Integrality theorems in the theory of topological strings

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Abstract

We give a simplified derivation of the expression of instanton numbers and of mirror map in terms of Frobenius map on p-adic cohomology and use this expression to prove integrality theorems. Modifying this proof we verify that the Aganagic-Vafa formulas for the number of holomorphic disks can be expressed in terms of Frobenius map on p-adic relative cohomology; this expression permits us to prove integrality of this number.

1 Introduction

The original goal of this paper was to prove that the formula for the number of holomorphic disks given in [1],[[2] always gives an integer (as expected from physical considerations). Pursuing this goal we simplified and generalized the proof of integrality of instanton numbers given in [8] and [21] in such a way that it can be applied also to the situation of holomorphic disks. We will start with the simplified proof of integrality of instanton numbers and of some results of [20] ; later we will discuss the modifications necessary to analyze holomorphic disks. We have tried to make our paper self-contained; in particular, the paper does not depend on [8] and [20].

The starting point of the integrality proof is the consideration of de Rham cohomology over the ring of integers $\mathbb{Z}$ and over the ring of p-adic integers $\mathbb{Z}_p$. Defining de Rham cohomology over a ring we consider differential forms with coefficients in this ring. The definition can be applied only to manifold defined over this ring.[1] For example, to define de Rham cohomology over integers we should work with manifolds over integers. (In other words, the transition from one coordinate chart to another one should have integrality properties, that guarantee that a form having integer coefficients in one chart has integer coefficients in another chart.) Correspondingly, the definition of de Rham cohomology

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[1]The de Rham cohomology of a manifold over a ring R is defined as hypercohomology of a sheaf of differential forms with coefficients in R.

over $\mathbb{Z}_p$ (the ring of p-adic integers) can be applied only to the manifold defined over this ring; however, one can prove that for compact manifolds this cohomology depends only on corresponding $\mathbb{F}_p$-manifold. (The canonical homomorphism of $\mathbb{Z}_p$ onto $\mathbb{F}_p$, corresponding to the maximal ideal, generated by p, permits us to consider a manifold over $\mathbb{Z}_p$ as a manifold over $\mathbb{F}_p$.) The map $x \to x^p$ (Frobenius map) is a homomorphism of a field of characteristic p. It is not a homomorphism of $\mathbb{Z}_p$, however, it generates a homomorphism Fr (Frobenius transformation) on corresponding cohomology groups. (The construction of this homomorphism is complicated, but the origin of it can be traced to the fact that cohomology depends only of $\mathbb{F}_p$-manifold and the Frobenius map is an endomorphism of $\mathbb{F}_p$-manifold. See [17], Sec 1.3, for a short description of the construction of Fr.)

Notice that knowing de Rham cohomology over integers we can calculate de Rham cohomology over any other ring R. This follows immediately from Kuenneth theorem and the remark that the R-module of differential forms over R is a tensor product of the abelian group of differential forms over $\mathbb{Z}$ with R . In particular, if there is no torsion in the cohomology over $\mathbb{Z}$ (or if we neglect torsion) the cohomology over R is a tensor product of cohomology over $\mathbb{Z}$ with R. This crucial fact permits us to use Frobenius map on p-adic cohomology to obtain information about cohomology over complex numbers.

In what follows we neglect torsion in cohomology. This means that we consider an entire ring R without torsion as coefficient ring and factorize the cohomology with respect to the torsion subgroup; the quotient will be denoted $H_R(X)$. Notice, that $H_R(X)$ can be regarded as a subgroup of $H_{F(R)}(X)$ and specifies an integral structure in $H_{F(R)}(X)$ (we say that an element of $H_{F(R)}(X)$ is integral if it belongs to this subgroup). [2] In particular, $H_{\mathbb{Q}}(X)$ and $H_{\mathbb{C}}(X)$ have integral structures specified by $H_{\mathbb{Z}}(X)$. Notice that these integral structures (algebraic integral structures) are specified only if X is a manifold over integers. It is important to emphasize that that there exists another integral structure in $H_{\mathbb{C}}(X)$, that can be called topological integral structure. Recall, that for every topological space X one can define cohomology with coefficients in any abelian group G; we use the notation $H(X, G)$ for this cohomology. If X is a manifold over integers then $H_{\mathbb{C}}(X)$ is canonically isomorphic to $H(X, \mathbb{C})$; however, the integral structure in this vector space specified by means of $H_{\mathbb{Z}}(X)$ does not coincide with integral structure defined by the map of $H(X, \mathbb{Z})$ into $H(X, \mathbb{C})$ (topological integral structure). Expanding the system of generators in one these integral structures with respect to the generators in another structure we obtain the so called period matrix ; the entries of this matrix are in general transcendental .

The interplay between these two integral structures plays an important role in our considerations.

The proof of integrality of mirror map, of instanton numbers, of the number of holomorphic disks consists of two steps. First of all we prove that these quantities can be expressed in terms of matrix entries of Frobenius map on p-adic cohomology. Second, we show that under certain conditions these expressions give p-adic integers. Knowing p-integrality for

[2] We use the notation $F(R)$ for the field of fractions of the ring R.

all prime numbers p we can say that the quantities we are interested in are integers; if p-integrality can be violated for a finite set of "bad" primes, we can say that in fractions representing the quantities at hand the denominator contains only "bad" prime factors.

It seems that the first step relating arithmetic geometry to quantities arising in topological string can be used in both directions: to obtain information about physical quantities from number theory (as we do in present paper) or to find new ways of dealing with complicated questions of arithmetic geometry starting with ideas of physics. (A small step in the second direction was made in [17] where a construction of Frobenius map on p-adic cohomology based on the ideas of supergeometry was given.)

The proofs of this paper use many ideas of the rigorous proofs in [21]. However, we use a different definition of the mirror map; this permits us to weaken conditions of integrality statements. (In our definition we use algebraically integral sections instead of topologically integral sections. However, we should require that algebraically integral sections we are using are related to topologically rational sections.)

Proving p-integrality we will always consider a family of Calabi-Yau manifolds in a neighborhood of a maximally unipotent boundary assuming that this family is defined over integers and remains smooth after reduction mod p; we assume that the cohomology over Z_p are generated by the Calabi-Yau form and its covariant derivatives with respect to an integral coordinate system. We give an efficient way to check these assumptions. Under the above assumptions we prove the p-integrality of mirror map. The proof of the p-integrality of instanton numbers requires an additional condition $p > 3$.

Our proofs are not written in rigorous mathematical way, but it is not difficult to make rigorous almost all of them (with important exception of cases when a rigorous proof uses the theory of motives).

2 Frobenius map on cohomology of Calabi-Yau threefolds

Let us consider a family of Calabi-Yau smooth threefolds X_z over a base $\mathcal{M}$. We assume that the family is universal (contains all complex deformations), then the base can be identified with the moduli space of complex structures . We assume that the family is defined over integers; this means, in particular, that holomorphic 3-forms Ω entering the definition of Calabi-Yau manifold have integer coefficients. Moreover, we assume that Ω is minimal among the forms having these property (i.e. dividing it by non-invertible integer we obtain a form with fractional coefficients).

We suppose that the manifolds of our family remain smooth after reduction mod p where p is an arbitrary prime number. A rational number can be considered as a p-adic number for every p ; to prove that the rational number is an integer it is sufficient to check that the corresponding p-adic numbers are integral for all primes p. The assumption of smoothness after reduction mod p can be violated for finite set of prime numbers; then our methods give p-integrality only for the complement of this set. This means that the rational number we consider can have a denominator, but the prime factors of the denominator can belong only to a finite set of "bad" prime numbers.

At first we will consider the family over complex numbers. Then the cohomology $H(X_z, \mathbb{C})$ constitute a vector bundle over $\mathcal{M}$. [3]This vector bundle is equipped with flat connection ∇ (Gauss-Manin connection). We will consider the cohomology bundle in the neighborhood of a boundary point $z = 0$ of $\mathcal{M}$. More precisely, we assume that our smooth family over $\mathcal{M}$ can be extended to a non-smooth family over $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}} \setminus \mathcal{M}$ is a normal crossings divisor $z^1 \cdots z^r = 0$. We assume that the cohomology bundle can be extended to $\bar{\mathcal{M}}$ and the Gauss-Manin connection has first order poles with nilpotent residues on the extended bundle. [4]

We will restrict ourselves to the bundle of middle-dimensional cohomology $H^3(X_z, \mathbb{C})$; this is a bundle of symplectic vector spaces of dimension $2r + 2$ where $r = \dim \mathcal{M}$. (Recall that the Hodge numbers of Calabi-Yau threefold are $h^{3,0} = h^{0,3} = 1, h^{2,1} = h^{1,2} = r$.)

Let us fix a symplectic basis g^0, g^a, g_a, g_0 of flat sections of our bundle (these sections are in general multivalued, hence it would be more precise to consider sections of a vector bundle over the universal cover of $\mathcal{M}$). The holomorphic 3-form Ω can be written as linear combination

$$\Omega = X^0 g_0 + X^a g_a + X_a g^a + X_0 g^0 \tag{1}$$

Recall that for a given complex structure a holomorphic 3-form is defined up to a constant factor. We assume that Ω depends holomorphically on complex structure; it is defined up to a factor that is holomorphic on $\mathcal{M}$. The quotients

$$t^a = \frac{X^a}{X^0}$$

can be considered as coordinates on the moduli space $\mathcal{M}$; they are called special coordinates. (These coordinates are multivalued in general, but for the case we are interested in the coordinates $q^a = \exp t^a$ are one-valued and can be extended to the neighborhood of the point $z = 0$ in $\bar{\mathcal{M}}$. This point corresponds to $q = 0$.)

One can normalize the form Ω to obtain $X^0 = 1$; then the normalized form $e_0 = \frac{\Omega}{<g^0,\Omega>}$ is equal to

$$e_0 = g_0 + t^a g_a + \frac{\partial f_0}{\partial t^a} g^a - (2f_0 - t^a \frac{\partial f_0}{\partial t^a}) g^0,$$

where f_0 stands for genus zero free energy. (See, for example, [4] or [19]).[5]

[3]More precisely, we should regard the family as a map of the union X of X_z onto $\mathcal{M}$ and consider corresponding relative de Rham cohomology.

[4]These assumptions are satisfied if the family at hand is semistable.

[5]To obtain this formula from mathematical viewpoint one should work on the extended moduli space (moduli space of pairs (complex structure, Calabi-Yau form)). The Calabi-Yau forms Ω and their multiples specify a Lagrangian cone in the extended moduli space; the free energy $F_0(X^0, X^a)$ is defined as the generating function of this cone. It is a homogeneous function of degree 2; we represent it as $(X^0)^2 f_0(\frac{X^a}{X^0})$.

One can include e_0 into symplectic basis e^0, e^a, e_a, e_0 where

$$e^0 = g^0, e^a = g^a - t^a g^0, e_a = \nabla_a e_0,$$

Using that g^0, g^a, g_a, g_0 are flat (covariantly constant with respect to the Gauss-Manin connection ∇_a), we see that

$$\begin{array}{ll} \nabla_b e_0 = e_b, & \nabla_b e_a = C_{abc} e^c \\ \nabla_b e^a = \delta_{ab} e^0, & \nabla_b e^0 = 0, \end{array} \tag{2}$$

where $C_{abc} = \partial_a \partial_b \partial_c f_0$ are called Yukawa couplings. The Yukawa couplings are holomorphic with respect to $q^a = \exp t^a$; the holomorphic part of free energy can be expressed in terms of them. Notice, that one can add a polynomial of degree ≤ 2 with respect to $t^a = \log q^a$ to the free energy without changing any physical quantities; this addition corresponds to the change of the basis g^0, g^a, g_a, g_0. We will assume that f_0 contains only terms of degree 3 with respect to t.

Let us consider the family at hand in the neighborhood of maximally unipotent boundary point of the moduli space $\mathcal{M}$ of complex structures. Recall, that the coordinates in the neighborhood of the boundary point are denoted by $z^1, ..., z^r$; the points on the boundary divisor obey $z^1 \cdots z^r = 0$.

One classifies flat sections according their behavior at the boundary point $z = 0$. Our assumption that the point is maximally unipotent means that there exists one (up to a constant factor) holomorphic flat section, r linearly independent flat sections with logarithmic behavior at $z = 0$ (or, more precisely, $r + 1$ flat sections with at most logarithmic singularities),r flat sections behaving as $\log^2$ and one flat section behaving as $\log^3$. Instead of flat sections one can talk about solutions of Picard-Fuchs equations. (Recall, that the Picard-Fuchs equation is the equation for periods, i.e. for expresions $< g, \Omega >$ where g is a flat section.) Notice, that under our conditions the holomorphic solution $< g^0, \Omega >$ of the Picard-Fuchs equation can be written as a series with integer coefficients and the constant term equal to 1. This can be checked directly in all interesting cases (for example, for hypersurfaces in toric varieties this follows from expressions given in [3], see also [5]). The general proof can be based on Poincare duality in (algebraically and topologically) integer cohomology [7].

One can expand flat sections into series with respect to $z^i, \log z^i$; we can consider filtration on the space of flat sections $W^0 \subset W^1 \subset W^2 \subset W^3$ (no logarithms, at most linear in $\log z^i$, at most quadratic in logarithms, etc). This filtration can be extended to the space of all sections using the remark that every section can be represented as a linear combination of flat sections with variable coefficients. It can be defined also in terms of monodromy operators M_i and their logarithms N_i; it is called monodromy weight filtration. (Notice, that the standard definition of monodromy weight filtration differs slightly of the above construction and our notations are not standard.)

We will take the symplectic basis g^0, g^a, g_a, g_0 of flat sections in such a way that $g^0 \in W^0, g^a \in W^1, g_a \in W^2, g_0 \in W^3$. More precisely, we assume that

$$g^a = g^0 \log z^a + h^a$$

where g^0 and h^a are holomorphic at $z = 0$. Then the special coordinates

$$t^a = \frac{< g^a, \Omega >}{< g^0, \Omega >} = \log z^a + \frac{< h^a, \Omega >}{< g^0, \Omega >}$$

are called canonical coordinates. The coordinates t^a are multivalued; therefore instead of them it is convenient to use coordinates

$$q^a = \exp t^a = z^a \exp \frac{< h^a, \Omega >}{< g^0, \Omega >}$$

that are also called canonical coordinates. The expression of canonical coordinates q^a in terms of original coordinates z^a is called mirror map. Notice, that the definition of canonical coordinates q^a specifies them only up to constant factors. (The canonical coordinate t^a does not depend on the choice of g^0, however, we have a freedom to add a multiple of g^0 to g^a; hence t^a is defined up an additive constant.) We will assume that q^a is expressed in terms of z^a as a series with rational coefficients. (It is sufficient to assume that the terms linear with respect to z^a have rational coefficients; then solving recursively the equations for the flat section we obtain that all other coefficients are also rational.)

The assumption that the family at hand is defined over integers can be used to fix canonical coordinates and mirror map uniquely . We require that q^a behaves like z^a as $z \to 0$; this means that

$$\exp \frac{< h^a, \Omega >}{< g^0, \Omega >} = 1 \tag{3}$$

at $z = 0$. (More generally one can assume that q^a behaves like $\pm z^a$ at $z = 0$; in other words $\exp \frac{<h^a,\Omega>}{<g^0,\Omega>} = \pm 1$ at $z = 0$). [6] We assume that the coordinates $z^1, ..., z^r$ agree with integral structure on the moduli space $\mathcal{M}$. It is natural to conjecture that canonical coordinates $q^1, ..., q^r$ also have this feature; in other words, the coefficients of expansion of these coordinates in terms of $z^1, ..., z^r$ are integers (recall that we assumed that the linear terms of these series are equal to $\pm z^a$). This conjecture (integrality of mirror map) was proven in many cases; see [15],[9],[10], [21] and the discussion in Sec 4.

In the present section and in Sec 3 we will freely use the integrality of mirror map proving integrality of other objects; in Sec 4 we will check the integrality of mirror map in the conditions we need.

[6]Notice, that there exists another approach to fixing of mirror map based on consideration topologically integral sections g^0, g^a. Our requirement agrees with the definition of monomial-divisor mirror map for hypersurfaces in toric varieties. It is important to emphasize that a toric variety can be considered naturally as a variety over $\mathbb{Z}$; the same is true for hypersurfaces therein if the coefficients of defining equation are integral.

It follows from integrality properties of holomorphic period $<g^0,\Omega>$ that $e_0 = \frac{\Omega}{<g^0,\Omega>}$ is an integral vector in algebraic sense (or more precisely, an integral section of the cohomology bundle). Using integrality of mirror map we can see that e_a also has this feature. (The covariant derivatives ∇_a with respect to canonical coordinates preserve algebraic integrality.) The integrality of normalized Yukawa couplings C_{abc} follows from the formula

$$C_{abc} = < e_0, \nabla_a \nabla_b \nabla_c e_0 > = < \nabla_b e_a, e_c > .$$

We will assume that e^c can be expressed as a linear combination of $\nabla_b e_a$ with integer coefficients; hence it is an integral vector. This means, in particular, that for fixed c the numbers $C_{abc}(0)$ (classical limits of Yukawa couplings) are relatively prime. If we are interested in p-integrality we should work with $\mathbb{Z}_p$-cohomology; the above assumption takes the form : $C_{abc}(0)$ reduced mod p is a matrix of rank r (we consider the pair a, b as one index).

We see that the basis e^0, e^a, e_a, e_0 consists of integral vectors (in algebraic sense); moreover, these vectors generate the de Rham cohomology over integers (this follows immediately from the fact that the basis is symplectic). However, in present section we need only much weaker statement that this basis consists of algebraically rational vectors. This statement follows from the rationality condition used in the construction of canonical map.

We will impose also an additional condition that the sections g^a are topologically rational. (Recall that topologically rational sections are always flat.) It seems that using the results of [3] one can prove that this condition is satisfied for hypersurfaces in toric varieties.

We assumed that the family at hand is defined over integers; this means that we can consider it over any other ring, in particular, over p-adic integers $\mathbb{Z}_p$ or over the field of p-adic rational numbers $\mathbb{Q}_p$. (If our family were defined over $\mathbb{Q}$ we still would be able to consider it over $\mathbb{Q}_p$.) Again we obtain a bundle of corresponding cohomology with flat connection (Gauss-Manin connection). [7] It is important that Gauss-Manin connections for different rings are compatible. (A homomorphism of rings induces a homomorphism of de Rham cohomology, commuting with connection.)

We neglect torsion in cohomology, therefore we can obtain de Rham cohomology with coefficients in any ring R tensoring the cohomology over integers with R; this operation is compatible with Gauss-Manin connection. Analogously, the natural map of the cohomology with coefficients in $\mathbb{Q}$ into cohomology with coefficients in $\mathbb{Q}_p$ commutes with Gauss-Manin connection. This means, in particular, that we can consider g^0, g^a, g_a, g_0 and e^0, e^a, e_a, e_0 as bases in p-adic cohomology; they are defined in a neighborhood of the point $z = 0$.

[7]Due to the fact that we are working with cohomology factorized with respect to torsion this statement is not trivial; moreover, it is not always true. However, it was proved in [7] that this statement is true in the case if the family at hand has a smooth reduction mod p and $p > 3$. We always assume that the $\mathbb{Z}_p$-cohomology form a bundle; then one can construct the Gauss-Manin connection on this bundle.

[8] (Recall that log makes sense as a one-valued function defined for any non-zero p-adic number.)

In the neighborhood of $z = 0$ we consider a map $z \to z^p$ transforming every coordinate to its p-th power: $(z^1, ..., z^r) \to ((z^1)^p, ..., (z^r)^p$. In the case when R is the ring of integer p-adic numbers $\mathbb{Z}_p$ we can lift the map $z \to z^p$ to the map Fr on cohomology $H_{\mathbb{Z}_p}(X_z)$. This map (relative Frobenius map) is compatible with Gauss-Manin connection and with the scalar product on cohomology:

$$\nabla_a \mathrm{Fr} = p \mathrm{Fr} \nabla_a, \tag{4}$$

$$< \mathrm{Fr}x, \mathrm{Fr}y >= p^3 \mathrm{Fr} < x, y > . \tag{5}$$

Where we denote by ∇_a the covariant derivative on the bundle, corresponding to the logarithmic derivative $\delta_a = z^a \frac{d}{dz^a}$ in the neighborhood of $z = 0$. We consider x and y as sections of the bundle of cohomology; hence $< x, y >$ is a function on the neighborhood of $z = 0$. The action of Fr on a function $f(z)$ transforms it into $f(z^p)$.

Notice, that Frobenius map is not linear in standard sense: if x is a section of cohomology bundle and f is function on the base then

$$\mathrm{Fr}(fx) = \mathrm{Fr}(f)\mathrm{Fr}(x).$$

Frobenius map is compatible with monodromy weight filtration:

$$\mathrm{Fr} W^a \subset W^a.$$

The map Fr can be extended to the vector spaces $H_{\mathbb{Q}_p}(X_z)$. Fixing bases $e_a(z)$ in these vector spaces we can talk about corresponding matrices $m_a^b(z)$. If for $z \in \mathbb{Z}_p$ the basis $e_a(z)$ can be considered as a set of generators of $H_{\mathbb{Z}_p}(X_z)$ then the matrix has integral entries (entries belonging to $\mathbb{Z}_p$). Moreover, these entries have some divisibility properties; this follows from the fact that applying the Frobenius map to Ω (to the cohomology class corresponding to the Calabi-Yau 3-form) we obtain a cohomology class divisible by p^3. If F^k denotes the Hodge filtration one can prove that applying the Frobenius map to a cohomology class belonging to F^k we obtain a class divisible by p^k. To understand these statements one should notice that the Hodge filtration is related to the number of differentials dz and $d(z^p) = pz^{p-1}dz$ is divisible by p. However, this remark is a far cry from a proof; this is clear from the fact that in general the divisibility properties are valid only for $p > 3$. (See [17], Sec 5 , for the proof of divisibility.) Notice, however, that in the case $k = 1$ the condition $p > 3$ is not necessary for divisibility by p. [9] (We will use this fact in Sec 4.)

[8]More precisely one should consider these bases in a formal neighborhood of $z = 0$. This means that we consider them as formal power series with respect to $z^a, \log z^a$.

[9]The condition $p > 3$ arises because we are working with threefolds. On n-dimensional manifolds one should require $p > n$.

If the elements of the basis are flat sections of the bundle of cohomology it follows immediately from (4) that the entries of the matrix $m_a^b(z)$ are constant (do not depend on z).

Notice, that the Frobenius map depends on the choice of the coordinate system on the moduli space. If the coordinate system behaves like z at the point $z=0$ (more precisely, the Jacobian matrix of the transformation of $z^1,...,z^r$ to the new coordinates is non-degenerate at $z=0$) one can say that the Frobenius map is regular at $z=0$ (the matrix entries are holomorphic at this point). In particular, this is true in canonical coordinates.

Let us consider the simplest situation when the three-dimensional Betti number is equal to four; then the moduli space of complex structures on Calabi-Yau manifold is one-dimensional ($r=1$). We will regard z as a coordinate on a punctured disk belonging to the moduli space; the logarithmic derivative with respect to z will be denoted by δ and the corresponding Gauss-Manin covariant derivative by ∇. In the situation at hand cohomology classes $\Omega_i=\nabla^i\Omega$ for i=0,1,2,3 constitute a basis in three-dimensional cohomology; hence

$$\nabla^4\Omega=c_3\nabla^3\Omega+c_2\nabla^2\Omega+c_1\nabla\Omega+c_0\Omega. \tag{6}$$

It follows from this formula that for any flat section g the scalar product $<g,\Omega>$ (period) obeys Picard-Fuchs equation

$$\delta^4<g,\Omega>=c_3\delta^3<g,\Omega>+c_2\delta^2<g,\Omega>+c_1\delta<g,\Omega>+c_0<g,\Omega>. \tag{7}$$

If all coefficients $c_i(z)$ vanish at the point $z=0$ then this point is a maximally unipotent boundary point.

Notice that Picard-Fuchs equation has the same form over any field of characteristic zero.

It is easy to relate the basis g^0,g^1,g_1,g_0 to the basis e^0,e^1,e_1,e_0 and to the basis Ω_i, $i=0,1,2,3$. The entries of the matrix of scalar products of g^0,g^1,g_1,g_0 with Ω_b (period matrix) are obtained from periods by means of differentiation: if g is a flat section then

$$<g,\Omega_b>=\delta^b<g,\Omega>$$

.

We have seen that

$$e^0=g^0,e^1=g^1+tg^0,e_1=g_1+f_0''g^1-(f_0'-tf_0'')g^0,e_0=g_0+tg_1+f_0'g^1-(2f_0-tf_0')g^0. \tag{8}$$

Where $'$ denotes the derivative with respect to t (=logarithmic derivative with respect to the canonical coordinate $q=e^t$).

Conversely,

$$g^0=e^0,g^1=e^1-te^0,g_1=e_1-f''e^1+f'e^0,g_0=e_0-te_1-(f'-tf'')e^1+(2f-tf')e^0. \tag{9}$$

The Gauss-Manin connection has the form

$$\begin{array}{ll}\nabla e_0 = e_1, & \nabla e_1 = Y e^1 \\ \nabla e^1 = e^0, & \nabla e^0 = 0,\end{array} \tag{10}$$

where $Y = \delta^3 f_0$ (Yukawa coupling) is the third derivative of the free energy f_0 with respect to $t = \log q$ (third logarithmic derivative with respect to q) and ∇ is covariant derivative corresponding to the logarithmic derivative δ.

We will consider more general situation when the Gauss-Manin connection in symplectic basis has the form

$$\begin{array}{ll}\nabla e_0 = Y_3 e_1, & \nabla e_1 = Y_2 e^1 \\ \nabla e^1 = Y_1 e^0, & \nabla e^0 = 0.\end{array} \tag{11}$$

It is easy to check that $Y_1 = Y_3$. (This follows from the assumption that the basis at hand is symplectic.)

Let us denote by $m_{a,b}$ the entries of the matrix of Frobenius map in the basis e^0, e^1, e_1, e_0.

$$\mathrm{Fr}e^0 = m_{1,1}e^0, \mathrm{Fr}e^1 = m_{2,2}e^1 + m_{1,2}e^0, \mathrm{Fr}e_1 = m_{3,3}e_1 + m_{2,3}e^1 + m_{1,3}e^0, \mathrm{Fr}e_0 = m_{4,4}e_0 + m_{3,4}e_1 + m_{2,4}e^1 + m_{1,4}e^0. \tag{12}$$

(The matrix is triangular, because the Frobenius map is compatible with the monodromy weight filtration.)

From the equation (4) we derive the equations for the matrix elements of Frobenius operator:

$$\delta m_{i,i} = 0 \tag{13}$$

$$\delta m_{i,i+1} = p m_{i,i} \mathrm{Fr} Y_i - m_{i+1,i+1} Y_i \tag{14}$$

$$\delta m_{i,i+2} = p m_{i,i+1} \mathrm{Fr} Y_{i+1} - m_{i+1,i+2} Y_i \tag{15}$$

$$\delta m_{1,4} = p m_{1,3} \mathrm{Fr} Y_3 - m_{2,4} Y_1. \tag{16}$$

It follows from the first equation that the diagonal entries are constant. The matrix entries are holomorphic at $z = 0$, hence the LHS of the above equations vanishes. We obtain that

$$p m_{i,i} = m_{i+1,i+1},$$

hence

$$m_{i,i} = p^{i-1} m_{1,1}.$$

Using (5) we conclude that

$$m_{1,1} = \pm 1.$$

In what follows we will assume that $m_{1,1} = 1$; the modifications necessary in the case when the entry $= -1$ are obvious. Using vanishing of LHS and (5) we can express all matrix entries of Frobenius map at $z = 0$ in terms of $\alpha = p^{-1} m_{1,2}(0)$ and $\beta = p^{-3} m_{1,4}(0)$. In particular, $m_{2,3}(0) = p^2 c_0 \alpha$ where $c_0 = Y_2(0)$,$m_{3,4} = p^3 \alpha$, $m_{1,3}(0) = \frac{1}{2} p^2 c_0 \alpha^2$, $m_{2,4}(0) = \frac{1}{2} p^3 c_0 \alpha^2$. Solving differential equations we can find the dependence of matrix entries of z.

In canonical coordinates (i.e. when $Y_1 = Y_3 = 1$) the solutions have the following form:

$$m_{1,2} = \alpha,$$

$$m_{2,3} = (\mathrm{Fr} f_0'' p - f_0'' p^2) + p^2 c_0 \alpha$$

$$m_{1,3} = (\mathrm{Fr} f_0' - p^2 f_0') + \frac{1}{2} p^2 c_0 \alpha^2 + \mathrm{Fr} \tilde{f}_0'' p \alpha$$

$$m_{3,4} = p^3 \alpha$$

$$m_{2,4} = -(\mathrm{Fr} f_0' p - p^3 f_0') + \frac{1}{2} p^3 c_0 \alpha^2 - p^3 \alpha \tilde{f}_0''$$

$$m_{1,4} = 2(\mathrm{Fr} f_0 - p^3 f_0) + p^3 \beta + p^2 \alpha \mathrm{Fr} \tilde{f}_0' + p^3 \alpha \tilde{f}_0'.$$

Here $\tilde{f}_0$ stands for the holomorphic (with respect to q) part of free energy. Recall, that the third derivative of f_0 is Yukawa coupling $Y = \sum_{k \geq 0} c_k q^k$, hence we can take

$$f_0 = c_0 \frac{t^3}{3!} + \tilde{f}_0$$

where

$$\tilde{f}_0 = \sum_{k>0} \frac{c_k}{k^3} q^k.$$

There exists a more direct way to obtain these formulas using the basis consisting of flat connections.

We denote by $\mu_{a,b}$ the entries of the matrix of Frobenius map in the basis g^0, g^1, g_1, g_0.

$$\mathrm{Fr} g^0 = \mu_{1,1} g^0, \mathrm{Fr} g^1 = \mu_{2,2} g^1 + \mu_{1,2} g^0, \mathrm{Fr} g_1 = \mu_{3,3} g_1 + \mu_{2,3} g^1 + \mu_{1,3} g^0, \mathrm{Fr} g_0 = \mu_{4,4} g_0 + \mu_{3,4} g_1 + \mu_{2,4} g^1 + \mu_{1,4} g^0. \tag{17}$$

This matrix is triangular because Fr is compatible with monodromy weight filtration; it is a constant matrix because the basis consists of flat sections.

Using the relations (8) and (9) we can express the matrix $m_{a,b}$ in canonical coordinates in terms of the constant matrix $\mu_{a,b}$.

We obtain

$$m_{i,i} = \mu_{i,i}$$

$$m_{1,2} = \mu_{1,2},$$

$$m_{2,3} = (\mathrm{Fr} f_0'' p - f_0'' p^2) + \mu_{2,3}$$

$$m_{1,3} = (\mathrm{Fr} f_0' - p^2 f_0') + p^2 \mu_{2,3} t + \mu_{1,3} + \mathrm{Fr} f_0'' p \mu_{12}$$

$$m_{3,4} = \mu_{34}$$

$$m_{2,4} = -(\mathrm{Fr} f_0' p - p^3 f_0') + \mu_{2,4} + t \mu_{2,3} - \mu_{3,4} f_0''$$

$$m_{1,4} = 2(\mathrm{Fr} f_0 - p^3 f_0) + \mu_{1,4} + t \mu_{2,4} + t \mu_{1,3} p + t^2 \mu_{2,3} + \mathrm{Fr} f_0' \mu_{1,2} + f_0' \mu_{3,4}$$

It follows from these formulas that

$$\mu_{i,j} = m_{i,j}(0).$$

Notice, that in the derivation we used the assumption that f_0 contains only cubic term with respect to t.

As we mentioned the terms in $m_{a,b}$ that contain $t = \log q$ should vanish. This means that we can replace everywhere f_0 with its holomorphic (with respect to q) part $\tilde{f}_0$ and omit all terms with explicit dependence of t . This leads to the formulas we obtained solving differential equations. (In some expressions containing f_0 nonholomorphic terms cancel , hence there is no necessity to replace f_0 with its holomorphic part.)

The calculation of $m_{i,j}(0)$ (of the behavior of entries of Frobenius matrix at the maximally unipotent boundary point) is much more difficult. It is based on relation between flat sections g^0, g^1 that *we assume to be topologically rational* with algebraically rational sections e^0, e^1. If we are working in a coordinate system z compatible with integral structure of family of varieties at hand one can prove that $m_{1,2}(0) = 0$, hence $m_{2,3}(0) = 0$, $m_{3,4}(0) = 0$. (For the case of quintic this can be proved by means of direct calculation [18].) If coordinate u is expressed in terms of z by a series with rational coefficients and the series starts with cz (i.e. u is compatible with the structure of the family over $\mathbb{Q}$ and in the neighborhood of the boundary point u behaves like cz) then the matrix element $m_{1,2}(0)$ of the Frobenius operator calculated in the coordinate u is equal to $Y_3(0)\log(c^{p-1})$ where the logarithm is understood in p-adic sense.

The proof is based on the theory of motives. (The theory of motives can be regarded as a kind of "universal cohomology theory".) We will skip the proof referring to [21], but we will mention some salient steps. The main point is a construction of a motive over $\mathbb{Q}$ that has realizations as complex Hodge structure and "p-adic Hodge structure" (Fontaine-Laffaille structure). (In both cases we have in mind the limiting Hodge structures, i.e. Hodge structures over the boundary point.) The p-adic Hodge structure has Frobenius map as one of ingredients, hence the motive can be used to relate the matrix of Frobenius to the period matrix over complex numbers.

The general statement looks as follows.

Let us consider a symplectic rational basis e^0, e^1, e_1, e_0 where e_0 is a Calabi-Yau form and the Gauss- Manin connection is given by the formula (11) where covariant derivatives correspond to logarithmic derivatives with respect to the coordinate u. Let us assume that $Y_3(0) = 1$. Then at $u = 0$ the superdiagonal entry of the matrix of periods has the form $\frac{1}{2\pi i}\log c$ where c is a product of a rational number and a root of unity. [10]The entry $m_{1,2}(0)$

[10]This means that for topologically rational logarithmic section g^1 the period $< g^1, e_0 >$ behaves like $\frac{1}{2\pi i}(\log u + \log c)$ as $u \to 0$. (We normalize g^1 imposing the condition $< g^1, e_1 >= 1$.) Notice, that g^1 is defined up to a summand αg^0 where $g^0 = e^0$ is a topologically (and algebraically) integral holomorphic section and α is a rational number. Hence c is defined only up to a factor $\exp(2\pi i\alpha)$. This factor is a root of unity, p-adic logarithm of it is equal to zero, hence it does not contribute to the expression for $m_{1,2}(0)$.

of Frobenius matrix in the coordinate u can be expressed in terms of c as $\pm\log c^{1-p}$ where c should be understood as a p-adic number.

In canonical coordinates $c=1$, hence $m_{1,2}(0)=0$. We obtain the same answer for this entry in any integral coordinate system, because the answer depends only on behavior of coordinates at the boundary point.

Notice, that as long we are interested in the behavior of Frobenius operator only at boundary point it is sufficient to calculate this behavior in one coordinate system, because there exists a general formula relating the matrices of Frobenius operators in different coordinates (see [21]).

Applying the above consideration to the matrix of Frobenius operator in canonical coordinate q we obtain that, in particular,

$$m_{1,4}=2(\mathrm{Fr} f_0-p^3 f_0)+p^3\beta. \tag{18}$$

The proof of (18) was given in the case when the dimension r of the moduli space of complex structures is equal to 1. If $r>1$ we can calculate the matrix of Frobenius map in symplectic basis e^0, e^a, e_a, e_0 of the space of sections of the cohomology bundle if the Gauss-Manin connection in this basis has the form

$$\begin{array}{ll}\nabla_a e_0=(Y_3)_a^b e_b, & \nabla_a e_b=(Y_2)_{abc}e^c \\ \nabla_a e^b=(Y_1)_a^b e^0, & \nabla_a e^0=0.\end{array} \tag{19}$$

where $Y_1=Y_3$.

Canonical coordinates are defined by the condition $(Y_1)_a^b=(Y_3)_a^b=\delta_a^b$.

Basically the same arguments as for $r=1$ can be used to prove (18) in canonical coordinates in the basis e^0, e^a, e_a, e_0.

Notice that $(Y_1)_a^b=(Y_3)_a^b$ coincides with y_a^b introduced in Sec 4 where we prove p-integrality of the mirror map under the condition that at the boundary point the reduction of this matrix mod p is nondegenerate. It is important to emphasize that this condition follows from our assumptions about normalized Yukawa couplings reduced mod p. To check this we consider non-normalized Yukawa couplings defined by the formula $< e_0, \nabla_a\nabla_b\nabla_c e_0 >$ where the covariant derivatives are taken in the original coordinate system (as in (19)). At the boundary point canonical coordinates coincide with the original ones, hence the difference between normalized and non-normalized Yukawa couplings disappears. From the other side using (19) we can express non-normalized Yukawa couplings in terms of Y_i; it follows immediately from this expression that for degenerate $y_a^b(0)$ mod p our assumptions about Yukawa couplings are violated.

In the next section we will use (18) to express the instanton numbers in terms of Frobenius map.

3 Instanton numbers

In the case when there is one Kaehler parameter (in terms of A-model) or the moduli space of complex structures on Calabi-Yau manifold is one-dimensional (in mirror B-model) the instanton numbers n_k are related to the genus zero free energy f_0 in the following way:

$$\tilde{f}_0(q) = \sum_{k=1} \sum_{d=1} d^{-3} n_k q^{dk} \tag{20}$$

where $\tilde{f}_0$ stands for the holomorphic part of free energy and q denotes the canonical coordinate.

Equivalently one can relate the instanton numbers to the (normalized) Yukawa coupling:

$$Y(q) = const + \sum_{k=1} n_k k^3 \frac{q^k}{1-q^k} = const + \sum_{k=1} \sum_{d=1} n_k k^3 q^{dk} \tag{21}$$

The following lemma permits us to use p-adic methods in the analysis of integrality of instanton numbers.

Lemma 1 The numbers n_k defined in terms of $Y(q)$ by the formula (21) are integers if and only if for every prime p there exists such a series $\psi(q) = \sum s_k q^k$ having p-adic integer coefficients that

$$\mathrm{Fr} Y - Y = \delta^3 \psi$$

(or, in other words, $Y(q^p) - Y(q) = \delta^3 \psi(q)$.

We will give a proof of this lemma based on the notion of Dirichlet product of arithmetic functions.(Another proof that does not use Dirichlet product is given in [8].) Recall, that an arithmetic function is a function defined on the set $\mathbb{N}$ of natural numbers and taking values in complex numbers (or in any other commutative ring). The Dirichlet product of two arithmetic functions is defined by the formula

$$(f * g)(s) = \sum_{dk=s} f(d) g(k) = \sum_{d|s} f(d) g(\frac{s}{d}) \tag{22}$$

Where d, k, s are natural numbers, $d|s$ means that d divides s.

Arithmetic functions form a commutative, associative unital ring with respect to Dirichlet product.

The relations (20) and (21) can be written as

$$(\tilde{f}_0) = \{\frac{1}{k^3}\} * \{n_k\},$$

$$(Y) = \{1\} * \{n_k k^3\}.$$

Here we use the notation $\{c_k\}$ for the arithmetic function, corresponding to the sequence $c_1, ..., c_k, ...$ and the notation (f) for the arithmetic function, corresponding to the sequence of coefficients of power expansion of f. (The equivalence of (20) and (21) follows from the remark that $\delta((f)*(g)) = \delta(f)*\delta(g)$.)

Using the new form of (21) we can express the instanton numbers in terms of Yukawa coupling:

$$n_k k^3 = \mu * (Y). \tag{23}$$

Where $\mu(k)$ stands for the Moebius function that is equal to $(-1)^s$ if k is represented as a product of s distinct primes and vanishes if k is not square-free. (This expression follows from the remark that $\mu * 1 = (q)$ is the unit of Dirichlet ring.)

The proof of lemma 1 can be based on the following statement

Lemma 2. If $g = \mu * (h)$ and $m = (\mathrm{Fr}h - h)$ then

$$g(p^a t) = -\sum_{d|t} \mu(d) m(\frac{p^a t}{d}).$$

Here t is a natural number that is not divisible by p.

Let us denote the set of such numbers by $\mathbb{N}_p$. It easy to check that the definition of Dirichlet product makes sense for functions defined on $\mathbb{N}_p$ and that Lemma 2 can be formulated in terms of this product:

$$g^{(a)} = \mu * (h - \mathrm{Fr}h)^{(a)}$$

where $g^{(a)}$ is a function on $\mathbb{N}_p$ defined by the formula $g^{(a)}(r) = g(p^a r)$.

The proof of Lemma 2 follows immediately from the properties of the Moebius function: we use that $\mu(pt) = -\mu(t), \mu(p^a t) = 0$ for $a > 1$.

Lemma 2 combined with (23) gives an expression of instanton numbers (or, more precisely, of their p-adic reduction) in terms of $\mathrm{Fr}Y - Y$. If $\mathrm{Fr}Y - Y = \delta^3\psi$ where ψ is a series with p-adic integral coefficients we obtain p-adic integrality of instanton numbers. To prove the remaining statement of Lemma 1 we notice that it follows from (21) that

$$\mathrm{Fr}Y - Y = \sum_{k=1}\sum_{d=1} n_k k^3 q^{dpk} - \sum_{k=1}\sum_{d=1} n_k k^3 q^{dk}.$$

In the RHS the first summand cancels the terms of the second summand that correspond to d divisible by p. We obtain that the coefficient in front of q^t is divisible by t^3 in p-adic sense. This proves the statement of Lemma 1.

One can reformulate Lemma 2 in the following way

Lemma 3. If $v = \{\frac{1}{k^s}\} * g$ and $m = (p^{-s}\mathrm{Fr}v - v)$, then

$$g(p^a t) = -\sum_{d|t} \frac{1}{d^s} \mu(d) m(\frac{p^a t}{d}).$$

Here t is a natural number that is not divisible by p.

Lemma 3 follows from Lemma 2 applied to $\delta^s v = \{1\} * \delta^s g$ and $\delta^s m = (\mathrm{Fr}\delta^s v - \delta^s v)$.

Combining (23) and Lemma 3 with the formula (18) of Sec 2 one can calculate the instanton numbers in terms of the matrix of Frobenius map in the basis (e^0, e^1, e_1, e_0). Corresponding expression will be written later in more general situation (see (25)).

Similar statements can be proved in the case of multidimensional moduli spaces. In this case instanton numbers specify a multidimensional arithmetic function $\{n_{\mathbf{k}}\}$ where $\mathbf{k} = (k^1, ..., k^r)$ is a multiindex having natural numbers as components; the same is true for the coefficients of power series for $\tilde{f}_0$(holomorphic part of free energy). They are related by the formula

$$\tilde{f}_0(\mathbf{q}) = \sum_{d|\mathbf{k}} \frac{\mathbf{q}^{\mathbf{k}}}{d^3} n_{\frac{\mathbf{k}}{d}}$$

or, equivalently,

$$(\tilde{f}_0) = \{\frac{1}{d^3}\} * \{n_{\mathbf{k}}\}. \tag{24}$$

We use here the fact that the multidimensional arithmetic functions constitute a module over Dirichlet ring: if g is an element of this ring and h is a multidimensional arithmetic function we define

$$g * h = \sum_{d|\mathbf{k}} g(d) h(\frac{\mathbf{k}}{d}).$$

Lemma 3 can be generalized immediately to the case of multidimensional arithmetic functions v, g, m. We obtain

Lemma 3′. If $v = \{\frac{1}{k^s}\} * g$ and $m = (p^{-s}\mathrm{Fr}v - v)$, then

$$g(p^a \mathbf{t}) = -\sum_{d|\mathbf{t}} \frac{1}{d^s} \mu(d) m(\frac{p^a \mathbf{t}}{d}).$$

Here $\mathbf{t}$ is a multiindex that is not divisible by p.

Using Lemma 3′ we obtain the expression of instanton numbers in terms of the coefficients $M_{\mathbf{k}}$ of the power series for the entry $m_{1,4}$ of the matrix of Frobenius map in the basis e^0, e^a, e_a, e_0. Namely,

$$n_{p^a \mathbf{k}} = -\frac{1}{2} \sum_{d|\mathbf{k}} \frac{1}{p^3 d^3} \mu(d) M_{\frac{p^a \mathbf{k}}{d}} \tag{25}$$

where $\mathbf{k}$ is a vector that is not divisible by p.

To check the p-integrality of instanton numbers it is sufficient to prove that the coefficients of matrix entry $m_{1,4}$ are integers divisible by p^3. If $p > 3$ the integrality of $p^{-3} m_{1,4}$ would follow from the p-adic integrality of the basis e^0, e^a, e_a, e_0(more precisely, we should prove that the basis consists of sections that are integral in algebraic sense and these sections constitute a system of generators over $\mathbb{Z}_{|}$).

We have assumed that Ω is an integral cohomology class; to check that e_0 is also an integral class we should verify that the holomorphic period (holomorphic solution to the Picard-Fuchs equation) $< g^0, \Omega >$ is a series with respect to q having integer coefficients and that the zeroth order term in this series is equal to ± 1. We mentioned already that this statement can be verified by means of direct calculation in the case when the Calabi-Yau manifold is realized as a complete intersection in toric variety and that in general case it follows from Faltings' results [7].

Integrality of vectors $e_a = \nabla_a e_0$ follows from the integrality of the mirror map ([15],[9],[10],[21] and the next section), or, equivalently, from the fact that the canonical coordinates q^a are integral. (It follows from this fact that covariant derivatives with respect to these coordinates preserve integrality.) The same arguments prove that the basis e^0, e^a, e_a, e_0 becomes integral after reduction to p-adic numbers if the expression of e^a in terms of $\nabla_b e_c$ is invertible mod p at $z = 0$. [11] If $r = 1$ this means that p does not divide $Y(0)$; then $Y(0)$ is an invertible element of of $\mathbb{Z}_p$. (See Sec 2 for more detailed discussion of our assumptions about Yukawa couplings $C_{abc}(0) = Y_{abc}(0)$ in the case $r > 1$.) Notice, that the Yukawa couplings at the boundary point correspond to the structure constants of classical cohomology ring on A-model side and can be easily calculated; for quintic $Y(0) = 5$. [12]

4 Integrality of mirror map

Let us consider the Hodge filtration $F^3 \subset F^2 \subset F^1 \subset F^0$ on the cohomology with complex coefficients. Recall that F^p is a direct sum of all groups $H^{k,l}$ where $k \geq p$. We restrict ourselves to the middle-dimensional cohomology H^3, hence $\dim F^3 = h^{3,0} = 1, \dim F^2/F^3 = h^{2,1} = r, \dim F^1/F^2 = h^{1,2} = r, \dim F^0/F^1 = h^{0,3} = 1$. Recall that on the sections of the vector bundle of middle-dimensional cohomology in a neighborhood of the maximally unipotent boundary point we have also the monodromy weight filtration $W^0 \subset W^1 \subset W^2 \subset W^3$ with $\dim W^0 = 1, \dim W^1/\dim W^0 = \dim W^2/W^1 = r, \dim W^3/W^2 = 1$. One can represent the space of sections as a direct sum of F^k and W^{k-1}; hence this space is a direct sum of $W^k \bigcap F^k$ where $k = 0, 1, 2, 3$; in particular, W^1 is a direct sum of W^0 and $W^1 \bigcap F^1$. The quotient $Gr^k_W = W^k/W^{k-1}$ is isomorphic to $W^k \bigcap F^k$.The Gauss-Manin connection descends to this quotient and has trivial monodromy there. We can take a symplectic basis e^0, e^a, e_a, e_0 of the space of sections of the cohomology bundle in such a way that the Gauss-Manin connection in this symplectic basis has the form (19). (To con-

[11] One can formulate this condition in more transparent, but less explicit way: the Calabi-Yau form and its covariant derivatives with respect to coordinate system compatible with integral structure should generate the de Rham cohomology over $\mathbb{Z}_p$.

[12] For $p > 3$ one can prove the p-adic integrality of the basis e^0, e^a, e_a, e_0 without this additional assumption; the proof is based on theorem by Faltings [7] stating that inner product is a perfect pairing on the cohomology group $H_{\mathbb{Z}_p}$. (We have mentioned this theorem in relation to integrality of holomorphic solution to Picard-Fuchs equation.) However, one can conjecture that our assumptions follow from smoothness of reduction mod p required in [7].

struct this basis one can take flat sections of $Gr^k_W = W^k/W^{k-1}$ and lift them to $W^k \bigcap F^k$. The formula for Gauss-Manin connection follows from Griffiths transversality.) All these statements are proven, for example, in [16],[[6] and [5].

It is a non-trivial fact, that under certain conditions these statements remain correct also for de Rham cohomology with integer coefficients. (This means, in particular, that the entries of the matrix (19) are series with integral coefficients.) This fact follows from the results of [21]. The proof uses reduction to p-adic numbers; it gives p-integrality for $p > 3$ (as always we assume that the family remains smooth after reduction mod p).

To prove the integrality of mirror map we need only a part of these statements. We will formulate the necessary facts and sketch the proof. The most important step is the expression of mirror map in terms of the matrix of Frobenius transformation.

To analyze the mirror map we take a basis of W_1 consisting of algebraically integral sections e^0, e^a that can be extended to the boundary point $z = 0$ and obey $e^0 \in W_0, e^a \in F^1 \bigcap W_1$,

$$\nabla_a e^0 = 0, \nabla_a e^b = y^b_a(z) e^0. \tag{26}$$

Here the functions $y^b_a(z)$ are holomorphic at $z = 0$. To construct such a basis we notice that both W_0 and W_1/W_0 are trivial bundles; more precisely, they can be characterized as trivial Hodge structures $\mathbb{Z}$ and $\mathbb{Z}(1)^r$. We define e^0 as an algebraically (and topologically) integral generator of W_0 and obtain e^a by lifting an algebraically integral basis of flat sections of W_1/W_0 to $F^1 \bigcap W_1$.

We can consider e^0, e^a also as sections of $\mathbb{Q}_p$ cohomology bundle; they constitute a basis in the corresponding version of W_1. (These vectors can be considered also as sections of $\mathbb{Z}_p$- cohomology, but it is not clear whether they generate W_1 in this cohomology.)

The matrix of Frobenius operator in the basis e^0, e^a has the form

$$\mathrm{Fr} e^0 = m_{1,1} e^0, \mathrm{Fr} e^a = (m_{2,2})^a_b e^b + m^a_{1,2} e^0. \tag{27}$$

The entries of this matrix obey equations coming from (19)

$$\delta_a m_{1,1} = 0 \tag{28}$$

$$\delta_a (m_{2,2})^b_c = 0 \tag{29}$$

$$\delta_a m^b_{1,2} = p m_{1,1} \mathrm{Fr} y^b_a - (m_{2,2})^c_a y^b_c \tag{30}$$

It follows from these equations that the diagonal entries are constant. One can prove that $m_{1,1} = \pm 1$, $(m_{2,2})^a_b = \pm p \delta^a_b$. This follows immediately from the theory of motives (from the fact that there are motives with realizations W_0 and W_1/W_0). However, it is possible to give a more elementary proof of this fact. We see that the last equation takes the form

$$\delta_a m^b_{1,2} = p \mathrm{Fr} y^b_a - p y^b_a \tag{31}$$

(We have assumed that $m_{1,1} = 1$.)

Applying (31) we can express the mirror map in terms of the entry $m^b_{1,2}$ of the matrix of Frobenius transformation. We are using in this expression the Artin-Hasse exponential function

$$E_p(x) = \exp(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + ...).$$

The expansion of this function with respect to x has integer p-adic coefficients: $E_p(x) \in \mathbb{Z}_p[[x]]$. It is easy to check that

$$\delta \log E_p(x^{pk}) - \delta \log E_p(x^k) = -kx^k = -\delta x^k.$$

Using this identity we obtain the following

Lemma 3

Let us suppose that $r^i(z) \in \mathbb{Q}_p[[z_1, ..., z_r]]$. Then for the infinite product

$$Q^i(z) = z_i \Pi_k E_p(z_i^k)^{-r^i_k},$$

where r^i_k are coefficients of the power expansion of $r^i(z)$ we have

$$\mathrm{Fr}\nu^i_j - \nu^i_j = \delta_j r^i$$

where $\nu^i_j = \delta_j \log Q^i$. If $r^i(z) \in \mathbb{Z}_p[[z_1, ..., z_r]]$ then $Q^i(z) \in \mathbb{Z}_p[[z_1, ..., z_r]]$.

Here δ_j stands for the logarithmic derivative with respect to z_j and the Frobenius map Fr transforms $f(z_1, ..., z_r)$ into $f(z_1^p, ..., z_r^p)$. We consider k as multiindex.

Let us notice now that the mirror map $q(z)$ obeys:

$$\delta_a \log q^b(z) = y^b_a(z).$$

(This follows immediately from the fact that in canonical coordinates q we can write an analog of the formula (33) with y^a_b replaced by δ^a_b.) Taking into account that at $z = 0$ the canonical coordinate q^a should behave like z_a we obtain from the Lemma applied to $r^a(z) = \frac{1}{p} m^a_{1,2}$ an expression for the mirror map. To prove p-adic integrality of mirror map it is sufficient to check that e^0, e^a generate W_1 in the cohomology with coefficients in $\mathbb{Z}_p$. (It is well known that the Frobenius map on F^1 is divisible by p .) This can be easily checked if the matrix $y^b_a(0)$ is nondegenerate after reduction mod p.

Notice that at least for $p > 3$ the nondegeneracy of the matrix $y^b_a(0)$ reduced mod p can be derived from the conditions imposed on the classical limit of Yukawa couplings in Sec 2. (See the discussion at the end of Sec 2.)

The condition of integrality of mirror map can be expressed also in terms of topologically integral sections.

The space W^1 is generated by topologically integral sections $g^0 \in W^0, g^a \in W^1, a = 1, ..., r$. Corresponding solutions to the Picard-Fuchs equations have the form $f^0(z)$ and $f^a(z) = \frac{1}{2\pi i} m^a_b f^0(z) \log z^b + r^a(z)$ where f^0 and r^a are holomorphic in the neighborhood of

$z = 0$ and m^a_b are integers. (This form of f^a follows from the remark that the monodromy preserves topological integrality.) We will assume that $\det m^a_b = \pm 1$; this equation was called small monodromy condition in [21] and Morrison integrality conjecture in [5]. If it is satisfied then without loss of generality one can assume that $m^a_b = \delta^a_b$: one should work in another integral coordinate system. Assuming that $m^a_b = \delta^a_b$ we define the mirror map by the formula

$$\log q^a = 2\pi i \frac{< g^a, \Omega >}{< g^0, \Omega >} \tag{32}$$

If the constant term of the series for q^a is integral one can prove p-adic integrality of all coefficients of the series for such p that our family remains smooth after reduction mod p [21]. (It is sufficient to assume that the constant term is rational; than one can check that it is an invertible integer.)

5 Counting holomorphic disks.

If we consider an A-model on a Calabi-Yau threefold we can try to calculate the number of holomorphic disks with boundary on a Lagrangian submanifold. This problem is ambiguous from mathematical viewpoint (one needs additional information called framing) , but the corresponding mirror problem is well defined [1], [2].

Let us consider a compact Calabi-Yau complex threefold X with a one-dimensional (not necessarily connected) complex submanifold Y. We assume that X is a mirror of some threefold $\tilde{X}$ and Y is a mirror of a Lagrangian submanifold of $\tilde{X}$. Then the number of holomorphic disks can be expressed in terms of potential $\mathcal{W}$ defined as an integral of the Calabi-Yau 3-form Ω over a relative 3-cycle with boundary in Y. More precisely, one should consider the moduli space $\mathcal{M}$ of deformations of the pair (X, Y); we assume that the moduli space is defined over $\mathbb{Z}$ and the coordinates on it agree with this structure. The relative cohomology $H^3(X, Y)$ specifies a vector bundle over $\mathcal{M}$; this bundle is equipped with a flat connection ∇ (Gauss-Manin connection). Relative homology $H_3(X, Y)$ is dual to the relative cohomology; it is also equipped with Gauss-Manin connection. On can express the number of holomorphic disks in terms of Gauss-Manin connection on the relative homology (see [11], [12], [13]), [14]) ; we will use this expression.

Let us assume that we are working in the neighborhood of the boundary point of $\mathcal{M}$ with maximally unipotent monodromy. This means that in appropriate coordinate system $(z^1, ..., z^r)$ there exist a unique (up to a factor) holomorphic at the boundary point $z = 0$ flat section g^0 of the bundle of relative homology, r linearly independent flat sections g^a of this bundle with logarithmic behavior at the boundary point (here $r = \dim \mathcal{M}$),etc. We consider a basis $(g^0, g^a, g^\alpha, ...)$ of algebraically integral flat sections of homology bundle classified according to the power of logarithm at the boundary point (= agreeing with monodromy weight filtration). The omitted elements of the basis (if they are necessary) behave as $\log^3$ as $q \to 0$; they do not play any role in the calculations below.

We specify the cohomology class $e_0 = \frac{\Omega}{<g^0,\Omega>}$ normalizing the form Ω; as in Sec 2 this class is algebraically integral. Now we can define canonical coordinates t^a on $\mathcal{M}$ by the formula $t^a =< g^a, e_0 >$. The canonical coordinates depend on the choice of flat sections g^a; we can fix this choice requiring that in the neighborhood of the boundary point the coordinates $q^a = e^{t^a}$ are single-valued and the Jacobian matrix $\frac{\partial q^a}{\partial z^b}$ with respect to original coordinates is a unit matrix at this point. The coordinates q^a are also called canonical coordinates; the expression of them in terms of original coordinates z^a is called mirror map. We define the potential $\mathcal{W}^\alpha$ as $< g^\alpha, e_0 >$. The number of holomorphic disks can be expressed in terms of the coefficients n in the decomposition of holomorphic part of the potential

$$\tilde{\mathcal{W}}^\alpha = \sum_{d|\mathbf{k}} \frac{\mathbf{q}^{\mathbf{k}}}{d^2} n^\alpha_{\frac{\mathbf{k}}{d}} \tag{33}$$

Here $\mathbf{q} = (q^1, ..., q^r)$,$\mathbf{q}^{\mathbf{k}} = (q^1)^{k^1} \cdots (q^r)^{k^r}$. [13] As in Sec 3 we can express 33 in terms of Dirichlet product:

$$\tilde{\mathcal{W}}^\alpha = \{\frac{1}{k^2}\} * \{n^\alpha_k\}. \tag{34}$$

Under certain conditions we will prove that the coefficients n are integers. As in Sec 3 the proof will be based on the expression of these numbers in terms of Frobenius map in p-adic cohomology.

Let us consider a basis $(g_0, g_a, g_\alpha, ...)$ consisting of flat sections of the cohomology bundle that is dual to the basis $(g^0, g^a, g^\alpha, ...)$ in homology. It follows from the above definitions that

$$e_0 = g_0 + t^a g_a + \mathcal{W}^\alpha g_\alpha + ...$$

Let us define $e_a = \nabla_a e_0$ where ∇_a stands for the covariant derivative corresponding to $\delta_a = \frac{\partial}{\partial t^a} = q^a \frac{\partial}{\partial q^a}$. We obtain

$$e_a = g_a + \delta_a \mathcal{W}^\alpha g_\alpha + ...$$

Taking covariant derivative of this equation we see that

$$\nabla_a e_b = \delta_a \delta_b \mathcal{W}^\alpha g_\alpha + ...$$

Imposing some conditions [14] on the behavior of the matrix $\delta_a \delta_b \mathcal{W}^\alpha$ at the boundary point we can construct sections $e_\alpha = g_\alpha + ...$ as linear combinations with integral coefficients of the sections $\nabla_a e_b$. We obtain

$$\nabla_a e_b = \delta_a \delta_b \mathcal{W}^\alpha e_\alpha + ...$$

[13] One can choose the coordinates $q^1, ..., q^r$ in such a way that the first of them describe the variation of complex structure (closed string sector). Corresponding part of the potential is equal to the derivative of genus zero free energy and can be expressed in terms of instanton numbers. The remaining part of the potential is responsible for holomorphic disks.

[14] We can assume that the basis $(g^0, g^a, g^\alpha, ...)$ and, hence, the dual basis $(g_0, g_a, g_\alpha, ...)$ are integral in algebraic sense. Then we can require that the Calabi-Yau form together with its covariant derivatives with respect to integral coordinate system generates de Rham cohomology over $\mathbb{Z}$ or over $\mathbb{Z}_p$.

We will work in the basis $(e_0, e_a, e_\alpha, ...)$.

Notice, that in the case when the manifold X is compact instead of relative cohomology one can consider de Rham cohomology of $X\backslash Y$ with compact support and relative homology is isomorphic to the de Rham cohomology of $X \setminus Y$. This remark simplifies the analysis of the relation with p-adic picture.

We assumed that the moduli space $\mathcal{M}$ is defined over integers and that the sections of homology and cohomology bundles we consider are defined over $\mathbb{Q}$; this means that we can consider all these sections over $\mathbb{Q}_p$. The Frobenius map acts on the de Rham cohomology with coefficients in $\mathbb{Q}_p$; the relations (19) and (5) remain correct in the present situation. (In (5) we should consider the pairing between cohomology of $X \setminus Y$ and the cohomology with compact support.)

As earlier the matrix of the Frobenius map in the basis of flat sections is a constant matrix. The Frobenius map in the basis $(g_0, g_a, g_\alpha, ...)$ has the form

$$\mathrm{Fr} g_0 = \mu_0^0 g_0 + \mu_0^a g_a + \mu_0^\alpha g_\alpha + ...,$$

$$\mathrm{Fr} g_b = \mu_b^a g_a + \mu_b^\alpha g_\alpha + ...,$$

$$\mathrm{Fr} g_\beta = \mu_\beta^\alpha g_\alpha + ...,$$

(This matrix is triangular, because the dual basis $(g^0, g^a, g^\alpha, ...)$ agrees with monodromy weight filtration and Frobenius map preserves this filtration.)

Now we can express the matrix of Frobenius map in the basis $(e_0, e_a, e_\alpha, ...)$ in terms of constant matrix μ. We obtain that

$$\mathrm{Fr} e_0 = \mu_0^0 e_0 + \mu_0^b e_b + \mu_0^0 (p^{-2} \mathrm{Fr} \tilde{\mathcal{W}}^\alpha - \tilde{\mathcal{W}}^\alpha) e_\alpha - \mu_0^b \partial_b \tilde{\mathcal{W}}^\alpha e_\alpha + ...,$$

$$\mathrm{Fr} e_a = p^{-1} \mu_0^0 e_a + \mu_0^0 (p^{-2} \mathrm{Fr} \partial_a \mathcal{W}^\alpha - p^{-1} \partial_a \mathcal{W}^\alpha) e_\alpha + ...,$$

$$\mathrm{Fr} e_\alpha = p^{-2} \mu_0^0 e_\alpha + ...$$

Let us denote the matrix entries by $m_0^0 = \mu_0^0, m_0^a = \mu_0^a, m_0^\alpha, ...$ If $\mu_0^a = 0$ we see that

$$m_0^\alpha = m_0^0 (p^{-2} \mathrm{Fr} \tilde{\mathcal{W}}^\alpha - \tilde{\mathcal{W}}^\alpha) \tag{35}$$

Using Lemma $3'$ we can express the number of holomorphic disks in terms of this matrix element. We obtain

$$n^\alpha_{p^a \mathbf{k}} = -\frac{1}{m_0^0} \sum_{d|\mathbf{k}} \frac{1}{d^2} \mu(d) M^\alpha_{\frac{p^a \mathbf{k}}{d}} \tag{36}$$

where $\mathbf{k}$ is a multiindex that is not divisible by p and $M^\alpha_{\mathbf{t}}$ denotes a coefficient in the power expansion : $m_0^\alpha = \sum M^\alpha_{\mathbf{t}} q^{\mathbf{t}}$. To check that (36) gives a p-adic integer we prove first of all the p-integrality of mirror map using the methods of Sec 4. It follows that canonical coordinates are compatible with integral structure, hence in our assumptions the sections $(e_0, e_a, e_\alpha, ...)$ generate $\mathbb{Z}_p$-cohomology. This means that matrix entries of Frobenius are integral, moreover, taking into account that $e_0 \in F^3$ we see that $M_{\mathbf{t}}^{\alpha}$ is divisible by p^3 for $p > 3$. To prove the p-adic integrality of (36) we should check that m_0^0 divides p^3; this can be derived from (5) applied to the pairing between $\mathbb{Z}_p$-homology and $\mathbb{Z}_p$-cohomology. (One can prove also a more precise statement: $m_0^0 = \pm p^3$.)

It remains to justify the assumption that $\mu_0^a = 0$. The proof is completely analogous to the (motivic) proof of the relation $m_{1,2}(0) = 0$ (see Sec 2).

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