RADIAL DIVERGENCE IN BMOA

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ABSTRACT

The set of boundary points at which a BMOA function in the unit ball of $\mathbb{C}^d$ fails to have a radial limit may be an arbitrary G_δ of measure zero. Similar results are given for the Bloch space and for VMOA. As an application it is shown that there exist functions with no radial limits in the holomorphic Besov spaces interpolating between the Hardy space H^2 and the Bloch space.

0. *Introduction*

Notation and terminology will be for the most part as in [**20**]: the notation $H^p = H^p(B)$ $(0 < p \leqslant \infty)$ will refer to the usual Hardy spaces of holomorphic functions in $B_d = B$, the (Euclidean) unit ball of $\mathbb{C}^d$, and the boundary of B will be denoted $S_d = S$. We will set $D = B_1$. The rotation-invariant Borel probability measure on S is σ.

It is very well known that a bounded holomorphic function in D must possess a radial limit at almost every point of the boundary. In fact this theorem of Fatou is optimal: if $E \subset S_1$ has measure zero then there exists a bounded holomorphic function with a radial limit at no point of E (cf. the introduction to [**17**]).

The situation in B_d $(d \geqslant 2)$ is not quite so simple. While it is certainly still true that a bounded holomorphic function must have a radial limit almost everywhere (this follows easily from the corresponding fact in one variable), this is far from a characterization of the sets of radial divergence for H^∞-functions. For example, any function in $H^\infty(B_2)$ must have a limit at the point $(e^{it}, 0)$ for almost every value of t, although these points form a set of measure zero; this is because the points $(e^{it}, 0)$ form the boundary of a (smoothly) embedded holomorphic disc. Nagel and Rudin [**14**; **20**, Theorem 11.2.4] have given a remarkable generalization: if γ is a $C^{1+\varepsilon}$ curve in S which is transverse to the 'complex-tangential' directions, then γ is the boundary of a disc which is in a sense 'almost holomorphic', so that any $f \in H^\infty(B)$ must have a radial limit at almost every point of γ. (This was extended to rectifiable γ in [**15**].) But basic questions concerning the structure of the sets of divergence for H^∞-functions remain open. In particular, it is not known whether $f \in H^\infty(B)$ must have a radial limit almost everywhere with respect to an arbitrary 'representing measure' for the ball algebra [**20**, p. 247], even when the representing measure arises from a proper holomorphic map from the disc to the ball; one may obtain a positive result here if the image of the disc has finite area [**27**].

It appears that H^∞ is the only one of the H^p-spaces in which the structure of the sets of divergence depends in this way on the splitting of the tangent space into complex tangential and transverse directions. This is easy to see: take $d = 2$ and set

$$f(z_1, z_2) = \sum_{j=1}^{\infty} (z_1)^{2^j}. \tag{0.0}$$

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Then $f \in H^p$ for all $p < \infty$, although f has a radial limit at none of the points $(e^{it}, 0)$. (In fact f lies in the space called 'BMOA_2' in [**10**]; if the exponents in (0.0) are replaced by a sequence tending to infinity somewhat faster then f lies in 'BMOA_1'. See § 1 for definitions and comments.)

Given a function f defined in B, let E_f be the set of boundary points at which f fails to have a radial limit. We shall show that if $E \subset S$ is a G_δ then there exists a Bloch function f such that $E_f = E$; if $\sigma(E) = 0$, we may achieve $f \in \mathrm{BMOA}$ (Theorem 1). This does not quite characterize $\{E_f\colon f \in \mathcal{B}\}$ or $\{E_f\colon f \in \mathrm{BMOA}\}$, because in general E_f need be only a $G_{\delta\sigma}$. (In fact the function constructed in Theorem 1 has infinite radial variation on E and finite radial variation elsewhere; note that the set where an arbitrary function has infinite radial variation must be a G_δ.)

The proof of Theorem 1 would be much simpler if we restricted attention to compact sets; in this regard we point out that Theorem 1(i) is not sufficient for the application given in Theorem 3. Let X_p $(2 \leq p \leq \infty)$ denote the Besov space in B interpolating between the Bloch space $\mathcal{B}(B)$ and H^2 (so that $X_2 = H^2(B)$ and $X_\infty = \mathcal{B}(B)$; see (1.10) below). We show in Theorem 3 that for $2 < p \leq \infty$ there exists $f \in X_p$ with $E_f = S$. The proof proceeds by first constructing $g \in X_p$ with $\sigma(E_g) = 1$; this is relatively simple (Proposition 0). Then we apply Theorem 1(ii) to find $h \in \mathrm{BMOA}$ such that $f = g + h \in X_p$ has no radial limits.

One cannot expect a similar pointwise statement for VMOA or $\mathcal{B}_0$, because [**8**] a function in $\mathcal{B}_0(D)$ must have a radial limit at each point of a dense subset of the circle; in fact this set must have Hausdorff dimension 1 [**12, 13**]. (Similarly, it is known that a Bloch function must have radial limits on a dense set if the value infinity is allowed; indeed, Theorem 1 of [**5**] shows that if $f \in \mathcal{B}(D)$ and $\sigma(E_f) = 1$ then the real part of f must have radial limit $+\infty$ on a dense set. For us the word 'limit' will mean 'finite limit'.) It is shown in Theorem 2 below that if μ is a (regular Borel) measure on S then there exists an $f \in \mathcal{B}_0(B)$ with $\mu(S \backslash E_f) = 0$, and that if μ is singular then there exists a VMOA function with this property. Various special cases of Theorems 1 and 2 appear in [**1**; **17**; **18**; **21**, Theorem 5.2; **22**; **26**].

The proofs of Theorems 1 and 2 are substantially simpler if E is *circled*; that is, if $e^{it}\zeta \in E$ whenever $\zeta \in E$. This is largely because if E is circled then the function f in the two theorems may be constructed as a lacunary sum of *homogeneous* polynomials, so that the restriction to a complex line through the origin is a lacunary power series; a great deal is known about questions of convergence for such series. We shall see that most of Theorem 1 can be reduced to this case by a device known in some circles as *homogenization*; we prove the theorem for circled sets in § 3 and then indicate in § 4 how the general case follows.

Related results for $d = 1$ may be found in [**11**].

It is a pleasure to thank Alan Noell and Wade Ramey for stimulating discussions on these topics.

1. *Definitions and statement of results*

If $f\colon B \to \mathbb{C}$ and $\zeta \in S$, we will define $f_\zeta\colon D \to \mathbb{C}$ by

$$f_\zeta(\lambda) = f(\lambda\zeta) \quad (\lambda \in D). \tag{1.0}$$

The Bloch space $\mathcal{B}(B)$ can be defined in terms of these 'slice functions' f_ζ [**25**].

We begin with $d = 1$. We write $f \in \mathcal{B}(D)$ if f is holomorphic in D and

$$|f'(\lambda)| \leq \frac{c}{1-|\lambda|^2} \quad (\lambda \in D) \tag{1.1}$$

for some finite constant c; for $f \in \mathcal{B}(D)$ we set

$$\|f\|_{\mathcal{B}(D)} = \sup_{\lambda \in D}(1-|\lambda|^2)\,|f'(\lambda)|. \tag{1.2}$$

(Of course (1.2) defines only a seminorm on $\mathcal{B}(D)$; one might regard (1.2) as defining a norm on $\mathcal{B}(D)/\mathbb{C}$.)

The space $\mathcal{B}_0(D)$ is the set of all $f \in \mathcal{B}(D)$ with

$$\lim_{|\lambda| \to 1}(1-|\lambda|^2)\,|f'(\lambda)| = 0. \tag{1.3}$$

It is easy to see that $\mathcal{B}_0(D)$ is precisely the closure in $\mathcal{B}(D)$ of the polynomials.

We will say that $f \in \mathcal{B}(B)$ if f is holomorphic in B and $\|f_\zeta\|_{\mathcal{B}(D)}$ is bounded for $\zeta \in S$; we set

$$\|f\|_{\mathcal{B}(B)} = \sup_{\zeta \in S} \|f_\zeta\|_{\mathcal{B}(D)}. \tag{1.4}$$

Thus a function f holomorphic in B is a Bloch function if and only if

$$(1-|z|^2)\,|Rf(z)| \leq c, \tag{1.5}$$

where R denotes the radial derivative [**20**, p. 103]; as in the case where $d = 1$, we say that $f \in \mathcal{B}_0(B)$ if

$$\lim_{|z| \to 1}(1-|z|^2)\,|Rf(z)| = 0, \tag{1.6}$$

and we note that $\mathcal{B}_0(B)$ is the closure of the polynomials in $\mathcal{B}(B)$.

We turn now to the definition of BMOA. If $f \in L^1(S) = L^1(\sigma)$ and $\mathcal{Q} \subset S$ with $\sigma(\mathcal{Q}) > 0$ then we define $f_{\mathcal{Q}} = \sigma(\mathcal{Q})^{-1}\int_{\mathcal{Q}} f\,d\sigma$. For $\zeta \in S$ and $\delta > 0$ we define

$$\mathcal{Q}_\delta(\zeta) = \{\xi \in S\colon\ |\xi - \zeta| < \delta\}; \tag{1.7}$$

finally, $\mathrm{BMO} = \mathrm{BMO}(S)$ denotes the space of all functions $f \in L^2(S)$ such that the quantity $\sigma(\mathcal{Q})^{-1}\int_{\mathcal{Q}} |f - f_{\mathcal{Q}}|\,d\sigma$ is bounded for $\mathcal{Q} = \mathcal{Q}_\delta(\zeta)$, $\zeta \in S$, $\delta > 0$.

As usual, $\mathrm{BMOA} = \mathrm{BMO} \cap H^2$, and VMOA is the closure of the (holomorphic) polynomials in BMOA; it is easy to see that $f \in \mathrm{VMOA}$ if and only if $f \in \mathrm{BMOA}$ and

$$\lim_{\delta \to 0} \sup_{\zeta \in S} \sigma(\mathcal{Q}_\delta(\zeta))^{-1} \int_{\mathcal{Q}_\delta(\zeta)} |f - f_{\mathcal{Q}_\delta(\zeta)}|\,d\sigma = 0. \tag{1.8}$$

Our notation 'BMOA' is not consistent with the notation in [**2, 19**], etc. Following [**10**], let BMOA_1 be the space defined above, and set BMOA_2 equal to the space of holomorphic functions satisfying a BMO condition with respect to the well-known 'non-isotropic balls' on the boundary (see (2.2) below). It often seems that BMOA_2 is the more 'natural' of the two spaces, because, for example, the dual of the (holomorphic) Hardy space $H^1(B)$ is BMOA_2. It is shown in [**10**] that BMOA_1 is a proper subspace of BMOA_2, so that our results for BMOA_1 contain the corresponding results for BMOA_2.

NOTE. The proof in [10] does not explicitly exhibit an element of $\mathrm{BMOA}_2 \backslash \mathrm{BMOA}_1$. It may be of some interest to see that a lacunary series gives a simple example of such a function.

Let $g(\lambda) = \sum_{j=1}^{\infty} \lambda^{10^j}$ $(\lambda \in D)$. Then $g \in \mathcal{B}(D)$, because g is a lacunary series with bounded coefficients [4]. Now set $f(z) = g(z_1)$ $(z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{d-1})$. The fact that $g \in \mathcal{B}(D)$ implies that $f \in \mathrm{BMOA}_2$ (for much stronger results of this sort see [19]). However $f \notin \mathrm{BMOA}_1$: it is easy to see that $|g'(\lambda)| \geq c10^j$ for all λ in the annulus defined by the inequality $|1 - 10^{-j} - |\lambda|| \leq c10^{-j}$, and this shows that $(1 - |z|^2)\,|\nabla f|^2$ cannot be a Carleson measure. (Define $C_h = \{(z_1, z')\colon |z'| < h, z_1 = re^{it}, |t| < h, 1 - h < r < 1 - 2h^2\}$. Note that $|z'|^2 \leq \frac{1}{2}(1 - |z_1|^2)$ for $z \in C_h$ $(0 < h < \frac{1}{2})$. It follows that $\int_{C_h} (1 - |z|^2)\,|\nabla f|^2 \geq ch^{2d-1} \log(1/h)$.)

We define E_f to be the set of all $\zeta \in S$ such that f does not have a radial limit at ζ. Theorem 1 below states in part that if $E \subset S$ is a G_δ with $\sigma(E) = 0$ then there exists $f \in \mathrm{BMOA}$ with $E = E_f$. For the proof of Theorem 3 we will need explicit estimates on the rate of oscillation of f on radii terminating at points of E; this requires another definition.

We set

$$\omega_j(f) = \omega_j(f, \zeta) = \sup\{|f(t\zeta) - f(s\zeta)|\colon s, t \in I_j\},$$

where $I_j = [1 - 2^{-(j-1)}, 1 - 2^{-(j+1)}]$, and we define

$$\omega(f) = \omega(f, \zeta) = \limsup_{j \to \infty} \omega_j(f, \zeta). \tag{1.9}$$

(Thus $\omega(f)$ might be called the (radial) *dyadic oscillation* of f.)

THEOREM 1. (i) *Suppose $K \subset S$ is compact. There exists $f \in \mathcal{B}(B)$ such that f extends to a holomorphic function in a neighbourhood of $\bar{B} \backslash K$ but f has a radial limit at no point of K; in fact $\omega(f, \xi) > 0$ for $\xi \in K$.*

(ii) *Suppose $E \subset S$ is a G_δ. There exists $f \in \mathcal{B}(B)$ such that $\omega(f, \xi) > 0$ for $\xi \in E$ while all of the first-order partial derivatives of f have non-tangential limits at every point of $S \backslash K$.*

(iii) *If $\sigma(K) = 0$ then we may obtain $f \in \mathrm{BMOA}(B)$ in part* (i)*; similarly if $\sigma(E) = 0$, we may obtain $f \in \mathrm{BMOA}(B)$ in part* (ii).

In particular $E_f = K$ in (i) and $E_f = E$ in (ii). This may be compared with previous results in this direction. It was shown in [23] that there exists a function $f \in \mathcal{B}(B)$ with $\sigma(E_f) = 1$, while the result of [26] states that there exists $f \in \mathcal{B}(B)$ with $E_f = S$.

The proof of Theorem 1(ii) will show that we could obtain non-tangential convergence of the partial derivatives of any given order on $S \backslash E$; further we could obtain admissible convergence, or indeed convergence in essentially any 'approach region', in place of non-tangential convergence.

Since any set of measure zero is contained in a G_δ of measure zero, we obtain the following corollary.

COROLLARY. *Suppose $E \subset S$ and $\sigma(E) = 0$. Then there exists $f \in \mathrm{BMOA}(B)$ such that f has a radial limit at no point of E.*

Theorem 2 gives a similar result for $\mathcal{B}_0$ and VMOA.

THEOREM 2. (i) *Let μ be a (regular positive Borel) measure on S. There exists $f \in \mathcal{B}_0(B)$ with $\mu(S \backslash E_f) = 0$.*

(ii) *If μ is singular then we may obtain $f \in$ VMOA in part* (i).

For f holomorphic in B and $1 \leq p < \infty$ we will write $f \in X_p$ if

$$\int_B |Rf(z)|^p \, (1 - |z|^2)^{p-1} \, dm(z) < \infty, \tag{1.10}$$

where m is Lebesgue measure in $\mathbb{C}^d$ and R is the radial derivative. As p tends to infinity (1.10) becomes the condition that $|Rf(z)| \, (1 - |z|^2)$ be bounded, so that we define $X_\infty = \mathcal{B}(B)$. Note that X_2 is the Hardy space H^2; the spaces X_p $(2 < p < \infty)$ are simply the holomorphic Besov spaces interpolating between H^2 and $\mathcal{B}(B)$. (The fact that $X_2 = H^2(B)$ follows from the case where $d = 1$ by the trivial identity $\|f\|^2_{H^2(B)} = \int_S \|f_\zeta\|^2_{H^2(D)} \, d\sigma(\zeta)$.)

We shall prove the following theorem.

THEOREM 3. *If $2 < p < \infty$ then there exists $f \in X_p$ with $E_f = S$.*

We will prove Theorem 3 in the next section, assuming the validity of Theorem 1; the manipulations with 'Ryll–Wojtaszczyk polynomials' involved may serve as motivation for the somewhat more delicate constructions required in the proof of Theorem 1.

2. *Proof of Theorem* 3

LEMMA 0. *Suppose that P_k is a polynomial in $\mathbb{C}^d$ homogeneous of degree 2^k, $2 \leq p \leq \infty$, and suppose that*

$$\sum_{k=1}^{\infty} \|P_k\|_p^p < \infty \tag{2.0}$$

(with the obvious interpretation for $p = \infty$). Define $f = \sum_{k=1}^{\infty} P_k$. Then $f \in X_p$.

It will be clear to readers familiar with the various characterizations of the classical Besov spaces [**16**; **24**, § V.5] that the lemma is valid for $p \geq 1$. It is very easy to give a direct proof for $2 \leq p < \infty$.

Proof. It suffices to consider the two cases $p = 2$ and $p = \infty$, by interpolation. The lemma is trivial for $p = 2$, since $X_2 = H^2$.

Suppose that (2.0) holds with $p = \infty$; that is, $\|P_k\|_\infty$ is bounded. This implies that each slice function f_ζ is a lacunary series with bounded coefficients; thus $f_\zeta \in \mathcal{B}(D)$ [**4**], and in fact $\|f_\zeta\|_{\mathcal{B}(D)}$ is bounded for $\zeta \in S$. This implies that $f \in \mathcal{B}(B) = X_\infty$, by definition.

We will begin the proof of Theorem 3 by constructing a function $f \in X_p$ $(2 < p < \infty)$ with $\sigma(E_f) = 1$, using the so-called Ryll–Wojtaszczyk polynomials. As in [**21**], we define

$$d(\zeta, \xi) = (1 - |\langle \zeta, \xi \rangle|^2)^{\frac{1}{2}} \quad (\zeta, \xi \in S). \tag{2.1}$$

It is shown in [**21**, Lemma 2.2] that d satisfies the triangle inequality on S; d is not quite a metric, because $d(\zeta, \xi) = 0$ if and only if $\xi = e^{it}\zeta$ $(\zeta, \xi \in S)$. We define

$$\mathcal{Q}_\delta(\zeta) = \{\xi \in S: \ d(\zeta, \xi) < \delta\} \quad (\zeta \in S, \ \delta > 0). \tag{2.2}$$

Lemma 2.2 of [**21**] shows that

$$\sigma(\mathcal{Q}_\delta(\zeta)) = \delta^{2d-2} \quad (0 < \delta \leq 1). \tag{2.3}$$

We will say that the set $\Gamma \subset S$ is 'δ-separated' if the sets $\mathcal{Q}_\delta(\zeta)$ $(\zeta \in \Gamma)$ are pairwise disjoint.

Fix a positive integer k for a moment. Let $\delta = (Ak)^{-\frac{1}{2}}$; here A is a small positive number to be determined. Now let $\Gamma \subset S$ be a maximal δ-separated subset of S. (Note that Γ is a singleton if $d = 1$.) Define

$$W_k(z) = \sum_{\zeta \in \Gamma} \langle z, \zeta \rangle^k. \tag{2.4}$$

It is shown in [**21**, Theorem 2.1] that there exists a constant $c = c(d, A)$ such that $|W_k| \leq c$ in B.

The sequence of homogeneous polynomials $W_1, W_2, \ldots$ defined in the previous paragraph is called an 'RW sequence' in [**21**], where the existence of such polynomials is applied to a wide variety of problems. One of the original applications in [**23**] was to show that there exists $f \in \mathcal{B}(B)$ with $\sigma(E_f) = 1$; in fact $f = \sum_{j=1}^{\infty} W_{2^j}$ has this property. We shall see that if $2 < p < \infty$ then one may obtain $f \in X_p$ with $\sigma(E_f) = 1$ by a similar construction. We begin by showing that if A is small enough then the terms in (2.4) have almost disjoint support, in the following sense.

LEMMA 1. *Suppose k is a positive integer and $\delta = (Ak)^{-\frac{1}{2}}$, as above. Suppose Γ is a maximal δ-separated subset of S. If $A > 0$ is small enough then there exists a constant $\gamma = \gamma(A, d)$ such that*

$$|\langle \xi, \zeta \rangle^k| \geq \tfrac{3}{4} \quad (\zeta \in \Gamma, \ \xi \in \mathcal{Q}_{\gamma\delta}(\zeta)) \tag{2.5}$$

and

$$\sum_{\substack{\zeta' \in \Gamma \\ \zeta' \neq \zeta}} |\langle \xi, \zeta' \rangle^k| \leq \tfrac{1}{4} \quad (\zeta \in \Gamma, \ \xi \in \mathcal{Q}_{\gamma\delta}(\zeta)). \tag{2.6}$$

Proof. The proof of Theorem 2.1 in [**21**] shows that (2.6) holds with $\gamma = \frac{1}{2}$, say, if A is small enough. Once A is fixed so that (2.6) holds with $\gamma = \frac{1}{2}$, it is easy to see that (2.5) holds as well, with perhaps a smaller value of γ. Suppose first that $\gamma^2 \leq \frac{1}{2}A$, so that $\gamma^2\delta^2 \leq \frac{1}{2}$. Since $1 - u \geq e^{-cu}$ for $0 \leq u \leq \frac{1}{2}$, we obtain

$$|\langle \zeta, \xi \rangle^k|^2 \geq (1 - \gamma^2\delta^2)^k \geq e^{-ck\gamma^2\delta^2} = e^{-c\gamma^2/A} \tag{2.7}$$

for $\xi \in \mathcal{Q}_{\gamma\delta}(\zeta)$; we need merely choose γ so that $\exp(-c\gamma^2/A) \geq \frac{9}{16}$.

The values of A and γ will be fixed from now on so that the statement of Lemma 1 holds.

Note that (2.5) and (2.6) imply that

$$|W_k(\xi)| \geq \tfrac{1}{2} \quad \Big(\xi \in \bigcup_{\zeta \in \Gamma} \mathcal{Q}_{\gamma\delta}(\zeta)\Big). \tag{2.8}$$

Together with (2.3) and the maximality of Γ this gives the following corollary.

COROLLARY 1. *For each $p > 0$ there exists $c_p > 0$, depending only on d and p, such that*

$$\|W_k\|_p \geq c_p \quad (k = 1, 2, \ldots). \tag{2.9}$$

We will use the following 'scrambling lemma'.

LEMMA 2 [**20**, Lemma 7.2.7]. *Suppose that (χ_k) is a uniformly bounded sequence of non-negative measurable functions on S with $\sum_{k=1}^{\infty} \int_S \chi_k \, d\sigma = \infty$. Then there exist unitary maps $U_1, U_2, \ldots$ such that*

$$\sum_{k=1}^{\infty} \chi_k(U_k\zeta) = \infty \tag{2.10}$$

for almost every $\zeta \in S$.

(The lemma is valid under weaker hypotheses; see, for example, [**7**, Theorem 5.1.2]. The hypothesis of uniform boundedness is needed for the particularly elegant proof given in [**20**].)

We can now prove the 'almost everywhere' version of Theorem 3.

PROPOSITION 0. *If $2 < p \leq \infty$, there exists $f \in X_p$ such that $\sigma(E_f) = 1$.*

Proof. As mentioned earlier, the case where $p = \infty$ is due to Timoney [**25**]; we will simplify the notation by supposing that $2 < p < \infty$.

It follows from (2.9) that

$$\sum_{k=1}^{\infty} k^{-1} \int_S |W_{2^k}|^2 \, d\sigma = \infty. \tag{2.11}$$

Now Lemma 2 shows that there exist unitary maps $U_1, U_2, \ldots$ such that

$$\sum_{k=1}^{\infty} k^{-1} |W_{2^k}(U_k\zeta)|^2 = \infty \tag{2.12}$$

almost everywhere on S. Fix such a sequence and define

$$f(z) = \sum_{k=1}^{\infty} k^{-\frac{1}{2}} W_{2^k}(U_k z). \tag{2.13}$$

Lemma 0 shows that $f \in X_p$. Fix $\zeta \in S$ as in (2.12). Since $W_{2^k} \circ U_k$ is homogeneous of degree 2^k, we see that

$$f_\zeta(\lambda) = f(\lambda\zeta) = \sum_{k=1}^{\infty} a_k \lambda^{2^k}, \tag{2.14}$$

where $a_k = k^{-\frac{1}{2}} W_{2^k}(U_k \zeta)$. Thus f_ζ is given by a lacunary power series with coefficients which are not square summable (2.12); hence $\sigma_1(E_{f_\zeta}) = 0$ [**28**, v.I, p. 203].

Since f_ζ has radial limits almost nowhere for almost every value of ζ, Fubini's theorem shows that $\sigma(E_f) = 1$.

Proof of Theorem 3. By Proposition 0 we may find $f_1 \in X_p$ such that $\sigma(E_{f_1}) = 1$. Choose a G_δ-set $E \subset S$ with $\sigma(E) = 0$ and $S \backslash E_{f_1} \subset E$.

Theorem 1(ii) shows that there exists $f_2 \in \mathrm{BMOA}$ with $E_{f_2} = E$. Certainly $f_2 \in X_p$, since

$$\mathrm{BMOA} \subset H^2 \cap \mathcal{B} = X_2 \cap X_\infty \subset X_p.$$

Let $f = f_1 + f_2$. Then $f \in X_p$; we will show that $E_f = S$. Note first that $S \backslash E \subset E_{f_1} \cap S \backslash E_{f_2} \subset E_f$.

Now suppose that $\zeta \in E$. Then f_2 diverges at ζ, but f_1 may diverge at ζ as well. However, Theorem 1 tells us not only that f_2 has no radial limit at ζ but that in fact $\omega(f_2, \zeta) > 0$. On the other hand, if f_1 is as in (2.13) then $f_1 \in \mathcal{B}_0(B)$, because the coefficients in (2.13) tend to zero. This shows that $\omega(f_1) = 0$ on all of S; in particular, $\omega(f, \zeta) > 0$, so that $\zeta \in E_f$.

3. *Theorem* 1 *for circled sets*

Recall that A and γ are constants with values fixed so that Lemma 1 holds. We can obtain pointwise results instead of just 'a.e.' results by using the following lemma instead of measure-theoretic arguments like Lemma 2.

LEMMA 3 [**3, 26**]. *There exists a constant N, depending only on γ and the dimension d, such that if $\Gamma \subset S$ and Γ is $(\frac{1}{4}\gamma\delta)$-separated then there exist pairwise disjoint δ-separated sets Γ_ν $(1 \leq \nu \leq N)$ with $\Gamma = \bigcup_{\nu=1}^N \Gamma_\nu$.*

We fix a value of N so that Lemma 3 holds; we also fix a number $\alpha > 1$ such that $\alpha^{2N} < 2$, and a positive integer k^0 such that if $k \geq k^0$ then there exists an integer k' with $\alpha \leq k'/k \leq \alpha^2$.

Our next task is to define a certain polynomial W_k^K associated with each integer $k \geq k^0$ and circled compact set $K \subset S$. We will show that there exists a lacunary sum of N uniformly bounded homogeneous polynomials such that the sum of the absolute value of the terms gives a rough approximation to the characteristic function of K.

Fix a circled compact set $K \subset S$ and an integer $k \geq k^0$, and set $\delta = (Ak)^{-\frac{1}{2}}$ as before. Now let $\Gamma \subset K$ be a maximal $(\frac{1}{4}\gamma\delta)$-separated subset of K, and write $\Gamma = \bigcup_{\nu=1}^N \Gamma_\nu$ as in Lemma 3. Because $k \geq k^0$, we may find integers $k = k^{(1)} < k^{(2)} < \ldots < k^{(N)}$ such that $\alpha \leq k^{(\nu+1)}/k^{(\nu)} \leq \alpha^2$ for $1 \leq \nu \leq N-1$. (Note that $k^{(N)} \leq 2k$.) Define

$$W_k^K(z) = \sum_{\nu=1}^N \sum_{\zeta \in \Gamma_\nu} \langle z, \zeta \rangle^{k^{(\nu)}}. \tag{3.0}$$

We note first that if $\delta_\nu = (Ak^{(\nu)})^{-\frac{1}{2}}$ then $\delta_\nu \geq \frac{1}{2}\delta$, so that our choice of Γ implies

that

$$K \subset \bigcup_{\nu=1}^{N} \bigcup_{\zeta \in \Gamma_\nu} \mathcal{Q}_{\gamma\delta_\nu}(\zeta). \tag{3.1}$$

It follows that

$$\max_{1 \leq \nu \leq N} \left| \sum_{\zeta \in \Gamma_\nu} \langle \xi, \zeta \rangle^{k^{(\nu)}} \right| \geq \tfrac{1}{2} \quad (\xi \in K) \tag{3.2}$$

by Lemma 1.

We shall see that $f = \sum_{j=1}^{\infty} W_{k_j}^K$ is a Bloch function satisfying the conclusion of Theorem 1(i) if (k_j) is a sequence tending to infinity sufficiently rapidly. The proof is a simple 'gliding hump' argument, using the following lemma. As usual, $rC = \{rz: z \in C\}$.

LEMMA 4. *Suppose that K is a circled compact subset of S, and C is a compact subset of the closed ball $\bar{B}$ with $K \cap C = \varnothing$. Then there exists $r > 1$ such that W_k^K tends to zero uniformly on rC as $k \to \infty$. (Hence all the partial derivatives of W_k^K tend to zero uniformly on C.)*

Proof. We may find a circled relatively open set $V \subset S$ and a number $\tau < 1$ such that

$$C \subset \tau B \cup \{s\xi: \xi \in S \backslash V, 0 \leq s \leq 1\}.$$

Since W_k^K is a sum of N uniformly bounded homogeneous polynomials of degree at least k, it is clear that $W_k^K \to 0$ uniformly on $r\tau B$ for $r < 1/\tau$; we may therefore assume that $C = \{s\xi: \xi \in S \backslash V, 0 \leq s \leq 1\}$.

Since K is circled, we have $C \cap e^{it} K = \varnothing$ for all t. Thus by compactness there exists $\beta < 1$ such that

$$|\langle z, \zeta \rangle| \leq \beta \quad (\zeta \in K, z \in C); \tag{3.3}$$

hence there exists $\rho > 0$ with

$$d(\xi, \zeta) \geq \rho \quad (\zeta \in K, \xi \in S \backslash V). \tag{3.4}$$

Fix $k \geq k^{(1)}$ and $\xi \in S \backslash V$; let $\delta = (Ak)^{-\frac{1}{2}}$, as always. Let $\Gamma = \bigcup_{\nu=1}^{N} \Gamma_\nu$ as in the definition of W_k^K, and (as in [**21**, Theorem 2.1]) define

$$H_m = \{\zeta \in \Gamma: m\delta \leq d(\xi, \zeta) < (m+1)\delta\}.$$

As in [**21**], the disjointness of the sets $\mathcal{Q}_{\gamma\delta/4}(\zeta)$ $(\zeta \in \Gamma)$ shows that the cardinality of H_m is at most $(4(m+2)/\gamma)^{2d-2}$. Because of (3.4) we also have $H_m = \varnothing$ for $m < (\rho/\delta) - 1$, that is, for $m < \rho(Ak)^{\frac{1}{2}} - 1$. And for $\zeta \in \Gamma_\nu \cap H_m$ we have

$$\begin{aligned} |\langle \xi, \zeta \rangle|^2 &= 1 - d^2(\xi, \zeta) \leq 1 - m^2\delta^2 \\ &= 1 - m^2/Ak \leq \exp(-m^2/Ak), \end{aligned}$$

so that

$$|\langle \xi, \zeta \rangle|^{k^{(\nu)}} \leq |\langle \xi, \zeta \rangle|^k \leq \exp(-m^2/2A) \quad (\zeta \in \Gamma_\nu \cap H_m, \xi \in S \backslash V). \tag{3.5}$$

Let m_k be the least integer greater than or equal to $\rho(Ak)^{\frac{1}{2}} - 1$. Recalling that $k^{(\nu)} \leq 2k$ in the definition of W_k^K, we obtain finally

$$|W_k^K(z)| \leq r^{2k} \sum_{m=m_k}^{\infty} (4(m+2)/\gamma)^{2d-2} e^{-m^2/2A} \quad (z \in rC). \tag{3.6}$$

Since the ratio of successive terms in (3.6) tends to zero, the sum is no larger than a constant times the first term, so that

$$|W_k^K(z)| \leq cr^{2k}k^{d-1}\exp(-\tfrac{1}{2}\rho^2 k) \quad (z \in rC). \tag{3.7}$$

If we choose $r > 1$ so that $r^2 e^{-\rho^2/2} < 1$ then (3.7) shows that $W_k^K \to 0$ uniformly on rC.

We can now prove Theorem 1(i) (for circled sets). Suppose $K \subset S$ is a circled compact set. Choose compact sets C_j, with $j \geq 1$, such that $C_j \subset C_{j+1}$, $\bar{B}\backslash K = \bigcup_{j=1}^{\infty} C_j$, and C_j is contained in the interior of rC_j for $r > 1$. By Lemma 4 we may choose positive integers k_j and constants $r_j > 1$ ($j \geq 1$) such that $k_{j+1} \geq 4k_j$ and

$$|W_{k_j}^K(z)| \leq 2^{-j} \quad (z \in r_j C_j). \tag{3.8}$$

Define

$$f(z) = \sum_{j=1}^{\infty} W_{k_j}^K(z). \tag{3.9}$$

Then $f \in \mathcal{B}(B)$ as in Proposition 0, and (3.8) shows that f is holomorphic in the interior of $\bigcup_{j=1}^{\infty} r_j C_j$.

It remains only to show that $\omega(f, \xi) > 0$ for every $\xi \in K$. However, if $\xi \in K$ then (3.2) shows that f_ξ is a lacunary power series with (bounded) coefficients which do not tend to zero (f_ξ is lacunary because $k_{j+1} \geq 4k_j$ in (3.9), while $k^{(\nu+1)}/k^{(\nu)} \geq \alpha > 1$ and $k^{(N)}/k \leq 2$ in the definition of W_k^K.) Thus the fact that $\omega(f) > 0$ on K is a consequence of the following lemma.

LEMMA 5. *Suppose $n_{j+1}/n_j \geq \alpha > 1$ and $|c_j| \leq 1$ for all j. Let $g(\lambda) = \sum_{j=1}^{\infty} c_j \lambda^{n_j}$ for $\lambda \in D$. If the coefficients c_j do not tend to zero then $\omega(g, e^{it}) > 0$ for every t.*

Theorem 1 of [**6**] shows that g has no radial limits (see also [**9**]). Although the statement of the theorem in [**6**] says nothing about the *rate* of divergence, the fact that $\omega(g, e^{it}) > 0$ for all t follows from the proof given in [**6**]. (The lemma is proved in [**6**] by 'the method of repeated differentiation' [**9**] together with an argument involving finite differences; an application of Landau's inequalities or a normal-families argument may be used instead.)

We have proved Theorem 1(i) (for circled sets); we will show next that if $\sigma(K) = 0$ then the function f defined by (3.9) lies in BMOA if k_j tends to infinity rapidly enough.

Lemma 4 shows that if $K \subset S$ is circled and compact then

$$\limsup_{k \to \infty} \|W_k^K\|_2 \leq c\sigma(K). \tag{3.10}$$

Thus if $K \subset S$ is a circled compact set with $\sigma(K) = 0$ then $\lim_{k\to\infty} \|W_k^K\|_2 = 0$. This implies that the function f given by (3.9) lies in BMOA if the k_j tend to infinity rapidly enough, by the following proposition.

PROPOSITION 1. *Suppose that $f_1, f_2, \ldots \in \mathrm{VMO}(S)$, $\|f_k\|_{\mathrm{BMO}} \leq 1$ for all k, and $\lim_{k\to\infty} \|f_k\|_1 = 0$. Then the function*

$$f = \sum_{j=1}^{\infty} f_{k_j} \tag{3.11}$$

lies in BMO *if (k_j) is a sequence tending to infinity fast enough.*

(More precisely: there exist functions $\psi_j\colon \mathbb{N}^j \to \mathbb{N}$ *such that* $f \in \mathrm{BMO}$ *if* (k_j) *is any sequence with* $k_{j+1} \geq \psi_j(k_1, \ldots, k_j)$ *for all* j.)

Proof. Set $s_n = \sum_{j=1}^{n} f_{k_j}$. We need only prove the following statement.

If $k_1 < k_2 < \ldots < k_n$ *are such that* $\|f_{k_j}\|_1 \leq 2^{-j}$ $(1 \leq j \leq n)$ *and* $\|s_n\|_{\mathrm{BMO}} < 2$, *then* $\|f_{k_{n+1}}\|_1 \leq 2^{-(n+1)}$ *and* $\|s_{n+1}\|_{\mathrm{BMO}} < 2$ *for large enough* k_{n+1}.

Let us define

$$\mathrm{osc}_1(g, \mathcal{Q}) = \sigma(\mathcal{Q})^{-1} \int_{\mathcal{Q}} |g - g_{\mathcal{Q}}|\, d\sigma \tag{3.12}$$

for $g \in L^1(S)$ and $\mathcal{Q} = \mathcal{Q}_\delta(\zeta)$ (recall definition (1.7)). Let $2 - \|s_n\|_{\mathrm{BMO}} = t > 0$. Since $s_n \in \mathrm{VMO}$, we may choose $\rho > 0$ so that if $\mathcal{Q} = \mathcal{Q}_\delta(\zeta)$ with $0 < \delta < \rho$ then

$$\mathrm{osc}_1(s_n, \mathcal{Q}) < \tfrac{1}{2}. \tag{3.13}$$

If k_{n+1} is large enough then $\|f_{k_{n+1}}\|_1 < 2^{-(n+1)}$ and

$$\|f_{k_{n+1}}\|_1 < \tfrac{1}{4} t \sigma(\mathcal{Q}_\rho(\zeta)). \tag{3.14}$$

It follows that $\|s_{n+1}\|_{\mathrm{BMO}} < 2$. If $0 < \delta < \rho$, we have

$$\begin{aligned} \mathrm{osc}_1(s_{n+1}, \mathcal{Q}) &\leq \mathrm{osc}_1(s_n, \mathcal{Q}) + \mathrm{osc}_1(f_{k_{n+1}}, \mathcal{Q}) \\ &\leq \tfrac{1}{2} + \|f_{k_{n+1}}\|_{\mathrm{BMO}} < 2, \end{aligned}$$

by (3.13). If, on the other hand, $\delta \geq \rho$ then

$$\begin{aligned} \mathrm{osc}_1(s_{n+1}, \mathcal{Q}) &\leq \|s_n\|_{\mathrm{BMO}} + 2\sigma(\mathcal{Q}_\rho(\zeta))^{-1} \|f_{k_{n+1}}\|_1 \\ &\leq 2 - t + \tfrac{1}{2} t < 2, \end{aligned}$$

by (3.14).

We can now prove Theorem 1(ii) (for circled sets E). We have already established the estimates we need in the course of proving Theorem 1(i); it is only the outline which is a bit more involved.

Proof of Theorem 1(ii) *(when* E *is circled).* Suppose the circled set $E \subset S$ is a G_δ. Choose circled open sets $V_n \subset S$ $(n \geq 1)$ with $V_{n+1} \subset V_n$ and $E = \bigcap_{n=1}^{\infty} V_n$. For each n we may write $V_n = \bigcup_{j=1}^{\infty} K_{n,j}$, where $K_{n,j}$ is a circled compact set and $K_{n,j} \cap K_{n,j'} = \varnothing$ $(|j - j'| > 1)$. Define $K_{n,0} = \varnothing$. We will show that if the integers $k_{n,j}$ are chosen properly then

$$f = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} W_{n,j} \tag{3.15}$$

defines a Bloch function satisfying the conclusion of Theorem 1(ii); here $W_{n,j} = W_k^K$ as in (3.0), with $K = K_{n,j}$ and $k = k_{n,j}$.

We begin by partitioning the set of all positive powers of 4 into infinitely many infinite subsets:

$$\{4^m\colon\ m = 1, 2, \ldots\} = \bigcup_{n=1}^{\infty} J_n, \tag{3.16}$$

where the J_n are infinite and pairwise disjoint. The $k_{n,j}$ will be chosen so that

$$k_{n,j} \in J_n, \quad k_{n,j+1} > k_{n,j}. \tag{3.17}$$

This will ensure that the exponents appearing in $W_{n,j}$ are distinct from those in $W_{n',j'}$ unless $(n, j) = (n', j')$.

We need some notation for non-tangential approach regions. For $\rho \in (0, 1)$ and $\xi \in S$ let $\Omega_\rho(\xi)$ denote the convex hull of $\{\xi\} \cup \{z: |z| < \rho\}$. If $C \subset S$, we define $\Omega_\rho(C) = \bigcup_{\xi \in C} \Omega_\rho(\xi)$.

Since $K_{n,j}$ is a compact subset of $S \setminus \bigcup_{\iota=j-1}^{j+1} K_{n,\iota}$, Lemma 4 shows that if $k_{n,j}$ is large enough then

$$|W_{n,j}(z)| + \sum_{l=1}^{d} |D_l W_{n,j}(z)| \leq 2^{-(n+j)} \quad (z \in C_{n,j}), \tag{3.18}$$

where $C_{n,j} = \Omega_{\rho_{n+j}}(S \setminus \bigcup_{\iota=j-1}^{j+1} K_{n,\iota})$, with $\rho_n = 1 - 1/n$. (Here D_l is the partial derivative with respect to z_l.)

Suppose now that $\xi \in S \setminus E$. Then $\xi \in V_n$ for at most finitely many values of n, since $V_{n+1} \subset V_n$, and for each value of n such that $\xi \in V_n$ we have $\xi \in C_{n,j}$ except for at most three values of j. Thus (3.18) holds for all $z \in \Omega_{\rho_{n+j}}(\xi)$, except for at most finitely many values of (n, j). This shows that the series (3.15) converges uniformly on $\Omega_\rho(\xi)$ for every $\rho \in (0, 1)$, as does the corresponding series for $D_l f$, so that $D_l f$ has a non-tangential limit at ξ.

One may show that $f \in \mathcal{B}(B)$ and that $\omega(f) > 0$ on E exactly as in the proof of part (i) of the theorem.

The proof that we may obtain $f \in \mathrm{BMOA}$ if E is a G_δ of measure zero is somewhat more technical; this is because the sets V_n in the proof above cannot have measure zero, being open. One may proceed as follows.

Suppose E is a circled G_δ of measure zero. Choose a preliminary collection of circled open sets $\tilde{V}_n$ with $E = \bigcap_{n=1}^{\infty} \tilde{V}_n$ and $\tilde{V}_{n+1} \subset \tilde{V}_n$. Write $\tilde{V}_1 = \bigcup_{j=1}^{\infty} K_{1,j}$ and set

$$f_1 = \sum_{j=1}^{\infty} W_{1,j} \tag{3.19}$$

where $W_{1,j} = W_k^K$, $K = K_{1,j}$, $k = k_{1,j}$ as before.

Certainly if $k_{1,j}$ tends to infinity fast enough then $\|f_1(z)\|_\infty < c$ by Lemma 4, so $\|f_1\|_{\mathrm{BMO}} < c$. However $f_1 \notin \mathrm{VMO}$, so we cannot simply apply Proposition 1 as before.

It is clear, however, that f_1 is continuous on $\tilde{V}_1$ if the sequence $k_{1,j}$ grows fast enough, because $K_{1,j} \cap K_{1,j'} = \varnothing$ for $|j - j'| > 1$. This implies that f_1 is 'locally VMO' on V_1 in the following sense. We will say that $g \in \mathrm{BMO}(S)$ is locally VMO on the open set V if $\mathrm{osc}_1(g, \mathcal{Q}_j) \to 0$ whenever $\mathcal{Q}_j = \mathcal{Q}_{\delta_j}(\zeta_j)$ is a sequence with $\delta_j \to 0$, such that $\bigcup_{j=1}^{\infty} \mathcal{Q}_j$ is contained in some compact subset of V. (Recall definition (3.12).)

We will choose a new open set $V_2 \subset \tilde{V}_2$ with $E \subset V_2$. Since $\sigma(E) = 0$, we are free to choose V_2 so that each of the numbers $\sigma(V_2 \cap K_{1,j})$ is as small as we wish. We will then write $V_2 = \bigcup_{j=1}^{\infty} K_{2,j}$ and define f_2 as in (3.19); we will then choose $V_3 \subset V_2 \cap \tilde{V}_3$ such that each $\sigma(V_3 \cap K_{2,j})$ is very small, and so forth, finally setting $f = \sum_{n=1}^{\infty} f_n$.

The proof that $D_l f$ converges on $S \setminus E$ and $\omega(f) > 0$ on E will be exactly as before; these facts depended only on our choosing each $k_{n,j}$ large enough. We

need only show that if at each stage we choose V_{n+1} so that each $\sigma(V_{n+1} \cap K_{n,j})$ is small enough then the BMO norm of the partial sum $s_n = \sum_{m=1}^{n} f_m$ remains bounded. This follows from (3.10), by Proposition 2.

PROPOSITION 2. *Suppose that* $g \in \mathrm{BMO}(S)$ *and* $\|g\|_{\mathrm{BMO}} < 2$. *Suppose* g *is locally* VMO *on the open set* $V \subset S$, *and write* $V = \bigcup_{j=1}^{\infty} K_j$, *where the* K_j *are compact sets with* $K_j \cap K_{j'} = \varnothing$ *for* $|j - j'| > 1$. *Then there exists a sequence* $\varepsilon_j > 0$ $(j \geq 0)$ *such that if* $h \in L^{\infty}(S)$, $\|h\|_{\infty} \leq 1$, $|h(\xi)| \leq \varepsilon_0$ *for all* $\xi \in S \backslash V$, *and* $\int_{K_j} |h| < \varepsilon_j$ *for* $j \geq 1$, *then* $\|g + h\|_{\mathrm{BMO}} < 2$.

Proof. By hypothesis we may choose $\beta_j \geq 0$ $(j \geq 1)$ such that if $\mathcal{Q} = \mathcal{Q}_{\delta}(\xi)$ with $\delta \leq \beta_j$ and $\mathcal{Q} \cap K_j \neq \varnothing$ then

$$\mathrm{osc}_1(g, \mathcal{Q}) \leq \tfrac{1}{2}. \tag{3.20}$$

Let $t = 2 - \|g\|_{\mathrm{BMO}} > 0$. Below, we will suppose that $t < 1$ for convenience. Let $\varepsilon_0 = \frac{1}{4}t$, and for $j \geq 1$ choose $\varepsilon_j > 0$ so that

$$\varepsilon_j \sigma(\mathcal{Q}_{\beta_j}(\xi))^{-1} = 2^{-(j+2)}t. \tag{3.21}$$

Suppose h is as in the statement of the proposition. Write $h = h_1 + h_2$, where h_1 is supported on V and h_2 is supported on $S \backslash V$. Our choice of ε_0 shows that

$$\|h_2\|_{\mathrm{BMO}} \leq \|h_2\|_{\infty} \leq \tfrac{1}{4}t; \tag{3.22}$$

we will show that

$$\|g + h_1\|_{\mathrm{BMO}} \leq 2 - \tfrac{1}{2}t, \tag{3.23}$$

so that $\|g + h\|_{\mathrm{BMO}} \leq 2 - \frac{1}{4}t < 2$, as required.

Suppose now that $\mathcal{Q} = \mathcal{Q}_{\delta}(\xi)$. The argument is split into two cases.

Case I: $\delta \leq \beta_j$ *for some* $j \geq 1$ *with* $\mathcal{Q} \cap K_j \neq \varnothing$. In this case we see that

$$\begin{aligned} \mathrm{osc}_1(g + h_1, \mathcal{Q}) &\leq \mathrm{osc}_1(g, \mathcal{Q}) + \mathrm{osc}_1(h_1, \mathcal{Q}) \\ &\leq \tfrac{1}{2} + \|h_1\|_{\infty} \leq \tfrac{3}{2} \leq 2 - \tfrac{1}{2}t, \end{aligned} \tag{3.24}$$

by (3.20) and our hypotheses on h.

Case II: $\delta > \beta_j$ *whenever* $\mathcal{Q} \cap K_j \neq \varnothing$ $(j \geq 1)$. In Case II we define

$$J = \{j:\ \mathcal{Q} \cap K_j \neq \varnothing\}.$$

Then

$$\begin{aligned} \mathrm{osc}_1(g + h_1, \mathcal{Q}) &\leq \mathrm{osc}_1(g, \mathcal{Q}) + \mathrm{osc}_1(h_1, \mathcal{Q}) \\ &\leq \|g\|_{\mathrm{BMO}} + 2\sigma(\mathcal{Q})^{-1} \int_{\mathcal{Q}} |h_1|\, d\sigma \\ &\leq \|g\|_{\mathrm{BMO}} + 2 \sum_{j \in J} \sigma(\mathcal{Q}_{\beta_j}(\xi))^{-1} \int_{K_j} |h_1|\, d\sigma \\ &\leq \|g\|_{\mathrm{BMO}} + 2 \sum_{j=1}^{\infty} \varepsilon_j \sigma(\mathcal{Q}_{\beta_j}(\xi))^{-1} \\ &= 2 - t + \tfrac{1}{2}t = 2 - \tfrac{1}{2}t. \end{aligned} \tag{3.25}$$

4. *Theorem* 1

In § 3 we proved Theorem 1 for circled sets; now we will see how the general case follows.

We begin with Theorem 1(i). Suppose that $K \subset S_d$ is compact. Fix $\alpha, \beta > 0$ with $\alpha^2 + \beta^2 = 1$, and define $K' \subset S_{d+1}$ by $K' = \{(e^{it}\alpha, e^{it}\beta\zeta)\colon \zeta \in K, t \in \mathbb{R}\}$. Then K' is a compact circled subset of S_{d+1}, so by the results in the previous section we may obtain $F \in \mathcal{B}(B_{d+1})$ such that $E_F = K'$, and in fact $\omega(F) > 0$ on K', while F extends holomorphically past every point of $S_{d+1}\backslash K'$. We now define $f\colon B_d \to \mathbb{C}$ by $f(z) \in F(\alpha, \beta z)$.

It is clear that f extends to a function holomorphic in a neighbourhood of $\bar{B}_d\backslash K$, and the fact that $f \in \mathcal{B}(B_d)$ follows from the inequality $|\nabla F(z)| \leqslant c(1 - |z|^2)^{-1}$; the radial derivative satisfies this inequality by definition (1.5), while if v is a unit vector with $\langle v, z\rangle = 0$ and D_v denotes differentiation in the direction v, then Lemma 4.8 of [**25**] shows that

$$|D_v F(z)| \leqslant c(1 - |z|^2)^{-\frac{1}{2}}. \tag{4.0}$$

Finally, the fact that $\omega(f) > 0$ on K follows from the fact that $\omega(F) > 0$ on K' by the following lemma.

LEMMA 6. *Suppose* $F \in \mathcal{B}(B_{d+1})$, *for* $d \geqslant 1$, *and fix* $\xi = (\xi_1, \xi') \in S_{d+1}$. *Then*

$$\lim_{r \to 1}\{F(\psi_1(r)) - F(\psi_2(r))\} = 0, \tag{4.1}$$

if $\psi_1(r) = (\xi_1, r\xi')$ *and* $\psi_2(r) = \langle \psi_1(r), \xi\rangle\xi = (|\xi_1|^2 + r|\xi'|^2)\xi$.

Lemma 6 follows from (4.0).

The proof of Theorem 1(ii) from the circled case is exactly the same; if F and f are related as above then it is clear that non-tangential convergence for the partial derivatives of F implies non-tangential convergence for the partial derivatives of f.

We turn now to Theorem 1(iii). Supposing that $K \subset S_d$ is a compact set with $\sigma(K) = 0$ we define $K' \subset S_{d+1}$ as above. Then K' is a circled compact set of measure zero, so the previous section gives us $F \in \mathrm{BMOA}(B_{d+1})$ such that $\omega(F) > 0$ on K' and F extends to a neighbourhood of $\bar{B}_{d+1}\backslash K'$. A new difficulty arises at this point: the fact that $F \in \mathrm{BMOA}(B_{d+1})$ does not imply that $f \in \mathrm{BMOA}(B_d)$, if $f(z) = F(\alpha, \beta z)$ as above.

It seems likely that one could show that in fact $f_t \in \mathrm{BMOA}(B_d)$ for almost every t, if $F \in \mathrm{BMOA}(B_{d+1})$ and $f_t(z) = F(e^{it}\alpha, e^{it}\beta z)$. However this is not necessary. Define

$$\|F\|_* = \sup_{t \in \mathbb{R}} \|f_t\|_{\mathrm{BMO}(S_d)}. \tag{4.2}$$

It is clear that the proof of Proposition 1 remains valid with $\|\cdot\|_*$ in place of $\|\cdot\|_{\mathrm{BMO}}$, so that we may obtain $\|F\|_* < \infty$ above.

The second half of Theorem 1(iii) follows similarly.

5. *Theorem* 2

Theorem 2(i) is very simple. Suppose that μ is a (regular Borel) measure on S

(so that $\mu(S)<\infty$), and define f as in (2.13). It follows from Proposition 0 that $\sigma(E_f)=1$, while the proof of Theorem 3 shows that $f\in\mathcal{B}_0(B)$. As in the proof of Theorem 2.1 of [**21**], Fubini's theorem shows that we may find a unitary map U such that $\mu(E_g)=\mu(S)$, if $g(z)=f(Uz)$.

The second part of the theorem requires only a little more work. Suppose μ is a singular measure on S_d, and define a measure μ' on S_{d+1} by

$$\int_{S_{d+1}}\varphi\,d\mu'=(2\pi)^{-1}\int_0^{2\pi}\int_{S_d}\varphi(e^{it}\alpha,e^{it}\beta\zeta)\,d\sigma_n(\zeta)\,dt, \tag{5.0}$$

where α and β are as in the previous section.

We may choose a circled G_δ set $E\subset S_{d+1}$ such that $\sigma_{n+1}(E)=0$ and $\mu'(S\backslash E)=0$. Now Theorem 1 shows that there exists $G\in\mathrm{BMOA}(B_{d+1})$ with $E_G=E$, and in fact the construction in § 3 gives

$$G=\sum_{j=1}^{\infty}P_j \tag{5.1}$$

where P_j is a polynomial homogeneous of degree n_j, $n_{j+1}/n_j\geqslant\alpha>1$, and

$$\sum_{j=1}^{\infty}|P_j(\zeta)|^2=\infty\quad(\zeta\in E). \tag{5.2}$$

As noted at the end of the previous section, we may actually obtain $g_t\in\mathrm{BMOA}(B_d)$ for all t here, where $g_t(z)=G(e^{it}\alpha,e^{it}\beta z)$.

We will set

$$F=\sum_{j=1}^{\infty}a_jP_j, \tag{5.3}$$

where $a_j>0$ and $\lim_{j\to\infty}a_j=0$.

Now, the fact that $a_j\to0$ shows that $F\in\mathrm{VMOA}(B_{d+1})$ and in fact $f_t\in\mathrm{VMOA}(B_d)$ for every t (where $f_t(z)=F(e^{it}\alpha,e^{it}\beta z)$), because the estimates showing that $G\in\mathrm{BMO}$ show as well that

$$\Big\|\sum_{j=n}^{\infty}a_jP_j\Big\|_{\mathrm{BMO}}\leqslant c\max_{j\geqslant n}a_j, \tag{5.4}$$

and similarly for $\|\cdot\|_*$ (see (4.2)).

However, it is easy to see that (5.2) implies that

$$\sum_{j=1}^{\infty}a_j^2\,|P_j(\xi)|^2=\infty\quad\text{a.e. }[\mu'], \tag{5.5}$$

if a_j tends to 0 slowly enough.

As in § 2, set $F_\xi(\lambda)=F(\lambda\xi)$ for $\xi\in S_{d+1}$ and $\lambda\in D$. Then F_ξ is a lacunary power series with coefficients equal to $a_jP_j(\xi)$ (see (2.14) above). It follows that F_ξ has radial limits at almost no points of the unit circle whenever ξ satisfies (5.5). This shows that for almost every t we have $\mu(E_t)=\mu(S)$, where $E_t=\{\zeta\in S_d\colon(e^{it}\alpha,e^{it}\beta\zeta)\in E_F\}$. Now Lemma 6 shows that f_t has radial limits almost nowhere $[\mu]$, for almost every t.

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