THE BEST QUANTITATIVE KRONECKER'S THEOREM

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Abstract

The paper gives the best quantitative forms of Kronecker's theorem.

1. *Introduction*

Kronecker's classical theorem on Diophantine approximation is well known. It has two forms as follows. (The vector version is similar.)

THEOREM A. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any given real numbers and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be real numbers which are linearly independent over the rational number field. Then for any $\varepsilon > 0$ there exists a real number t such that*

$$\|\lambda_v t - \alpha_v\| < \varepsilon, \quad v = 1, 2, \dots, n,$$

where $\|x\|$ denotes the distance from x to the nearest integer.

THEOREM B. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any given real numbers and let $1, \lambda_1, \lambda_2, \dots, \lambda_n$ be real numbers which are linearly independent over the rational number field. Then for any $\varepsilon > 0$ there exists an integer t such that*

$$\|\lambda_v t - \alpha_v\| < \varepsilon, \quad v = 1, 2, \dots, n.$$

Theorems A and B are equivalent. For a proof see Hardy and Wright [**19**]. Many proofs of Theorems A and B have been given [**3–6**, **17**, **19**, **21–23**, **25**, **26**, **28–30**]. Although these theorems prove the existence of a suitable t, they give no bound on how big t is (and there is no such bound in general). If the condition 'linear independent' is relaxed, then we can obtain the quantitative forms of Theorems A and B. The quantitative form of Theorem A was given by Bacon [**1**] in 1934, as follows.

THEOREM A_1 (H. M. Bacon [**1**]). *Let α_j, λ_j $(1 \leqslant j \leqslant n)$ be real numbers and $M \geqslant 1$. If*

$$u_1 \lambda_1 + u_2 \lambda_2 + \dots + u_n \lambda_n = 0,$$

where u_j are integers with $|u_1| + |u_2| + \dots + |u_n| \leqslant M$, implies that $u_1 = u_2 = \dots = u_n = 0$, then there exists a real number t such that

$$\|\lambda_v t - \alpha_v\| < \frac{c(n)}{M},$$

where

$$c(n) = \frac{1}{2}(n-1)^{3/2}\left(\frac{125}{48}\right)^{(n^3-n)/12}.$$

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The quantitative form of Theorem B has appeared in [**6**, **20**, **21**]. This can be formulated as follows (in different notations, refer to [**6**]).

THEOREM B_1. *Let α_j, λ_j $(1 \leqslant j \leqslant n)$ be real numbers and $M \geqslant 1$. If*

$$u_1 \lambda_1 + u_2 \lambda_2 + \ldots + u_n \lambda_n$$

is an integer, where u_j are integers with $|u_j| \leqslant M$, implies that

$$u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_n \alpha_n$$

is an integer, then there exists an integer t such that

$$\|\lambda_v t - \alpha_v\| < \frac{((n+1)!)^2}{2^n} \frac{1}{M}, \quad v = 1, 2, \ldots, n.$$

Theorem A_1 was improved by Chen [**10**]. In this paper I obtain best possible results. The following two theorems are corollaries of our conclusions.

THEOREM A_2. *Let α_j, λ_j $(1 \leqslant j \leqslant n)$ be real numbers and $M \geqslant 1$ be an integer. If*

$$u_1 \lambda_1 + u_2 \lambda_2 + \ldots + u_n \lambda_n = 0,$$

where u_j are integers with $|u_j| \leqslant M$, implies that

$$u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_n \alpha_n$$

is an integer, then there exists a real number t such that

$$\sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2 < \frac{\pi^2}{16} \frac{n}{(M+1)^2}.$$

THEOREM B_2. *Let α_j, λ_j $(1 \leqslant j \leqslant n)$ be real numbers and $M \geqslant 1$ be an integer. If*

$$u_1 \lambda_1 + u_2 \lambda_2 + \ldots + u_n \lambda_n$$

is an integer, where u_j are integers with $|u_j| \leqslant M$, implies that

$$u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_n \alpha_n$$

is an integer, then there exists an integer t such that

$$\sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2 < \frac{\pi^2}{16} \frac{n}{(M+1)^2}.$$

Theorems A_2 and B_2 are tight up to a multiplicative constant independent of M and the dimension n. As we will show, for any $n \geqslant 2$ and $M \geqslant 1$ we may take λ_j, α_j $(1 \leqslant j \leqslant n)$ satisfying the conditions of Theorem A_2, but for any real number t,

$$\sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2 \geqslant \frac{2}{21} \frac{n}{(M+1)^2}.$$

The same is true for Theorem B_2. In Section 5 we give an application to view-obstruction problems and several different quantitative forms of Kronecker's theorem.

2. *Notations and the main results*

In the following $\lambda_j, \boldsymbol{\lambda}_j, \alpha_j$ are real vectors or numbers, δ_j are positive real numbers and M is an integer with $M \geqslant 1$. Let $\mathbf{R}$ denote the set of all real numbers and $\mathbf{Z}$ the set of all integers.

Now $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in A^{mn}$ means that $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n$ are m-dimensional vectors with $\boldsymbol{\lambda}_j \in A^m$ and $\alpha_j \in A$ $(1 \leqslant j \leqslant n)$.

For $\boldsymbol{\lambda}, \mathbf{t} \in A^m$ with $\boldsymbol{\lambda} = (a_1, a_2, \ldots, a_m)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_m)$, we define

$$\boldsymbol{\lambda}\mathbf{t} = a_1 t_1 + a_2 t_2 + \ldots + a_m t_m.$$

$$R_{mn}(M) = \{(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in \mathbf{R}^{mn} : \text{for integers } u_j \text{ with } |u_j| \leqslant M \ (1 \leqslant j \leqslant n),\ u_1 \boldsymbol{\lambda}_1 + \ldots + u_n \boldsymbol{\lambda}_n = 0 \text{ implies that } u_1 \alpha_1 + \ldots + u_n \alpha_n \text{ is an integer}\}.$$

$$T_{mn}(M) = \{(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in \mathbf{R}^{mn} : \text{for integers } u_j \text{ with } |u_j| \leqslant M \ (1 \leqslant j \leqslant n),\ u_1 \boldsymbol{\lambda}_1 + \ldots + u_n \boldsymbol{\lambda}_n \text{ is an integral vector implies that } u_1 \alpha_1 + \ldots + u_n \alpha_n \text{ is an integer}\}.$$

$$\Lambda_1 = \Lambda_1(M, \boldsymbol{\lambda}) = \min\{\max_j |u_1 \lambda_{1j} + u_2 \lambda_{2j} + \ldots + u_n \lambda_{nj}| : u_s \text{ are integers with } |u_s| \leqslant M \ (1 \leqslant s \leqslant n) \text{ and } u_1 \boldsymbol{\lambda}_1 + \ldots + u_n \boldsymbol{\lambda}_n \neq 0\},$$

$$\Lambda_2 = \Lambda_2(M, \boldsymbol{\lambda}) = \min\{\max_j \|u_1 \lambda_{1j} + u_2 \lambda_{2j} + \ldots + u_n \lambda_{nj}\| : u_s \text{ are integers with } |u_s| \leqslant M \ (1 \leqslant s \leqslant n) \text{ and } u_1 \boldsymbol{\lambda}_1 + \ldots + u_n \boldsymbol{\lambda}_n \text{ is not an integral vector}\},$$

where $\boldsymbol{\lambda}_s = (\lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{sm})$, $s = 1, 2, \ldots, n$. If all $\boldsymbol{\lambda}_j = 0$, let $\Lambda_1 = 1$. If all $\boldsymbol{\lambda}_j$ are integral vectors, let $\Lambda_2 = 1$.

$$S(T) = \{(t_1, t_2, \ldots, t_m) : T_j \leqslant t_j < T_j' \ (1 \leqslant j \leqslant m)\}.$$

$$\|S(T)\| = \min_j (T_j' - T_j). \qquad \Delta = \sum_{v=1}^{n} \delta_v.$$

Now we state our main results.

Theorem 1. (i) *If* $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in R_{mn}(M)$, *then*

$$\inf_{t \in S(T)} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 < \frac{\Delta}{4} \sin^2 \frac{\pi}{2(M+1)} + \frac{\Delta M^n}{4\pi \Lambda_1 \|S(T)\|}.$$

(ii) *If* T_j *and* T_j' $(1 \leqslant j \leqslant n)$ *are integers and if* $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in T_{mn}(M)$, *then*

$$\inf_{t \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 < \frac{\Delta}{4} \sin^2 \frac{\pi}{2(M+1)} + \frac{\Delta M^n}{8\Lambda_2 \|S(T)\|}.$$

THEOREM 2.

$$\sup_{R_{mn}(M)} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 = \sup_{R_{1n}(M)} \inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \delta_v \|\lambda_v t - \alpha_v\|^2,$$

$$\sup_{T_{mn}(M)} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 \geqslant \sup_{T_{1n}(M)} \inf_{t\in\mathbf{Z}} \sum_{v=1}^{n} \delta_v \|\lambda_v t - \alpha_v\|^2.$$

THEOREM 3. *Let $n \geqslant 2$. Then*

$$\frac{2}{21}\frac{n}{(M+1)^2} \leqslant \sup_{R_{mn}(M)} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 < \frac{\pi^2}{16}\frac{n}{(M+1)^2},$$

$$\frac{2}{21}\frac{n}{(M+1)^2} \leqslant \sup_{T_{mn}(M)} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 < \frac{\pi^2}{16}\frac{n}{(M+1)^2}.$$

NOTE 1. Theorems A_2 and B_2 follow from Theorem 3 immediately. If we consider $u_1 \boldsymbol{\lambda}_1 + \ldots + u_n \boldsymbol{\lambda}_n$ with $|u_j| \leqslant M_j$ $(1 \leqslant j \leqslant n)$, and define corresponding notations, then Theorems 1 and 2 are also true. The changes are only

$$\frac{\Delta}{4}\sin^2\frac{\pi}{2(M+1)} \quad \text{to} \quad \frac{1}{4}\sum_{v=1}^{n} \delta_v \sin^2\frac{\pi}{2(M_v+1)}$$

and

$$M^n \quad \text{to} \quad \prod_{v=1}^{n} M_v.$$

Theorem 3 is true for the upper bound

$$\frac{\pi^2}{16}\sum_{v=1}^{n}\frac{1}{(M_v+1)^2}$$

and the lower bound

$$\frac{1}{10}\frac{n-1}{n}\sum_{v=1}^{n}\frac{1}{(M_v+1)^2}.$$

Proofs of these are similar to those in this paper.

3. *Lemmas*

LEMMA 1. *There exist real numbers b_v such that $b_0 = 1$, $b_1 = \cos\pi/(M+1)$, $b_v = b_{-v}$ and $b_v > 0$ for $|v| \leqslant M-1$, $b_v = 0$ for $|v| \geqslant M$ and*

$$\sum_{v=-M+1}^{M-1} b_v e^{ivt} \geqslant 0 \qquad \textit{for all real numbers } t,$$

$$b_{u-1} + b_{u+1} \geqslant 2b_1 b_u \quad \textit{for all } u,$$

$$\sum_{v=-M+1}^{M-1} b_v \leqslant M.$$

Proof. Let $\theta = \pi/(M+1)$ and x_v be non-negative real numbers such that

$$x_v = x_{v+1}\frac{\sin(v+1)\theta}{\sin(v+2)\theta}, \quad \text{if } v = 0, 1, 2, \dots, M-2,$$

$$\sum_{v=0}^{M-1} x_v^2 = 1, \quad x_v = 0 \qquad \text{for } v \leqslant -1 \text{ or } v \geqslant M.$$

Then (note that $\sin M\theta = \sin\theta$)

$$x_v = x_{M-1}\frac{\sin(v+1)\theta}{\sin\theta}, \quad \text{if } v = -1, 0, 1, 2, \dots, M,$$

$$\frac{x_{M-1}^2}{\sin^2\theta}\sum_{v=0}^{M-1}\sin^2(v+1)\theta = 1,$$

$$\begin{aligned} x_{v-1}+x_{v+1} &= \frac{x_{M-1}}{\sin\theta}(\sin v\theta + \sin(v+2)\theta) \\ &= 2\cos\theta\frac{\sin(v+1)\theta}{\sin\theta}x_{M-1} \\ &= 2x_v\cos\theta, \quad \text{if } v = 0, 1, 2, \dots, M-1, \\ x_{v-1}+x_{v+1} \geqslant 0 &= 2x_v\cos\theta, \quad \text{if } v \leqslant -1 \text{ or if } v \geqslant M. \end{aligned}$$

Let

$$\sum_{v=-M+1}^{M-1} b_v e^{ivt} = \left|\sum_{v=0}^{M-1} x_v e^{ivt}\right|^2,$$

$$b_v = 0 \quad \text{for } |v| \geqslant M.$$

Then

$$\begin{aligned} b_0 &= \sum_{v=0}^{M-1} x_v^2 = 1, \quad b_{-v} = b_v, \\ b_v &= \sum_k x_k x_{k+v} > 0, \quad \text{if } |v| \leqslant M-1, \\ b_1 &= \sum_v x_v x_{v+1} = \frac{x_{M-1}^2}{\sin^2\theta}\sum_{v=0}^{M-1}\sin(v+1)\theta\sin(v+2)\theta \\ &= \frac{x_{M-1}^2}{\sin^2\theta}\sum_{v=0}^{M-1}\sin(v+1)\theta(\sin(v+1)\theta\cos\theta + \cos(v+1)\theta\sin\theta) \\ &= \cos\theta\frac{x_{M-1}^2}{\sin^2\theta}\sum_{v=0}^{M-1}\sin^2(v+1)\theta \\ &= \cos\theta, \\ b_{u-1}+b_{u+1} &= \sum_v (x_v x_{v+u-1} + x_v x_{v+u+1}) \\ &\geqslant 2\cos\theta\sum_v x_v x_{v+u} \\ &= 2b_1 b_u, \\ \sum_{v=-M+1}^{M-1} b_v &= \left|\sum_{v=0}^{M-1} x_v\right|^2 \leqslant M\sum_{v=0}^{M-1} x_v^2 = M. \end{aligned}$$

This completes the proof of Lemma 1. □

LEMMA 2. *Let*

$$\lambda_j = \frac{1}{(M+1)^{n-j+1}} \quad (1 \leqslant j \leqslant n).$$

Then, for any $(\alpha_1, \ldots, \alpha_n)$,

$$(\lambda_1, \lambda_2, \ldots, \lambda_n; \alpha_1, \ldots, \alpha_n) \in T_{1n}(M).$$

Proof. Suppose that $u_1, u_2, \ldots, u_n$ are integers with $|u_j| \leqslant M$ $(1 \leqslant j \leqslant n)$ such that

$$u_1 \lambda_1 + u_2 \lambda_2 + \ldots + u_n \lambda_n = d$$

is an integer. Then

$$u_n(M+1)^{n-1} + u_{n-1}(M+1)^{n-2} + \ldots + u_1 = d(M+1)^n.$$

This implies that $M+1 \mid u_1$. By $|u_1| \leqslant M$ we have $u_1 = 0$. Hence

$$u_n(M+1)^{n-2} + u_{n-1}(M+1)^{n-3} + \ldots + u_2 = d(M+1)^{n-1}.$$

Similarly, we have $u_2 = 0$. Continuing this procedure, we have $u_1 = u_2 = \ldots = u_n = 0$. Therefore

$$(\lambda_1, \lambda_2, \ldots, \lambda_n; \alpha_1, \ldots, \alpha_n) \in T_{1n}(M).$$

This completes the proof of Lemma 2. □

4. *Proofs of theorems*

Proof of Theorem 1. We only give a detailed proof of Theorem 1(ii). We give an outline proof of Theorem 1(i). For this refer to the proofs of [**9**, Lemma 3; **10**, Lemma 3].

If some $\boldsymbol{\lambda}_v$ is an integral vector, then by the assumption α_v is an integer. Thus for integral vector variable $\mathbf{t}$, $\|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\| = 0$, so, without loss of generality, we may assume that none of $\boldsymbol{\lambda}_v$ is an integral vector. In the following let $\mathbf{t}$ be an integral vector variable. We take b_v as given in Lemma 1. Now let

$$F(\mathbf{t}) = \sum_{v=1}^{n} \delta_v (e^{-2\pi i(\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v)} + e^{2\pi i(\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v)}),$$

$$G(t) = \sum_{v=-M+1}^{M-1} b_v e^{2\pi i v t}$$

and

$$H(\mathbf{t}) = G(\boldsymbol{\lambda}_1 \mathbf{t} - \alpha_1) \ldots G(\boldsymbol{\lambda}_n \mathbf{t} - \alpha_n).$$

Let

$$S_2 = \{(u_1, u_2, \ldots, u_n) : u_j \ (1 \leqslant j \leqslant n) \text{ are integers and } u_1 \boldsymbol{\lambda}_1 + u_2 \boldsymbol{\lambda}_2 + \ldots + u_n \boldsymbol{\lambda}_n \equiv 0 \pmod 1\},$$

$$S_2' = \{(v_1, v_2, \ldots, v_n) : v_j \ (1 \leqslant j \leqslant n) \text{ are integers and there exists a } (u_1, u_2, \ldots, u_n) \in S_2 \text{ such that } |u_1 - v_1| + \ldots + |u_n - v_n| = 1\},$$

$$a = 2 \sum_{v=1}^{n} \delta_v \sum_{\substack{(v_1, \ldots, v_n) \in S_2' \\ v_1 \boldsymbol{\lambda}_1 + \ldots + v_n \boldsymbol{\lambda}_n \equiv \boldsymbol{\lambda}_v \pmod 1}} b_{v_1} \ldots b_{v_n},$$

$$b = \sum_{(u_1, \ldots, u_n) \in S_2} b_{u_1} \ldots b_{u_n}.$$

Then $S_2 \cap S_2' = \varnothing$ for none of $\boldsymbol{\lambda}_v$ being an integral vector. Since $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n) \in T_{mn}(M)$, we have (noting the fact that $b_v = 0$ for $|v| \geqslant M$ and $b_{-v} = b_v$ for all v)

$$H(\mathbf{t}) = b + \sum_{(v_1, \ldots, v_n) \in S_2'} b_{v_1} \ldots b_{v_n} e^{2\pi i((v_1 \boldsymbol{\lambda}_1 + v_2 \boldsymbol{\lambda}_2 + \ldots + v_n \boldsymbol{\lambda}_n)\mathbf{t} - (v_1 \alpha_1 + \ldots + v_n \alpha_n))} + S(\mathbf{t}),$$

$$\begin{aligned} H(\mathbf{t})\, F(\mathbf{t}) &= \sum_{v=1}^{n} \delta_v \sum_{\substack{(v_1, \ldots, v_n) \in S_2' \\ v_1 \boldsymbol{\lambda}_1 + \ldots + v_n \boldsymbol{\lambda}_n \equiv -\boldsymbol{\lambda}_v \pmod 1}} b_{v_1} b_{v_2} \ldots b_{v_n} \\ &\quad + \sum_{v=1}^{n} \delta_v \sum_{\substack{(v_1, \ldots, v_n) \in S_2' \\ v_1 \boldsymbol{\lambda}_1 + \ldots + v_n \boldsymbol{\lambda}_n \equiv \boldsymbol{\lambda}_v \pmod 1}} b_{v_1} b_{v_2} \ldots b_{v_n} + R(\mathbf{t}) \\ &= a + R(\mathbf{t}) \end{aligned}$$

where $S(\mathbf{t})$ is a trigonometrical polynomial whose exponents are all different from 0, $\pm 2\pi \boldsymbol{\lambda}_v$ $(1 \leqslant v \leqslant n)$ modulo 2π, and $R(\mathbf{t})$ is a trigonometrical polynomial whose exponents are all different from 0 modulo 2π. Similarly to the proof of [**9**, Lemma 3] by $\delta_v > 0$ $(1 \leqslant v \leqslant n)$ we have

$$F(\mathbf{t}) \leqslant 2\Delta - 16 \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2$$

holds for all integral vectors $\mathbf{t} \in S(T) \cap \mathbf{Z}^m$. Since $H(\mathbf{t}) \geqslant 0$ for all integral vectors $\mathbf{t}$, we have

$$H(\mathbf{t})\, F(\mathbf{t}) \leqslant 2\Delta H(\mathbf{t}) - 16 H(\mathbf{t}) \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2$$

holds for all $\mathbf{t} \in S(T) \cap \mathbf{Z}^m$. That is,

$$\begin{aligned} &16b \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 \\ &\quad \leqslant 2\Delta b - a + (H(\mathbf{t}) - b)\left(2\Delta - 16 \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2\right) - (H(\mathbf{t})\, F(\mathbf{t}) - a). \quad (1) \end{aligned}$$

Noting that

$$0 \leqslant \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 \leqslant \tfrac{1}{4}\Delta,$$

taking summation of (1) by

$$\sum = \left(\prod_{j=1}^{m} \frac{1}{T_j' - T_j}\right) \sum_{T_1 \leqslant t_1 < T_1'} \cdots \sum_{T_m \leqslant t_m < T_m'},$$

we have

$$16b \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 \leqslant 2\Delta b - a + 2\Delta \left|\sum (H(\mathbf{t}) - b)\right| + \left|\sum (H(\mathbf{t})\, F(\mathbf{t}) - a)\right|. \quad (2)$$

Noting that for $\alpha \in \mathbf{R}$ and integers s and s' with $s < s'$,

$$\left|\frac{1}{s'-s}\sum_{s \leqslant t < s'} e^{2\pi i \alpha t}\right| \leqslant 1$$

and for $\alpha \in \mathbf{R} \backslash Z$,

$$\left|\frac{1}{s'-s}\sum_{s \leqslant t < s'} e^{2\pi i \alpha t}\right| = \frac{1}{s'-s}\left|\frac{\sin \pi\alpha(s'-s)}{\sin \pi\alpha}\right| \leqslant \frac{1}{2(s'-s)\,\|\alpha\|},$$

we have

$$\begin{aligned} \left|\sum e^{2\pi i(a_1 t_1 + a_2 t_2 + \ldots + a_m t_m)}\right| &= \left|\prod \frac{1}{T_j' - T_j}\sum_{T_j \leqslant t_j < T_j'} e^{2\pi i a_j t_j}\right| \\ &\leqslant \frac{1}{2\min_j(T_j' - T_j)\max_j \|a_j\|}. \end{aligned}$$

Hence, by Lemma 1 and $b \geqslant b_0^n = 1$,

$$\begin{aligned} \left|\sum (H(\mathbf{t}) - b)\right| &\leqslant \frac{1}{2\min_j(T_j' - T_j)\,\Lambda_2}(G(0)^n - b) \\ &< \frac{1}{2\|S(T)\|\,\Lambda_2} M^n, \end{aligned} \tag{3}$$

$$\left|\sum (H(\mathbf{t})\,F(\mathbf{t}) - a)\right| \leqslant \frac{\Delta}{\|S(T)\|\,\Lambda_2} M^n. \tag{4}$$

Again, by Lemma 1 (note that $b_{-v} = b_v$ for all v),

$$\begin{aligned} a &= 2\sum_{v=1}^{n} \delta_v \sum_{\substack{(v_1, \ldots, v_n) \in S_2' \\ v_1\lambda_1 + \ldots + v_n\lambda_n \equiv \lambda_v (\mathrm{mod}\,1)}} b_{v_1} b_{v_2} \ldots b_{v_n} \\ &= \sum_{v=1}^{n} \delta_v \sum_{(u_1, \ldots, u_n) \in S_2} b_{u_1} \ldots b_{u_{v-1}} (b_{u_v - 1} + b_{u_v + 1})\, b_{u_{v+1}} \ldots b_{u_n} \\ &\geqslant 2b_1 \sum_{v=1}^{n} \delta_v \sum_{(u_1, \ldots, u_n) \in S_2} b_{u_1} b_{u_2} \ldots b_{u_n} \\ &= 2\Delta b b_1. \end{aligned} \tag{5}$$

Thus by (2), (3), (4) and (5) we have

$$16b \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 < 2\Delta b - 2\Delta b b_1 + \frac{2\Delta}{\|S(T)\|\,\Lambda_2} M^n. \tag{6}$$

By (6), $b \geqslant b_0^n = 1$ and Lemma 1 we have

$$\begin{aligned} \inf_{\mathbf{t} \in S(T) \cap \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 &< \frac{\Delta}{8}(1 - b_1) + \frac{\Delta M^n}{8\|S(T)\|\,\Lambda_2} \\ &= \frac{\Delta}{4}\sin^2 \frac{\pi}{2(M+1)} + \frac{\Delta M^n}{8\|S(T)\|\,\Lambda_2}. \end{aligned}$$

For a proof of Theorem 1(i) we need only make some adjustments:

(I) Without loss of generality, assume that none of $\boldsymbol{\lambda}_v$ is a zero vector.

(II) Let

$$S_1 = \{(u_1, u_2, \ldots, u_n)\colon u_j\ (1 \leqslant j \leqslant n) \text{ are integers and } u_1 \boldsymbol{\lambda}_1 + u_2 \boldsymbol{\lambda}_2 + \ldots + u_n \boldsymbol{\lambda}_n = 0\},$$

$$S_1' = \{(v_1, v_2, \ldots, v_n)\colon v_j\ (1 \leqslant j \leqslant n) \text{ are integers and there exists a } (u_1, u_2, \ldots, u_n) \in S_1 \text{ such that } |u_1 - v_1| + \ldots + |u_n - v_n| = 1\},$$

$$a = 2 \sum_{v=1}^{n} \delta_v \sum_{\substack{(v_1, \ldots, v_n) \in S_1' \\ v_1 \boldsymbol{\lambda}_1 + \ldots + v_n \boldsymbol{\lambda}_n = \boldsymbol{\lambda}_v}} b_{v_1} \ldots b_{v_n},$$

$$b = \sum_{(u_1, \ldots, u_n) \in S_1} b_{u_1} \ldots b_{u_n}.$$

(III) Use

$$\int = \left(\prod_{j=1}^{m} \frac{1}{T_j' - T_j} \right) \int_{T_1}^{T_1'} \ldots \int_{T_m}^{T_m'} dt_1 \ldots dt_m$$

instead of $\sum$.

(IV) Let $\mathbf{t}$ be a real vector variable.

The other parts of the proof are completely similar. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $\boldsymbol{\lambda} = (1, 1, \ldots, 1) \in \mathbf{R}^m$. For any $(\lambda_1, \ldots, \lambda_n; \alpha_1, \ldots, \alpha_n) \in R_{1n}(M)$ we have $(\lambda_1 \boldsymbol{\lambda}, \ldots, \lambda_n \boldsymbol{\lambda}; \alpha_1, \ldots, \alpha_n) \in R_{mn}(M)$. Thus

$$\inf_{t \in \mathbf{R}} \sum_{v=1}^{n} \delta_v \| \lambda_v t - \alpha_v \|^2 = \inf_{\mathbf{t} \in \mathbf{R}^m} \sum_{v=1}^{n} \delta_v \| \lambda_v \boldsymbol{\lambda} \mathbf{t} - \alpha_v \|^2$$

$$\leqslant \sup_{R_{mn}(M)} \inf_{\mathbf{t} \in \mathbf{R}^m} \sum_{v=1}^{n} \delta_v \| \boldsymbol{\lambda}_v \mathbf{t} - \alpha_v \|^2.$$

Hence

$$\sup_{R_{1n}(M)} \inf_{t \in \mathbf{R}} \sum_{v=1}^{n} \delta_v \| \lambda_v t - \alpha_v \|^2 \leqslant \sup_{R_{mn}(M)} \inf_{\mathbf{t} \in \mathbf{R}^m} \sum_{v=1}^{n} \delta_v \| \boldsymbol{\lambda}_v \mathbf{t} - \alpha_v \|^2.$$

Similarly, we have

$$\sup_{T_{1n}(M)} \inf_{t \in \mathbf{Z}} \sum_{v=1}^{n} \delta_v \| \lambda_v t - \alpha_v \|^2 \leqslant \sup_{T_{mn}(M)} \inf_{\mathbf{t} \in \mathbf{Z}^m} \sum_{v=1}^{n} \delta_v \| \boldsymbol{\lambda}_v \mathbf{t} - \alpha_v \|^2.$$

Now let

$$(\boldsymbol{\lambda}_1', \ldots, \boldsymbol{\lambda}_n'; \alpha_1, \ldots, \alpha_n) \in R_{mn}(M)$$

and

$$A = \{u_1 \boldsymbol{\lambda}_1' + \ldots + u_n \boldsymbol{\lambda}_n'\colon u_j \text{ are integers with } |u_j| \leqslant M\ (1 \leqslant j \leqslant n)\}.$$

Then A is a finite set. Let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_k$ be all nonzero vectors in A. For each $\boldsymbol{\beta}_j$ there exists an l_j such that, if $l \geqslant l_j$, then $\boldsymbol{\beta}_j(1, l, \ldots, l^{m-1}) \neq 0$. Hence there exists a vector $\mathbf{L}_0 = (1, l, \ldots, l^{m-1})$ such that

$$\boldsymbol{\beta}_j \mathbf{L}_0 \neq 0, \quad j = 1, 2, \ldots, k.$$

If $A = \{0\}$, we take $\mathbf{L}_0 = (1, 1, \ldots, 1)$. Thus

$$(\boldsymbol{\lambda}_1' \mathbf{L}_0, \ldots, \boldsymbol{\lambda}_n' \mathbf{L}_0 ; \alpha_1, \ldots, \alpha_n) \in R_{1n}(M),$$

whence

$$\inf_{\mathbf{t} \in \mathbf{R}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v' \mathbf{t} - \alpha_v\|^2 \leqslant \inf_{t \in \mathbf{R}} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v' \mathbf{L}_0 t - \alpha_v\|^2$$
$$\leqslant \sup_{R_{1n}(M)} \inf_{t \in \mathbf{R}} \sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2.$$

Thus

$$\sup_{R_{1n}(M)} \inf_{t \in \mathbf{R}} \sum_{v=1}^{n} \delta_v \|\lambda_v t - \alpha_v\|^2 \geqslant \sup_{R_{mn}(M)} \inf_{\mathbf{t} \in \mathbf{R}^m} \sum_{v=1}^{n} \delta_v \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2.$$

From the above arguments we obtain a proof of Theorem 2. □

Proof of Theorem 3. The upper bounds are immediate results of Theorem 1. For the lower bounds, by Theorem 2 we need only consider the case $m = 1$. Take

$$\lambda_v = \frac{1}{(M+1)^{n-v+1}}, \quad \alpha_v = \frac{1}{2M}((M+1)^v - 1), \quad v = 1, 2, \ldots, n.$$

Then for any real number t we have

$$(M+1)(\lambda_v t - \alpha_v) - (\lambda_{v+1} t - \alpha_{v+1}) = \tfrac{1}{2}, \quad v = 1, 2, \ldots, n-1.$$

Hence (note that M is an integer with $M \geqslant 1$)

$$\tfrac{1}{2} = \|(M+1)(\lambda_v t - \alpha_v) - (\lambda_{v+1} t - \alpha_{v+1})\|$$
$$\leqslant (M+1)\|\lambda_v t - \alpha_v\| + \|\lambda_{v+1} t - \alpha_{v+1}\|,$$

so

$$\frac{n-1}{2} \leqslant (M+1)\|\lambda_1 t - \alpha_1\| + \sum_{v=2}^{n-1} (M+2)\|\lambda_v t - \alpha_v\| + \|\lambda_n t - \alpha_n\|,$$

whence

$$\left(\frac{n-1}{2}\right)^2 \leqslant ((M+1)^2 + (n-2)(M+2)^2 + 1) \sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2.$$

Thus

$$\sum_{v=1}^{n} \|\lambda_v t - \alpha_v\|^2 \geqslant \frac{1}{4} \frac{(n-1)^2}{(M+1)^2 + (n-2)(M+2)^2 + 1} \geqslant \frac{2}{21} \frac{n}{(M+1)^2}$$

holds for all real numbers t and $n \geqslant 2$, $M \geqslant 1$. By Lemma 2 we have

$$(\lambda_1, \lambda_2, \ldots, \lambda_n ; \alpha_1, \ldots, \alpha_n) \in T_{1n}(M).$$

Since $T_{1n}(M) \subseteq R_{1n}(M)$, we also have

$$(\lambda_1, \lambda_2, \ldots, \lambda_n ; \alpha_1, \ldots, \alpha_n) \in R_{1n}(M).$$

This completes the proof of Theorem 3. □

5. *Remarks*

5.1. *On* b_1

If b_v are real numbers with $b_0 = 1$ such that

$$\sum_{v=-M+1}^{M-1} b_v e^{ivt} \geqslant 0 \quad \text{for all real numbers } t,$$

then

$$|b_1| \leqslant \cos\frac{\pi}{M+1},$$

whence by Lemma 1, $\sup|b_1| = \cos\pi/(M+1)$.

For a proof see [**16**, **18**, **24**, **27**]. In fact by a theorem of Fejér and Riesz there exist numbers $x_0, x_1, \ldots, x_{M-1}$ such that

$$\sum_{v=-M+1}^{M-1} b_v e^{ivt} = \left|\sum_{v=0}^{M-1} x_v e^{ivt}\right|^2.$$

It is clear that

$$\begin{aligned}
2b_0\cos\alpha \pm 2b_1 &= (2\cos\alpha)\sum_v |x_v|^2 \pm \sum_v (\bar{x}_v x_{v+1} + x_v \bar{x}_{v+1}) \\
&= \frac{\sin 2\alpha}{\sin\alpha}\left|x_0 \pm x_1\frac{\sin\alpha}{\sin 2\alpha}\right|^2 + \frac{\sin 3\alpha}{\sin 2\alpha}\left|x_1 \pm x_2\frac{\sin 2\alpha}{\sin 3\alpha}\right|^2 + \ldots \\
&\quad + \frac{\sin M\alpha}{\sin(M-1)\alpha}\left|x_{M-2} \pm x_{M-1}\frac{\sin(M-1)\alpha}{\sin M\alpha}\right|^2 + \frac{\sin(M+1)\alpha}{\sin M\alpha}|x_{M-1}|^2.
\end{aligned}$$

Thus

$$2b_0\cos\frac{\pi}{M+1} \pm 2b_1 \geqslant 0,$$

whence

$$|b_1| \leqslant \cos\frac{\pi}{M+1}.$$

Similarly, we have

$$\sup|b_k| = \cos\frac{\pi}{[(M+k-1)/k]+1}, \quad k = 1, 2, \ldots, M-1.$$

5.2. *View-obstruction problems*

Now we make some remarks on view-obstruction problems. View-obstruction problems for cubes and spheres are to determine the values of $\lambda(n)$ and $\nu(n)$, where

$$\lambda(n) = 2 \sup_{\substack{a_1, \ldots, a_n \\ a_j > 0}} \inf_{t\in\mathbf{R}} \max_{1\leqslant j\leqslant n} \|a_j t - \tfrac{1}{2}\|,$$

$$\tfrac{1}{4}\nu(n)^2 = \sup_{\substack{a_1, \ldots, a_n \\ a_j > 0}} \inf_{t\in\mathbf{R}} \sum_{v=1}^{n} \|a_j t - \tfrac{1}{2}\|^2.$$

By the well-known Dirichlet theorem it suffices to consider the case when $a_1, a_2, \ldots, a_n$ are positive integers. The view-obstruction problem was given its name by T. W. Cusick [**12**] in 1973. The related problem corresponding to cubes on Diophantine approximation was considered in 1968 by J. M. Wills [**31**]. For $n = 2$ we have $\lambda(2) = 1/3$ and $\nu(2) = 1/\sqrt{5}$ (see [**12**]). Up to now, for true values of $\lambda(n)$ and $\nu(n)$, we only know that $\lambda(3) = 1/2$ (Cusick [**12**]), $\lambda(4) = 3/5$ (Cusick and Pomerance [**13**]), $\nu(3) = \sqrt{3/7}$ (Dumir and Hans-Gill [**14**]), $\nu(4) = \sqrt{11/15}$ (Chen [**7**]), and $\nu(5) = \sqrt{41/42}$ (Dumir, Hans-Gill and Wilker [**15**]). Cusick [**12**] conjectured that

$$\lambda(n) = \frac{n-1}{n+1}.$$

In [**8**, **9**] I proved that

$$\lim_{n\to\infty} \frac{\nu(n)^2}{n}$$

exists and its value is a positive number. For the other references see [**2**, **11**]. Let $n \geqslant 2$. The proof of Theorem 1 in this paper and the results in [**9**, **10**] imply the following theorem.

THEOREM 4. (i) *If $a_1, a_2, \ldots, a_n$ are positive integers such that*

$$\inf_{t\in[0,1]} \sum_{v=1}^{n} \|a_v t - \tfrac{1}{2}\|^2 = \tfrac{1}{4}\nu(n)^2,$$

then there exist integers $c_1, c_2, \ldots, c_n$ with $|c_v| \leqslant 3$ $(1 \leqslant v \leqslant n)$ and at most one of c_v is 3 or -3 such that

$$c_1 a_1 + c_2 a_2 + \ldots + c_n a_n = 0,$$
$$2 \nmid c_1 + c_2 + \ldots + c_n.$$

(ii) *If $a_1, a_2, \ldots, a_n$ are positive integers such that*

$$2 \inf_{t\in[0,1]} \max_{1\leqslant v\leqslant n} \|a_v t - \tfrac{1}{2}\| = \lambda(n),$$

then there exist integers $c_1, c_2, \ldots, c_n$ with $|c_v| \leqslant (\pi/2)\sqrt{n} + \pi$ $(1 \leqslant v \leqslant n)$ such that

$$c_1 a_1 + c_2 a_2 + \ldots + c_n a_n = 0,$$
$$2 \nmid c_1 + c_2 + \ldots + c_n.$$

Currently, the problem has not been reduced to a finite search in any dimension except the above known cases. We do not know whether Theorem 4 can be used to reduce the search for the optimal $(a_1, \ldots, a_n)$ to a finite search.

5.3. *Quantitative forms of Kronecker's theorem*

Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \in [-M, M]^n \cap \mathbf{Z}^n,$$
$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_n) \in \mathbf{R}^{mn},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{R}^n.$$

Write

$$\mathbf{u}\boldsymbol{\lambda} = u_1 \boldsymbol{\lambda}_1 + u_2 \boldsymbol{\lambda}_2 + \ldots + u_n \boldsymbol{\lambda}_n,$$
$$\mathbf{u}\boldsymbol{\alpha} = u_1 \alpha_1 + u_2 \alpha_2 + \ldots + u_n \alpha_n.$$

We say that

$$\mathbf{u}\boldsymbol{\lambda} = 0 \Rightarrow \mathbf{u}\boldsymbol{\alpha} \in \mathbf{Z}$$

means that for any $\mathbf{u} \in [-M, M]^n \cap \mathbf{Z}^n$, $\mathbf{u}\boldsymbol{\lambda} = 0$ implies that $\mathbf{u}\boldsymbol{\alpha} \in \mathbf{Z}$. Let

$$\kappa_0(m,n,M) = \sup_{\mathbf{u}\lambda \in \mathbf{Z}^m \Rightarrow \mathbf{u}=0} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

$$\kappa_1(m,n,M) = \sup_{\mathbf{u}\lambda = 0 \Rightarrow \mathbf{u}=0} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

$$\kappa_2(m,n,M) = \sup_{\mathbf{u}\lambda \in \mathbf{Z}^m \Rightarrow \mathbf{u}\alpha \in \mathbf{Z}} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

$$\kappa_3(m,n,M) = \sup_{\mathbf{u}\lambda = 0 \Rightarrow \mathbf{u}\alpha \in \mathbf{Z}} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

$$\kappa_4(m,n,M) = \sup_{\mathbf{u}\lambda \in \mathbf{Z}^m \Rightarrow \mathbf{u}=0} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

$$\kappa_\infty(m,n,M) = \sup_{\mathbf{u}\lambda \in \mathbf{Z}^m \Rightarrow \mathbf{u}\alpha \in \mathbf{Z}} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,$$

where each supremum is taken over all $(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n; \alpha_1, \ldots, \alpha_n)$ which satisfy the corresponding conditions. Then we have the relations in the following theorem.

THEOREM 5.

$$\begin{aligned}
\kappa_j(1,n,M) &= \kappa_j(m,n,M), && \text{for } j = 0,1,2,3,\\
\kappa_0(m,n,M) &= \kappa_1(m,n,M), &&\\
\kappa_2(m,n,M) &= \kappa_3(m,n,M), &&\\
\kappa_j(1,n,M) &\leqslant \kappa_j(m,n,M), && \text{for } j = 4, \infty,\\
\kappa_0(m,n,M) &\leqslant \kappa_2(m,n,M) \leqslant \kappa_\infty(m,n,M), &&\\
\kappa_0(m,n,M) &\leqslant \kappa_4(m,n,M) \leqslant \kappa_\infty(m,n,M). &&
\end{aligned}$$

Proof. First we give a detailed proof of $\kappa_0(m,n,M) = \kappa_1(m,n,M)$. It is obvious that $\kappa_0(m,n,M) \leqslant \kappa_1(m,n,M)$. Now fix $\boldsymbol{\lambda}, \boldsymbol{\alpha}$ such that $\mathbf{u}\boldsymbol{\lambda} = 0 \Rightarrow \mathbf{u} = 0$. Since $\{\mathbf{u}\boldsymbol{\lambda} : \mathbf{u} \in [-M, M]^n \cap \mathbf{Z}^n\}$ is a finite set, there exists a real number $s \neq 0$ such that

$$\mathbf{u}\boldsymbol{\lambda}s \in \mathbf{Z}^m \Leftrightarrow \mathbf{u}\boldsymbol{\lambda} = 0.$$

Thus

$$\mathbf{u}\boldsymbol{\lambda}s \in \mathbf{Z}^m \Rightarrow \mathbf{u} = 0.$$

Hence

$$\begin{aligned}
\inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 &= \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v s\mathbf{t} - \alpha_v\|^2\\
&\leqslant \sup_{\mathbf{u}\lambda \in \mathbf{Z}^m \Rightarrow \mathbf{u}=0} \inf_{\mathbf{t}\in\mathbf{R}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2,
\end{aligned}$$

so

$$\kappa_1(m,n,M) \leqslant \kappa_0(m,n,M).$$

Therefore

$$\kappa_0(m,n,M) = \kappa_1(m,n,M).$$

Similarly, we can prove that

$$\kappa_2(m,n,M) = \kappa_3(m,n,M).$$

Similarly to the proof of Theorem 2 we have

$$\kappa_1(1,n,M) = \kappa_1(m,n,M), \quad \kappa_3(1,n,M) = \kappa_3(m,n,M),$$
$$\kappa_j(1,n,M) \leqslant \kappa_j(m,n,M), \quad \text{for } j = 4, \infty.$$

Thus

$$\kappa_0(1,n,M) = \kappa_1(1,n,M) = \kappa_1(m,n,M) = \kappa_0(m,n,M),$$
$$\kappa_2(1,n,M) = \kappa_3(1,n,M) = \kappa_3(m,n,M) = \kappa_3(m,n,M).$$

The other relations are clear. This completes the proof of Theorem 5. □

NOTE 2. By taking $\boldsymbol{\lambda}_v = ((M+1)^v, (M+1)^v, \ldots, (M+1)^v)$, $\alpha_v = \frac{1}{2}$ we have

$$\sup_{\mathbf{u}\boldsymbol{\lambda}=0 \Rightarrow \mathbf{u}=0} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 = \tfrac{1}{4}n,$$

$$\sup_{\mathbf{u}\boldsymbol{\lambda}=0 \Rightarrow \mathbf{u}\boldsymbol{\alpha}\in\mathbf{Z}} \inf_{\mathbf{t}\in\mathbf{Z}^m} \sum_{v=1}^{n} \|\boldsymbol{\lambda}_v \mathbf{t} - \alpha_v\|^2 = \tfrac{1}{4}n,$$

so we have not introduced the corresponding κ.

NOTE 3. By the proof of Theorem 3 and Theorem 5 we have

$$\frac{2}{21}\frac{n}{(M+1)^2} \leqslant \kappa_j(m,n,M) < \frac{\pi^2}{16}\frac{n}{(M+1)^2}, \quad j = 0,1,2,3,4,\infty.$$

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