ORDER STATISTICS APPROACH FOR DETECTING A TRANSIENT SIGNAL OF UNKNOWN ARRIVAL TIME IN NOISE

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ABSTRACT

Detecting a transient signal of unknown arrival time in noise is actually a binary hypothesis test problem, where the null hypothesis (noise only) is a simple one, while the alternative hypothesis is composite. The generalized likelihood ratio test (GLRT) is a common tool to solve such problems. In this paper we show how order statistics (OS) approach can be used to solve the same problem. We show that the two hypothesis becomes simple using the OS approach so a likelihood ratio test (LRT) can be applied, and we discuss the trade-offs between the two solutions. In particular, we point out cases where the OS detector outperforms the GLRT.

1. SUMMARY

The first step in most application is to decide on the presence of a signal of interest in noisy measurements. Naturally, this problem has been studied deeply from the early days of statistical signal processing. If the additive noise is Gaussian and the arrival time, t_0, is known, the LRT detector is implemented by comparing the match filter (MF) output at t_0 to a threshold. If t_0 is unknown the GLRT detector is still the MF, but now the maximal value of the MF output is to be compared to a threshold [7]. However, if the noise is non-Gaussian, and/or if other signal parameters are unknown, the MF is no longer optimal. The modern digital signal processing (DSP) technology enables implementing the LRT or the GLRT even if they don't get the simple form of a linear filter. However, implementation of other, not necessarily simple processors, can also be considered.

The general problem of detecting a transient signal of unknown arrival time in noise can be formalized as a binary hypothesis test, where:

$$\begin{array}{rc} H_0: & \underline{x} = \underline{n} \\ H_1(L): & \underline{x} = \underline{s} + \underline{n} \end{array} \qquad (1)$$

where $\underline{x} = [x_1\ x_2 ... x_N]^t$ is a vector of the N measurements, $\underline{n} = [n_1\ n_2 ... n_N]^t$ is the noise vector and $\underline{s} = [\underline{0}_L^t\ s_1\ s_2 ... s_M\ \underline{0}_{N-M-L}^t]^t$ is the signal vector. $\underline{0}_m$ is a vector of m zeros and it is assumed that $N \geq L + M$. For a known transient signal of unknown time of arrival, the scalar coefficients $s_1, ..., s_M$ are known but L is unknown.

In the GLRT approach, the likelihood ratio between $H_1(L)$ and H_0 is evaluated, for $L = 1, ..., N-M$, and the maximal value is comapred to a threshold.

$$GLRT(\underline{x}) = \max_L \frac{f(\underline{x}|H_1(L))}{f(\underline{x}|H_0)} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \eta \qquad (2)$$

In this paper we suggest to order the vector $\underline{x}$ first. We show that the ordered measurement vector under all different hypothesis, $H_1(L); L = 1, ..., N - M$ has the same $p.d.f$, and therefore the LRT on the ordered vector can be implemented, instead of the GLRT on the non-ordered vector. It can be shown that the resulting processor is the same as the one based on the min-max approach as suggested in [5].

Denote by $\tilde{\underline{x}}$ the ordered measurements vector, the proposed processor is:

$$OSLRT(\tilde{\underline{x}}) = \frac{f(\tilde{\underline{x}}|H_1)}{f(\tilde{\underline{x}}|H_0)} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} \eta' \qquad (3)$$

We compare the two processors in terms of performance, complexity and robustness. We show that for cases where $L << N$ the OS detector outperform the GLRT even in the traditional case of a white, Gaussian vector. We also show that the OS detector can also be used (as is) for cases where the symbols $s_1, s_2, \ldots, s_M$ are known but their order and/or their distribution over the vector $\underline{x}$ is arbitrary and unknown. In such cases, the detector of (2) does not hold and a GLRT detector become much more complicated.

1.1. Basic Order Statistics Distributions

We first presents the basic notation and results from order statistics theory relevant for this paper. Complete tutorial

on the topic can be found in [1]. Let $\underline{X} = [X_1 \; \ldots X_N]$ be a random vector of length N. Let the $p.d.f$ of a random sample $\underline{x} = [x_1 \ldots x_N]$ from $\underline{X}$ be: $f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$. That is, the random variables $X_i; i = 1 \ldots N$ are statistically independent but not identically distributed. Denote by $x_{r:n}$ the $r-th$ smallest value of $\underline{x}$, that is $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$. Denote by $\underline{\tilde{x}}$ the ordered sample,
$\underline{\tilde{x}} \triangleq [x_{1:n} \; x_{2:n} \ldots x_{n:n}]$, and by $\underline{\tilde{X}}$ the random vector corresponds to the ordered sample.

The $p.d.f$ of the random vector $\underline{\tilde{X}}$ relates to the $p.d.f$ of the random vector $\underline{X}$ as follows [6]:

$$f_{\underline{\tilde{X}}}(\underline{\tilde{x}}) = \sum_{i=1}^{N!} \prod_{j=1}^{N} f_{X_{L_i(j)}}(x_{j:n}) \tag{4}$$

where L_i is the $i-th$ permutation of the vector $[1, \ldots, N]$ which is a vector of integers of length N. $L_i(j)$ is the $j-th$ element of the vector L_i. Recall that the vector $[1, \ldots, N]$ has $N!$ different permutation. If $X_1, \ldots, X_N$ are $i.i.d$ random variables than the probability density function of $\underline{\tilde{X}}$ becomes:

$$f_{\underline{\tilde{X}}}(\underline{\tilde{x}}) = N! f_{\underline{X}}(\underline{\tilde{x}}). \tag{5}$$

1.2. Detectors for a Known Signal with Unknown Arrival Time

Detecting known signal of unknown arrival time can be seen as a binary hypnosis test problem, the two hypnosis are: (1) signal does not exists; (2) signal exists. The second hypothesis is, actually, composed of $N-M$ different hypothesis, each one correspond to one of the $N-M$ possible arrival time of the signal.

In general, there is no uniformly most powerful (UMP) test for deciding between two hypothesis when at least one of the hypothesis is composite. The $GLRT$ is a common tool, although no nonasymptoticly optimality is claimed [7]. In cases where the two hypothesis are simple, optimal (UMP) solution exists and is given by the Nweman-Person, or the LRT. In our problem, the OS approach transforms the composite alternative hypothesis into a simple one, and therefore an LRT can be formulated.

Let $\underline{x}$ be the vector of N measurements. The likelihood of $\underline{\tilde{x}}$ under the null hypothesis is given in (5), that is $f_{\underline{\tilde{X}}}(\underline{\tilde{x}}|H_0) = N! f_{\underline{X}}(\underline{\tilde{x}}|H_0)$. The likelihood of $\underline{\tilde{x}}$ under the hypothesis that signal exists is given in (4). As can be seen from (4), this likelihood does not depends on the arrival time of the signal. The detection problem can be reformulated as deciding between two hypothesis: (1) $\underline{\tilde{x}}$ contains noise samples only - namely no signal; (2) $\underline{\tilde{x}}$ contains noise plus signal samples - namely signal exists. Since the likelihood of the sample, $\underline{\tilde{x}}$ can be written explicitly under both hypothesis, they are simple hypothesis and thus LRT test can be formulated. We define this detector as the $OSLRT$ detector and following (3) it is given by:

$$OSLRT(\underline{\tilde{x}}) = \frac{\sum_{i=1}^{N!} \prod_{j=1}^{N} f_{X_{L_i(j)}}(x_{j:n}|H_1)}{N! \prod_{j=1}^{N} f_X(x_{j:n}|H_0)} =$$

$$\frac{1}{N!} \sum_{i=1}^{N!} \prod_{j=1}^{N} \frac{f_{X_{L_i(j)}}(x_{j:n}|H_1)}{f_X(x_{j:n}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta' \tag{6}$$

The OSLRT is more complex than the GLRT - it needs to average over $\binom{N}{M}$ likelihood ratios instead on $N-M$ for the GLRT. This is due to the fact that the OSLRT makes no use on the information about the structure of the signal, e.g., the fact that the signal symbols form a continuous sequence $s_1, s_2, ..., s_M$. That is, when $N >> M$ we expect the OSLRT to work better than the GLRT because of its UMP property. However, if $N - M << N$ due to the use of the prior information on the structure of the signal, the GLRT needs to search over a small number of possibilities and therefore it should perform better than the OSLRT. This trade-off is demonstrated in the following example:

Let $N = 20$ $L = 1$ and $n(t_i)$ be a Gaussian R.V. with zero mean and unit variance. The signal symbols are $\sqrt{\frac{2}{M}}$. Fig. 1 present the probability of detection as function of P_{fa} (ROC curve) for the OS detector and the $GLRT$ detector for $M = 2$ and for $M = 18$. It shows that, as expected, for $M = 2$ the OSLRT outperforms the GLRT while for $M = 18$ the opposite holds.

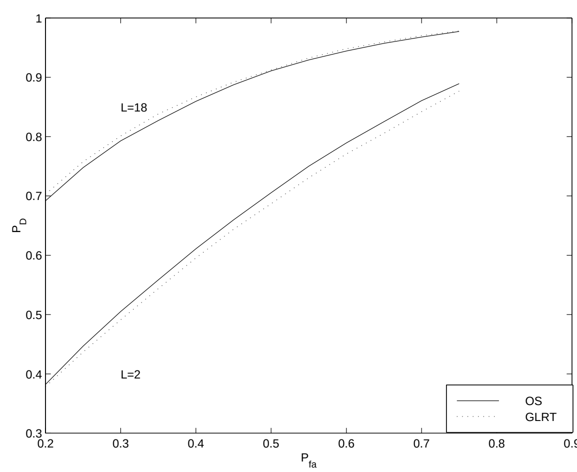

Figure 1: Probability of Detection

In Fig. 2 we presents the ROCs of the OSLRT and the GLRT where the symbols are not continuous in time. As expected, the performance of the OSLRT is the same as in the case of Fig. 1, while that of the GLRT is much worse. In this case, the OSLRT which is robust, outperforms the GLRT even for $M = 18$. A different GLRT, which takes into consideration the fact that the structure of the signal is unknown, can be designed. However, this later GLRT is no longer simple.

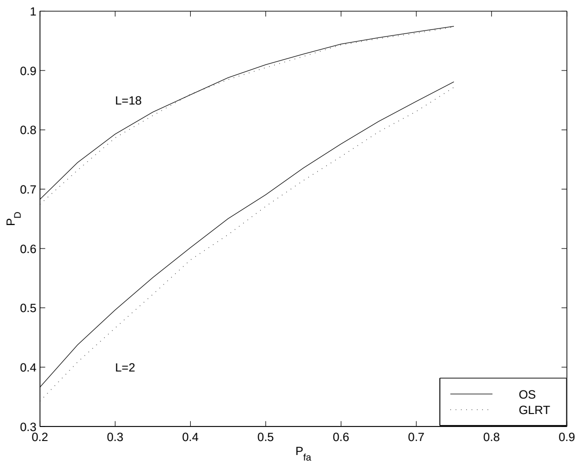

Figure 2: Probability of Detection

1.3. Concluding Remarks and Conclusions

The $OSLRT$, as presented here, demonstrates the ability to reduce problem dimensionality using order statistics. The OS approach trades the use of structural information for the ability to perform a UMP test. In cases where the signal duration is small compared with the observation time, the information carried in the time domain is small compared to the potential gain in using optimal procedure like the LRT, and thus it is profitable to ignore this information and to use the OS approach.

As expected, the OS detector is robust to the temporal structure of the signal and therefore it outperforms the GLRT if such modeling errors exist.

2. REFERENCES

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