SOME EXPERIMENTS ON THE BALANCING OF SMALL FLEXIBLE ROTORS: PART I—THEORY

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A theoretical balancing technique was proposed by Bishop and Gladwell (1)§ and an experimental investigation of its usefulness is reported. Experiments conducted on the laboratory scale were believed necessary before full-scale tests on large industrial rotors (such as alternator rotors) could be properly justified. The thin shafts which were used clearly showed the importance of initial bend (as opposed to mass unbalance) as a source of forced vibration. This required some additions to the original theory, and they are the subject of Part I of this paper. Part II ‖ contains an account of the actual experiments and of the test apparatus. It is concluded that the theory is fully borne out on the laboratory scale and that comparable tests on large rotors would be justified.

INTRODUCTION

A FRESH APPROACH to the theory of vibration of rotating shafts has recently been made in a series of papers (Bishop (2), Bishop and Gladwell (1), Gladwell and Bishop (3) and (4)). In the second of these papers, in particular, a theory of balancing is given for high-speed flexible rotors. That theory is extended in the third and fourth papers (by implication) to shafts of non-uniform section and shafts mounted in flexible bearings. An experimental investigation of the usefulness of this balancing technique has been made on the laboratory scale and the purpose of this paper is to present the results of that investigation, and the additional theoretical work prompted by these results.

The forced vibration of a rotating flexible shaft may evidently arise from two defects. In the first place the shaft may lack mass balance and this state of affairs will be referred to as that of 'mass unbalance'. Secondly, the shaft may possess an initial bend; this will be referred to as the case of 'elastic unbalance'. Any given rotor may suffer to some extent from both defects at the same time.

Part of the paper by Bishop and Gladwell (1) was devoted to the theory of balancing a shaft suffering from mass unbalance. A brief resumé of that theory will be presented here and the effects will be examined of small errors in the process of balancing. It was stated in the paper referred to that the vibration of a shaft suffering from elastic unbalance is qualitatively somewhat different from that with only mass unbalance. It was discovered in the experiments to be reported here that, on the laboratory scale, the shafts concerned suffered predominantly from elastic unbalance. For this reason the theory of balancing a shaft with elastic unbalance (i.e. with an initial bend) is developed in this paper and the effects of small errors in this process of balancing are investigated. From these discussions, the general balancing of unspecified defects is developed. It should perhaps be stated that the balancing of a shaft with both defects, when treated in all its generality, is somewhat complicated. This is especially true where the analysis of errors is concerned. These theoretical matters are dealt with in Part I of the paper, while Part II deals with the experimental results.

Notation

The notation, in general, is that of the references (1) and (2). The following additions and modifications are necessary.

$\bar{A}_r$	Residual mass unbalance.
$b(x)$	Mass distribution per unit mass of shaft equivalent to an isolated balancing mass.
b_r	Component of $b(x)$ in the rth mode (see equation (65)).
$\bar{b}_r$	Component of balance in the rth mode.
$\bar{f}(x, \Omega)$	General force distribution (see equation (62)).
$f_r(\Omega), f'_r(\Omega)$	Modal components of $\bar{f}(x, \Omega)$ (see equation (62)).

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‖ *The second part of this paper will be published in the next issue of this Journal.*

$\bar{f}_r(\Omega)$	Term representing unspecified defects in the shaft, $=f_r(\Omega)+if'_r(\Omega)$.
$r_1, r_2 \ldots$	Distances of mass centres of balancing masses $m_1, m_2, \ldots$ from the elastic axis of the shaft.
α_r	Angular error in location of the balancing plane.
α'_r	Equivalent angular error in location of the balancing plane (see equations (52) and (60)).
Γ_r	See equation (19).
γ_r	Real multiplier denoting the error in magnitude of the balancing masses.
γ'_r	Equivalent error in magnitude of the balancing masses (see equations (52) and (60)).
Θ_r	See equation (19).
θ_r	See equations (17) and (42).
λ_r	Real multiplier governing the magnitude of the balancing masses.
ρ_r	$\left\lvert\frac{\bar{a}_r}{\bar{\epsilon}_r}\right\rvert$, see equation (77).
ψ_r	See equation (77).
Ω'	Angular velocity at which vibration of a partially balanced shaft is a minimum.

MASS UNBALANCE

Consider a flexible shaft whose mass axis does not coincide exactly with its geometric axis. The balancing of such a shaft is dealt with by Bishop and Gladwell (1). The previous notation and the specific results which are derived in that paper will be used here so as to obviate much tedious repetition.

The shaft gives rise to a series of orthogonal characteristic functions $\phi_1(x), \phi_2(x), \ldots$ which define the shapes of its principal modes in flexural vibration in the absence of damping and rotation. The departure of the mass axis from the geometric axis may be resolved at any transverse section on to longitudinal planes XOU, XOV where OXUV is a set of axes fixed in the shaft. OX coincides with the geometric axis when the shaft is not rotating (the effects of dead weight may be excluded from the present argument), and with the line of bearings during rotation. Each of the projected curves may be expressed in the form of a series of the characteristic functions. Thus, in the planes OXU and OXV,

$$a(x) = \sum_{r=1}^{\infty} a_r\phi_r(x); \quad a'(x) = \sum_{r=1}^{\infty} a'_r\phi_r(x) \quad (1)$$

respectively. And if the plane OUV is regarded as an Argand diagram this becomes

$$\bar{a}(x) = a+ia' = \sum_{r=1}^{\infty} \bar{a}_r\phi_r(x) \quad . \quad . \quad (2)$$

When the shaft is rotating with angular velocity Ω its geometric axis moves out of coincidence with the axis OX, taking components of displacement v and v' in the directions OU, OV respectively. This is illustrated in Fig. 1 in

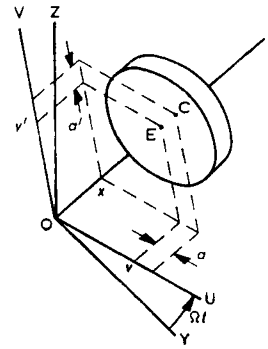


Fig. 1. *Slice of shaft showing location of centre of mass of section relative to fixed and rotating axes*

which C represents the centre of mass of the slice shown and E is the intersection of the elastic axis with the plane faces of the slice. It is here assumed that the elastic axis coincides with the geometric axis in the undeformed shaft (so that E is the centre of the circular cross-section). Both $v(x)$ and $v'(x)$ may be expressed in the series form; thus

$$\eta(x,t) = v(x,t)+iv'(x,t) = \sum_{r=1}^{\infty} \eta_r(t)\phi_r(x) \quad (3)$$

so that the complex multipliers η_r give a real series representing v and an imaginary series giving iv'.

It was shown previously (equations (19) and (20) of Bishop and Gladwell (1)) that, in the rth mode,

$$\eta_r = \frac{\Omega^2\bar{a}_r e^{-i\zeta_r}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} = \text{(a complex constant)} \quad . \quad . \quad . \quad (4)$$

where

$$\zeta_r = \tan^{-1}\left(\frac{2\mu_r\omega_r\Omega}{\omega_r^2-\Omega^2}\right) \quad . \quad . \quad . \quad (5)$$

In these equations, ω_r is the rth natural (circular) frequency of bending vibration and μ_r is the damping factor for external viscous damping of vibration in the rth mode. The interpretation of this result in physical terms was given previously.

A balancing technique was developed theoretically in the previous paper whereby the shaft can be balanced mode by mode. As many components $\eta_1, \eta_2, \ldots$ are balanced separately by nullifying the corresponding multipliers $\bar{a}_1, \bar{a}_2, \ldots$ as may be required by the operational speed of the shaft. To efface the vibration in the rth mode a distribution of masses is attached to the shaft in the radial plane containing the component $\bar{a}_r\phi_r(x)$ of the mass unbalance so as to produce a compensating mass unbalance of $-\bar{a}_r\phi_r(x)$. Generally speaking, the added distribution is composed of isolated masses and a method was described by which these could be selected in such a way as to leave the (previously achieved) balance in all the modes 1, 2, . . ., r−1 undisturbed.

In the experiments to be described, the flexural distortion of the shaft was measured by transducers which were fixed (rather than rotating). The deflection in the

rth mode was thus measured relative to the axes OXYZ shown in Fig. 1. It is therefore desirable to obtain an expression for the forced motion in terms of the displacement components $u(x, t)$ and $u'(x, t)$. If the axes OYZ define an Argand diagram, then

$$u(x,t)+iu'(x,t) = \xi(x,t) = \sum_{r=1}^{\infty} \xi_r(t)\phi_r(x) \quad (6)$$

where

$$\xi_r = \eta_r e^{i\Omega t} = \frac{\Omega^2 \bar{a}_r e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad (7)$$

It was pointed out by Bishop and Gladwell (1) that, in theory, a shaft which suffers from mass unbalance only and which is completely balanced in its rth mode at, say, the rth critical running speed, is balanced in that mode at all other running speeds. This is in contradistinction to a shaft with elastic unbalance; the latter will display the 'residual vibration', the possibility of whose existence was mentioned previously and which will be taken up here.

It is instructive, especially in the interpretation of experimental data, and in estimating the sources of forced vibration, to consider the effect on the vibration of a shaft of two of the errors which can arise in the balancing procedure. Suppose that it is wished to balance out a component $\bar{a}_r$ of unbalance. Ideally this is achieved by imposing a new component of unbalance $\bar{b}_r$ where

$$\bar{b}_r = -\bar{a}_r \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In practice the technique which is employed in doing this is first to identify the plane of $\bar{a}_r$ and then to adjust the balancing masses so as to achieve the correct magnitude of $|\bar{b}_r|$. For this reason it is convenient, here, to identify two distinct conditions: the first in which the plane of $\bar{a}_r$ is found exactly (so that only the intensity of the balancing masses need be at issue) and the second in which an error is made in determining the plane of unbalance.

Case (a): error in the magnitude only of the balancing masses

Suppose that the plane is correctly found and that the error resides in the magnitude of the balancing masses so that equation (8) becomes

$$\bar{b}_r = -\gamma_r \bar{a}_r \quad . \quad . \quad . \quad . \quad (9)$$

where $\gamma_r = 1$ would represent perfect balance. Clearly a residual vibration will be left by this error, its magnitude being given by

$$\xi_r = \frac{\Omega^2 \bar{a}_r(1-\gamma_r) e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (10)$$

At no finite speed Ω is this motion nil.

Case (b)(i): best balance with angular error

In general some error will be made in the initial determination of the plane of $\bar{a}_r$. Let this error be of amount α_r so that the line $\Omega^2\bar{b}_r$ in Fig. 2 represents the direction of the balancing distribution. The process of balancing (once

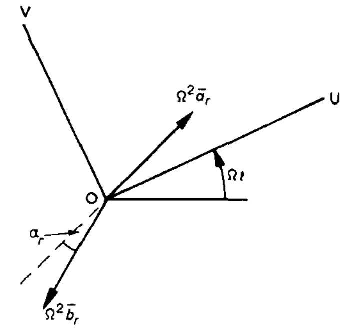


Fig. 2. *Error in angular position of balancing mass*

this error has been committed) lies in the adjustment of $|\bar{b}_r|$ and it is clear that no value of $|\bar{b}_r|$ can completely remove the lack of balance. The process of balancing must now be that of seeking a minimum of residual vibration by adjustment of the real multiplier λ_r in the relation

$$\bar{b}_r = -\lambda_r \bar{a}_r e^{i\alpha_r} \quad . \quad . \quad . \quad . \quad (11)$$

The residual vibration is now

$$\xi_r = \frac{\Omega^2 \bar{a}_r(1-\lambda_r e^{i\alpha_r}) e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (12)$$

and adjustment of λ_r gives a minimum amplitude of vibration *at all speeds* Ω, when

$$\lambda_r = \cos\alpha_r \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The corresponding minimum vibration is given by equation (12) with this value for λ_r, which result may be written in the alternative form

$$\xi_r = \frac{\Omega^2 \bar{a}_r \sin\alpha_r . e^{i(\Omega t-\zeta_r+\alpha_r-\pi/2)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (14)$$

As already mentioned the vibration in the rth mode is not entirely eliminated.

Case (b)(ii): general error in balancing

In general, one can neither detect the plane of unbalance with complete precision nor obtain an exact minimum for the residual vibration so that the resulting motion in the rth mode is of the form (12). If the previous notation (by which incorrect adjustment of the masses means a departure of γ_r from unity) is used in this result then it gives

$$\xi_r = \frac{\Omega^2 \bar{a}_r(1-\gamma_r \cos\alpha_r . e^{i\alpha_r}) e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (15)$$

which may be written in the form

$$\xi_r = \frac{\Omega^2 \bar{a}_r \sqrt{[1-2\gamma_r\cos^2\alpha_r+\gamma_r^2\cos^2\alpha_r]} . e^{i(\Omega t-\zeta_r-\theta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad . \quad . \quad (16)$$

where

$$\theta_r = \tan^{-1}\left(\frac{\gamma_r\cos\alpha_r\sin\alpha_r}{1-\gamma_r\cos^2\alpha_r}\right) \quad . \quad . \quad (17)$$

These various residual vibrations may remain as a consequence of the errors which can arise during the process of balancing the rth component of unbalance so far as the rth mode is concerned. By affixing balancing masses to the shaft with the wrong ratios between the masses, it is possible to impair the balance in a lower mode than the rth.

The residual vibrations (10), (14) and (16), which remain as a result of inaccurate balancing, are all of the form

$$\xi_r = \frac{\Omega^2 \bar{A}_r e^{\iota(\Omega t - \zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (18)$$

where

$$\bar{A}_r = \bar{a}_r(1-\Gamma_r)e^{-\iota\Theta_r} \quad . \quad . \quad . \quad (19)$$

and, as may be seen from equations (10), (14) and (16),

$$\Gamma_r = \gamma_r, \ (1-\sin\alpha_r) \text{ and } [1-(1-2\gamma_r\cos^2\alpha_r+\gamma_r^2\cos^2\alpha_r)]$$

$$\Theta_r = 0, \quad (\pi/2-\alpha_r) \text{ and } \theta_r \quad . \quad . \quad (20)$$

respectively. The vector $\bar{A}_r$ represents effectively a 'residual mass unbalance'. In all cases the vibration suffers a phase change of about $(-\pi)$ radians as Ω passes through the value ω_r rad/sec. The plane of the residual mass unbalance is fixed relative to the shaft—or, rather, its direction is fixed relative to the axes OXUV—though it will not generally make a small angle with the initial component of mass unbalance determined by $\bar{a}_r$. This angular displacement, for what it is worth, is expressed by Θ_r in equations (20) via equations (18) and (19).

It will be shown later that the nature of this residual vibration is of some interest. Fig. 3 shows some computed curves for ζ_r—the phase of the vibration vector ξ_r with respect to the forcing vector $\bar{a}_r$ or $\bar{A}_r$—for various values of the external damping coefficient, μ_r. The points marked in this figure represent some experimental values extracted from a complete set of experimental data which will be reported separately in Part II. These points indicate that $\mu_r = 0{\cdot}025$ is a reasonable value to assume for the damping coefficient in the critical speed range. The value chosen for μ_r in these theoretical illustrations is not over-important and it is only to preserve some reality that this value for the external damping coefficient has been used in computing the remaining curves, not only in this section, but in all subsequent sections. The ζ_r curve for $\mu_r = 0{\cdot}025$ therefore represents, in a qualitative fashion the variation in the direction of the vibration vector through the critical speed range for all cases considered in this section. It will be seen in later sections that, with different sources of unbalance, this variation will take an entirely different form.

The variation of vibration amplitude with angular velocity for a partially balanced shaft suffering from the error of case (a), is shown in Fig. 4, for a couple of values (0·9 and 1·1) of the error factor γ_r, which values both happen to require the same curve. Considering these as values of Γ_r instead of γ_r, it is possible, through equations (18), (19) and (20), to reinterpret Fig. 4 as an amplitude curve for a general, partially balanced shaft suffering from mass unbalance. Thus all the results of this section can be depicted readily in two figures.

ELASTIC UNBALANCE

A shaft whose axis contains an initial bend, but no mass unbalance, also gives rise to a forced motion. Let the components of the bend be $v_0(x)$ and $v'_0(x)$ relative to the

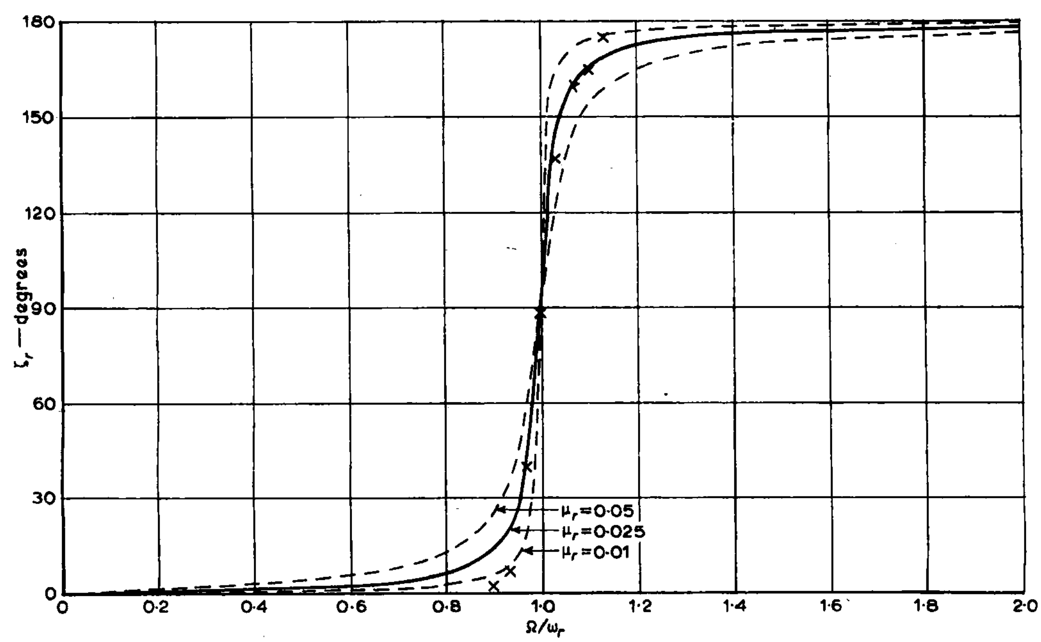
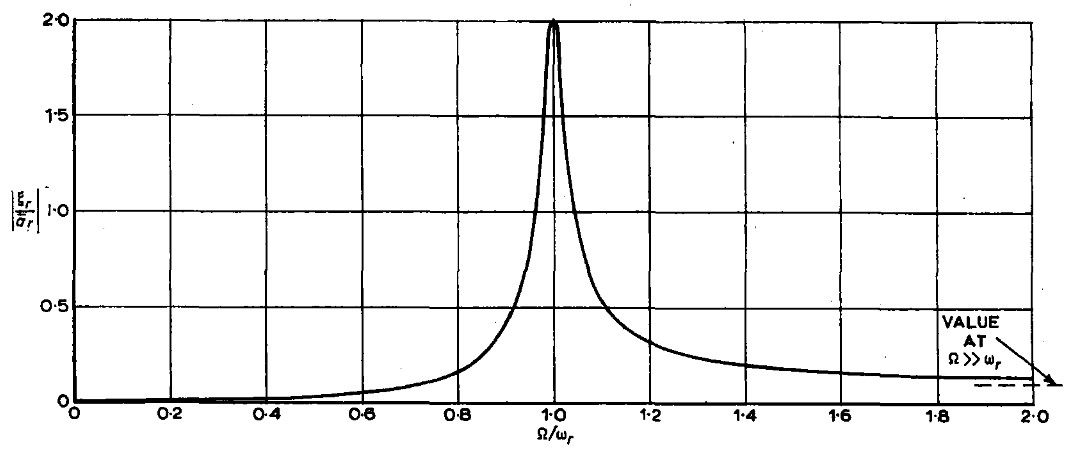


Fig. 3. *Phase–speed curves for a shaft having mass unbalance for three values of μ_r, the external damping factor*



$\gamma = 1{\cdot}1$ or $0{\cdot}9$ if error of case (*a*).

Fig. 4. *Amplitude–speed curves for a shaft having mass unbalance*

rotating axes XOU and XOV respectively and let these deflections be expressed as series of the othogonal functions $\phi_r(x)$ so that

$$\bar{\epsilon}(x) = v_0(x)+iv'_0(x) = \sum_{r=1}^{\infty} \epsilon_r\phi_r(x)+i\sum_{r=1}^{\infty} \epsilon'_r\phi_r(x) = \sum_{r=1}^{\infty} \bar{\epsilon}_r\phi_r(x) \quad (21)$$

When the shaft rotates, its axis moves relative to the fixed system of axes OXYZ and its forced motion has been shown to be given by equation (6), though now

$$\xi_r = \frac{\omega_r^2\bar{\epsilon}_r e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (22)$$

The derivation and physical interpretation of this result have been given previously by Bishop (**2**).

The replacement of Ω^2 in the numerator of the expression (7) by ω_r^2 makes a fundamental difference to the balancing problem. The reason for this is that the elastic unbalance must be nullified as nearly as may be by the addition of a mass unbalance and not by bending the shaft. The disturbing force due to elastic unbalance, which is independent of the speed of rotation, Ω, is thus balanced by a centrifugal force which is a function of Ω^2.

Suppose that, in order to obtain balance in the rth mode, a mass distribution such that

$$\bar{b}_r = -\bar{\epsilon}_r \quad . \quad . \quad . \quad . \quad . \quad (23)$$

is added to the shaft. The resultant vibration is given by

$$\xi_r = \frac{(\omega_r^2-\Omega^2)\bar{\epsilon}_r e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (24)$$

The balancing masses will annul the vibration when $\Omega = \omega_r$, but will leave a residual vibration in the rth mode at any other speed, even though the balance is 'perfect'. The nature of this form of residual vibration will now be examined.

At very low speeds, the distortion of the balanced shaft relative to the rotating axes OXUV is given by

$$\eta_r = \xi_r e^{-i\Omega t} \rightarrow \bar{\epsilon}_r \quad (\Omega \ll \omega_r) \quad . \quad . \quad (25)$$

The distortion is small in this mode and what little there is occurs in phase with the elastic unbalance in this mode; it is the rth component of initial bend with no dynamic magnification.

As the speed is increased the amplitude of the residual vibration ξ_r decreases; this may be proved by expanding equation (24) in a series form. As the critical speed is approached the plane of the distortion lags behind that of the elastic unbalance $\bar{\epsilon}_r$ by an increasing amount. Finally, at the critical speed, this distortion vanishes completely since

$$\eta_r \rightarrow \lim_{\Omega\to\omega_r}\left[\frac{(\omega_r^2-\Omega^2)\bar{\epsilon}_r e^{-i\pi/2}}{2\mu_r\omega_r^2}\right] = 0 \quad (\Omega \rightarrow \omega_r) \quad (26)$$

Further increase of speed causes a reappearance of the residual vibration and, for $\Omega \gg \omega_r$,

$$\eta_r \rightarrow -\bar{\epsilon}_r e^{-i\pi} = \bar{\epsilon}_r e^{-i2\pi} \quad . \quad . \quad (27)$$

That is to say, at very high speeds the distortion in the rth mode increases to a small, but non-zero, magnitude; this magnitude is that of the initial bend. This distortion occurs in a plane which is, once again, almost coincident with that of the rth component of the elastic unbalance.

The nature of this residual vibration of a perfectly balanced shaft is illustrated by certain of the curves of Figs 5, 6, 7 and 8. Fig. 5 (for $\gamma_r = 1$) or Fig. 6 (for $\alpha_r = 0°$) shows the phase–speed relation, while Fig. 7 (for $\gamma_r = 1$) or Fig. 8 (for $\alpha_r = 0°$) illustrates the amplitude–speed variation. It will be noted that the 180° phase 'jump' in Fig. 5 occurs as the amplitude passes through zero in Fig. 7.

Once again it is instructive to consider the effects on the vibration characteristics of a 'balanced' shaft of the two types of error discussed under the heading of 'mass unbalance'. This allows an illuminating comparison to be made with the previous case. Once more the balancing technique—which will be described later—requires first the determination of the plane of the unbalance; when this is known, the magnitude of the balancing masses is adjusted so as to achieve balance. It is therefore convenient again to break the discussion of errors down into the previous subdivisions.

Case (a): error in the magnitude only of the balancing masses

Assuming that the plane of the elastic unbalance has been accurately identified, suppose that a mass distribution

$$\bar{b}_r = -\gamma_r \bar{\epsilon}_r \quad . \quad . \quad . \quad . \quad . \quad (28)$$

is attached. Perfect balancing would be achieved if $\gamma_r = 1$. The residual vibration in the rth mode is

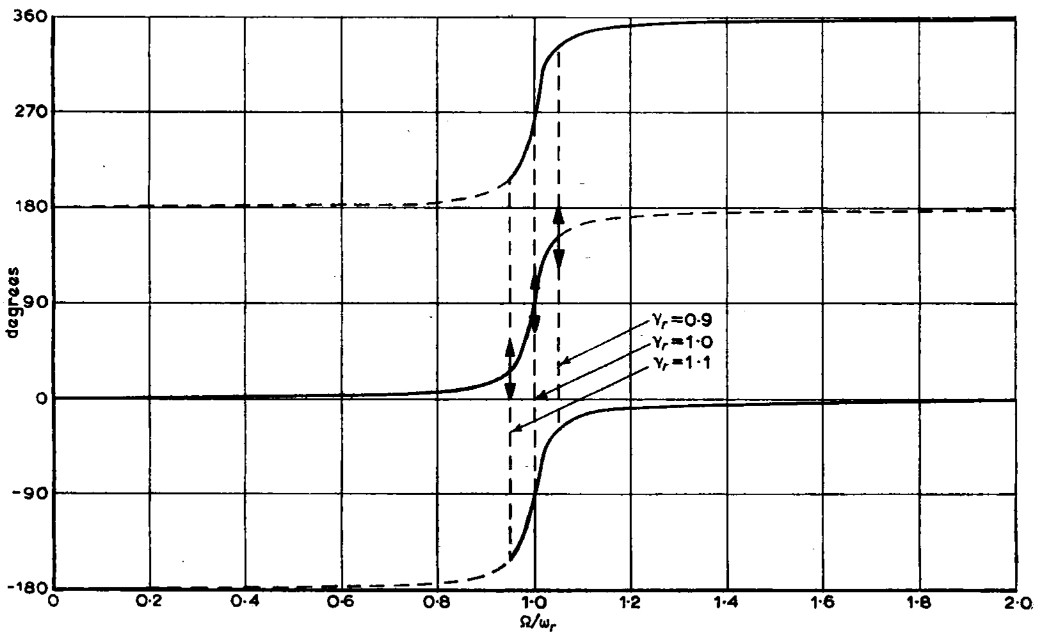
$$\xi_r = \frac{(\omega_r^2 - \Omega^2\gamma_r)\bar{\epsilon}_r e^{i(\Omega t - \zeta_r)}}{\sqrt{[(\omega_r^2 - \Omega^2)^2 + 4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (29)$$

and it is useful to consider two distinct cases.

Case (a)(i): shaft overbalanced ($\gamma_r > 1$)

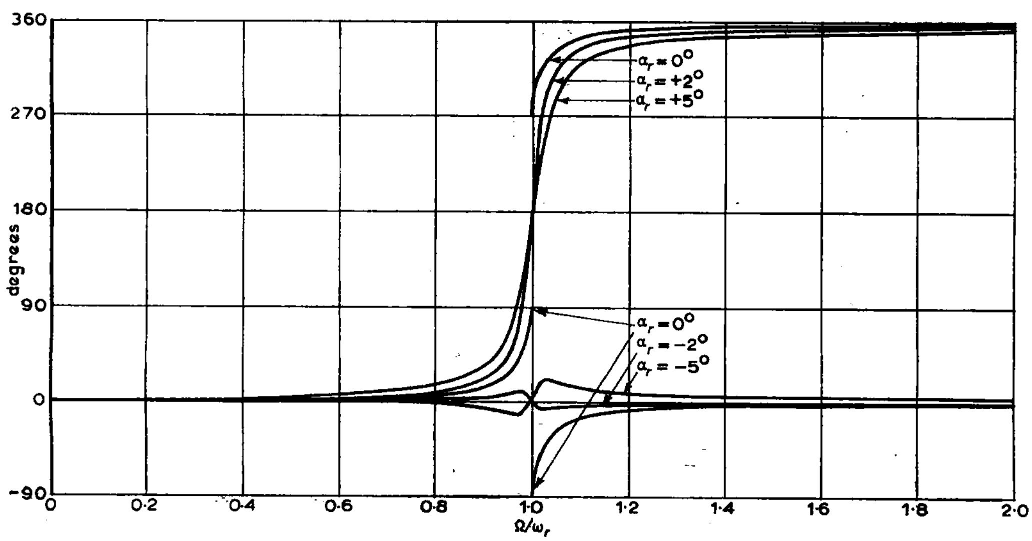
If the error is such that $\gamma_r > 1$, then

$$\xi_r = 0 \quad \text{when} \quad \Omega = \Omega' = \frac{\omega_r}{\sqrt{\gamma_r}} < \omega_r \quad (30)$$



Errors of case (a).

Fig. 5. *Phase–speed curves for a shaft having elastic unbalance*



Errors of case (b)(i).

Fig. 6. *Phase–speed curves for a shaft having elastic unbalance*

The behaviour at very low speeds is similar to that described for perfect balance, for at these speeds the term containing Ω is unimportant. Increasing the speed causes the amplitude in the rth mode to decrease, until it vanishes at the speed Ω' defined in equation (30). Above this speed, the amplitude increases so that, at the critical speed ω_r there is a distortion in the rth mode

$$\eta_r \rightarrow \lim_{\Omega \to \omega_r} \frac{(\omega_r^2 - \gamma_r \Omega^2)\bar{\epsilon}_r e^{-i\pi/2}}{2\mu_r \omega_r^2}$$

$$= \frac{(1-\gamma_r)\bar{\epsilon}_r e^{-i\pi/2}}{2\mu_r} = \frac{(\gamma_r - 1)\bar{\epsilon}_r e^{i\pi/2}}{2\mu_r} \quad (31)$$

There is thus a resonant residual vibration in the rth mode whose phase is $\pi/2$ radians in advance of the plane of the rth component of the elastic unbalance.

At higher speeds, the amplitude of the distortion decreases until

$$\eta_r \rightarrow -\gamma_r \bar{\epsilon}_r e^{-i\pi} = \gamma_r \bar{\epsilon}_r e^{-i2\pi} \quad (\Omega \gg \omega_r) \quad (32)$$

Once more the distortion is very slight and is practically in phase with the rth component of the elastic unbalance.

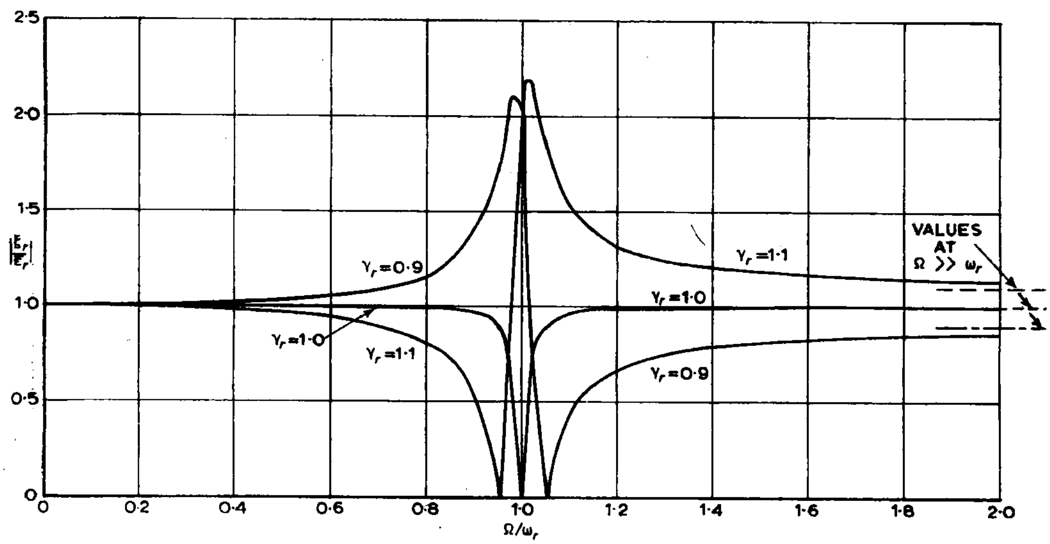
The phase–speed relation is illustrated in Fig. 5 and the amplitude–speed relation is illustrated in Fig. 7 by the curves for $\gamma_r = 1{\cdot}1$.

Case (a)(ii): shaft underbalanced ($\gamma_r < 1$)

When the shaft is underbalanced,

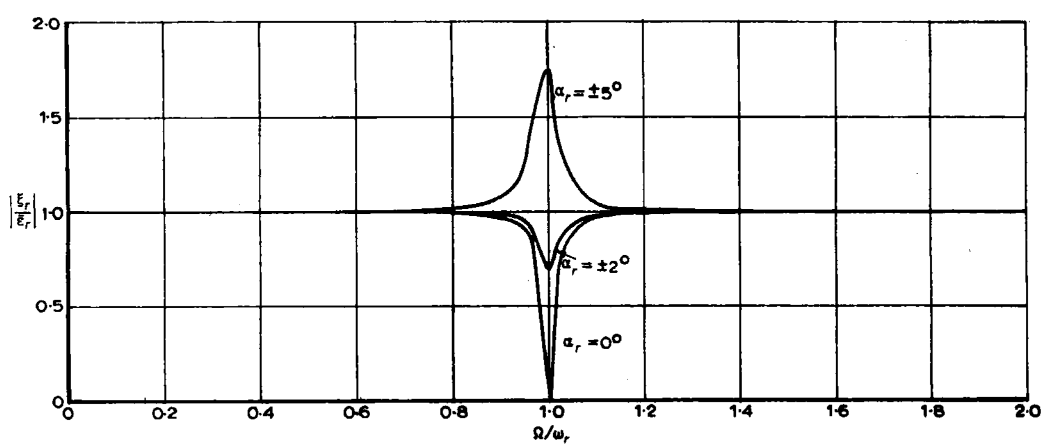
$$\xi_r = 0 \quad \text{when} \quad \Omega = \Omega' = \frac{\omega_r}{\sqrt{\gamma_r}} > \omega_r \quad (33)$$

The characteristics of the vibration are similar to those of



Errors of case (a).

Fig. 7. *Amplitude–speed curves for a shaft having elastic unbalance*



Errors of case (b)(i).

Fig. 8. *Amplitude–speed curves for a shaft having elastic unbalance*

the overbalanced shaft, save in that the amplitude increases from its value at low speeds to the resonant value

$$\eta_r \to \frac{(1-\gamma_r)\bar{\epsilon}_r e^{-i\pi/2}}{2\mu_r} \quad (\Omega = \omega_r) \quad . \quad (34)$$

There is thus again some resonant vibration, though now its plane is $\pi/2$ radians behind the plane of the component of elastic unbalance.

The vibration decreases to zero at the speed $\Omega = \Omega'$ defined in equation (33). Finally, at very high speeds ($\Omega \gg \omega_r$) the vibration is indistinguishable from that when the shaft is overbalanced.

The natures of the phase–speed and amplitude–speed relations are illustrated in Figs 5 and 7 respectively, by the curves for $\gamma_r = 0{\cdot}9$.

Case (b): error in the location of the balancing plane

An error will normally be made in the initial determination of the plane of $\bar{\epsilon}_r$. Let the angular error be of magnitude α_r as shown in Fig. 9. Its existence means that there must be a residual vibration at all speeds Ω. Suppose that, with this angular error, the magnitude of the balancing component is

$$\bar{b}_r = -\lambda_r \bar{\epsilon}_r e^{i\alpha_r} \quad . \quad . \quad . \quad . \quad (35)$$

The residual forcing function is therefore of the form $(\omega_r^2 - \lambda_r \Omega^2 e^{i\alpha_r})\bar{\epsilon}_r$ so that

$$\xi_r = \frac{(\omega_r^2 - \lambda_r \Omega^2 e^{i\alpha_r})\bar{\epsilon}_r e^{i(\Omega t - \zeta_r)}}{\sqrt{[(\omega_r^2 - \Omega^2)^2 + 4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (36)$$

This expression gives the general vibration of an incompletely balanced shaft with elastic unbalance.

If the criterion of minimum amplitude at the critical speed ω_r is used, then λ_r takes the value $\cos \alpha_r$*. If this is achieved exactly, then the residual vibration in the rth mode is given by equation (36) with this value for λ_r. If this adjustment is not made exactly, then the residual vibration may be expressed in the form

$$\xi_r = \frac{(\omega_r^2 - \gamma_r \Omega^2 \cos \alpha_r . e^{i\alpha_r})\bar{\epsilon}_r e^{i(\Omega t - \zeta_r)}}{\sqrt{[(\omega_r^2 - \Omega^2)^2 + 4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad (37)$$

where $\gamma_r = 1$ represents the best balance that the error α_r will allow.

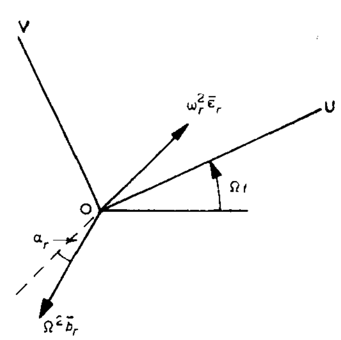


Fig. 9. *Error in angular position of balancing mass for shaft with an initial bend*

* *This may be shown by differentiating* $|1-\lambda_r e^{i\alpha_r}|$ *with respect to* λ_r *and equating the derivative to zero.*

Case (b)(i): best balance with angular error ($\gamma_r = 1$)

The distortion of the shaft at very low speeds is given by

$$\eta_r \to \bar{\epsilon}_r \quad (\Omega \ll \omega_r) \quad . \quad . \quad . \quad (38)$$

and this will be found to be common to all shafts possessing elastic unbalance.

At the critical speed ω_r, the distortion takes the form

$$\eta_r \to \frac{(1-\cos \alpha_r . e^{i\alpha_r})\bar{\epsilon}_r e^{-i\pi/2}}{2\mu_r} = \frac{\bar{\epsilon}_r \sin \alpha_r}{2\mu_r} e^{i(\alpha_r - \pi)} \quad (\Omega = \omega_r) \quad . \quad . \quad . \quad (39)$$

This represents a vibration of a magnitude which is either slightly less than, or slightly greater than, that at low speed (see equation (38)) depending on the relative magnitudes of α_r and μ_r. This is illustrated, for $\mu_r = 0{\cdot}025$, in Fig. 8. What motion there is occurs in a plane which is approximately $(\pi - \alpha_r)$ radians behind the plane of the elastic unbalance $\bar{\epsilon}_r$.

At very high speeds the distortion in the rth mode becomes

$$\eta_r \to -\bar{\epsilon}_r \cos \alpha_r . e^{i(\alpha_r - \pi)} = \bar{\epsilon}_r \cos \alpha_r . e^{i(\alpha_r - 2\pi)} \quad (\Omega \gg \omega_r) \quad . \quad . \quad . \quad (40)$$

The residual vibration in the rth mode thus leads the elastic unbalance by the small phase angle α_r.

The variations of the phase with speed and of the amplitude with speed are illustrated in Figs 6 and 8 respectively.

Case (b)(ii): general error in balancing ($\gamma_r \neq 1$)

Equation (37) gives the residual vibration of a shaft with an error α_r in the balancing plane and (if $\gamma_r \neq 1$) an error in the magnitude of the balancing mass distribution. For very low speeds, the distortion is again that of equation (18).

If γ_r differs from unity, then at the critical speed ω_r the distortion is given by

$$\eta_r = \frac{\sqrt{[1-2\gamma_r \cos^2 \alpha_r + \gamma_r^2 \cos^2 \alpha_r]}\bar{\epsilon}_r e^{-i(\pi/2 + \theta_r)}}{2\mu_r} \quad (\Omega = \omega_r) \quad . \quad . \quad . \quad (41)$$

where

$$\theta_r = \tan^{-1}\left(\frac{\gamma_r \cos \alpha_r \sin \alpha_r}{1-\gamma_r \cos^2 \alpha_r}\right) \quad . \quad . \quad (42)$$

That is, the distortion occurs in a plane which lags $(\pi/2 + \theta_r)$ radians behind that of the rth component of unbalance. This distortion is no longer the least for all speeds Ω since the minimum vibration occurs at a speed which differs from ω_r (the difference depending on the value of γ_r). In fact the minimum distortion occurs at a speed Ω' which satisfies the quartic

$$\gamma_r \cos^2 \alpha_r [1-\gamma_r(1-2\mu_r^2)]\Omega'^4 + (\gamma_r^2 \cos^2 \alpha_r - 1)\omega_r^2 \Omega'^2 + (1-2\mu_r^2 - \gamma_r \cos^2 \alpha_r)\omega_r^4 = 0 \quad (43)$$

and, if the external damping is slight and the speed Ω' sufficiently far removed from the critical speed for the damping terms to be negligible, this becomes approximately

$$\left(\frac{\Omega'}{\omega_r}\right)^2 \doteqdot \frac{1-\gamma_r \cos^2 \alpha_r}{\gamma_r(1-\gamma_r)\cos^2 \alpha_r} \quad . \quad . \quad (44)$$

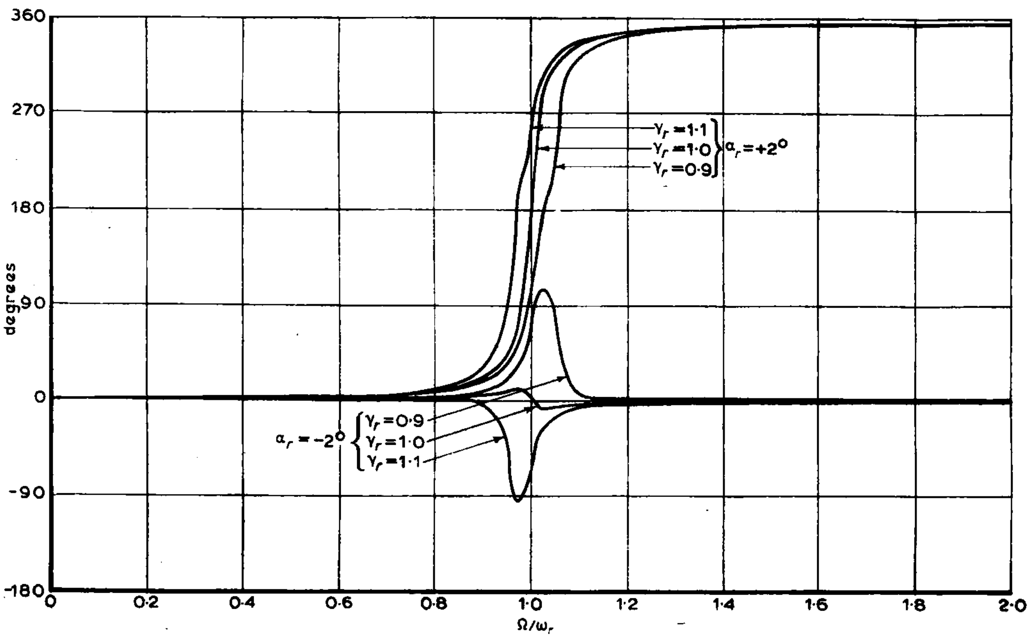
The expression (43) is derived by considering the minimum value of the square of the amplitude deduced from equation (37). It is of interest to note that for $\alpha_r = 0$, the approximation (44) reduces to the exact expressions (30) and (33).

Finally at very high speeds the vibration in the rth mode becomes

$$\eta_r \rightarrow -\gamma_r \bar{\epsilon}_r \cos \alpha_r . e^{i(\alpha_r - \pi)} = \gamma_r \bar{\epsilon}_r \cos \alpha_r . e^{i(\alpha_r - 2\pi)} \quad (\Omega \gg \omega_r) \quad . \quad . \quad . \quad (45)$$

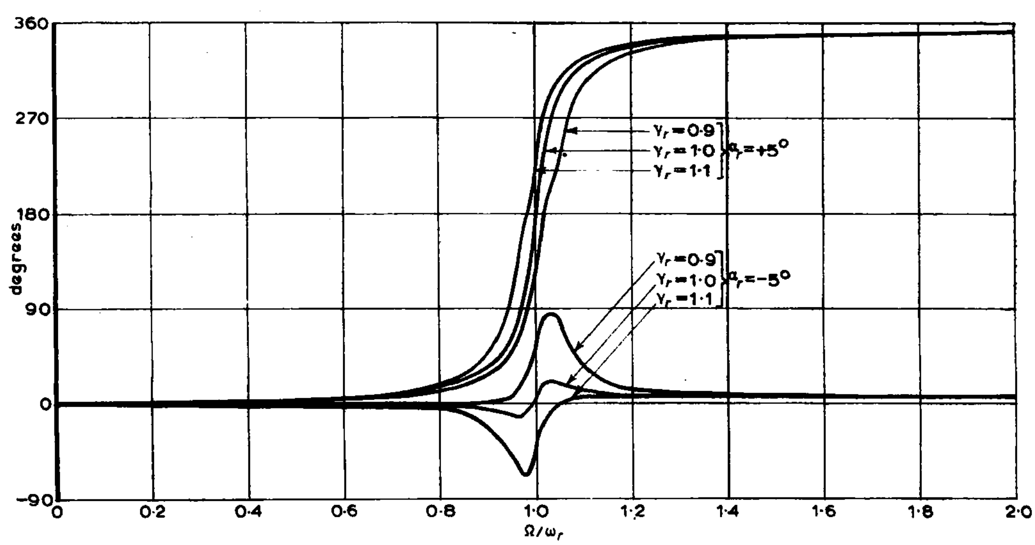
so that residual vibration in the rth mode leads the elastic unbalance by the small phase angle α_r.

The phase–speed and amplitude–speed relations are illustrated in Figs 10 and 11 and Figs 12 and 13 respectively, for $\gamma_r < 1$, $\gamma_r = 1$, and $\gamma_r > 1$.



Errors of case (b)(ii).

Fig. 10. *Phase–speed curves for a shaft having elastic unbalance*



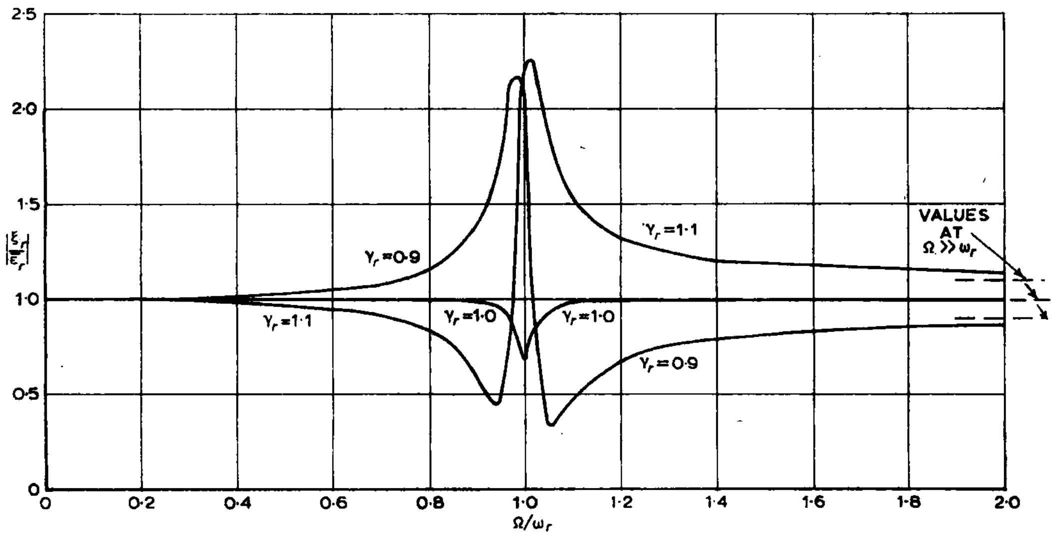
Errors of case (b)(ii).

Fig. 11. *Phase–speed curves for a shaft having elastic unbalance*

To sum up, the vibration of a shaft with elastic unbalance is different from that with mass unbalance. This remains true (though the nature of the vibration is altered) if the elastic unbalance is 'balanced'—either exactly or inaccurately—by an imposed mass unbalance. The behaviour of the balanced shaft is a special case of that of a shaft with both elastic unbalance and mass unbalance; this is treated in the next section.

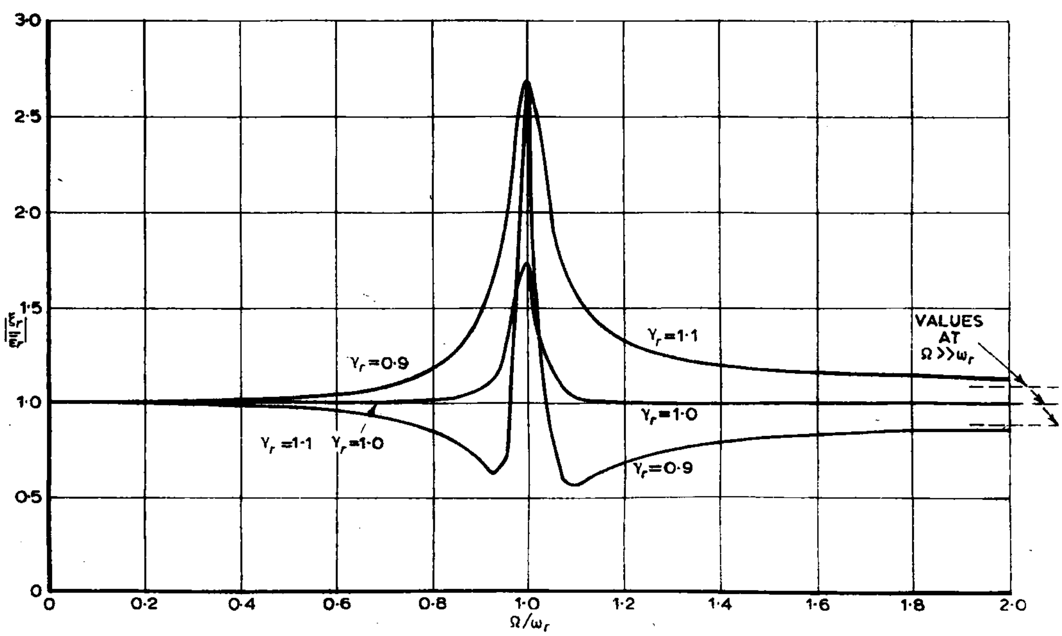
COMBINED MASS AND ELASTIC UNBALANCE

In the previous two sections the problems of balancing a shaft with defects resulting in either mass unbalance or elastic unbalance have been considered separately. This treatment is correct when only one of the defects is present, though it may be adequate when one of these predominates. It is, however, possible for a shaft to suffer



Errors of case (b)(ii); $\alpha_r = \pm 2°$.

Fig. 12. *Amplitude–speed curves for a shaft having elastic unbalance*



Errors of case (b)(ii); $\alpha_r = \pm 5°$.

Fig. 13. *Amplitude–speed curves for a shaft having elastic unbalance*

from both defects to a comparable degree and it is the purpose of the present section to investigate the problems arising from this situation.

When the shaft rotates, its axis moves relative to the fixed system of axes OXYZ and its forced motion has been shown to be given by equation (6) though now

$$\xi_r = \frac{(\bar{a}_r\Omega^2+\bar{\epsilon}_r\omega_r{}^2)e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r{}^2-\Omega^2)^2+4\mu_r{}^2\omega_r{}^2\Omega^2]}} \quad . \quad (46)$$

There is an immediate, and important, difference between this and the two preceding problems. In this case the forcing function, $(\bar{\epsilon}_r\omega_r{}^2+\bar{a}_r\Omega^2)$, varies continuously with speed, not only in magnitude but also in direction. Thus at speeds well below ω_r this forcing function is practically in phase with its elastic unbalance component, whereas at speeds well above the critical value, ω_r, the mass unbalance component predominates. Because of this, the location of the rth modal balancing plane is considerably harder than it was in the previous cases, wherein this plane remained fixed relative to the shaft at all speeds.

Assuming, however, for the present that this plane is known when $\Omega = \omega_r$, then a balance may be achieved in the usual manner by the addition of a mass distribution such that

$$\bar{b}_r = -(\bar{\epsilon}_r+\bar{a}_r) \quad . \quad . \quad . \quad . \quad (47)$$

A shaft balanced in this manner has a resultant vibration in the rth mode expressed by

$$\xi_r = \frac{\bar{\epsilon}_r(\omega_r{}^2-\Omega^2)e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r{}^2-\Omega^2)^2+4\mu_r{}^2\omega_r{}^2\Omega^2]}} \quad . \quad (48)$$

The result of balancing is thus to remove the mass unbalance term completely, leaving a residual vibration which is identical to that of a shaft in which the initial mass unbalance is absent (see equation (24)). This would be expected from the nature of the balancing technique which removes sources of mass unbalance but only counteracts the elastic forces due to elastic unbalance. The physical interpretation of this result has been discussed in detail in the previous section, and as before the phase–speed and the amplitude–speed curves are shown in Fig. 5 ($\gamma_r = 1$) or Fig. 6 ($\alpha_r = 0°$) and Fig. 7 ($\gamma_r = 1$) or Fig. 8 ($\alpha_r = 0°$) respectively.

It is possible, once more, to estimate the effects on the vibration characteristics of a 'balanced' shaft of the two sources of error, although the analysis becomes rather complicated. Once more in the discussion of errors it is convenient to consider several distinct possibilities.

Case (a): error in the magnitude only of the balancing masses

Assuming that the plane of unbalance is known, suppose that a mass distribution

$$\bar{b}_r = -\gamma_r(\bar{\epsilon}_r+\bar{a}_r) \quad . \quad . \quad . \quad (49)$$

is added to the shaft, where a perfect balance would be achieved if $\gamma_r = 1$. The resultant vibration in the rth mode is expressed by

$$\xi_r = \frac{\bar{\epsilon}_r(\omega_r{}^2-\gamma_r\Omega^2)+\bar{a}_r\Omega^2(1-\gamma_r)}{\sqrt{[(\omega_r{}^2-\Omega^2)^2+4\mu_r{}^2\omega_r{}^2\Omega^2]}}\, e^{i(\Omega t-\zeta_r)} \quad (50)$$

which may be compared with equations (10) and (29). It should again be noted that the direction of vibration changes not only with ζ_r, but also with the residual forcing function contained in the numerator of expression (50).

Some progress is possible in the physical interpretation of this result if equation (50) is rewritten in the form

$$\xi_r = \frac{(\omega_r{}^2-\Omega^2\gamma'_re^{i\alpha'_r}\cos\alpha'_r)\bar{\epsilon}_r}{\sqrt{[(\omega_r{}^2-\Omega^2)^2+4\mu_r{}^2\omega_r{}^2\Omega^2]}}\, e^{i(\Omega t-\zeta_r)} \quad (51)$$

where

$$\gamma'_re^{i\alpha'_r}\cos\alpha'_r = \gamma_r-\frac{\bar{a}_r}{\bar{\epsilon}_r}(1-\gamma_r) \quad . \quad (52)$$

Now the expression (51) is identical with (37) which describes the residual vibration of a shaft, initially having elastic unbalance only, suffering from both balancing errors, if the substitutions $\gamma_r = \gamma'_r$ and $\alpha_r = \alpha'_r$ are made. The general nature of the phase–speed and amplitude–speed relations are once more given by Figs 10 and 11 and Figs 12 and 13 with the substitution of α'_r and γ'_r for α_r and γ_r; but, as is shown by equations (75) and (76) of Appendix I, the exact relation of these curves via γ'_r and α'_r with the actual error, γ_r, in this case is not a simple one. The approximate treatment that is given in Appendix I demonstrates, however, that α'_r is small (see inequality (82)) and of a magnitude comparable to those for which Figs 10 and 11 were constructed, provided that the initial mass unbalance is not much greater than the initial elastic unbalance. Further the approximate inequality (84) for γ'_r shows that the possible range of values of this factor does not diverge considerably from those for which Figs 12 and 13 were drawn. In fact it is reasonable to expect these figures and the description at the end of the previous section, case (b)(ii), to furnish a good qualitative picture of the present situation. In addition, although the vibration never vanishes, it may be shown on substituting $\gamma_r = \gamma'_r$ and $\alpha_r = \alpha'_r$ into the expressions (43) and (44) that the distortion has a minimum value at a speed Ω' given by

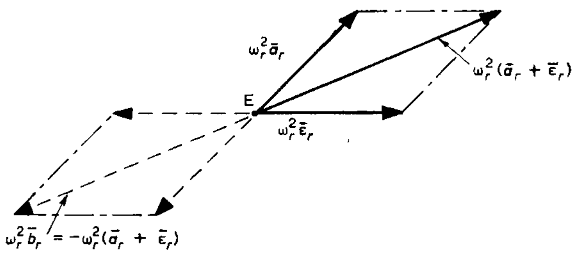
$$\gamma'_r\cos^2\alpha'_r[1-\gamma'_r(1-2\mu_r{}^2)]\Omega'^4+(\gamma'_r{}^2\cos^2\alpha'_r-1)\omega_r{}^2\Omega'^2 + (1-2\mu_r{}^2-\gamma'_r\cos^2\alpha_r)\omega_r{}^4 = 0 \quad (53)$$

or approximately by

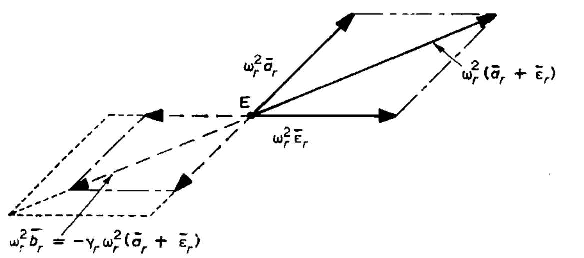
$$\left(\frac{\Omega'}{\omega_r}\right)^2 \doteqdot \frac{1-\gamma'_r\cos^2\alpha'_r}{\gamma'_r(1-\gamma'_r)\cos^2\alpha'_r} \quad . \quad . \quad (54)$$

The approximation inherent in the result (54) is explained in the development of the analogous expression in equation (44).

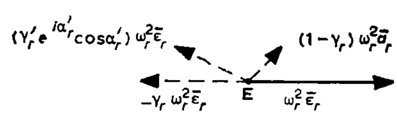
The identity between the present case (a) and case (b)(ii) for elastic unbalance may also be established vectorially. Fig. 14a represents the state of affairs with perfect balancing, so that $\gamma_r = 1$. The balancing force, represented by $\bar{b}_r\omega_r{}^2 = -(\bar{a}_r+\bar{\epsilon}_r)\omega_r{}^2$ is in equilibrium with the resultant driving force, $(\bar{a}_r+\bar{\epsilon}_r)\omega_r{}^2$, at the critical speed and there is no vibration. Fig. 14b portrays the situation when there is an error, $\gamma_r < 1$, in the magnitude of the balancing mass distribution and hence in the



a Perfect balance in the general case ($\gamma_r = 1$).



b Error in magnitude of balancing masses ($\gamma_r < 1$).



c Alternative form of *b*, equivalent to configuration of Fig. 9.

Fig. 14. *Equivalence of error of type (a) for general unbalance and error of type (b)(ii) for a shaft with initial bend only*

balancing force. The configuration of Fig. 14*c* is equivalent to that of Fig. 14*b*, being merely an alternative representation; the 'disturbing force' is $\bar{\epsilon}_r\omega_r^2$ and the 'balancing force' is supplied by the resultant of $-\gamma_r\bar{\epsilon}_r\omega_r^2$ and $(1-\gamma_r)\bar{a}_r\omega_r^2$; this is a valid standpoint since the factor ω_r^2 in these two components of the balancing force is replaced by Ω^2 at any speed other than the resonant, for which the vector figures are drawn. Thus the situation is equivalent to that of a shaft with initial elastic unbalance suffering from both balancing errors, γ'_r and α'_r. The vibration is thus of the type (37) and

$$-\gamma'_r e^{i\alpha'_r}\cos\alpha'_r\bar{\epsilon}_r\omega_r^2 = -\gamma_r\bar{\epsilon}_r\omega_r^2+(1-\gamma_r)\bar{a}_r\omega_r^2$$

or

$$\gamma'_r e^{i\alpha'_r}\cos\alpha'_r = \gamma_r-(1-\gamma_r)\frac{\bar{a}_r}{\bar{\epsilon}_r} \quad . \quad . \quad (55)$$

and expression (55) is identical to that of equation (52).

Case (b): error in the location of the balancing plane

It is probable that an error will be made in the initial determination at $\Omega = \omega_r$ of the unbalance plane, $\bar{a}_r+\bar{\epsilon}_r$. Let this angular error be of magnitude α_r as shown in Fig. 15. Its existence means that there must be a residual vibration at all speeds Ω. Accepting this angular error, consider a balancing mass distribution given by

$$\bar{b}_r = -\lambda_r(\bar{a}_r+\bar{\epsilon}_r)e^{i\alpha_r} \quad . \quad . \quad . \quad (56)$$

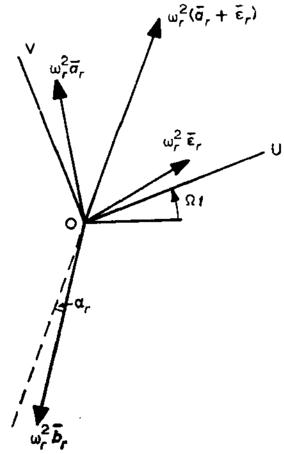


Fig. 15. *Error in angular position of balancing mass for shaft with mass unbalance and initial bend*

attached to the shaft. The residual forcing function is therefore of the form $\bar{\epsilon}_r\omega_r^2+\bar{a}_r\Omega^2-\lambda_r\Omega^2(\bar{a}_r+\bar{\epsilon}_r)e^{i\alpha_r}$ so that the resultant vibration is

$$\xi_r = \frac{\bar{\epsilon}_r(\omega_r^2-\lambda_r e^{i\alpha_r}\Omega^2)+\bar{a}_r\Omega^2(1-\lambda_r e^{i\alpha_r})}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}}\,e^{i(\Omega t-\zeta_r)} \quad (57)$$

which describes the general vibration of an incompletely balanced shaft with both mass and elastic unbalance.

As before, on applying the balancing criterion of minimum amplitude at the critical speed, ω_r, a value $\cos\alpha_r$ is found for λ_r. If this value of λ_r is incorrectly determined, then the residual vibration may be expressed in the form

$$\xi_r = \frac{[\bar{\epsilon}_r(\omega_r^2-\Omega^2\gamma_r\cos\alpha_r e^{i\alpha_r})+\bar{a}_r\Omega^2(1-\gamma_r\cos\alpha_r e^{i\alpha_r})]e^{i(\Omega t-\zeta_r)}}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}} \quad . \quad . \quad . \quad (58)$$

This is found by substituting $\lambda_r = \gamma_r\cos\alpha_r$ in equation (57), so that $\gamma_r = 1$ represents the best balance that the error α_r admits. This expression may be compared with the corresponding ones in the previous two sections, namely equations (15) and (37) and it is again clear that the presence of both defects—mass unbalance and elastic unbalance—renders the analysis more complicated.

It is fruitful, however, to rewrite equation (58) in the form

$$\xi_r = \frac{\bar{\epsilon}_r(\omega_r^2-\Omega^2\gamma'_r e^{i\alpha'_r}\cos\alpha'_r)}{\sqrt{[(\omega_r^2-\Omega^2)^2+4\mu_r^2\omega_r^2\Omega^2]}}\cdot e^{i(\Omega t-\zeta_r)} \quad (59)$$

where

$$\gamma'_r e^{i\alpha'_r}\cos\alpha'_r = \gamma_r e^{i\alpha_r}\cos\alpha_r+(\gamma_r e^{i\alpha_r}\cos\alpha_r-1)\frac{\bar{a}_r}{\bar{\epsilon}_r} \quad (60)$$

from which the analogy between the present shaft and a shaft with only elastic unbalance is evident. In fact equation (59) is identical with the expression (37) for a shaft with elastic unbalance and with both balancing errors, when γ'_r and α'_r are substituted for γ_r and α_r respectively. Once more the exact relations between these factors are not simple (see equations (78) and (79) of Appendix I) but the Figs 10–13 of the phase–speed and amplitude–speed relations are again relevant for those values of γ_r and α_r which yield $\gamma'_r = 0{\cdot}9,\ 1{\cdot}0,\ 1{\cdot}1$ and $\alpha'_r = 0°$, $\pm 2°$, $\pm 5°$ for which the figures were constructed.

Further curves for other values of γ_r and α_r could easily be constructed in a similar manner with the aid of equations (78) and (79). In addition the crude approximations of Appendix I, expressed in the inequalities (86) and (89), demonstrate that the Figs 10–13 provide a good qualitative picture of the present problem, although quantitative differences must be expected. As in case (a), the residual vibration of equation (59) has a minimum value at a speed Ω' given by the expressions (53) and (54) and thus depends on the value of γ_r and α_r.

In conclusion, the vibration of a shaft with both mass unbalance and elastic unbalance differs from both of the previous cases considered, when only one defect was present. In a state of balance or partial balance the behaviour of the present shaft in vibration remains different from that of a shaft with initial mass unbalance (see an earlier section); but, for the reasons outlined above, it possesses considerable similarity with that of the shaft suffering from elastic unbalance (described in an earlier section).

THEORY OF BALANCING

The theory underlying the balancing procedure for a uniform shaft with mass unbalance was given by Bishop and Gladwell (1). It was extended—by implication only—in two later papers (3) and (4) to shafts of non-uniform section and shafts supported in flexible bearings. For two reasons (apart from completeness of presentation) it seems desirable to develop the theory afresh here. The first of these is the rather prosaic fact that the radial dimensions of the balancing masses which were used in the laboratory were comparable in magnitude to the radius of the shaft; this produced an effect which would not be expected in larger industrial rotors but which could not be ignored in the experiments. The second reason is that the slenderness of the laboratory shafts threw emphasis on the importance of elastic unbalance, that is on the case not dealt with specifically before. The authors are, in fact, unaware of any published method aimed specifically at the balancing of a bent shaft.

While the treatment given by Bishop and Gladwell (1) needs only very slight modification to cover these two matters, the balancing technique will be explained here in a slightly different way. The theory will be given in terms of a general, unspecified, initial defect in the shaft. By analogy with the expressions (4), (22) and (46) the forced vibration which is to be balanced out (as far as this is possible) is given by

$$\xi_r = \frac{\bar{f}_r(\Omega)e^{i(\Omega t - \zeta_r)}}{\sqrt{[(\omega_r^2 - \Omega^2)^2 + 4\mu_r^2\omega_r^2\Omega^2]}} \quad (r = 1, 2, \ldots) \quad (61)$$

This represents the motion in the rth mode, $\bar{f}_r(\Omega)$ being a term representing the defects in the shaft. This term can be treated as the component in the rth mode of a general force distribution of the form

$$\begin{aligned}\rho A\bar{f}(x,\Omega) &= \rho A \sum_{r=1}^{\infty} \bar{f}_r(\Omega)\phi_r(x) \\ &= \rho A \sum_{r=1}^{\infty} [f_r(\Omega) + if'_r(\Omega)]\phi_r(x) \quad . \quad . \quad (62)\end{aligned}$$

by analogy with equations (1) and (21). In fact equation (61) could be derived on this basis either by solving the appropriate equations of motion or by using the appropriate receptance expressions contained in Gladwell and Bishop (3). For example, in the case of a shaft suffering from both mass and elastic unbalance, a comparison of equations (46) and (61) shows that

$$\bar{f}_r(\Omega) = \bar{a}_r\Omega^2 + \bar{\epsilon}_r\omega_r^2 \quad . \quad . \quad . \quad (63)$$

The physical interpretation of equation (61) follows the usual pattern whereby the vibration vector ξ_r, varies from being almost in phase to being in anti-phase with $\bar{f}_r(\Omega)e^{i\Omega t}$, as the shaft speed is changed from a value less than the rth critical speed to one greater. It has to be remembered, however, that $\bar{f}_r(\Omega)$ being a complex, or vector, function may itself vary in direction as the shaft speed is altered. In addition, each mode of vibration predominates in turn, as the shaft speed is increased through the corresponding critical value.

This possible variation in the direction of the disturbing forces with speed requires some modification of the concept of balancing alluded to in the section on elastic unbalance. Now, balancing masses must be added until the vibration in a mode at the appropriate critical speed is reduced to zero, although at other speeds some vibration may remain in this balanced mode. That is, the modal balancing procedure must now be extended to vibrations of the form

$$\xi(x, t) = \sum_{r=1}^{\infty} \xi_r\phi_r(x), \quad . \quad . \quad . \quad (64)$$

where ξ_r is given by equation (61).

Let m_i be a balancing mass which, when attached to the shaft at an axial station x_i, has its mass centre at a distance r_i from the elastic, or geometric, axis of the shaft. Then (from equation (39) of Bishop and Gladwell (1)) this discrete mass can be represented by a mass distribution $\rho Ab(x)$ such that, for a uniform shaft,

$$b(x) = \sum_{r=1}^{\infty} b_r\phi_r(x) = \sum_{r=1}^{\infty} \frac{m_i r_i}{A\rho Z}\phi_r(x_i)\phi_r(x) \quad (65)$$

where

$$Z = \int_0^l [\phi_r(x)]^2\,\mathrm{d}x \quad . \quad . \quad . \quad (66)$$

The plane of the unbalance at the critical speed, ω_1, in the first mode—that is the plane of $\bar{f}_1(\omega_1)\phi_1(x)$ relative to the shaft—must be determined by observation; let this plane be the OXU plane. The vibration in the first mode may then be balanced by selecting m_1, r_1 and x_1 such that, from equation (65),

$$f_1(\omega_1) + \frac{m_1 r_1}{A\rho Z}\omega_1^2\phi_1(x_1) = 0 \quad . \quad . \quad (67)$$

is satisfied (cf. equation (56) of Bishop and Gladwell (1)). As noted before, the product m_1r_1 has its minimum value when x_1 is chosen to make $\phi_1(x_1)$ a maximum—that is when x_1 is the anti-node of the first mode. This balancing mass may alter the unbalance in the other modes; that is it may alter $\bar{f}_r(\Omega)$ (where $r \geqslant 2$) through equation (65)*. Consequently the functions $\bar{f}_r(\Omega)$ (where $r \geqslant 2$) can be considered to include this extra component, although to avoid complications in the experimental work this mass may be placed at a node of the deflection in the second mode.

The second mode of vibration can now be balanced ensuring at the same time that the previously attained balance in the first mode is not upset. This is accomplished by selecting m_2, r_2, x_2 and m_3, r_3, x_3 such that, again from equation (65),

$$m_2r_2\phi_1(x_2)+m_3r_3\phi_1(x_3) = 0 \quad . \quad . \quad (68)$$

$$f_2(\omega_2)+\frac{m_2r_2}{A\rho Z}\,\omega_2{}^2\phi_2(x_2)+\frac{m_3r_3}{A\rho Z}\,\omega_2{}^2\phi_2(x_3) = 0 \quad . \quad . \quad . \quad (69)$$

are satisfied, with the slight restriction (identical with equation (59) of Bishop and Gladwell (1)) that

$$\begin{vmatrix}\phi_1(x_2) & \phi_1(x_3)\\ \phi_2(x_2) & \phi_2(x_3)\end{vmatrix} \neq 0 \quad . \quad . \quad . \quad (70)$$

This latter requirement is always fulfilled if x_2 and x_3 are chosen on opposite sides of a node in the second mode. It is assumed here for convenience that the axes fixed in the shaft are chosen such that $\bar{f}_2(\omega_2)$ lies in the OXU plane.

In a similar manner the vibration in the third mode may now be balanced by selecting m_i, r_i, x_i (i = 4, 5, 6) such that

$$m_4r_4\phi_1(x_4)+m_5r_5\phi_1(x_5)+m_6r_6\phi_1(x_6) = 0 \quad (71)$$

$$m_4r_4\phi_2(x_4)+m_5r_5\phi_2(x_5)+m_6r_6\phi_2(x_6) = 0 \quad (72)$$

$$f_3(\omega_3)+\frac{m_4r_4}{A\rho Z}\,\omega_3{}^2\phi_3(x_4)+\frac{m_5r_5}{A\rho Z}\,\omega_3{}^2\phi_3(x_5) +\frac{m_6r_6}{A\rho Z}\,\omega_3{}^2\phi_3(x_6) = 0 \quad (73)$$

are satisfied, with the restriction on x_4, x_5, x_6 that

$$\begin{vmatrix}\phi_1(x_4) & \phi_1(x_5) & \phi_1(x_6)\\ \phi_2(x_4) & \phi_2(x_5) & \phi_2(x_6)\\ \phi_3(x_4) & \phi_3(x_5) & \phi_3(x_6)\end{vmatrix} \neq 0 \quad . \quad (74)$$

Again the axes have been chosen such that $\bar{f}_3(\omega_3)$ defines the OXU plane.

The method can be extended to as many modes as necessary and most of the comments made previously remain relevant. It will be noted, however, that the equations are in terms of m_ir_i, and not merely m_i as before. Both here and in the paper by Bishop and Gladwell (1) it is the centrifugal force given by $m_ir_i\Omega^2$ which constitutes the balancing effect of the mass m_i. Previously, however, $r_i \doteqdot r$, the shaft radius, and hence the r's cancelled from the equations.

* *This will not be the case with those modes of which x_1 is a node of deflection.*

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APPENDIX I

ANALOGY BETWEEN ELASTIC UNBALANCE AND COMBINED MASS AND ELASTIC UNBALANCE; APPROXIMATE TREATMENT

The expressions for γ'_r and α'_r arising in the analogy between the problem of elastic unbalance and that of mass and elastic unbalance combined, as deduced from either equation (52) or (60) are not simple. In fact, from equation (52) it is found that

$$\tan\alpha'_r = \frac{\rho_r(\gamma_r-1)\sin\psi_r}{\gamma_r+\rho_r(\gamma_r-1)\cos\psi_r} \quad . \quad . \quad . \quad (75)$$

and

$$\gamma'_r\cos^2\alpha'_r = \gamma_r+\rho_r(\gamma_r-1)\cos\psi_r \quad . \quad . \quad (76)$$

where

$$\frac{\bar{a}_r}{\bar{\epsilon}_r} = \rho_r e^{i\psi_r} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

Similarly, equation (60) leads to the results

$$\tan\alpha'_r = \frac{\gamma_r\sin\alpha_r\cos\alpha_r(1+\rho_r\cos\psi_r)+\rho_r(\gamma_r\cos^2\alpha_r-1)\sin\psi_r}{\gamma_r\cos^2\alpha_r-\gamma_r\rho_r\sin\alpha_r\cos\alpha_r\sin\psi_r+\rho_r(\gamma_r\cos^2\alpha_r-1)\cos\psi_r} \quad . \quad . \quad . \quad (78)$$

and

$$\gamma'_r\cos^2\alpha'_r = \gamma_r\cos^2\alpha_r-\gamma_r\rho_r\sin\alpha_r\cos\alpha_r\sin\psi_r +\rho_r(\gamma_r\cos^2\alpha_r-1)\cos\psi_r \quad (79)$$

In view of the complexity of the relations (75), (76), (78) and (79) a crude, approximate treatment is attempted below, so that some idea of the orders of magnitude of γ'_r and α'_r may be attained. It should be stressed, however, that in this approach no attempt is made to achieve exactness and results should be regarded as having a qualitative significance only. It must be remembered in this approximate analysis that the balancing errors are assumed to be small so that γ_r and α_r are of the order

$$\left.\begin{matrix}0{\cdot}9 \leqslant \gamma_r \leqslant 1{\cdot}1\\ -5^\circ \leqslant \alpha_r \leqslant +5^\circ\end{matrix}\right\} \quad . \quad . \quad . \quad . \quad (80)$$

For simplicity, a limitation is also placed on ρ_r, defined in equation (77), by restricting it to values not much greater than unity, or, if more precision is desired, the analysis is certainly valid for $\rho_r \leqslant 1$. Effectively this restriction eliminates the problem in which the mass unbalance factor is dominant. The discussion is developed in two parts, these corresponding to the balancing errors described under case (a) and case (b).

Case (a): $\gamma_r \neq 1$, $\alpha_r = 0^\circ$

For the values of γ_r and ρ_r under consideration, equation (75) shows that

$$\tan\alpha'_r \doteqdot \rho_r\left(\frac{\gamma_r-1}{\gamma_r}\right)\sin\psi_r \quad . \quad . \quad . \quad (81)$$

and therefore an approximate range of values of α'_r is

$$-6^\circ \leqslant \alpha'_r \leqslant 6^\circ \quad . \quad . \quad . \quad . \quad (82)$$

Similarly, from equation (76) it may be shown that

$$1-\rho_r\left|\frac{\gamma_r-1}{\gamma_r}\right| \leqslant \frac{\gamma'_r}{\gamma_r}\cos^2\alpha'_r \leqslant 1+\rho_r\left|\frac{\gamma_r-1}{\gamma_r}\right| \quad . \quad (83)$$

and for α'_r given by the inequality (82) γ'_r is given approximately by

$$0{\cdot}89\,\gamma_r \leqslant \gamma'_r \leqslant 1{\cdot}11\,\gamma_r \quad . \quad . \quad . \quad . \quad (84)$$

It must, however, be observed that the extreme values of γ'_r and α'_r are not attained for the same values of ψ_r.

Case (b): $\alpha_r \neq 0°$

Again, approximating to the relation (78) leads to

$$\tan \alpha'_r \doteqdot \tan \alpha_r(1+\rho_r \cos \psi_r) \quad . \quad . \quad . \quad (85)$$

and therefore an estimate for α'_r is contained in the inequality

$$-2\alpha_r \leqslant \alpha'_r \leqslant 2\alpha_r, \quad . \quad . \quad . \quad . \quad (86)$$

for the small values of α_r admitted. Similarly from the expression (79) γ'_r must satisfy

$$\gamma'_r \cos^2 \alpha'_r \doteqdot \gamma_r \cos^2 \alpha_r - \gamma_r \rho_r \sin \alpha_r \cos \alpha_r \sin \psi_r \quad (87)$$

and therefore

$$1-\rho_r \tan \alpha_r \leqslant \frac{\gamma'_r \cos^2 \alpha'_r}{\gamma_r \cos^2 \alpha_r} \leqslant 1+\rho_r \tan \alpha_r \quad . \quad (88)$$

In view of the orders of magnitude which γ_r, α_r, α'_r and ρ_r may take, this leads to

$$0{\cdot}91\gamma_r \leqslant \gamma'_r \leqslant 1{\cdot}09\gamma_r \quad . \quad . \quad . \quad . \quad (89)$$

for the order of magnitude of γ'_r. The approximate expressions (85) and (87) are reasonable unless $\psi_r \sim \pi$ in the former or $\psi_r \sim 0$ or π in the latter, but in these cases the corresponding α'_r and γ'_r still satisfy the inequalities (86) and (88) respectively. It cannot be overstressed, however, that all these results are very rough approximations indeed and serve only as a guide to the probable values of γ'_r and α'_r in each case.

APPENDIX II

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COMMUNICATIONS

The following Communication relates to the paper 'Dimensionless Kinetic Energy: A New Parameter for Gas Flow in Ducts with Application to Nuclear-power Reactors', by J. PATTERSON and K. J. DURRANDS, which was published in 1961, vol. 3, No. 3, p. 267.

Mr A. R. Pickering (Winfrith) and **Mr R. Staniforth** (*Associate Member*)—The method used to solve the fundamental equations of mass continuity, energy and momentum conservation is to combine these into one single equation, and to solve this single equation. This equation is of such a form that for all but a few simple cases it must be solved numerically. An alternative method of solution of the equations is the simultaneous solution of the three original equations using finite difference techniques. This method has the advantage that the friction factor may be varied along the channel as the temperature-dependent fluid properties and hence Reynolds number change. This method has the additional advantage that effects such as changes in mass flow in the channel mass flow due to leakages into and through the channel wall may be accommodated, and that additional pressure losses caused by flow obstructions can be allowed for at the appropriate flow condition and gas density.

The authors have demonstrated how the channel pressure drop can alter as the form of the heat generation curve changes. The concept of an 'equivalent concentrated heat source' is an interesting one, but it is difficult to see how it can be of any practical use, since there appears to be no simple method of determining its position.

In the experiments performed to substantiate the theory, the pressure drop through a heated test section was measured and compared with calculated values. However, the friction factors used in calculating this pressure drop were obtained from tests performed without heat addition using the same methods in reverse so that only that part of the pressure drop caused by heat addition depends upon the method of prediction. In case (1), this is only 6 per cent of the total pressure drop so that the apparent error of the prediction is about 25 per cent. Clearly this is due to experimental inaccuracies and so the experiments do not provide the necessary conditions for proving the theory.

It is a pity the authors have not compared their accurate pressure drop calculations with those obtained by simple but approximate methods, for example, Guggenheim's method. We find that for cases of practical interest, this method is within about 3 per cent, surely quite adequate for practical applications.

AUTHORS' REPLY

Dr J. Patterson and **Mr K. J. Durrands**—It is thought that Mr Pickering and Mr Staniforth should distinguish between what was put forward as an original contribution and the well known (but perhaps 'non-U') numerical procedures which they seem to favour. They should appreciate: (1) that the simultaneous numerical integrations of three differential equations can be no different from the numerical integration of the single combined equation; (2) that whereas numerical integration is a valuable technique, an analytical solution represents a more successful application of mathematics—even though a numerical iterative procedure may be required to obtain a particular result, as for example with Guggenheim's equation; and (3) that very fine numerical integrations were obtained from an electronic computer solely in order to check the approximate analytical solutions actually obtained for a number of non-simple cases.

The paper has demonstrated the method for calculating the position of an equivalent concentrated heat source and further discussion at this stage would simply be repetition. It is surprising to read, however, that a concept can be of no practical use unless a 'simple' method exists for evaluating it.

It seems necessary to point out that the more general results in the paper have wider application than nuclear engineering. The case of continuous burning in a combustion chamber is an example which comes to mind as an approximation to a concentrated heat source where the position is known and does not require calculating.

The use of 'cold' friction factors was considered in a previous reply (1962, vol. 4, No. 2, p. 195) where it was stated that future work would endeavour to express f in terms of annular dimensions. In the cases considered, however, f is substantially independent of the Reynolds number so that there would be little merit in allowing for its variation along the channel.

A detailed reply has also been given on the matter of stringent tests and reference should be made to that issue of the *Journal*. However, an undue emphasis on quick and approximate solutions seems to indicate a serious lack of regard for the qualitative value in theoretical work.

It is quite true that the small test rig used in the experiments could not produce a large pressure drop due to heating alone. It is to be hoped, however, that other workers with less limited facilities at hand will provide experimental data for such cases, rather than conjecture.

Nevertheless in test 1 the experimental (and hence theoretical) pressure drop due to friction with no heating is

0·51 lb/in^2, whereas with heating the experimental and theoretical values are 0·59 and 0·6 respectively. The theoretical value for the heating effect therefore is 0·09 compared with the measured result of 0·08. This gives an apparent accuracy of about 88 per cent and not 75 per cent.

It is unfortunate that Mr Pickering and Mr Staniforth are interested only in simple methods and show little concern for the limitations and shortcomings in Guggenheim's analysis which have now been overcome. A complete analytical solution is always useful in that it allows comparison to be made with the approximate solution to the problem. This in fact has been done by the writers in their communication who show that the pressure drop error using Guggenheim is 3 per cent. If the comparison is extended to asymmetrical heat inputs that error is considerably increased. Whether an error of 3 per cent or more in pressure drop is important can only be judged by the equivalent power consumption, which for a nuclear power station could be quite large.

The following Communication relates to the paper 'Dimensionless Kinetic Energy: A New Parameter for Gas Flow in Ducts with Application to Nuclear-power Reactors', by J. PATTERSON and K. J. DURRANDS, which was published in 1961, vol. 3, No. 3, p. 267.

Dr B. N. Cole and **Dr B. Mills** (Birmingham)—We wish first to refer to the contribution by Mr A. J. Ward Smith (1962, vol. 4, No. 2, p. 193) to the discussion of the above paper.

In that contribution, it was suggested that 'the analytical solution in closed form for the frictional flow in a constant area duct with constant heat flux was . . . new and vindicated the authors' choice of the variable ϕ.' This view has not been declined by the authors of the main paper. Mr Ward Smith further quoted references to work by R. N. Noyes in 1960 and 1961 which dealt with the same problem by a different algebraic arrangement.

The present contributors wish to mention their own paper* which includes amongst much other information an analytical solution of this same problem. This is given as equation (27) in that paper, and adjoining equations facilitate the calculation of all parameters likely to be of practical interest. There is no doubt that this particular system, in which the effects of heat transfer are distributed hypothetically at a uniform rate over the whole pipe, is of great practical importance. There are many cases where the overall loss (or gain) of heat is known with reasonable accuracy, the the precise variation of loss or gain along the pipe is uncertain. In these circumstances, in attempting to estimate overall effects, a simple hypothesis is clearly preferable to an elaborate one.

The present contributors' own results have been obtained straightforwardly without recourse to 'dimensionless kinetic energy', and anticipate both equation (13) of the paper by Noyes† and equation (13) of the paper by Patterson and Durrands. While our own work has been included in the references of the last named paper, it seems that the equivalence of the solutions under present discussion has not been appreciated.

We wish further to take this opportunity of correcting an error of wording in our own paper: this concerns the last of three observations offered on p. 627 of that paper. This observation should have stated that the overall pressure drop is greater if the bulk of the total heat transfer is located at the upstream, rather than the downstream, end of the pipe. This conclusion then accords with results implicit in the work of Dr Patterson and Mr Durrands.

* MILLS, B. and COLE, B. N. 'Compressible gas flow in commercial pipes'. *Proc. Instn mech. Engrs, Lond.*, 1957 **171**, 617.

† NOYES, R. N. *Trans. Amer. Soc. mech. Engrs*, series C, 1961 (November), 454.

AUTHORS' REPLY

Dr J. Patterson and **Mr K. J. Durrands**—Dr Cole and Dr Mills are thanked for their communication which perhaps focused attention upon a less serious feature of the work.

The particular equations obtained by them gave a result for the special case of a linear rise in total head temperature, which corresponded to uniform heating along a pipe. In addition they obtained results for the special case of a linear rise in static temperature although details of these were not given in their paper.

In the former case they remarked: 'a clear perception of the qualitative effects of heat transfer relative to adiabatic conditions is obscured by the complex nature of [the] equations'; and in the latter case: 'the resulting equations are considered far too cumbersome to be of real practical value and a fuller record is, therefore, omitted'. In view of those remarks it is interesting to note that Dr Cole and Dr Mills regard their results as having been obtained 'straightforwardly'. Indeed, under the circumstances it must have been exceedingly difficult to draw general conclusions from such particular 'solutions'.

However, they came, unfortunately, to the erroneous conclusion that the overall pressure drop would be greater where the bulk of the heat input was located towards the downstream (rather than the upstream) end of the pipe. In a private communication the attention of Dr Cole and Dr Mills was drawn to this matter.

On the other hand, when expressed in 'dimensionless kinetic energy' form, the exact solution not only solved the problem once and for all, but paved the way for further solutions to more complex problems involving asymmetrical heat distributions. It could be that as a result of this work a chapter in the field of steady adiabatic flow, which had hitherto remained uncompleted, has now been closed.

One of the main aims in the work was to find solutions to problems where the heat input was distributed asymmetrically, and especially in those forms relevant to nuclear power applications. Neither the results of Dr Cole and Dr Mills nor the equation obtained later by Noyes provide solutions to such problems.

Thus, the suggestion by Mr Ward Smith—that the choice (albeit Hobsons!) of the variable, ϕ, had been vindicated—was, therefore, not without some justification.