}_e\big(\Pi[B(g(0), 2\sigma)]\big) \leq 6r_g \ .$$

Now Sullivan's Shadow Lemma 5.11 says that

$$\mu\big(\Pi[B(g(0), 2\sigma)]\big) \asymp_\sigma \big(\mathrm{diam}_e\Pi[B(g(0), 2\sigma)]\big)^\delta$$

which in turn implies that

$$\mu\big(B_e(\xi, r_g)\big) \lesssim_\sigma r_g^\delta \text{ and that } \mu\big(B_e(\xi, 3r_g)\big) \gtrsim_\sigma r_g^\delta \ .$$

Now since $\xi \in L_{ur}$, there exists a constant $K = K(\sigma)$ such that *for every* $r > 0$ one can find a g with $r \asymp_K r_g$ and this allows us to get uniformly that

$$r^\delta \lesssim_\sigma \mu\big(B_e(\xi, r)\big) \lesssim_\sigma r^\delta$$

after which one can apply Theorem 5.13 to get

$$H^\delta_{ur,\sigma} \ \asymp_\sigma \ \mu_{ur,\sigma} \ \asymp_\sigma \ P^\delta_{ur,\sigma} \ .$$

(ii) Note that in Theorem 5.13 the $\circledast$ decorations on the lim sup and lim inf inequalities need to hold just on subsequences to get the corresponding decorated inequalities about measures. In fact, for $\xi \in L_{r,\sigma}$ we have that

$$r_n^\delta \ \lesssim_\sigma \ \frac{\mu(B_e(\xi, r_n))}{r_n^\delta} \ \lesssim_\sigma \ r_n^\delta$$

holds only on some subsequence $r_n \to 0$. Therefore by the corresponding decorated inequalities about measures from Theorem 5.13 we have that

$$H^\delta_{r,\sigma} \ \lesssim_\sigma \ \mu_{r,\sigma} \ \lesssim_\sigma \ P^\delta_{r,\sigma} \ .$$

(iii) This follows from (1) and the fact that $L_{ur}(G) = \cup_{\sigma>0} L_{ur,\sigma}(G)$.

(iv) Similarly, this follows from (2) and the fact that $L_r(G) = \cup_{\sigma>0} L_{r,\sigma}(G)$.

(v) The proof follows almost immediately from Sullivan's Shadow Lemma. Suppose otherwise, i.e. that $\mu_0(\xi) > 0$ for some $\xi \in L_r(G)$. Then we have a sequence of $g_n(0)$ converging to ξ. Now by the Shadow Lemma, we have that

$$\begin{aligned} 1 \geq \mu_0(g_n^{-1}(\xi)) &= \int_\xi P(g_n(0), \eta)^\delta d\mu_0(\eta) \\ &= P(g_n(0), \eta)^\delta \mu_0(\xi) \\ &\asymp_r e^{\delta d(0, g_n(0))} \ . \end{aligned}$$

Note that the last estimate comes from $P(z, \xi) \asymp_r \frac{1}{1-|z|} \asymp_r e^{d(0,z)}$ for $\xi \in \Pi[B(z, r)]$. But this is absurd as $g_n(0) \to \xi$ and therefore μ_0 is atomless on $L_r(G)$.

(vi) This follows from $H_r^\delta \ll \mu_r$, since any measure absolutely continuous to a finite measure is σ-finite .

(vii) By Theorem 3.42 we have that $L(G) = L_{ur,\sigma}(G)$ for some σ and therefore by (i) we have that $0 < \ H^\delta \restriction_{L(G)} < \infty$ since μ is Ahlfors regular.

□

Remark 5.15. Note that in the proof of (5), it may be more geometrically intuitive to think in terms of the derivative. Recall as in the proof of the Shadow Lemma that the diameter of its complement $\mathrm{diam}_e\big(\partial\mathbb{B} - g^{-1}\Pi[B(g(0), r)]\big) = \mathrm{diam}_e\big(\partial\mathbb{B} - \Pi_{g^{-1}(0)}[B(0, r)]\big)$ goes to zero as we increase the radius r and thereby the distance $d(0, g(0))$. Intuitively the derivative of g^{-1} scales like "diameter of image" over "diameter of domain", i.e.

$$|(g^{-1})'(\xi)| \asymp_r \frac{1}{\mathrm{diam}_e\big(\Pi[B(g(0), r)]\big)} \asymp_r \frac{1}{1 - |g(0)|}.$$

This tells us the derivative blows up as $g(0)$ approaches ξ on the boundary, which then leads to an easy contradiction given that we are dealing with probability measures.

CHAPTER 6

CONCLUDING REMARKS

We continue our exploration of discrete group actions based primarily in arbitrary hyperbolic metric spaces rather than just in infinite-dimensional hyperbolic space in [8]. We also have not included our results about Schottky groups and hope that the reader will have come to this work via [8] where such objects and various other examples are explored thoroughly. To give a taste of what is accomplished in [8] we present some results from that work which should help indicate some of the future directions we have been and will be taking. We emphasize that this work continues to be worked out together with David Simmons and Mariusz Urbański.

Recall that to define conformal measures we need the notion of a derivative. In general, this notion may not be well-defined, but it will be well-defined up to a constant, as the following proposition demonstrates:

PROPOSITION 6.1. *Let X be a hyperbolic metric space and let $a > 1$ be as in Proposition 2.43. Then there exists a constant $\kappa > 0$ such that for every isometry $g : X \to X$ and for every point $\xi \in \partial X$, we have*

$$a^{-\kappa} a^{\mathcal{B}_\xi(g^{-1}(0),0)} \leq \liminf_{\eta\to\xi} \frac{\mathrm{Dist}_a(g(\xi), g(\eta))}{\mathrm{Dist}_a(\xi,\eta)} \leq \limsup_{\eta\to\xi} \frac{\mathrm{Dist}_a(g(\xi), g(\eta))}{\mathrm{Dist}_a(\xi,\eta)} \leq a^{\kappa} a^{\mathcal{B}_\xi(g^{-1}(0),0)}, \tag{29}$$

where both limits are taken over the boundary ∂X.

This proposition justifies calling the expression $a^{\mathcal{B}_\xi(g^{-1}(0),0)}$ "the derivative of g at ξ"; however, we should keep in mind that the geometric significance of this expression is only valid up to a constant. We now define the notion of a quasiconformal measure, a generalization of the notion of a conformal measure. The generalization is necessary due to the fact that "the derivative is only well-defined up to a constant".

DEFINITION 6.2. Let (X, dist) be hyperbolic metric space and let G be a strongly discrete group of isometries acting on X. For each $t \geq 0$, a Borel probability measure m on Λ is

called t-quasiconformal if

$$m(g(A)) \asymp \int_A a^{t\mathcal{B}_\xi(0,g^{-1}(0))} dm(\xi) \tag{30}$$

for all $g \in G$ and all $\xi \in \Lambda$.

Recall the result of Patterson [25] and Sullivan [37]; although they were working in the context of conformal measures on the boundaries of standard finite-dimensional hyperbolic spaces, their proof easily generalizes to the following (see also [8]):

THEOREM 6.3. *If (X, dist) is a proper hyperbolic metric space and if G is a strongly discrete group of isometries acting on X, then there exists a quasiconformal meaure for G with the exponent equal to δ_G.*

In [8], in the context of non-proper hyperbolic metric spaces, with the help of the Čech-Stone compactification, we prove the following generalization:

THEOREM 6.4. *If (X, dist) is an arbitrary hyperbolic metric space and if G is a strongly discrete group of isometries of divergence type acting on X, then there exists a quasiconformal meaure m for G with the exponent equal to δ_G. It is unique up to a multiplicative constant in the sense that if m_1, m_2 are two such measures then $m_1 \asymp m_2$ and the Radon-Nikodym derivative $\delta m_1/\delta m_2$ is bounded from above and below. In addition, the quasiconformal measure is supported on the radial limit set of G, meaning that $m(\Lambda \setminus \Lambda_{\mathrm{r}}) = 0$.*

An obvious question is about the geometrical interpretation of a quasiconformal measure. And indeed, due to the work of Dennis Sullivan, it is known that if G is a convex-cocompact group acting on a standard finite-dimensional space, then the δ_G-Hausdorff and packing meaures on Λ are δ_G-quasiconformal. The more general proper case is surveyed in [3]. In [8] the following is proved.

THEOREM 6.5. *If (X, dist) is a hyperbolic metric space and if G is a strongly discrete convex co-bounded group of isometries acting on X, then a unique (up to multiplicative boundedness) δ_G-quasiconformal measure exists and coincides, up to multiplicative boundedness, with both δ_G-Hausdorff and packing meaures on $\Lambda(G)$.*

To end with, let us indicate a few more of our goals with regard to Schottky systems. Here is a brief summary of IFS's, GDMS's and pseudo-Markov systems. *Iterated function systems* (IFSs) generalize the process of observing the backward trajectories of dynamical systems. They started out as finite collections of contracting similitudes (called generators) of a metric space, say a compact region X of some $\mathbb{R}^n$. They each generate, via a recursive procedure, a unique fractal set called an attractor or limit set and may be used to encode and generate fractal images in computer simulations. The theory underwent a rigorous treatment in the 1980's and 90's by Hutchinson [15], Falconer [11] and Barnsley [1] among others with an emphasis on finitely-generated IFSs. In time they were applied to a variety of new situations via the generalization to *infinite conformal IFS*s as developed in the paper [19] by Mauldin and Urbański in 1996. Areas of application included the metric theory of continued fractions with real and complex entries that could be restricted to a particular sets [20]; the Hausdorff dimensions of Julia sets arising from exponential functions [46]; harmonic measure on various fractals [45] and also Diophantine approximation, e.g. [34] proves that conformal measures for convex cocompact Kleinian groups are extremal in the sense of Kleinbock-Lindenstrauss-Weiss. The theory was further generalized in 2003 by Mauldin and Urbański via the theory of *Graph Directed Markov Systems* (GDMSs) [21]. The idea is natural: every IFS may be thought of as a graph with a single vertex (with X attached to it) and countably many loops (each a generator) about that vertex. The limit set generated by all infinite paths on this graph can be thought of as being attached to the vertex. GDMS's could have finitely many vertices, and countably infinite edges between them. In this setting there is a unique fractal set associated to each vertex and the limit set for the system is the disjoint union of these sets. Finally, in 2007 Stratmann and Urbański generalized GDMS's to *pseudo-Markov systems* [36], where GDMS's are allowed to have infinitely many vertices and applied the theory to infinitely generated Schottky groups. Mauldin-Szarek-Urbański began the investigation of GDMSs in Hilbert spaces in [18].

One of our goals in [8] is to apply the thermodynamic formalism à lá Bowen [4] and

Ruelle [30] to our various infinite-dimensional settings and then build appropriate forms of thermodynamic formalism, ergodic theory, and geometric measure theory to investigate the geometry of limit sets. A special role is played by infinitely-generated discrete groups of Schottky type since they can be treated by methods of pseudo-Markov and graph directed Markov systems. In particular, we clarify and extend the results of [36] to get strong geometric and stochastic properties for large classes of infinitely generated Schottky groups and in particular for groups that are not of compact type, e.g. strongly discrete Schottky groups for which the orbit of 0 under generators does not have compact closure.

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