Free Vibration Analysis of Laminated Cylindrical Shells by Using Fourier Series Expansion Method

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ABSTRACT: Free vibration analysis of laminated cylindrical shells with arbitrary classical boundary conditions is presented in this article. Strain-displacement relations from Love's shell theory are employed in the analysis. Using Fourier series expansion and Stokes' transformation, frequency determinant of laminated cylindrical shells is derived. Comparisons between the results of the present method and previous studies are performed. Excellent agreement is obtained when enough terms of the infinite Fourier series are considered. As a numerical example, an antisymmetric angle-ply laminated cylindrical shell is considered. The effects of boundary conditions, length-to-radius ratio, thickness-to-radius ratio, and fiber orientation on the natural frequencies are analyzed and graphically presented.

KEY WORDS: laminated cylindrical shell, free vibration, Fourier series expansion method, Stokes' transformation.

INTRODUCTION

CYLINDRICAL SHELL IS one of the most important structural components which has been widely used in many engineering applications. In order to avoid structural failure aroused from dynamic loadings, free vibration characteristics of various cylindrical shells have attracted many researchers' attentions. Meeting with this engineering requirement, many investigations on this subject have been carried out by using theoretical, numerical, and experimental approaches.

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Following the pioneer work of Arnold and Warburton [1], theoretical studies on the vibration of cylindrical shells have been extensively conducted. The vibration characteristic of laminated structures is one of the most attractive topics in the past years. Unlike the homogeneous materials, the properties of composite materials can be tailored to have a very high strength and yet be very light. The natural frequency of such laminate structures can be adjusted by designing a suitable lamination scheme. Earlier investigations of vibration characteristic of laminated structures are performed by Reddy [2], Sinha and Rath [3], Noor [4], etc. Kapania [5] and Reddy [6] gave a comprehensive review for such earlier studies. Following these pioneer works, many of the theoretical methods have been developed to solving such vibration problems. The most popular theoretical method is the energy method, including Galerkin's method, Ritz method, and Rayleigh's method. Lam and Loy [7] discussed the influence of boundary conditions for a rotating laminated cylindrical shell by using the Galerkin's method. They [8] also investigated the influence of fiber orientation and boundary conditions on the natural frequencies of orthotropic laminated cylindrical shells by using the Ritz method. Based on the Flügge shell theory and the Rayleigh's energy method, Lee et al. [9] analyzed the free vibration characteristics of a joined cylindrical shell structure and individual shell components. The second popular theoretical method is the spline function approximation method. Using this method, Viswanathan and Lee [10] investigated the free vibration of symmetric and anti-symmetric angle-ply laminated plates based on shear deformation theory. Viswanathan and Navaneethakrishnan [11] studied the free vibration of a laminated truncated conical shell frusta of differently varying thickness by using the method of collocation with cubic and quintic splines. The third popular theoretical method is the differential quadrature method (DQM). Using this method, Haftchenari et al. [12], Sharma et al. [13], and Shu and Du [14] analyzed the free vibration of laminated cylindrical shells with different boundary conditions. Bert and Malik [15] gave a comprehensive review of the application of differential quadrature method for the analysis of composite structures.

Fourier series expansion method is also very promising in conducting free vibration analysis of cylindrical shells, in which Stokes' transformation is exploited to obtain correct expressions for the derivatives of the Fourier series. This method was firstly introduced by Chung [16] to analyze the free vibration characteristics of a cylindrical shell. Based on some unknown boundary values, Fourier series expansions and Stokes' transformation were employed to derive the frequency determinant for different boundary conditions. Using Fourier series expansion method, Kabir and Al-Khaleefi [17] studied the free vibration of thin arbitrary laminated anisotropic plates;

Jeong and Lee [18,19] analyzed the vibration characteristics of both partially liquid-filled and partially liquid-surrounded circular cylindrical shells.

In the present study, free vibration characteristics of laminated cylindrical shells are analyzed by using the Fourier expansion method. Stokes' transformation will be employed to derive the frequency determinants for some classical boundary conditions. Effects of boundary conditions, fiber orientation, length-to-radius ratio, and thickness-to-radius ratio on the natural frequencies of laminated cylindrical shells will be investigated. The advantages of this method are that the free vibration behavior of laminated shells can be analyzed in a uniform and simple manner. Because the frequency determinant is derived on the basis of unknown boundary values, it is convenient for the analysis of free vibration characteristics of shells and plates under more complicated surroundings.

FORMULATIONS OF THE PROBLEM

We consider a thin laminated cylindrical shell with mean radius R, axial length L, and thickness h as shown in Figure 1. Under the orthogonal coordinate system (x, φ, z) as shown in Figure 1, the equilibrium equations of motion for a thin cylindrical shell can be written as

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\varphi}}{\partial \phi} = \rho_t \frac{\partial^2 u}{\partial t^2} \tag{1a}$$

$$\frac{1}{R}\frac{\partial N_\varphi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + \frac{1}{R}\frac{\partial M_{x\varphi}}{\partial x} + \frac{1}{R^2}\frac{\partial M_\varphi}{\partial \phi} = \rho_t \frac{\partial^2 v}{\partial t^2} \tag{1b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{2}{R}\frac{\partial^2 M_{x\varphi}}{\partial x \partial \phi} + \frac{1}{R^2}\frac{\partial^2 M_\varphi}{\partial \phi^2} - \frac{1}{R} N_\varphi = \rho_t \frac{\partial^2 w}{\partial t^2} \tag{1c}$$

where u, v, and w are deformations of the shell in x, φ, and z directions, respectively. ρ_t is the mass per unit area and can be calculated as

$$\rho_t = \int_{-h/2}^{h/2} \rho \mathrm{d}z$$

in which ρ is the mass density. The force resultants $\{\boldsymbol{N_x}, N_\varphi, \boldsymbol{N_{x\varphi}}\}$ and moment resultants $\{\boldsymbol{M_x}, M_\varphi, \boldsymbol{M_{x\varphi}}\}$ are defined as

$$\left\{N_x, N_\varphi, N_{x\varphi}\right\} = \int_{-h/2}^{h/2} \left\{\sigma_x, \sigma_\varphi, \sigma_{x\varphi}\right\} \mathrm{d}z \tag{2a}$$

$$\left\{M_x, M_\varphi, M_{x\varphi}\right\} == \int_{-h/2}^{h/2} \left\{\sigma_x, \sigma_\varphi, \sigma_{x\varphi}\right\} z \mathrm{d}z \tag{2b}$$

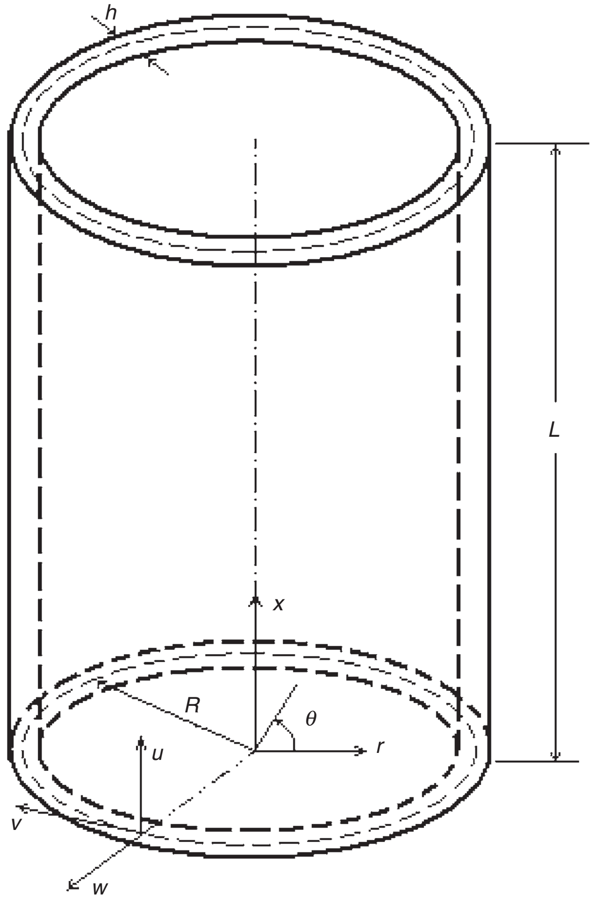

Figure 1. Geometry and coordinate system of a laminated cylindrical shell.

where $\{\sigma_x, \sigma_\varphi, \sigma_{x\varphi}\}$ are the stresses on the x-φ plane. For laminated structures, the relationships between stresses and strains in each layer are defined as

$$\{\sigma_x, \sigma_\varphi, \sigma_{x\varphi}\}^T = [\bar{Q}]\{\varepsilon_x, \varepsilon_\varphi, \varepsilon_{x\varphi}\}^T \tag{3}$$

and the transformed stiffness matrix $[\bar{Q}]$ is defined as

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-T}. \tag{4}$$

In the above Equation (4), [T] is the transformation matrix between the principle material coordinate system and the structural coordinate system. It is defined as

$$[\mathrm{T}] = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \tag{5}$$

in which α is the fiber orientation in each layer. [Q] is the reduced stiffness matrix for each layer and defined as

$$[\mathrm{Q}] = \begin{bmatrix} \mathrm{Q}_{11} & \mathrm{Q}_{12} & 0 \\ \mathrm{Q}_{12} & \mathrm{Q}_{22} & 0 \\ 0 & 0 & \mathrm{Q}_{66} \end{bmatrix} \tag{6}$$

For orthotropic materials, the reduced stiffness Q_{ij} are defined as

$$\mathrm{Q}_{11} = \frac{E_{11}}{(1-\mu_{12}\mu_{21})} \tag{7a}$$

$$\mathrm{Q}_{12} = \frac{\mu_{12}E_{22}}{(1-\mu_{12}\mu_{21})} \tag{7b}$$

$$\mathrm{Q}_{22} = \frac{E_{22}}{(1-\mu_{12}\mu_{21})} \tag{7c}$$

$$\mathrm{Q}_{66} = G_{12} \tag{7d}$$

where E_{11} and E_{22} are the Young's moduli in the principal directions, G_{12} is the shear modulus, μ_{12} and μ_{21} are corresponding Poisson's ratios.

Based on the first order approximation theory, the strain components $\{\varepsilon_x, \varepsilon_\varphi, \varepsilon_{x\varphi}\}$ in Equation (3) are defined as

$$\varepsilon_x = e_1 + z\kappa_1 \tag{8a}$$

$$\varepsilon_\varphi = e_2 + z\kappa_2 \tag{8b}$$

$$\varepsilon_{x\varphi} = e_{12} + 2z\kappa_{12} \tag{8c}$$

where e_1, e_2, and e_{12} are the reference surface strains. κ_1, κ_2, and κ_{12} are the reference curvatures. According to the Love shell theory, they are expressed as

$$\{e\} = \left\{\frac{\partial u}{\partial x}, \frac{1}{R}\left(\frac{\partial v}{\partial\varphi} + w\right), \left(\frac{\partial u}{R\partial\varphi} + \frac{\partial v}{\partial x}\right)\right\} \tag{9a}$$

$$\{\kappa\} = \left\{-\frac{\partial^2 w}{\partial x^2}, \frac{1}{R^2}\left(-\frac{\partial^2 w}{\partial\varphi^2} + \frac{\partial v}{\partial\varphi}\right), \frac{1}{R}\left(-\frac{\partial^2 w}{\partial x\partial\varphi} + \frac{\partial v}{\partial x}\right)\right\}. \tag{9b}$$

Substituting Equations (8a, b, c) into (3) and later into the definition of force and moment resultants (2), the constitutive equation of laminated shell can be obtained, i.e.,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{10}$$

where A, B, and D are the extensional, coupling, and bending stiffness matrices, respectively. For cross-ply laminates, the stiffness coefficients A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, and D_{26} are identically zero (see Whitney and Leissa [20] and Whitney and Paganao [21] for more details), other stiffness coefficients can be expressed as

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{k}(h_k - h_{k-1}), \quad i,j = 1, 2, 6 \tag{11a}$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{k}(h_k^2 - h_{k-1}^2), \quad i, j = 1, 2, 6 \tag{11b}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{k}(h_k^3 - h_{k-1}^3), \quad i, j = 1, 2, 60. \tag{11c}$$

Substituting Equations (9) into (10) and then the equilibrium Equation (1), after re-arrangement, we can obtain the following partial differential equations.

$$e_{11}\frac{\partial^2 u}{\partial x^2} + e_{12}\frac{\partial^2 u}{\partial \varphi^2} + e_{13}\frac{\partial^2 v}{\partial x \partial \varphi} + e_{14}\frac{\partial w}{\partial x} + e_{15}\frac{\partial^3 w}{\partial x^3} + e_{16}\frac{\partial^3 w}{\partial x^2 \partial \varphi} = \rho_t \frac{\partial^2 u}{\partial t^2} \tag{12a}$$

$$e_{21}\frac{\partial^2 u}{\partial x \partial \varphi} + e_{22}\frac{\partial^2 v}{\partial x^2} + e_{23}\frac{\partial^2 v}{\partial \varphi^2} + e_{24}\frac{\partial^3 w}{\partial x^2 \partial \varphi} + e_{25}\frac{\partial w}{\partial \varphi} + e_{26}\frac{\partial^3 w}{\partial \varphi^3} = \rho_t \frac{\partial^2 v}{\partial t^2} \tag{12b}$$

$$\begin{aligned} &e_{31}\frac{\partial^3 u}{\partial x^3} + e_{32}\frac{\partial u}{\partial x} + e_{33}\frac{\partial^3 u}{\partial x \partial \varphi^2} + e_{34}\frac{\partial^3 v}{\partial x^2 \partial \varphi} + e_{35}\frac{\partial^3 v}{\partial \varphi^3} + e_{36}\frac{\partial v}{\partial \varphi} + e_{37}\frac{\partial^4 w}{\partial x^4} \\ &\quad + e_{38}\frac{\partial^4 w}{\partial x^2 \partial \varphi^2} + e_{39}\frac{\partial^4 w}{\partial \varphi^4} + e_{310}\frac{\partial^2 w}{\partial x^2} + e_{311}\frac{\partial^2 w}{\partial \varphi^2} + e_{312} w = \rho_t \frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{12c}$$

where the coefficients e_{kl} are all listed in Appendix A. For the sake of convenience of the following analysis, it is necessary to express the membrane force N_x and $N_{x\varphi}$, the bending moment M_x, and the effective transverse shear force Q_x in terms of the displacement variables.

The relationships between these generalized forces and the displacement variables are given below.

$$N_x = e_{41}\frac{\partial u}{\partial x} + e_{42}\frac{\partial v}{\partial \varphi} + e_{43}w + e_{44}\frac{\partial^2 w}{\partial x^2} + e_{45}\frac{\partial^2 w}{\partial \varphi^2} \tag{13a}$$

$$M_x = e_{51}\frac{\partial u}{\partial x} + e_{52}\frac{\partial v}{\partial \varphi} + e_{53}w + e_{54}\frac{\partial^2 w}{\partial x^2} + e_{55}\frac{\partial^2 w}{\partial \varphi^2} \tag{13b}$$

$$N_{x\varphi} = e_{61}\frac{\partial u}{\partial \varphi} + e_{62}\frac{\partial v}{\partial x} + e_{63}\frac{\partial^2 w}{\partial x \partial \varphi} \tag{13c}$$

$$Q_x = e_{71}\frac{\partial^2 u}{\partial x^2} + e_{72}\frac{\partial^2 v}{\partial x \partial \varphi} + e_{73}\frac{\partial w}{\partial x} + e_{74}\frac{\partial^3 w}{\partial x \partial \varphi^2} + e_{75}\frac{\partial^3 w}{\partial x^3} \tag{13d}$$

where the coefficients e_{kl} are also listed in Appendix A.

The general relation for the displacement fields of thin cylindrical shells with general boundary conditions can then be written in the following form

$$u(x, \varphi, t) = u(x) \cdot \cos n\varphi \cdot \exp(i\omega t) \tag{14a}$$

$$v(x, \varphi, t) = v(x, \varphi) \cdot \sin n\varphi \cdot \exp(i\omega t) \tag{14b}$$

$$w(x, \varphi, t) = w(x, \varphi) \cdot \cos n\varphi \cdot \exp(i\omega t),\ i = \sqrt{-1} \tag{14c}$$

where $u(x)$, $v(x)$, and $w(x)$ are displacement modal functions. In the axial direction, these modal functions can be expressed by using of the following Fourier series,

$$u(x) = a_{n0} + \sum_{m=1}^{\infty} a_{nm}\cos\left(\frac{m\pi x}{L}\right) \tag{15a}$$

$$v(x) = \sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{m\pi x}{L}\right) \tag{15b}$$

$$w(x) = \sum_{m=1}^{\infty} c_{nm}\sin\left(\frac{m\pi x}{L}\right) \tag{15c}$$

As shown in Equations (15a,b,c), displacement modal functions of a cylindrical shell can be expressed by a Fourier sine and cosine series in the ranges of $0 < x < L$. In the following analysis, we use Stokes' transformation to evaluate the derivatives of these displacement modal functions. For the sake of brevity, the detailed deductive process is omitted in this article.

For more details of the Stokes' transformations and the exact expression of derivatives for displacement modal function (15), one can find in Chung [16]. Substituting Equations (14), displacement modal functions (15) and their derivatives into the governing Equations (12a,b,c), the following explicit relations for a_{n0}, a_{nm}, b_{nm}, c_{nm} can be obtained

$$a_{n0} = f_{o1}(\tilde{u}_0 + \tilde{u}_L) + f_{02}(v_0 + v_L) + f_{03}(w_0 + w_L) + f_{04}\left(\tilde{\tilde{w}}_0 + \tilde{\tilde{w}}_L\right) \tag{16a}$$

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{Bmatrix} a_{nm} \\ b_{nm} \\ c_{nm} \end{Bmatrix} = \begin{Bmatrix} f_{11}(\tilde{u}_0 + \tilde{u}_L(-1)^m) + f_{12}(v_0 + v_L(-1)^m) + f_{13}(w_0 + w_L(-1)^m) + f_{14}\left(\tilde{\tilde{w}}_0 + \tilde{\tilde{w}}_L(-1)^m\right) \\ f_{22}(v_0 + v_L(-1)^m) + f_{23}\left(w_{j0} + w_L(-1)^m\right) \\ f_{31}(\tilde{u}_0 + \tilde{u}_L(-1)^m) + f_{32}(v_0 + v_L(-1)^m) + f_{33}(w_0 + w_L(-1)^m) + f_{34}\left(\tilde{\tilde{w}}_0 + \tilde{\tilde{w}}_L(-1)^m\right) \end{Bmatrix} \tag{16b}$$

where g_{kl} and f_{kl} $(k, l = 1, 2, 3, 4)$ are listed in Appendix B. Symbols v_0, v_L, w_0, and w_L are boundary values of the cylindrical shell. All these boundary values are calculated as follows.

$$v_o = -\frac{2}{\pi \sin n\varphi} v(0, \varphi), \quad v_L = -\frac{2}{\pi \sin n\varphi} v(L, \varphi)$$
$$w_o = -\frac{2}{\pi \sin n\varphi} w(0, \varphi), \quad w_L = -\frac{2}{\pi \sin n\varphi} w(L, \varphi)$$

The symbols $\tilde{u}_0$, $\tilde{u}_L$, $\tilde{\tilde{w}}_0$, and $\tilde{\tilde{w}}_L$ are associated with the end forces N_x and M_x. Using Equations (5a,b), the relations between them can be obtained, i.e.,

$$\begin{Bmatrix} N_x(0,\varphi) \\ -N_x(L,\varphi) \end{Bmatrix} = \left[f_{41}\begin{Bmatrix} \tilde{u}_0 \\ \tilde{u}_L \end{Bmatrix} + f_{42}\begin{Bmatrix} v_0 \\ v_L \end{Bmatrix} + f_{43}\begin{Bmatrix} w_0 \\ w_L \end{Bmatrix} + f_{44}\begin{Bmatrix} \tilde{\tilde{w}}_0 \\ \tilde{\tilde{w}}_L \end{Bmatrix} \right] \cos n\varphi \tag{17a}$$

$$\begin{Bmatrix} M_x(0,\varphi) \\ -M_x(L,\varphi) \end{Bmatrix} = \left[f_{51}\begin{Bmatrix} \tilde{u}_0 \\ \tilde{u}_L \end{Bmatrix} + f_{52}\begin{Bmatrix} v_0 \\ v_L \end{Bmatrix} + f_{53}\begin{Bmatrix} w_0 \\ w_L \end{Bmatrix} + f_{54}\begin{Bmatrix} \tilde{\tilde{w}}_0 \\ \tilde{\tilde{w}}_L \end{Bmatrix} \right] \left(\frac{L}{\pi}\right) \cos n\varphi \tag{17b}$$

where f_{kl} ($k=4,5$; $l=1,2,3,4$) are also listed in Appendix B. Considering the relations (17a,b), Equations (16a,b) can then be rewritten in terms of the eight unknown boundary values v_0, v_L, w_0, w_L, N_{x0}, N_{xL}, M_{x0}, and M_{xL}.

$$a_{n0} = q_{02}(v_0 + v_L) + q_{03}(w_0 + w_L) + q_{04}(N_{x0} + N_{xL}) + q_{05}(M_{x0} + M_{xL}) \tag{18a}$$

$$\begin{Bmatrix} a_{nm} \\ b_{nm} \\ c_{nm} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \begin{Bmatrix} q_{12}[v_0 + (-1)^m v_L] + q_{13}[w_0 + (-1)^m w_L] + q_{14}[N_{x0} + (-1)^m N_{xL}] + q_{15}[M_{x0} + (-1)^m M_{xL}] \\ +q_{22}[v_0 + (-1)^m v_L] + q_{23}[w_0 + (-1)^m w_L] \\ +q_{32}(v_0 + v_L) + q_{33}(w_0 + w_L) \quad q_{34}[N_{x0} + (-1)^m N_{xL}] + q_{35}[M_{x0} + (-1)^m M_{xL}] \end{Bmatrix} \tag{18b}$$

where $[s_{kl}]$ is the inverse matrix of $[g_{kl}]$. g_{kl} and q_{kl} are listed in Appendix C. The boundary forces and moments, N_{x0}, N_{xL}, M_{x0}, and M_{xL} are defined as follows.

$$N_{xo} = \frac{N_x(0,\varphi)}{\cos n\varphi}, \quad N_{xL} = -\frac{N_x(L,\varphi)}{\cos n\varphi},$$
$$M_{xo} = \frac{\pi M_x(0,\varphi)}{L\cos n\varphi}, \quad M_{xL} = -\frac{\pi M_x(L,\varphi)}{L\cos n\varphi}.$$

In the above analysis, altogether 8 boundary values, i.e., v_0, v_L, w_0, w_L, N_{x0}, N_{xL}, M_{x0}, and M_{xL}, are involved. Substituting the explicit relations (18a,b) into different boundary conditions, the frequency determinant can then be obtained. In the following, the frequency determinants for four classical boundary conditions: clamped–clamped, clamped–simply supported, clamped–free, and simply supported–simply supported are derived and discussed.

BOUNDARY CONDITIONS

Clamped–Clamped Boundary Condition (C–C)

For clamp–clamped cylindrical shells, the boundary values v_0, v_L, w_0, and w_L at both ends must equal to zero. The other boundary conditions, which should be satisfied, can be expressed as follows.

$$u = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0, \quad \text{at} \quad x = 0, L. \tag{19}$$

Substituting Equations (18a,b) into boundary conditions (19), after a complex derivation, we can obtain the following algebraic equations.

$$\begin{bmatrix} t_{55} & t_{56} & t_{57} & t_{58} \\ t_{65} & t_{66} & t_{67} & t_{68} \\ t_{75} & t_{76} & t_{77} & t_{78} \\ t_{85} & t_{86} & t_{87} & t_{88} \end{bmatrix} \begin{Bmatrix} N_{x0} \\ N_{xL} \\ M_{x0} \\ M_{xL} \end{Bmatrix} = 0 \tag{20}$$

where t_{kl} are listed in Appendix D. Equations (20) are a set of homogeneous algebraic equation for the three unknown boundary values. In order to search for a nontrial solution of these equations, the coefficient determinant must vanish. So the following frequency determinant can now be constructed.

$$[t_{kl}] = 0, \quad k, l = 5, 6, 7, 8 \tag{21}$$

Clamped–simply supported boundary condition (C–S)

As a typical case, the axial displacement at the simply supported end of the shell is assumed to be constrained. So the boundary values v_0, w_0 at the clamped end and v_{L}, w_{L}, M_{xL} at the simply supported end must be zero. The other boundary condition can be expressed as

$$u = 0, \quad \text{and} \quad \frac{\partial w}{\partial x} = 0, \quad \text{at } x = 0; u = 0, \quad \text{at } x = L \tag{22}$$

Substituting Equations (18a,b) into boundary conditions (22), we can obtain

$$\begin{bmatrix} t_{55} & t_{56} & t_{57} \\ t_{65} & t_{66} & t_{67} \\ t_{75} & t_{76} & t_{77} \end{bmatrix} \begin{Bmatrix} N_{x0} \\ N_{xL} \\ M_{x0} \end{Bmatrix} = 0 \tag{23}$$

where t_{kl} are also listed in Appendix D. The frequency determinant can be obtained as

$$[t_{kl}] = 0, \quad k, l = 5, 6, 7 \tag{24}$$

Clamped–free boundary condition (C–F)

For clamp–free cylindrical shells, the boundary values v_0, w_0 at clamped end and N_{xL}, M_{xL}, at the free end vanish. The other boundary conditions are

$$u = 0, \text{ and } \frac{\partial w}{\partial x} = 0, \text{ at } x = 0; N_{x\varphi} = 0, \text{ and } Q_x = 0, \text{at } x = L. \tag{25}$$

Substituting Equations (18a,b) into boundary conditions (25), it is obtained

$$\begin{bmatrix} t_{22} & t_{24} & t_{25} & t_{27} \\ t_{42} & t_{44} & t_{45} & t_{47} \\ t_{52} & t_{54} & t_{55} & t_{57} \\ t_{72} & t_{74} & t_{75} & t_{77} \end{bmatrix} \begin{Bmatrix} v_L \\ w_L \\ N_{x0} \\ M_{x0} \end{Bmatrix} = 0 \tag{26}$$

where t_{kl} are listed in Appendix D. So the frequency determinant can be expressed as

$$[t_{kl}] = 0, \quad k, l = 2, 4, 5, 7 \tag{27}$$

Simply supported–simply supported boundary condition (S–S)

It should be noted that the axial displacements at both simply supported ends of the shell are constrained here. So the boundary values v_0, w_0, M_{x0}, v_L, w_L, and M_{xL} at both simply supported ends must be zero. This boundary condition can be expressed as

$$u = 0, \quad \text{at } x = 0, L \tag{28}$$

Substituting Equations (18a,b) into boundary conditions (28), it is obtained

$$\begin{bmatrix} t_{55} & t_{56} \\ t_{65} & t_{66} \end{bmatrix} \begin{Bmatrix} N_{x0} \\ N_{xL} \end{Bmatrix} = 0 \tag{29}$$

where t_{kl} are listed in Appendix D. So the frequency determinant can be expressed as

$$[t_{kl}] = 0, \quad k, l = 5, 6 \tag{30}$$

RESULTS AND DISCUSSION

Based on the infinite series of the algebraic terms involved in the frequency determinants, the convergence of the sums of the infinite series terms should be evaluated. Here we take into account a homogeneous cylindrical shell presented in Chung [16]. For the sake of brevity, we only compare the first three fundamental frequencies. The convergence and

Table 1. Convergence and accuracy of present method.

	$n=1$		$n=2$		$n=3$	
Number of terms	Chung's 855.1 Hz Present (Hz)	Error (%)	Chung's 403.72 Hz Present (Hz)	Error (%)	Chung's 223.34 Hz Present (Hz)	Error (%)
5	920.64	7.67	450.96	11.7	243.89	9.2
9	896.97	4.9	434.01	7.5	233.36	4.5
19	874.42	2.3	417.36	3.4	224.76	0.6
29	867.68	1.5	412.02	2.1	223.82	0.2
39	864.44	1.09	408.97	1.3	223.72	0.17
49	862.37	0.85	406.55	0.7	223.68	0.15
60	861.12	0.70	405.33	0.4	223.59	0.11
70	860.19	0.6	404.81	0.27	223.54	0.09

accuracy of the present analysis is shown in Table 1. It is seen that the present results show excellent agreement with Chung's result when enough series terms were taken into account. After 40 series terms, the error between the present result and the reference result became less than one percent. In the following calculations, an upper limit of 40 series terms is truncated in deriving the frequency determinant of laminated cylindrical shells. We secondly check the validity of the present method for the vibration analysis of composite shells. Therefore, we consider a three-layered laminated shell with fiber orientation sequence $45°/-45°/45°$. Comparison between the present result and Lam and Loy's result [8] is summarized in Figure 2. It is seen that excellent agreement between these two results is obtained with 40 terms of the infinite series being considered in the present method.

Now we consider an antisymmetric cross-ply laminated cylindrical shell with mean radius $R = 0.5$ m. The thickness of each layer is assumed to be $0.001R$. So the shell thickness increased with the increment of layer number. The mechanical properties of the laminated shell are listed in Table 2.

We firstly analyze the effect of boundary conditions on frequencies of the laminated cylindrical shell with dimensions: $L/R = 20$, $h/R = 0.004$, and fiber orientation $-45°/45°/-45°/45°$. Four classical boundary conditions are considered here, namely, clamped–clamped, clamped–simply supported (with axial constraints), clamped–free, and simply supported–simply supported (both with axial constraints). The numerical results are plotted in Figure 3. It is seen that dramatic effects of the boundary conditions occur for the first two fundamental frequencies. For the four classical boundary conditions considered here, the natural frequencies of the clamped–clamped boundary condition are the highest, and the natural frequencies associated with the clamped–free boundary condition are the lowest. For higher order frequencies, these effects gradually become minor.

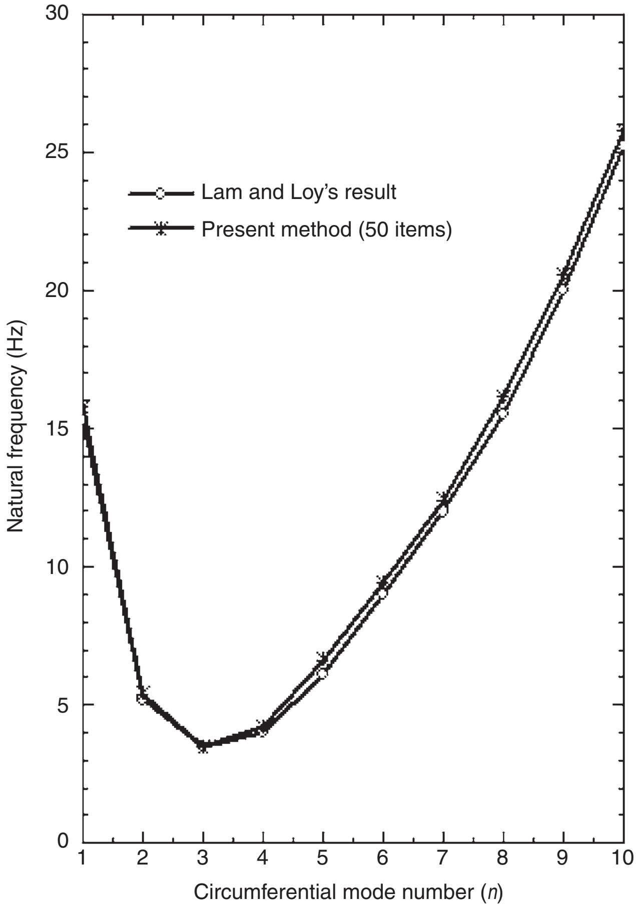

***Figure 2.** Comparison with Lam and Loy's result.*

Table 2. Properties of each layer of laminated cylindrical shell.

Properties	Lamina thickness
$E_1 = 19.0$ GPa $E_2 = 7.6$ GPa $G_{12} = 4.1$ GPa $\mu = 0.26$ $\rho = 1643$ kg/m^3	Each layer thickness $= 0.001R$

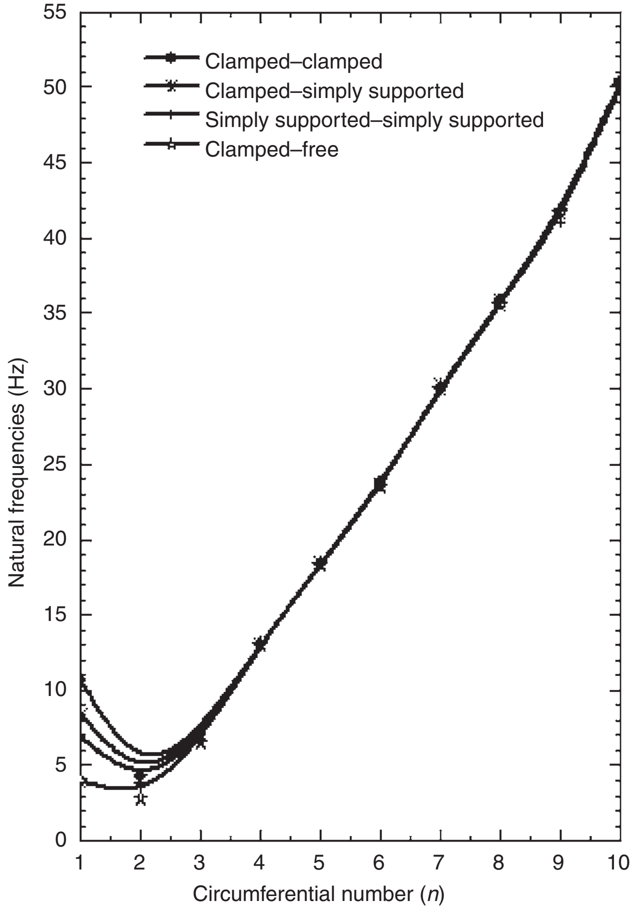

Figure 3. *Effects of boundary condition on natural frequencies (fiber orientation $-45°/45°/-45°/45°$, $h/R = 0.004$ and $L/R = 20$).*

Effects of the length-to-radius ratio on natural frequencies of the laminated shell for the four classical boundary conditions are summarized in Figure 4 ($h/R = 0.004$, fiber orientation $-45°/45°/-45°/45°$). It is seen that the influences for the four boundary conditions are all relatively significant at small length-to-radius ratios. With the increase of the length-to-radius ratio, these influences decreased gradually. At a ratio of $L/R = 20$, the influences become very small. Among the four boundary conditions, the most significant influence occurs for the clamped–clamped boundary condition.

Different from the published analysis, we here consider the case of constant thickness of each layer. For an antisymmetric cross-ply laminated shell, the

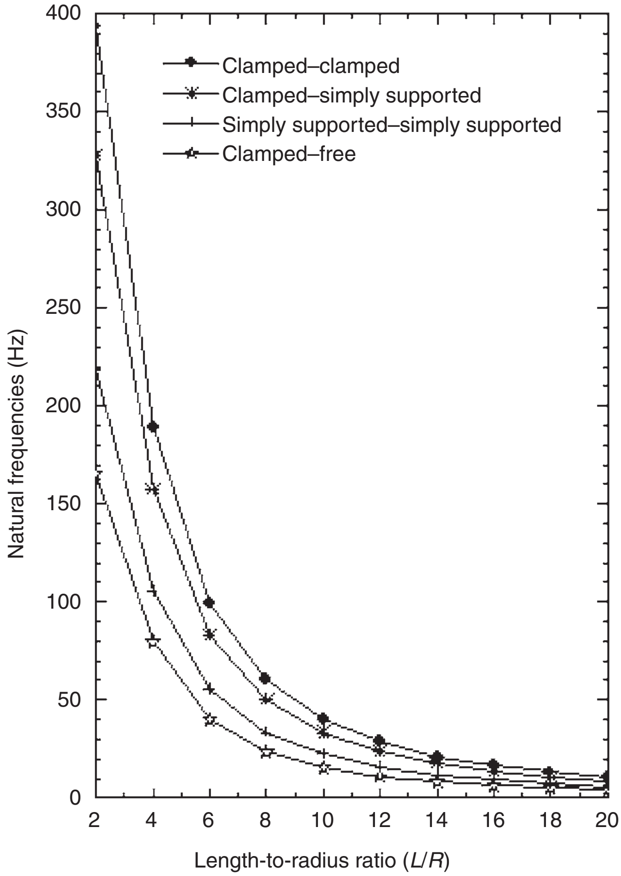

Figure 4. *Effects of length-to-radius ratio on natural frequencies (fiber orientation −45°/45°/−45°/45° and h/R = 0.004).*

analyzed thickness variations are two layers ($0.002R$, $-45°/45°/-45°/45°$), four layers ($0.004R$, $45°/-45°/45°/-45°/45°/-45°$), and so on up to 20 layers ($0.02R$). Numerical results for the aforementioned four boundary conditions are summarized in Figure 5. It's seen that the effects of shell thickness are significant in the considered thickness-to-radius ratio range ($0.002 < h/R < 0.02$) for the four boundary conditions. Frequencies increase with the increase of the shell thickness. However, these effects on frequencies decrease with the increase of the thickness-to-radius ratio. For each thickness-to-radius ratio, the frequency of clamped–clamped boundary condition is the highest and followed by sequence of clamped–simply supported, simply supported–simply supported, and clamped–free boundary conditions.

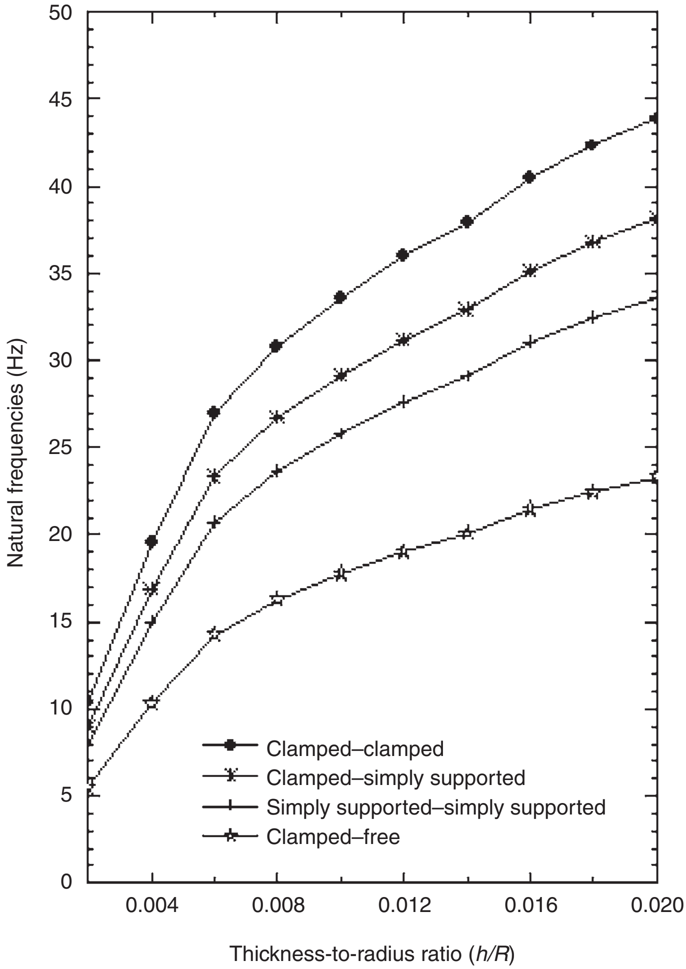

Figure 5. *Effect of thickness-to-radius ratio on natural frequencies ($L/R = 20$).*

The effects of fiber orientation on the natural frequencies are also analyzed. Three antisymmetric cross-ply $-30°/30°/-30°/30°$, $-45°/45°/-45°/45°$, and $-60°/60°/-60°/60$ are considered. All numerical results for clamped–free boundary condition are graphically presented in Figure 6. It is seen that the effects of fiber orientation on the natural frequencies are not significant with respect to a lower circumferential wave number n, for example, $n<4$. However, these effects become considerable when the circumferential wave number n is larger than 4. In this range, the frequencies

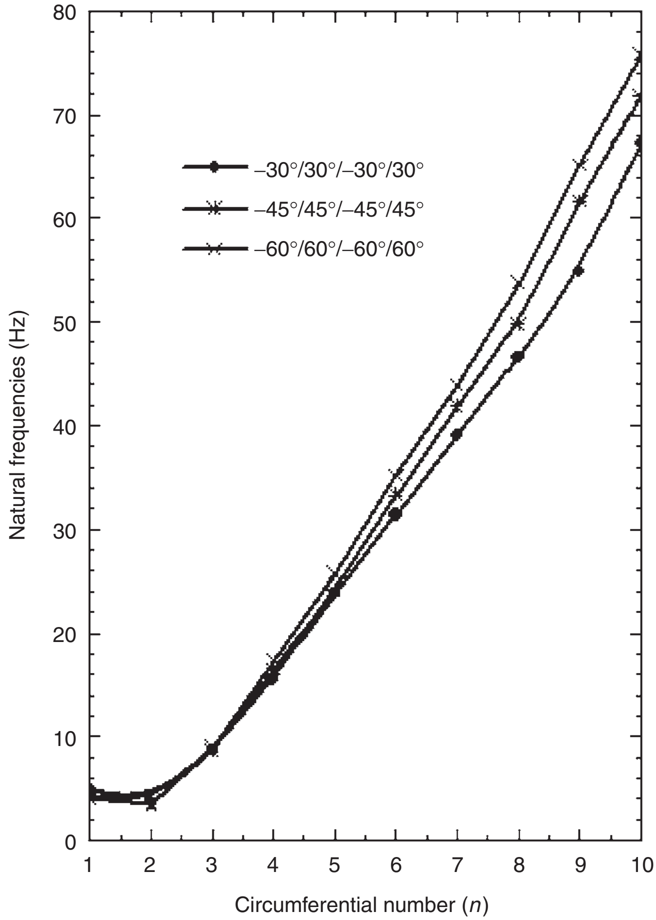

Figure 6. *Effect of fiber orientation on natural frequencies (h/R = 0.004, L/R = 20).*

of fiber orientation 60° are the highest and followed by sequence of fiber orientation 45° and 30°. The differences of frequencies caused by fiber orientation show increasing tendencies with the increase of the circumferential wave number *n*.

CONCLUSION

In this article, free vibration of laminated cylindrical shells was analyzed by using Fourier expansion method and Stokes' transformation. For the sake of brevity, only the frequency determinants for four classical boundary conditions, i.e., clamped–clamped, clamped–simply supported (with axial constraint), clamped–free, and simply supported–simply supported (both with axial constraints) were presented. Frequency determinants for other boundary conditions, such as free–free, free–simply supported, can also be derived by using the same method.

It has been shown that the present method has the advantages in analyzing the free vibration behavior of laminated shells in a uniform manner. Based on the unknown boundary displacements and forces, the frequency determinant for different boundary conditions can be easily obtained. It is worth noting that, because infinite terms of the Fourier series are used in deriving the frequency determinant, convergence of the Fourier series method must be evaluated. It has been found in the present study that 40 terms truncated from the infinite Fourier series gives an acceptable accuracy of the natural frequencies.

APPENDIX A

$$e_{11} = A_{11}, e_{12} = \frac{A_{66}}{R^2}, e_{13} = \frac{A_{12} + A_{66}}{R} + \frac{B_{12} + 2B_{66}}{R^2}, e_{14} = \frac{A_{12}}{R},$$
$$e_{15} = -B_{11}, e_{16} = -\frac{B_{12} + 2B_{66}}{R^2}$$
$$e_{21} = e_{13} - \frac{B_{66}}{R^2}, e_{22} = A_{66} + \frac{3B_{66}}{R} + \frac{2D_{66}}{R^2}, e_{23} = \frac{A_{22}}{R^2} + \frac{2B_{22}}{R^3} + \frac{D_{22}}{R^4}$$
$$e_{24} = -\left(\frac{B_{12} + 2B_{66}}{R} + \frac{D_{12} + 2D_{66}}{R^2}\right), e_{25} = \frac{A_{22}}{R^2} + \frac{B_{22}}{R^3}, e_{26} = -\left(\frac{B_{22}}{R^3} + \frac{D_{22}}{R^4}\right)$$
$$e_{31} = -e_{11}, e_{32} = -e_{14}, e_{33} = -e_{16}, e_{34} = -e_{24} + \frac{2D_{66}}{R^2}, e_{35} = -e_{26}, e_{36} = -e_{25}$$
$$e_{37} = -D_{11}, e_{38} = -\frac{2D_{12} + 4D_{66}}{R^2}, e_{39} = -\frac{D_{22}}{R^4}, e_{310} = \frac{2B_{12}}{R}, e_{311}$$
$$= \frac{2B_{22}}{R^3}, e_{312} = -\frac{A_{22}}{R^2}$$
$$e_{41} = e_{11}, e_{42} = \frac{A_{12}}{R} + \frac{B_{12}}{R^2}, e_{43} = e_{14}, e_{44} = B_{11}, e_{45} = \frac{B_{12}}{R^2}$$
$$e_{51} = B_{11}, e_{52} = \frac{B_{12}}{R} + \frac{D_{12}}{R^2}, e_{53} = \frac{B_{12}}{R}, e_{54} = D_{11}, e_{55} = \frac{D_{12}}{R^2}$$
$$e_{61} = \frac{A_{66}}{R} + \frac{B_{66}}{R^2}, e_{62} = e_{22}, e_{63} = \frac{2B_{66}}{R} + \frac{2D_{66}}{R^2}$$
$$e_{71} = B_{11}, e_{72} = e_{52}, e_{73} = \frac{B_{12}}{R}, e_{74} = \frac{D_{12}}{R^2}, e_{75} = D_{11}$$

APPENDIX B

$$d_1 = 2\left(n^2 e_{12} - \rho_t \omega^2\right), f_{01} = \frac{e_{11}}{d_1}\left(\frac{\pi}{L}\right)^2, f_{02} = \frac{ne_{13}}{d_1}\frac{\pi}{L},$$

$$f_{03} = \frac{e_{14} + n^2 e_{16}}{d_1}\frac{\pi}{L}, f_{04} = \frac{e_{15}}{d_1}\left(\frac{\pi}{L}\right)^3$$

$$g_{11} = m^2 e_{11}\left(\frac{\pi}{L}\right)^2 + e_{12}n^2 - \rho_t\omega^2, g_{12} = -nme_{13}\left(\frac{\pi}{L}\right),$$

$$g_{13} = -me_{14}\left(\frac{\pi}{L}\right) + m^3 e_{15}\left(\frac{\pi}{L}\right)^3 + n^2 me_{16}\left(\frac{\pi}{L}\right)$$

$$g_{21} = -nme_{21}\left(\frac{\pi}{L}\right), g_{22} = m^2 e_{22}\left(\frac{\pi}{L}\right)^2 + n^2 e_{23} - \rho_t\omega^2,$$

$$g_{23} = -nm^2 e_{24}\left(\frac{\pi}{L}\right)^2 + ne_{25} - n^3 e_{26}$$

$$g_{31} = -m^3 e_{31}\left(\frac{\pi}{L}\right)^3 + me_{32}\left(\frac{\pi}{L}\right) - n^2 me_{33}\left(\frac{\pi}{L}\right), g_{32} = nm^2 e_{34}\left(\frac{\pi}{L}\right)^2 + n^3 e_{35} - ne_{36}$$

$$g_{33} = -m^4 e_{37}\left(\frac{\pi}{L}\right)^4 - n^2 m^2 e_{38}\left(\frac{\pi}{L}\right)^2 - n^4 e_{39} + m^2 e_{310}\left(\frac{\pi}{L}\right)^2 + n^2 e_{311} - e_{312} - \rho_t\omega^2$$

$$f_{11} = e_{11}\left(\frac{\pi}{L}\right)^2, f_{12} = ne_{13}\left(\frac{\pi}{L}\right), f_{13} = e_{14}\left(\frac{\pi}{L}\right) - m^2 e_{15}\left(\frac{\pi}{L}\right)^3 - n^2 e_{16}\left(\frac{\pi}{L}\right),$$

$$f_{14} = e_{15}\left(\frac{\pi}{L}\right)^3$$

$$f_{22} = -me_{22}\left(\frac{\pi}{L}\right)^2, f_{23} = nme_{24}\left(\frac{\pi}{L}\right)^2$$

$$f_{31} = -me_{31}\left(\frac{\pi}{L}\right)^3, f_{32} = -nme_{34}\left(\frac{\pi}{L}\right)^2$$

$$f_{33} = m^3 e_{37}\left(\frac{\pi}{L}\right)^4 + n^2 me_{38}\left(\frac{\pi}{L}\right)^2 - me_{310}\left(\frac{\pi}{L}\right)^2, f_{34} = -me_{37}\left(\frac{\pi}{L}\right)^4$$

$$f_{41} = -A_{11}\frac{\pi^2}{2L}, f_{42} = -\left(\frac{A_{12}}{R} + \frac{B_{12}}{R^2}\right)\frac{\pi}{2}n, f_{43} = -\left(\frac{A_{12}}{R} + \frac{B_{12}}{R^2}n^2\right)$$

$$\times\frac{\pi}{2}, f_{44} = B_{11}\frac{\pi^3}{2L^2}$$

$$f_{51} = -B_{11}\frac{\pi^2}{2L}, f_{52} = -\left(\frac{B_{12}}{R} + \frac{D_{12}}{R^2}\right)\frac{\pi}{2}n, f_{53} = -\left(\frac{B_{12}}{R} + \frac{D_{12}}{R^2}n^2\right)$$

$$\times\frac{\pi}{2}, f_{54} = D_{11}\frac{\pi^3}{2L^2}$$

APPENDIX C

$$q_{02} = f_{02} + Y_1 + Y_5,\ q_{03} = f_{03} + Y_2 + Y_6,\ q_{04} = Y_3 + Y_7,\ q_{05} = Y_4 + Y_8$$

$$q_{12} = f_{12} + f_{11} Y_1 + f_{14} Y_5,\ q_{13} = f_{13} + f_{11} Y_2 + f_{14} Y_6,\ q_{14} = f_{11} Y_3 + f_{14} Y_7$$

$$q_{22} = f_{22},\ q_{23} = f_{23}$$

$$q_{32} = f_{32} + f_{31} Y_1 + f_{34} Y_5,\ q_{33} = f_{33} + f_{31} Y_2 + f_{34} Y_6,$$

$$q_{34} = f_{31} Y_3 + f_{34} Y_7,\ q_{35} = f_{31} Y_4 + f_{34} Y_8$$

$$Y_1 = \frac{(f_{52} f_{44} - f_{54} f_{42})}{(f_{54} f_{41} - f_{51} f_{44})},\ Y_2 = \frac{(f_{53} f_{44} - f_{54} f_{43})}{(f_{54} f_{41} - f_{51} f_{44})}$$

$$Y_3 = \frac{f_{54}}{(f_{54} f_{41} - f_{51} f_{44})},\ Y_4 = \frac{-f_{44}}{(f_{54} f_{41} - f_{51} f_{44})}$$

$$Y_5 = \frac{(f_{51} f_{42} - f_{52} f_{41})}{(f_{54} f_{41} - f_{51} f_{44})},\ Y_6 = \frac{(f_{51} f_{43} - f_{53} f_{41})}{(f_{54} f_{41} - f_{51} f_{44})}$$

$$Y_7 = \frac{-f_{51}}{(f_{54} f_{41} - f_{51} f_{44})},\ Y_8 = \frac{f_{41}}{(f_{54} f_{41} - f_{51} f_{44})}$$

$$s_{11} = \frac{(g_{22} g_{33} - g_{23} g_{32})}{\Delta},\ s_{12} = \frac{(g_{23} g_{31} - g_{21} g_{33})}{\Delta},\ s_{13} = \frac{(g_{21} g_{32} - g_{22} g_{31})}{\Delta}$$

$$s_{21} = \frac{(g_{13} g_{32} - g_{12} g_{33})}{\Delta},\ s_{22} = \frac{(g_{11} g_{33} - g_{13} g_{31})}{\Delta},\ s_{23} = \frac{(g_{12} g_{31} - g_{11} g_{32})}{\Delta}$$

$$s_{31} = \frac{(g_{12} g_{23} - g_{13} g_{22})}{\Delta},\ s_{32} = \frac{(g_{13} g_{21} - g_{11} g_{23})}{\Delta},\ s_{33} = \frac{(g_{11} g_{22} - g_{12} g_{21})}{\Delta}$$

$$\Delta = \det \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

APPENDIX D

$$t_{12} = q_{02} + \sum_{m=1}^{\infty} d_{11}(-1)^m,\ t_{14} = q_{03} + \sum_{m=1}^{\infty} d_{12}(-1)^m,\ t_{15} = q_{04} + \sum_{m=1}^{\infty} d_{13},$$

$$t_{17} = q_{05} + \sum_{m=1}^{\infty} d_{14}$$

$$t_{22} = \sum_{m=1}^{\infty} md_{31}(-1)^m,\ t_{24} = \frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{32})(-1)^m,\ t_{25} = \sum_{m=1}^{\infty} md_{33},$$

$$t_{27} = \sum_{m=1}^{\infty} md_{34}$$

$$t_{32} = -ne_{61}\left[q_{02} + \sum_{m=1}^{\infty} d_{11}\right] + e_{62}\left(\frac{\pi}{L}\right)\left[\frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{21})\right] + ne_{63}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{31}$$

$$t_{34} = -ne_{61}\left[q_{03} + \sum_{m=1}^{\infty} d_{12}\right] + e_{62}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{22} + ne_{63}\left(\frac{\pi}{L}\right)\left[\frac{1}{2} + \sum_{m=1}^{\infty} md_{32}\right]$$

$$t_{35} = -ne_{61}\left[q_{04} + \sum_{m=1}^{\infty} d_{13}(-1)^m\right] + e_{62}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{23}(-1)^m + ne_{63}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{33}(-1)^m$$

$$t_{37} = -ne_{61}\left[q_{05} + \sum_{m=1}^{\infty} d_{14}(-1)^m\right] + e_{62}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{24}(-1)^m + ne_{63}\left(\frac{\pi}{L}\right)\sum_{m=1}^{\infty} md_{34}(-1)^m$$

$$t_{42} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_1 + \sum_{m=1}^{\infty} (Y_1 - m^2 d_{11})\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{21})\right] + (e_{73} + n^2 e_{74})\sum_{m=1}^{\infty} md_{31} - e_{75}\left(\frac{\pi}{L}\right)^2\left[Y_5 + \sum_{m=1}^{\infty} (Y_5 - m^2 d_{31})\right]$$

$$t_{44} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_2 + \sum_{m=1}^{\infty} (Y_2 - m^2 d_{12})\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{22})\right] + (e_{73} + n^2 e_{74})\sum_{m=1}^{\infty} md_{32} - e_{75}\left(\frac{\pi}{L}\right)^2\left[Y_6 + \sum_{m=1}^{\infty} (Y_6 - m^2 d_{32})\right]$$

$$t_{45} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_3 + \sum_{m=1}^{\infty} (Y_3 - m^2 d_{13})(-1)^m\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{23})(-1)^m\right] + (e_{73} + n^2 e_{74})\sum_{m=1}^{\infty} md_{33}(-1)^m - e_{75}\left(\frac{\pi}{L}\right)^2\left[Y_7 + \sum_{m=1}^{\infty} (Y_7 - m^2 d_{33})(-1)^m\right]$$

$$t_{47} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_4 + \sum_{m=1}^{\infty} (Y_4 - m^2 d_{14})(-1)^m\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty} (1 + md_{24})(-1)^m\right]$$

$$t_{45} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_3 + \sum_{m=1}^{\infty}(Y_3 - m^2 d_{13})(-1)^m\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty}(1 + md_{23})(-1)^m\right]$$
$$+ (e_{73} + n^2 e_{74})\sum_{m=1}^{\infty} md_{33}(-1)^m - e_{75}\left(\frac{\pi}{L}\right)^2\left[Y_7 + \sum_{m=1}^{\infty}(Y_7 - m^2 d_{33})(-1)^m\right]$$

$$t_{47} = e_{71}\left(\frac{\pi}{L}\right)\left[Y_4 + \sum_{m=1}^{\infty}(Y_4 - m^2 d_{14})(-1)^m\right] + ne_{72}\left[\frac{1}{2} + \sum_{m=1}^{\infty}(1 + md_{24})(-1)^m\right]$$
$$+ (e_{73} + n^2 e_{74})\sum_{m=1}^{\infty} md_{34}(-1)^m - e_{75}\left(\frac{\pi}{L}\right)^2\left[Y_8 + \sum_{m=1}^{\infty}(Y_8 - m^2 d_{34})(-1)^m\right]$$

$$d_{11} = s_{11}q_{12} + s_{12}q_{22} + s_{13}q_{32}, d_{12} = s_{11}q_{13} + s_{12}q_{23} + s_{13}q_{33}$$
$$d_{13} = s_{11}q_{14} + ss_{13}q_{34}, d_{14} = s_{11}q_{15} + ss_{13}q_{35}$$
$$d_{21} = s_{21}q_{12} + s_{22}q_{22} + s_{23}q_{32}, d_{22} = s_{21}q_{13} + s_{22}q_{23} + s_{23}q_{33}$$
$$d_{23} = s_{21}q_{14} + ss_{23}q_{34}, d_{24} = s_{21}q_{15} + ss_{23}q_{35}$$
$$d_{31} = s_{31}q_{12} + s_{32}q_{22} + s_{33}q_{32}, d_{32} = s_{31}q_{13} + s_{32}q_{23} + s_{33}q_{33}$$
$$d_{33} = s_{31}q_{14} + ss_{33}q_{34}, d_{34} = s_{31}q_{15} + ss_{33}q_{35}$$

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