Integral Trees
and
Integral Graphs

Ligong Wang

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en
assistent-promotor
dr. Georg Still

Contents

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Preface

This thesis is the result of almost four years of research in the field of algebraic graph theory between September 2001 and March 2005. After an introductory chapter the readers will find seven chapters that contain four topics within this research field. These topics have, to varying extent, strong connections with each other. The first topic is on some facts in number theory and matrix theory. It is closely related to integral graphs or integral trees. The second topic deals with integral trees. The third topic is on integral graphs, cospectral graphs and cospectral integral graphs. The fourth topic deals with Laplacian integral and integral regular graphs. Some results of this thesis have been published in journals. See the following list:

[1] L.G. Wang, X.L. Li and S.G. Zhang, Families of integral trees with diameters 4, 6 and 8, *Discrete Appl. Math.* **136** (2004), no.2-3, 349-362.

[2] L.G. Wang, X.L. Li and C. Hoede, Integral complete r-partite Graphs, *Discrete Math.* **283** (2004), no.1-3, 231-241.

[3] L.G. Wang, X.L. Li and C. Hoede, Two classes of integral regular graphs, Accepted for publication in *Ars Combinatoria.*

Chapter 1

Introduction

This thesis has four parts. The first part treats some facts in number theory and matrix theory. The second part is on integral trees. The third part deals with integral graphs, cospectral graphs and cospectral integral graphs. The fourth part is on Laplacian integral and integral regular graphs.

The first part of the thesis consists of Chapter 2. In this part, we present several facts in number theory and matrix theory.

The second part of the thesis consists of Chapters 3 to 4. In this part, some new families of integral trees with diameters 4, 5, 6 and 8 are characterized by making use of number theory and computer search. All these classes are infinite. They are different from those in the literature. We also prove that the problem of finding integral trees of diameters 4, 5, 6 and 8 is equivalent to the problem of solving Diophantine equations. This is a new contribution to the research of integral trees. We believe that it is useful for constructing other integral trees. In particular some special structures of integral trees of diameters 5, 6 and 8 are obtained for the first time. At the same time, some new results which treat interrelations between integral trees of various diameters are also found. These results generalize some well-known results or theorems on integral trees.

The third part of the thesis consists of Chapters 5 to 7. In this part, firstly, we give a useful sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. These results generalize Roitman's result on the integral complete tripartite graphs. Secondly, fifteen classes of larger integral graphs are constructed from the known 21 smaller integral graphs. These classes consist of nonregular and bipartite graphs. Their spectra and characteristic polynomials are obtained from matrix theory. Their integral property is derived by using number theory and computer search. All these classes are infinite. These results generalize some results of Balińska and Simić. Thirdly, we determine the characteristic polynomials of four classes of graphs. We also obtain sufficient and necessary conditions for these graphs to be integral by using number theory and computer search. All these classes are infinite. We also give some new cospectral graphs and cospectral integral graphs.

The fourth part of the thesis consists of Chapter 8. In this part, the spectra and characteristic polynomials of two classes of regular graphs are given. We also obtain the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. These graphs are not only integral but also Laplacian integral. These results generalize some results of Harary and Schwenk.

We assume that the reader is familiar with the essentials of graph theory. Most of the terminology and notations can be found in Bondy & Murty [10], Cvetković, Doob & Sachs [22] or Harary [35].

In the remainder of this introductory chapter, we will present, together with the relevant terminology and notations, a survey of the main results of the thesis against a background of related results.

1.1 History of integral graphs and basic definitions

Throughout this thesis we shall consider only simple graphs (*i.e.* finite undirected graphs without loops or multiple edges). We use G to denote a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A = A(G) = [a_{ij}]$ of G is an $n \times n$ symmetric matrix of 0's and 1's with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. The *characteristic polynomial* of G is the polynomial $P(G) = P(G,x) = det(xI_n - A)$, where I_n denotes the $n \times n$ identity matrix. The spectrum of $A(G)$ is also called the *spectrum* of G. If the eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \cdots, m_r$, respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix} \text{ or } Spec(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_r^{m_r}\}.$$

The study of graphs by investigation of the characteristic polynomial is sometimes called algebraic graph theory. The general goal is to relate properties of graphs with properties of this polynomial.

In this thesis a specific property of the characteristic polynomial is studied, namely that of having integral zeroes. A graph is called *integral* if all zeroes of its characteristic polynomial are integer. The general goal here is to determine these graphs that are integral.

1.1.1 Basic definitions

First of all, we give some terminology and notations occurring in this thesis.

Two graphs G and H are *cospectral* if $P(G,x) = P(H,x)$. We say that G is *characterized by its spectrum* if every graph cospectral to G is isomorphic to G. Let $G \cup H$ denote the union of two disjoint graphs G and H, and let nG denote the disjoint union of n copies of G.

It is well known that the center $Z(T)$ of a tree T consists of either a central vertex, or a center edge, depending on whether the diameter of T is even, or odd. If all the vertices at the same distance from the center $Z(T)$ are of the same degree, then the tree T will be called *balanced*. Clearly, the structure of a balanced tree (without vertices of degree 2) is determined by the parity of its diameter and the sequence $(n_k, n_{k-1}, \cdots, n_1)$, where k is the radius of T and n_j $(1 \leq j \leq k)$ denotes the number of successors of a vertex at distance $k-j$ from the center $Z(T)$. In what follows, n_i $(i = 1, 2, \cdots)$ always stands for an integer ≥ 2. The balanced trees of diameter $2k$ will be encoded by the sequence $(n_k, n_{k-1}, \cdots, n_1)$ or the tree $T(n_k, n_{k-1}, \cdots, n_1)$, while those with diameter $2k+1$ by the sequence $(1; n_k, n_{k-1}, \cdots, n_1)$ or the tree $T(1; n_k, n_{k-1}, \cdots, n_1)$. Sequences $(n_k, n_{k-1}, \cdots, n_1)$ and $(1; n_k, n_{k-1}, \cdots, n_1)$ will be called *integral* if the corresponding balanced trees are integral. We use $T(1, n_k, n_{k-1}, \cdots, n_1)$ or $(1, n_k, n_{k-1}, \cdots, n_1)$ to denote a tree obtained by joining the center of the tree $T(n_k, n_{k-1}, \cdots, n_1)$ to a new vertex v. A *complete bipartite graph* K_{p_1,p_2} is a graph with vertex set V such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where the two vertex classes V_1, V_2 are nonempty disjoint sets, $|V_i| = p_i$ for $i = 1, 2$, and such that two vertices in V are adjacent if and only if they belong to different classes. Let $K_{1,n_k} \bullet T(n_{k-1}, n_{k-2}, \ldots, n_1)$ denote a tree of diameter $2(k-1)$, which is obtained by identifying the center z of K_{1,n_k} with the center u of $T(n_{k-1}, n_{k-2}, \cdots, n_1)$.

Let the tree $T(m,t)$ of diameter 4 be obtained by joining the centers of m copies of $T(t) = K_{1,t}$ to a new vertex v. Let the tree $T(r,m,t)$ of diameter 6 be obtained by joining the centers of r copies of $T(m,t)$ to a new vertex w. If $r = 1$, then let the tree $T(r,m,t) = T(1,m,t)$ of diameter 4 be formed by joining the center of $T(m,t)$ to a new vertex w. Let the tree $T(s,r,m,t)$ of diameter 8 be obtained by joining the centers of s copies of $T(r,m,t)$ to a new vertex u. Let the tree $T(s,q,r,m,t)$ of diameter 10 be obtained by joining the centers of s copies of $T(q,r,m,t)$ to a new vertex z.

Let the tree $K_{1,s} \bullet T(m,t)$ of diameter 4 be obtained by identifying the center z of $K_{1,s}$ with the center v of $T(m,t)$. Let the tree $K_{1,s} \bullet T(r,m,t)$ of diameter 6 be obtained by identifying the center z of $K_{1,s}$ with the center u of $T(r,m,t)$. Let the tree $K_{1,s} \bullet T(q,r,m,t)$ of diameter 8 be obtained by identifying the center z of $K_{1,s}$ with the center w of $T(q,r,m,t)$.

Let the tree $T(p,q) \ominus T(m,t)$ of diameter 5 be obtained by joining the center u of $T(p,q)$ and the center v of $T(m,t)$ with a new edge. Let the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 be obtained by joining the center u of $K_{1,s} \bullet T(m,t)$ and the center v of $T(q,r)$ with a new edge. Let the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 be obtained by joining the center u of $K_{1,s} \bullet T(m,t)$ and the center w of $K_{1,p} \bullet T(q,r)$ with a new edge.

Let the tree $T[m,r]$ of diameter 3 be formed by joining the centers of $K_{1,m}$ and $K_{1,r}$ with a new edge. Let the tree $T^t[m,r]$ of diameter 5 be obtained by attaching t new endpoints to each vertex of the tree $T[m,r]$, and let the tree $T^t(r,m)$ of diameter 6 be obtained by attaching t new endpoints to each vertex of the tree $T(r,m)$. Let G^t be obtained by attaching t new endpoints to each vertex of the graph G.

Let $T(p,q) \bullet T(r,m,t)$ be the tree obtained by identifying the center u of $T(p,q)$ with the center w of $T(r,m,t)$. Let $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ be the tree obtained by identifying the center z of $K_{1,s}$ with the center w of $T(p,q) \bullet T(r,m,t)$. If $r=1$, then the trees $T(p,q) \bullet T(r,m,t) = T(p,q) \ominus T(m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t) = [K_{1,s} \bullet T(p,q)] \ominus T(m,t)$ are trees with diameter 5. If we say the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ are trees with diameter 6, then this means that $r > 1$.

A graph G in which one vertex u is distinguished from the rest is called a *rooted graph*. The distinguished vertex u is called the *root-vertex*, or simply the root. $r * G$ denotes the graph formed by joining the roots of r copies of G to a new vertex w. $K_{1,r} \bullet G$ denotes the graph obtained by identifying the center z of $K_{1,r}$ with the root u of G.

Trees with diameter 4 can be formed by joining the centers of r stars K_{1,m_1}, $K_{1,m_2}, \cdots, K_{1,m_r}$ to a new vertex v. The tree is denoted by $S(r; m_1, m_2, \cdots, m_r)$ or simply $S(r; m_i)$. For convenience, let $m_1, m_2, \cdots, m_r$ be nonnegative integers such that $m_1 < m_2 < \cdots < m_s$, $1 \leq s \leq r$, $m_i \in \{m_1, m_2, \cdots, m_s\}$, $1 \leq i \leq r$, and let a_i denote the multiplicities of m_i in the set $\{m_1, m_2, \cdots, m_r\}$. The tree $S(r; m_i)$ is also denoted by $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = 1 + \sum_{i=1}^{s} a_i(m_i + 1)$. Let the tree $K_{1,a_0} \bullet S(r; m_i) = K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 be obtained by identifying the center w of K_{1,a_0} with the center v of $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$.

A complete r-partite graph $K_{p_1,p_2,\cdots,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ of $p_1 + p_2 + \cdots + p_r(= n)$ vertices, where the V_i's are nonempty disjoint sets, $|V_i| = p_i$ for $1 \le i \le r$, such that two vertices in V are adjacent if and only if they belong to different V_i's. For convenience, we assume that the number of distinct integers of $p_1, p_2, \cdots, p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < \cdots < p_s$. Suppose that a_i is the multiplicity of p_i for each $i = 1, 2, \cdots, s$. The complete r-partite graph $K_{p_1,p_2,\cdots,p_r}$=$K_{p_1,\cdots,p_1,\cdots,p_s,\cdots,p_s}$ is also denoted by $K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_i p_i$.

A graph is $(r, s)-$*semiregular* if it is bipartite with a bipartition (V_1, V_2) in which each vertex of V_1 has degree r and each vertex of V_2 has degree s.

For a graph G, let $\overline{G}$ be the complement graph of G and let $L(G)$ denote the *line graph* of G, in which $V(L(G)) = E(G)$, and where two vertices are adjacent if and only if they are adjacent as edges of G. The *m-iterated line graph* of G is defined recursively by $L^0(G) = G$ and $L^m(G)$= $L(L^{m-1}(G))$. A graph is said to be *regular of degree k (or k-regular)* if each of its vertices has degree k. $S(G)$ denotes the subdivision of a graph G obtained by inserting only a single vertex into each edge of G, while $L_2(G) = L(S(G))$ is the line graph of the subdivision graph S(G) of the graph G.

Let $K_{1,r} \bullet K_n$ be the graph obtained by identifying the center w of $K_{1,r}$ with one vertex v of K_n. Let $u \in V(K_n)$ be the root of K_n, and let $r * K_n$ be the graph obtained by joining the roots of r copies of K_n to a new vertex w. Let $K_{m,n}$ be a complete bipartite graph with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, m\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n\}$, let $K_{1,r} \bullet K_{m,n}$ be the graph obtained by identifying the center w of $K_{1,r}$ with the vertex u_1 of $K_{m,n}$. Let $u_1 \in V_1$ be the root of $K_{m,n}$, let $r * K_{m,n}$ be the graph obtained by joining the roots of r copies of $K_{m,n}$ to a new vertex w.

The $(n+1)$-regular graph $K_{n,n+1} \equiv K_{n+1,n}$ on $4n+2$ vertices is obtained by adding the edges $\{v_i w_i | i = 1, 2, \cdots, n + 1\}$ from two disjoint copies of $K_{n,n+1}$ with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, n\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n + 1\}$ and $U_1 = \{z_i | i = 1, 2, \cdots, n\}$, $U_2 = \{w_i | i = 1, 2, \cdots, n + 1\}$, respectively.

Let $K_{1,p}$ be a graph with vertex classes $V_1 = \{u_1\}$ and $V_2 = \{v_i | i = 1, 2, \cdots, p\}$. The i-th graph K_p of $(p - 1)K_p$ has the vertex set $\{w_{i_j} | j = 1, 2, \cdots, p\}$, where $i = 1, 2, \cdots, p-1$. Then the p-regular graph $K_{1,p}[(p-1)K_p]$ on $p^2 + 1$ vertices is obtained by adding the edges $\{v_i w_{j_i} | j = 1, 2, \cdots, p - 1\}$, for $i = 1, 2, \cdots, p$, between the graph $K_{1,p}$ and the graph $(p - 1)K_p$.

Let $D(G)$= $diag(d(v_1), d(v_2), \cdots, d(v_n))$ be the diagonal matrix of the vertex degrees of G. Then $Lap(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Clearly, $Lap(G)$ is a real symmetric matrix. If all the eigenvalues of the Laplacian matrix $Lap(G)$ of G are integers, we say that G is *Laplacian integral.* Mohar [54] argues that, because of its importance in various physical and chemical theories, the spectrum of $Lap(G) = D(G) - A(G)$ is more natural and important than the more widely studied adjacency spectrum. For background knowledge, see [32, 53, 54]. The characteristic polynomial of $Lap(G)$ is the polynomial $\sigma(G) = \sigma(G, \mu) = det(\mu I_n - Lap(G))$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ (or simply $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$) be all the eigenvalues of the Laplacian matrix $Lap(G)$ of G. The multiplicity of μ as an eigenvalue of $Lap(G)$ will then be denoted by $m_G(\mu)$.

All other notation and terminology can be found in [10, 22].

1.1.2 History of integral graphs

Next we shall introduce a brief history of the study on integral graphs.

The research on integral graphs was initiated by F. Harary and A.J. Schwenk in 1974 (see [36]). Since the spectrum of a disconnected graph is the union of the spectra of its components, in any investigation of integral graphs it is sufficient to consider connected graphs only.

There are many simple examples of integral graphs (some of them are given in [36]). For example, the complete graph K_n; the cocktail-party graph $CP(n)(=\overline{nK_2})$; the complete multipartite graph $K_{n/k,n/k,\ldots,n/k}$; the path P_2 (P_2 is the only integral path in the set of paths P_n with n vertices); the circuits C_3, C_4 and C_6 (the three circuits are the only integral circuits in the set of circuits C_n with n vertices); the complete bipartite graph $K_{m,n}$($K_{m,n}$ is integral if and only if mn is a perfect square); the stars $K_{1,n}$ with $n = p^2$ ($p = 1, 2, 3, \ldots$), and so on.

At the same time, some of the well known graph operations, when applied to integral graphs, result in new integral graphs and thus can be used to generate an arbitrary number of them. Let us look at the following three operations based on the Cartesian product of the sets of vertices. Let G and H be two graphs with vertex sets $V(G)$ and $V(H)$. The next three operations define graphs having $V(G) \times V(H)$ as its vertex set (see also [22] pp. 65-66):

The *product* (or *conjunction*) $G \times H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G \times H$ if and only if x is adjacent to y in G and a is adjacent to b in H.

The *sum* (or *Cartesian product*) $G + H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G + H$ if and only if either $x = y$ and a is adjacent to b in H or $a = b$ and x is adjacent to y in G.

The *strong sum* (or *strong product*) $G \bigoplus H$ of G and H: the vertices (x, a) and (y, b) are adjacent in $G \bigoplus H$ if and only if a is adjacent to b in H and either x is adjacent to y in G or $x = y$.

The following result can be found in [22].
If λ_i, $(i = 1, 2, \cdots, n)$ and μ_j $(j = 1, 2, \cdots, m)$ are the eigenvalues of G and H, respectively, then

(1) the *product* $G \times H$ has eigenvalues $\lambda_i \mu_j$,

(2) the *sum* $G + H$ has eigenvalues $\lambda_i + \mu_j$,

(3) the *strong sum* $G \bigoplus H$ has eigenvalues $\lambda_i \mu_j + \lambda_i + \mu_j$,

(in all these cases $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$). Thus, these three operations preserve the integrality. For example, the so called bipartite product $G \times K_2$ has eigenvalues $\pm\lambda_i$, where λ_i $(i = 1, 2, \ldots, n)$ are the eigenvalues of G. In addition, other integrality preserving graph operations can be found in [2, 36] or [22], such as the complementary graph (or the line graph) of an integral regular graph G.

In general, the problem of characterizing integral graphs seems to be very difficult. Since there is no general characterization (besides the definition) of these graphs, the problem of finding (or characterizing) integral graphs has to be treated in some special interesting classes of graphs.

So far, there are many results on some particular classes of integral graphs. All integral connected cubic graphs were obtained by D. Cvetković and F. C. Bussemaker [20, 14], and independently in 1976 by A. J. Schwenk [63]. There are exactly thirteen connected cubic integral graphs. In fact, D. Cvetković [20] proved that the set of connected regular integral graphs of a fixed degree is finite. Similarly, the set of connected integral graphs with bounded vertex degrees is finite. An infinite family of integral complete tripartite graphs was constructed by M. Roitman in 1984 (see also [60]). Z. Radosavljević and S. Simić determined all 13 connected nonregular nonbipartite integral graphs with maximum degree four (it was the title of the report published in 1986 [58], the full version appeared in [65]).

In 1998, 4-regular integral graphs began to attract some attention. D. Stevanović [66] (see also [24]) determined all 24 connected 4-regular integral graphs avoiding ± 3 in the spectrum. D. Cvetković, S. Simić and D. Stevanović [24] found 1888 possible spectra of 4-regular bipartite integral graphs. (Due to the space limit, spectra with 9 distinct eigenvalues and more than 20 vertices are not shown in [24], for the complete list see [68]). The potential spectra of bipartite 4-regular integral graphs are also determined in [24]. They are quite numerous and its cannot be expected that all 4-regular integral graphs will be determined in the near future. In 1999, D. Stevanović [67] obtained nonexistence results for some of these potential spectra. It follows from these results that, except for 5 exceptional spectra, bipartite 4-regular integral graphs have at most 1260 vertices. As a corollary, a nonbipartite 4-regular integral graph G has at most 630 vertices, unless $G \times K_2$ has one of these exceptional spectra. In 2000, L.G. Wang, X.L. Li and S.G. Zhang [76] studied some constructions on integral graphs and obtained integral graphs K_n^t, $K_{a,b}^t$, $K_{a,a,\ldots,a}^t$, etc.

W. G. Bridges and R. A. Mena [12, 13] investigated some graphs with exact three distinct eigenvalues in 1979 and 1981. E.R. van Dam, W.H. Haemers *et al.* further studied nonregular or regular graphs with few different eigenvalues in [25, 26, 27, 28, 55].

K. Balińska *et al.* [1] proved in 1999 that there are exactly 150 connected integral graphs up to 10 vertices. The results of all connected integral graphs on 11 and 12 vertices can be found in [3, 4, 23]. K. Balińska and S. Simić [5, 6] also gave some results of the nonregular, bipartite, integral graphs with maximum degree 4. P. Hansen, H. Mélot, and D. Stevanovic [34] (see also [33]) gave in 2002 characterizations of integral graphs in the family of complete split graphs and a few related families of graphs. In particular, K. T. Balińska *et al.* [2] presented in 2002 a survey of results on integral graphs and on the corresponding proof techniques. Note that a few errata of the article [2] appeared in [23].

D.L. Zhang and H.W. Zhou [88] obtained in 2003 some new integral graphs based on the study of the bipartite semiregular graph. M. Lepović [41, 42, 43, 44] obtained in 2003 and 2004 some results on integral graphs which belong to the class $\overline{\alpha K_{a,b}}$, $\overline{\alpha K_a \cup \beta K_b}$ or $\overline{\alpha K_a \cup \beta K_{b,b}}$. K. T. Balińska, S. K. Simić and K. T. Zwierzyński [7] investigated in 2004 which non-regular bipartite integral graphs with maximum degree four do not have ± 1 as eigenvalues.

In Chapter 5 (see also [71]), a useful sufficient and necessary condition for complete r-partite graphs to be integral is given. In Chapter 6, fifteen classes of larger nonregular bipartite integral graphs are constructed from the known 21 smaller integral graphs of [5]. In Chapter 7, sufficient and necessary conditions for the graphs $K_{1,r} \bullet K_n$, $r * K_n$, $K_{1,r} \bullet K_{m,n}$ and $r * K_{m,n}$ to be integral are presented. Some new cospectral graphs and cospectral integral graphs also are obtained. In Chapter 8 (see also [72]), two classes of Laplacian integral and integral regular graphs $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]$ are studied.

Trees are another important and interesting family of graphs. In the initial paper of F. Harary and A.J. Schwenk [36] integral trees were mentioned as well, while considerable results on this topic were firstly published by M. Watanabe and A.J. Schwenk in [78] and [79]. Then, after several years of pause starting with the article [47] of X.L. Li and G.N. Lin, a group of Chinese mathematicians began to present their results. In 1987, X.L. Li and G.N. Lin [47] gave answers to the three questions proposed by A.J. Schwenk (see [19]), studied integral trees with diameters 4, 5 and 6, and discovered some new infinite sets of integral trees with diameters 4 and 6. Finally, they raised several open problems. In that paper [47], integral trees with diameter five were mentioned for the first time, where the authors considered the graph obtained by joining the centers of $S(r; m_i)$ and $S(s; n_j)$ and presented a theorem in the form of a necessary and sufficient condition for such a tree to be integral, but they were not able to find any example. The first integral tree with diameter five was constructed by R.Y. Liu in [50] in 1988, and it was proved that there are infinitely many such trees. He also studied integral trees of diameters 3, 4, 5 and 6, and obtained some new classes of integral trees of diameters 3, 4, 5 and 6 (see [50, 51]). In 1988, Z.F. Cao firstly found all integral trees of diameter 3 and further studied integral trees with diameters 4, 5 and 6. It was proved that there are infinitely many such trees (see [15, 16]). These results generalized some results of X.L. Li, G.N. Lin [47] and R.Y. Liu [50, 51].

In 1998, by using the solutions of some general quadratic Diophantine equations, Y. Li [49] generalized results of R.Y. Liu and Z.F. Cao on integral trees of diameter 5. In the same year, P. Híc and R. Nedela firstly constructed infinitely many integral trees with diameter 8, and obtained some positive and negative results about the questions on balanced integral trees (see [37, 38]). He also proved there are no balanced integral trees of diameter $4k + 1$ for $k \geq 1$. L.G. Wang, X.L. Li and R.Y. Liu also constructed independently some families of integral trees with diameter 8 by using a different method [74]. In Section 4.1 of Chapter 4, the structure of integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is investigated for the first time.

In 1998, P.Z. Yuan gave a necessary condition for trees $S(r; m_i)$ of diameter 4 to be integral, and constructed many new classes of such integral trees. In addition, some basic questions about integral trees with diameter 4 were posed in [85]. Then, D.L. Zhang, S.W. Tan [86] and M.S. Li, W.S. Yang, J.B. Wang [46] in 2000 gave a further useful sufficient and necessary condition for graphs to be such integral trees. Some questions proposed by P.Z. Yuan in [85] were answered in [46, 59, 80, 81, 86, 87]. In 2000, L.G. Wang and X.L. Li [69] constructed some new families of integral trees $K_{1,s} \bullet T(m,t)$ of diameter 4 (see also [78]) and $K_{1,s} \bullet T(r,m,t)$ of diameter 6 by identifying the centers of two trees. Then, in [69, 74, 75, 76, 77], L.G. Wang, X.L. Li *et al.* characterized some new families of integral trees with diameters 4, 6 and 8 by making use of number theory and computer search, and they proved that the problem of finding integral trees of diameters 4, 6 and 8 is equivalent to the problem of solving Diophantine equations. At the same time, some results which treat interrelations between integral trees of various diameters were firstly obtained in [76, 77] (see also Chapter 3).

Integral trees $T^t(r,m)$, $T(r,m,t)$, and $K_{1,s} \bullet T(r,m,t)$ of diameter 6 were investigated in [2, 16, 48, 51, 70, 77, 79], [2, 15, 37, 38, 39, 47, 48, 50, 75, 76, 77] and [2, 48, 69, 75, 76, 77], respectively. In Section 4.2 of Chapter 4, the structures of integral trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 were also investigated for the first time.

In 2003, P. Híc and and M. Pokorný [39] investigated integral balanced rooted trees of diameters 4, 6, 8 and 10. An infinite class of integral balanced rooted trees with diameter 10 were given. In Section 4.3 of Chapter 4, the structure of integral trees $K_{1,s} \bullet T(q,r,m,t)$ with diameter 8 are found for the first time. At the same time, some new results which treat interrelations among integral trees of various diameters were also studied. But the problem of the existence of integral balanced rooted trees of arbitrarily large diameter remains open.

In the remainder of this section, we will briefly introduce two other topics related to integral graphs.

Firstly, let us consider the Laplacian matrix $Lap(G) = D(G) - A(G)$ of a graph G. Graphs with integral Laplacian eigenvalues are called *Laplacian integral*.

When considering integral and Laplacian integral graphs, one can see great differences. A good example is the set of all 112 connected graphs on six vertices. Six of them are integral graphs, Of these six, five are regular: C_6 and its complement, K_6, $K_{3,3}$, and the cocktail party graph on six vertices. The sixth is the tree obtained by joining the centers of two copies of P_3 with a new edge. The first five of these are also Laplacian integral, while the only nonregular one is not. On the other hand, there are 37 other connected integral graphs on six vertices which are Laplacian integral (see also [2, 32]).

Let us consider regular graphs. In this case $Lap(G) + A(G) = rI$, where G is an $r-$regular graph. So μ is an eigenvalue of $Lap(G)$ if and only if $r - \mu$ is an eigenvalue of $A(G)$. That means that a regular graph is Laplacian integral if and only if it is integral (see also [32]).

However, the situation with trees is quite different. A tree is Laplacian integral if and only if it is a star $K_{1,n-1}$ (see also [32]).

Another great difference concerns complements. Since $Lap(G)+Lap(\overline{G}) = nI_n - J_n$ (J_n denotes the $n \times n$ matrix with all entries equal to 1), the eigenvalues of $Lap(\overline{G})$ are $\mu_i(\overline{G}) = n - \mu_{n-i}(G)$ ($1 \leq i \leq n-1$), and 0, which means that G and $\overline{G}$ can only be simultaneously Laplacian integral. For example, if we obtain the graph G_n by subdividing an edge of K_{n-1} $(n > 2)$, we immediately know it is Laplacian integral, since $\overline{G_n}$ consists of one copy of K_2 and one copy of $K_{1,3}$ (see also [32]).

Some graph operations, when applied to integral graphs, can also in the Laplacian case give rise to integral graphs. Thus, $G_1 \nabla G_2 := (\overline{\overline{G_1} \cup \overline{G_2}})$, *i.e.* the complete product of graphs, being the complement of the disjoint union (direct sum) of their complements, is one example (see also [2, 32]).

Some interesting results on Laplacian integral graphs can be found in [2, 32, 53, 54, 84], for example:

(1) (see [32]) Let G be a connected, r-regular, Laplacian integral graph on n vertices. Then its subdivision graph $S(G)$ is Laplacian integral if and only if $G = K_n$.

(2) (see [54]) Let G be a connected, (r, s)-semiregular, Laplacian integral graph. Then its line graph $L(G)$ is Laplacian integral.

(3) (see [53]) The most interesting and remarkable result concerning Laplacian integral spectra is expressed by a theorem about the so called maximal graphs. In particular, degree maximal graphs are Laplacian integral.

(4) (see [84]) Some conditions under which a Laplacian integral graph preserves this property when adding an edge were obtained.

Secondly, we consider digraphs. Contrary to (non-oriented) graphs, whose spectra are real, the eigenvalues of digraphs are complex numbers. Note that the adjacency matrix $A(G)$ of a digraph need no more be symmetric. A complex number $\lambda = \alpha + i\beta$ is called a Gaussian integer if α and β are integers. A digraph is called *Gaussian* if its spectrum consists only of Gaussian integers. Of course, if all of them are real integers, such a digraph will be called integral.

As in the case of integral graphs, given two Gaussian digraphs (*e.g.* C_4 and its complement) we can produce arbitrarily large families of Gaussian digraphs by means of well known graph operations [30].

As for integral digraphs, we note that there is an interesting example of two cospectral integral digraphs with four vertices (Fig.6 of [30]), which are the smallest integral digraphs. F. Esser and F. Harary [30] also proved the following result:

For any positive integer n we can find n cospectral strongly connected non-symmetric digraphs which are integral.

Other results on Gaussian or integral digraphs can be found in [2, 21, 30, 82, 83].

Hence, at present, the study of the theory of integral graphs has become a very active and important research field in graph theory.

1.2 Some formulae for the characteristic polynomials of graphs

In this section, we shall give some formulae for characteristic polynomials of graphs and some related results on integral trees.

The following lemmas can be found in the literature.

Lemma 1.2.1. *([31]) If $G \bullet H$ is the graph obtained from G and H by identifying the vertices $v \in V(G)$ with $w \in V(H)$, then*

$P(G \bullet H, x) = P(G, x)P(H_w, x) + P(G_v, x)P(H, x) - xP(G_v, x)P(H_w, x)$, *where G_v and H_w are the subgraphs of G and H induced by $V(G)\backslash\{v\}$ and $V(H)\backslash\{w\}$, respectively.*

Lemma 1.2.2. *([22]) Let $G_1 \cup G_2$ denote the union of two disjoint graphs G_1 and G_2. If $u \in V(G_1)$, $v \in V(G_2)$ and $G = G_1 \cup G_2 + uv$, then*

$P(G, x) = P(G_1, x)P(G_2, x) - P(G_1 - u, x)P(G_2 - v, x)$.

As a special case we obtain

Lemma 1.2.3. *([22]) Let G be a graph. If $u \in V(G), v \notin V(G)$ and $G^* = G + uv$, then $P(G^*, x) = xP(G, x) - P(G - u, x)$.*

Let G be a graph with n vertices, and G^t is obtained by attaching t new endpoints to each vertex of the graph G.

Lemma 1.2.4. *([22]) $P(G^t, x) = x^{nt}P(G, x - \frac{t}{x})$.*

Lemma 1.2.5. *([76]) If $u \in V(G)$, then we have*

(1) $P(r * G, x) = [P(G, x)]^{r-1}[xP(G, x) - rP(G - u, x)]$.

(2) $P(K_{1,r} \bullet G, x) = x^{r-1}[xP(G, x) - rP(G - u, x)]$.

Lemma 1.2.6. *([76])*

(1) Let $G_1 = (r-1)K_1 \cup r * G$, $G_2 = (r-1)G \cup [K_{1,r} \bullet G]$. Then G_1 and G_2 are cospectral.

(2) *Let* $G_1 = (n_k - 1)K_1 \cup T(n_k, n_{k-1}, \ldots, n_1)$, $G_2 = K_{1,n_k} \bullet T(n_{k-1}, n_{k-2}, \ldots, n_1) \cup (n_k - 1)T(n_{k-1}, n_{k-2}, \ldots, n_1)$. *Then* G_1 *and* G_2 *are cospectral forests.*

Lemma 1.2.7. *([76])*

(1) *If* G *and* $K_{1,r} \bullet G$ *are integral graphs, then* $r * G$ *is integral.*

(2) *If* G *and* $r * G$ *are integral graphs, then* $K_{1,r} \bullet G$ *is integral.*

Lemma 1.2.8. *([47])*

$$P[S(r; m_i), x] = x^{\sum_{i=1}^{r} m_i - (r-1)}[\prod_{i=1}^{r}(x^2 - m_i)][1 - \sum_{i=1}^{r}\frac{1}{x^2 - m_i}].$$

Lemma 1.2.9. *([47])*

(1) $P(K_{1,t}, x) = x^{t-1}(x^2 - t)$.

(2) $P(T(m,t), x) = x^{m(t-1)+1}(x^2 - t)^{m-1}[x^2 - (m+t)]$.

(3) $P(T(r,m,t), x) = x^{rm(t-1)+r-1}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1}[x^4 - (m+t+r)x^2 + rt]$.

Lemma 1.2.10. *([69])*

(1) $P[K_{1,s} \bullet T(m,t), x] = x^{m(t-1)+(s-1)}(x^2 - t)^{m-1}[x^4 - (m+t+s)x^2 + st]$.

(2) $P[K_{1,s} \bullet T(r,m,t), x] = x^{rm(t-1)+r+(s-1)}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1}[x^4 - (m+t+r+s)x^2 + rt + s(m+t)]$.

Lemma 1.2.11. *([69]) The tree* $K_{1,s} \bullet T(m,t)$ *of diameter 4 is integral if and only if* t *is a perfect square, and* $x^4 - (m+t+s)x^2 + st$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, where* a *and* b *are integers.*

Lemma 1.2.12. *([69]) The tree* $K_{1,s} \bullet T(r,m,t)$ *of diameter 6 (where* $r, m > 1$*) is integral if and only if* t *and* $m+t$ *are perfect squares, and* $x^4 - (m+t+r+s)x^2 + rt + s(m+t)$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, where* a *and* b *are integers.*

The following Corollaries 1.2.13 and 1.2.14 can also be found in [69].

Corollary 1.2.13. *([69]) If $s = t$, then the tree $K_{1,s} \bullet T(r, m, t)$ of diameter 6 is integral if and only if t, $m + t$ and $m + t + r$ are perfect squares.*

Corollary 1.2.14. *([69]) For the tree $K_{1,s} \bullet T(r, m, t)$ of diameter 6, let the numbers m_1, t_1, r_1, a, b, c and d be positive integers satisfying the following conditions*

$$m_1 + t_1 + r_1 = a^2 + b^2 = c^2 + d^2, \tag{1.2.1}$$

where $c > a > d$, $c > b > d$, $(a, b) = 1$, $(c, d) = 1$ and $a|cd$ or $b|cd$. Let $m_1 + t_1 + r_1$ be a perfect square, and let $m = m_1 n^2$, $s = t = t_1 n^2$ and $r = r_1 n^2$, where n is any positive integer. Then we have the following results.

(1) If $a|cd$, let $m_1 = b^2 - (\frac{cd}{a})^2$, $t_1 = (\frac{cd}{a})^2$ and $r_1 = a^2$, then $K_{1,s} \bullet T(r, m, t)$ is an integral tree.

(2) If $b|cd$, let $m_1 = a^2 - (\frac{cd}{b})^2$, $t_1 = (\frac{cd}{b})^2$ and $r_1 = b^2$, then $K_{1,s} \bullet T(r, m, t)$ is an integral tree.

Lemma 1.2.15. *([74] or [75])*
$P(T(q, r, m, t), x) = x^{qrm(t-1)+q(r-1)+1}(x^2 - t)^{qr(m-1)}[x^2 - (m + t)]^{q(r-1)}[x^4 - (m + t + r)x^2 + rt]^{q-1}[x^4 - (q + m + t + r)x^2 + rt + q(m + t)]$.

Lemma 1.2.16. *([76])*

(1) $P[T(s, q, r, m, t), x] = x^{sqrm(t-1)+sq(r-1)+s-1}(x^2 - t)^{sqr(m-1)}[x^2 - (m + t)]^{sq(r-1)}\ [x^4 - (m+t+r)x^2 + rt]^{s(q-1)}[x^4 - (q+m+t+r)x^2 + rt + q(m + t)]^{s-1}\{x^6 - (s+q+m+t+r)x^4 + [rt + q(m+t) + s(m+t+r)]x^2 - rst\}$.

(2) $P[K_{1,s} \bullet T(q, r, m, t), x] = x^{qrm(t-1)+q(r-1)+s-1}(x^2 - t)^{qr(m-1)}[x^2 - (m + t)]^{q(r-1)}\ [x^4 - (m + t + r)x^2 + rt]^{q-1}\{x^6 - (s + q + m + t + r)x^4 + [rt + q(m + t) + s(m + t + r)]x^2 - rst\}$.

Lemma 1.2.17. *([9])*

(1) $P(K_n, x) = (x + 1)^{n-1}[x - (n - 1)]$.

(2) $P(K_{a,b}, x) = x^{a+b-2}(x^2 - ab)$.

Lemma 1.2.18. *([9] or [64]) If G is a regular graph of degree k, then its line graph $L(G)$ is regular of degree $2k-2$.*

Lemma 1.2.19. *([9] or [64]) If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then $P(\overline{G}, x) = (-1)^n \frac{x-n+k+1}{x+k+1} P(G, -x-1)$.*

Lemma 1.2.20. *([9] or [64]) If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then $P(L(G), x) = (x+2)^{m-n} P(G, x+2-k)$.*

Lemma 1.2.21. *([9, 36, 64]) If a regular graph G is integral, then so is $\overline{G}$.*

Lemma 1.2.22. *([36] or [9, 64]) If a regular graph G is integral, then so is its line graph $L(G)$.*

Next we give some properties of characteristic polynomials of the Laplacian matrix of graphs.

Lemma 1.2.23. *([9] or [32]) If G is a regular graph of degree k with n vertices, then*

$\sigma(G, \mu) = (-1)^n P(G, k-\mu)$, *or* $\mu_i(G) = \mu_i = k - x_i \ (i = 1, 2, \cdots, n)$,

where the x_i's are the eigenvalues of $A(G)$, ordered in a weakly decreasing way.

Lemma 1.2.24. *([9] or [32]) Let G be a graph on n vertices, then the eigenvalues of the Laplacian matrix $Lap(\overline{G})$ are*

$\mu_i(\overline{G}) = n - \mu_{n-i}(G) \quad (1 \leq i < n)$ *and* 0.

Lemma 1.2.25. *([32]) G is Laplacian integral if and only if $\overline{G}$ is Laplacian integral.*

1.3 Survey of results

In this section, we shall give a survey of the main results on integral graphs. We distinguish between 3 special cases: trees, graphs, regular graphs.

1.3.1 Results on integral trees

First of all, we consider integral trees. In the second part of this thesis, in Chapters 3 and 4, we mainly investigate integral trees. Next we give a survey of former investigations, known main results and own main results concerning this topic.

In the initial paper of F. Harary and A.J. Schwenk [36] integral trees were considered in 1974. They showed that $T(n) = K_{1,n}$ is integral if and only if n is a perfect square. Moreover, they gave the following examples of integral trees: $T(1;2)$, $T(1;6)$, $T(3,1)$. Later, first considerable results on this topic were published by M. Watanabe and A.J. Schwenk in [78] and [79].

One of the first and very general results is the following theorem of M. Watanabe.

Theorem 1.3.1. *([78]) No integral tree except K_2 has a perfect matching.*

We say that a tree T is star-like if it is homeomorphic to a star $K_{1,m}$, which means that T has a unique vertex v of degree $m \geq 3$ such that $T - v$ is the (disjoint) union of m paths.

Theorem 1.3.2. *([79]) A star-like tree T is integral if and only if T is one of the following trees:*

(1) $T = K_1$ and $P(T,x) = x$;

(2) $T - v = k^2 P_1$ $(k \in N)$ and $P(T,x) = (x^2 - k^2)x^{k^2-1}$;

(3) $T - v = (k^2 + 2k)P_2$ $(k \in N)$ and $P(T,\lambda) = (x^2 - k^2)x(x^2 - 1)^{k^2+2k-1}$.

The next result concerns the trees homeomorphic to a double star, *i.e.* a tree obtained by joining the centers of two stars with an edge. Let a tree T have exactly two vertices u and v of degree greater than two, let u and v be adjacent and let T have m_i paths of length i at u and n_j paths of length j at v (then the number of vertices is clearly $n = 2 + \Sigma i m_i + \Sigma j n_j$).

Theorem 1.3.3. *([79]) If T is an integral tree having exactly two vertices u and v of degree exceeding two, and if u and v are adjacent, then T is either*

(1) a double star such that $T - u - v = (m_1 + n_1)P_1$ where the polynomial $x^4 - (m_1 + n_1 + 1)x^2 + m_1 n_1$ has only integral roots, or

(2) a tree determined by $T - u - v = m_1 P_1 + n_2 P_2$ where the polynomial $x^4 - (m_1 + n_2 + 2)x^2 + m_1 n_2 + m_1 + 1$ has only integral roots.

For example, if $m_1 = n_1 = a(a+1)$ $(a \in N)$, we get a whole family of solutions. The problem of finding all solutions (of the type of Theorem 1.3.3) was solved by R. L. Graham in 1978 (see also [79]). For the case $m_1 < n_1$ in (1) of Theorem 1.3.3, Z.F. Cao also gave in [15] all solutions of T to be an integral tree, different from the above case.

Another family of integral trees of diameter four has been constructed in [79] (see also Lemma 1.2.9 (2)).

Theorem 1.3.4. *([79]) The tree $T(r, m)$ of diameter 4 is integral if and only if both m and $r + m$ are perfect squares.*

For $m = 1$ we obtain the case in Theorem 1.3.2 (3), while $m = 4$, $r = 5$ is the smallest case for $m > 1$. We can prove that the set of the solutions is infinite.

Then, after a several years pause and having started by the article [47] of X.L. Li and G.N. Lin in 1987, a group of Chinese mathematicians began to present new results.

As a generalization of the above case, suppose that, instead of r copies of $K_{1,m}$, we take r stars $K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_r}$ and form the tree $S(r; m_1, m_2, \cdots, m_r)$ or $S(r; m_i)$ by joining their centers with a new vertex v. For the next result see Lemma 1.2.8.

Theorem 1.3.5. *([47]) A tree $S(r; m_i)$ is integral if and only if the equation*

$$(x^2 - m_1 - 1)(x^2 - m_2)\ldots(x^2 - m_r) - \sum_{j=2}^{r} \prod_{i=1, i\neq j}^{r} (x^2 - m_i) = 0$$

has only integral roots.

Another family of integral trees of diameter four can be obtained as follows.(see also Lemma 1.2.10 (1)).

Theorem 1.3.6. *([78] or [69]) The tree $K_{1,s} \bullet T(m, t)$ of diameter 4 is integral if and only if t is a perfect square and the polynomial $x^4 - (s + m + t)x^2 + st$ has only integral roots.*

For $s = t = 4$, $m = 9$, we have the smallest member of this family provided $s > 0$ (case $s = 0$ coincides with that in Theorem 1.3.4). The general problem of determining s, m, t is equivalent to turning the polynomial of Theorem 1.3.6 into the form $(x^2 - a^2)(x^2 - b^2)$ $(a, b \in N)$. It has been proved that this leads to solutions of the form:

$$s = \tfrac{1}{4}(A^2 + B^2 - 2m) + \tfrac{1}{2}C, t = \tfrac{1}{4}(A^2 + B^2 - 2m) - \tfrac{1}{2}C,$$

where integers A, B, C satisfy $(A^2 - m)(B^2 - m) = C^2$, and that there are infinitely many solutions. The authors of [47] showed by construction that even in case $s = t$ the number of solutions is infinite.

In [69], we give some other results on the integral tree $K_{1,s} \bullet T(m, t)$.

Theorem 1.3.7. *([69] or [77]) If the tree $K_{1,s} \bullet T(m, t)$ of diameter 4 is integral, and m (≥ 2) is a positive integer, then for any positive integer n the* *trees $K_{1,sn^2} \bullet T(mn^2, tn^2)$ and $K_{1,tn^2} \bullet T(mn^2, sn^2)$ of diameter 4 are integral.*

Theorem 1.3.8. *([69]) For any positive integers a, b and c, let $s = 4(a^2 + b^2)^4c^4$, $m = 64a^2b^2(a^2-b^2)^2c^4$ and $t = 4(a^4+b^4-6a^2b^2)^2c^4$, where $(a,b) = 1$ and $2 \not| (a+b)$. Then $K_{1,s} \bullet T(m,t)$ and $K_{1,t} \bullet T(m,s)$ are integral trees with diameter 4.*

Theorem 1.3.9. *([69]) Let m_1, t_1, s_1, a, b, c and d be positive integers satisfying the following conditions*

$$m_1 + t_1 + s_1 = a^2 + b^2 = c^2 + d^2,$$

where $c > a > d$, $c > b > d$, $(a,b) = 1$, $(c,d) = 1$ and $a|cd$ or $b|cd$. Let $m = m_1n^2$, $t = t_1n^2$ and $s = s_1n^2$, where n is a positive integer. Then the tree $K_{1,s} \bullet T(m,t)$ of diameter 4 is integral if one of the following conditions holds:

(1) $a|cd$, $m_1 = b^2 - (\frac{cd}{a})^2$, $t_1 = (\frac{cd}{a})^2$ and $s_1 = a^2$.

(2) $a|cd$, $m_1 = b^2 - (\frac{cd}{a})^2$, $s_1 = (\frac{cd}{a})^2$ and $t_1 = a^2$.

(3) $b|cd$, $m_1 = a^2 - (\frac{cd}{b})^2$, $t_1 = (\frac{cd}{b})^2$ and $s_1 = b^2$.

(4) $b|cd$, $m_1 = a^2 - (\frac{cd}{b})^2$, $s_1 = (\frac{cd}{b})^2$ and $t_1 = b^2$.

In [75], we obtain the following results on integral trees $K_{1,s} \bullet T(m,t)$ of diameter 4, $S(r; m_i)$ of diameter 4 and $T(r,m,t)$ of diameter 6.

Theorem 1.3.10. *([75]) Let k, q and n be positive integers, let $m = m_1n^2$,$t = t_1n^2$, $r = r_1n^2$. Suppose m_1, t_1 and r_1 satisfy one of the following conditions:*

(1) $m_1 = 3(6k+1)(72k^2-6k-1)$, $t_1 = 4(18k^2-6k-1)^2$, $r_1 = (18k^2+6k)^2$,

(2) $m_1 = 9(2k+1)(72k^2+42k+5)$, $t_1 = 4(18k^2+6k-1)^2$, $r_1 = (18k^2 + 18k+4)^2$,

(3) $m_1 = 12(2k+1)(k-1)(k+1)$, $t_1 = 4(k^2-2k-2)^2$, $r_1 = k^2(k+2)^2$, $k > 2$,

(4) $m_1 = (k^2-3q^2)^2 - (\frac{k^2+3q^2}{2})^2$, $t_1 = \frac{(k^2+3q^2)^2}{4}$, $r_1 = 16k^2q^2$, where $0 < k < q$ or $k > 3q$, and $2|(k+q)$.

(5) $m_1 = 3(k^2-q^2)(9q^2-k^2)$, $t_1 = 4(k^2-3q^2)^2$, $r_1 = 4k^2q^2$, where $q < k < 3q$.

Then $K_{1,r} \bullet T(m,t)$, $K_{1,t} \bullet T(m,r)$ and $T(r,m,t)$ are integral trees with diameter 4,4, and 6, respectively.

Next we give some notations on the Pell equation.

Let d be a positive integer that is not a square. The equation $x^2 - dy^2 = 1$ with variables x, y over integers is called *Pell equation.* If x_1, y_1 (or (x_1, y_1)) is a solution of the Pell equation, for convenience, then $x_1 + y_1\sqrt{d}$ is also called a solution of the Pell equation. A solution x_1, y_1 (or (x_1, y_1)) of the Pell equation is called positive if both x_1 and y_1 are positive integers. A positive solution x_1, y_1 (or (x_1, y_1)) is called the *fundamental solution* if it satisfies $x_1 < x$ and $y_1 < y$ for every other positive solution x, y (or (x, y)).

Theorem 1.3.11. *([75]) Suppose there is a solution for the Diophantine equation*

$$x^2 - (c^2 - h^2)y^2 = h^2w^2, \tag{1.3.1}$$

where $c > h$, c, h and w are positive integers, and $c^2 - h^2$ is not a perfect square. Let x_1, y_1 be the fundamental solution of the Diophantine equation

$$x^2 - (c^2 - h^2)y^2 = 1, \tag{1.3.2}$$

Let K be any associate class of solutions of the Diophantine equation (1.3.1)*(see Section 2.1.2), and let u_1, v_1 be the fundamental solution of the class K. Then all positive integral solutions x_k, y_k of the class K are given by*

$$x_k + y_k\sqrt{c^2 - h^2} = (x_1 + y_1\sqrt{c^2 - h^2})^k(u_1 + v_1\sqrt{c^2 - h^2}), \tag{1.3.3}$$

$k = 1, 2, 3, \cdots$

Theorem 1.3.12. *([75]) If there exists a solution for the Diophantine equation* (1.3.1)*, then we let c, h, w, u_1, v_1, x_k and y_k $(k = 1, 2, \cdots)$ be the same as in Theorem 1.3.11. For any positive integer n, if $t = (cwn)^2$, $m = [x_k^2 - (cw)^2]n^2$ and $r = (hy_kn)^2$, then $T(r,m,t)$ is an integral tree with diameter 6, and $K_{1,r} \bullet T(m,t)$, $K_{1,t} \bullet T(m,r)$ are integral trees with diameter 4.*

Theorem 1.3.13. *([75]) Let $m_1 = a(a+1) - \epsilon$, $r = a(a+1) + \epsilon - c^2 + 1$, $(\epsilon \in \{-c, c\})$, $m_2 = m_3 = \cdots = m_r = c^2$, $r > 1$, $m_1 > 0$, $c > 0$, and a, c are positive integers. Then $S(r; m_i)$ is an integral tree with diameter 4.*

Theorem 1.3.14. *([75]) Let $m_1 = b^2 + k$, $r = a^2 - c^2 - k (> 1)$, $m_2 = m_3 = \cdots = m_r = c^2$, where k is a positive integer. If there is a positive integer* *solution a, b and c for the Diophantine equation*

$$kx^2 - (k+1)y^2 + z^2 = k^2 + k. \tag{1.3.4}$$

Then $S(r; m_i)$ is an integral tree with diameter 4.

In Chapter 3 or [77], we obtain the following results on integral trees $K_{1,s} \bullet T(m,t)$ of diameter 4.

Theorem 1.3.15. *The tree $K_{1,s} \bullet T(m,t)$ of diameter 4 is integral if and only if $t = k^2$, $s = \frac{a^2b^2}{k^2}(\geq 1)$, and $m = a^2 + b^2 - k^2 - \frac{a^2b^2}{k^2}(> 1)$, where a, b and k are positive integers.*

In 1998, P.Z. Yuan gave a necessary condition for graphs to be integral trees $S(r; m_i)$ of diameters 4 based on [47]. He constructed many new types of integral trees of diameter 4 by using this necessary condition and he also answered some problems posed by Li and Lin [47].

Theorem 1.3.16. *([85]) A necessary condition for the tree $S(r; m_i)$=$S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameters 4 to be integral is that all solutions of the equation*

$$\sum_{i=1}^{s} \frac{a_i}{x^2 - m_i} = 1 \tag{1.3.5}$$

are integers. Moreover, there exist positive integers $u_1, u_2, \cdots, u_s$ satisfying

$$\sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \ldots < u_{s-1} < \sqrt{m_s} < u_s < +\infty \tag{1.3.6}$$

such that the following linear system in $a_1, a_2, \cdots, a_s$ has positive integral solutions $(a_1, a_2, \cdots, a_s)$.

$$\left\{ \begin{array}{l} \frac{a_1}{u_1^2 - m_1} + \frac{a_2}{u_1^2 - m_2} + \cdots + \frac{a_s}{u_1^2 - m_s} = 1, \\ \cdots\cdots\cdots \quad \cdots\cdots\cdots \\ \frac{a_1}{u_s^2 - m_1} + \frac{a_2}{u_s^2 - m_2} + \cdots + \frac{a_s}{u_s^2 - m_s} = 1. \end{array} \right. \tag{1.3.7}$$

In 2000, D.L. Zhang, S.W. Tan [86] and M.S. Li, W.S. Yang, J.B. Wang [46] gave a sufficient and necessary condition for graphs to be an integral tree. From this sufficient and necessary condition, they constructed some new types of integral trees of diameter 4. They also answered some basic open problems posed by Yuan [85].

Theorem 1.3.17. *([86] or [46]) The tree* $S(r; m_i)=S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ *of diameters 4 is integral if and only if there exist positive integers* u_i *and nonnegative integers* m_i $(i = 1, 2, \cdots, s)$ *such that* $\sqrt{m_1} < u_1 < \sqrt{m_2} < u_2 < \ldots < u_{s-1} < \sqrt{m_s} < u_s < +\infty$, *and such that*

$$a_k = \frac{\prod_{i=1}^{s}(u_i^2 - m_k)}{\prod_{i=1, i\neq k}^{s}(m_i - m_k)}, \quad (k = 1, 2, \cdots, s) \tag{1.3.8}$$

are positive integers, and such that $a_i = 1$ *must hold if* m_i *is not a perfect square.*

In Chapter 3 or [77], we also give a generalization of the integral tree of diameter 4, and obtain some new families of integral trees with diameter 4 by a computer search. Integral trees $K_{1,a_0} \bullet S(r; m_i) = K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 and $S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 are studied. In particular, integral trees $K_{1,s} \bullet S(m + q; t, r)$ of diameter 4 are obtained. The complete results can be found in Chapter 3.

Many other more or less particular results, in the form of necessary or sufficient conditions, for integral trees of diameter four can be found in [2, 15, 38, 39, 46, 47, 48, 51, 59, 69, 75, 76, 77, 80, 81, 82, 85, 86, 87].

The following three theorems of [38] contain important general results on balanced integral trees.

Theorem 1.3.18. *([38]) The tree* $T(n_k, n_{k-1}, \ldots, n_1)$ *of diameter* $2k$ *(where* $n_i > 1, i = 2, 3, \ldots, k$*) is integral if and only if for every* $n \in N$ *the tree* $T(n_k n^2, n_{k-1}n^2, \cdots, n_1 n^2)$ *of diameter* $2k$ *is integral.*

Theorem 1.3.19. *([38]) If the tree* $T(n_k, n_{k-1}, \cdots, n_1)$*(where* $n_i > 1, i = 2, 3, \ldots, k$*) is integral, then the tree* $T(n_j, n_{j-1}, \cdots, n_1)$ *is integral for every* $1 \leq j \leq k - 1$.

A branch of a tree T is a subtree T' of T such that every end-vertex of T' is an end-vertex of T.

Theorem 1.3.20. *([38]) Let* T *be an integral tree. If the balanced tree defined by* $T(2, n_k, n_{k-1}, \cdots, n_1)$ *is a branch of* T*, then the tree* $T(n_k, n_{k-1}, \cdots, n_1)$ *is integral.*

Integral trees with diameter five were mentioned for the first time in [47], but the authors were not able to find any example. The first integral tree with diameter five was constructed in [50], while in [16] it was also proved that there are infinitely many such trees. In [49], Y. Li generalized results of integral trees of diameter 5 of R.Y. Liu [50] and Z.F. Cao [16] by using the solutions of some general quadratic Diophantine equations. It is interesting that none of them is balanced.

The following lemmas contain results on Diophantine equations and will be used later on.

Lemma 1.3.21. *([17]) Let d (> 1) be a positive integer but not a perfect square. Then there exist solutions for the Pell equation*

$$x^2 - dy^2 = 1, \tag{1.3.9}$$

and all positive integral solutions x_k, y_k of Equation (1.3.9) *are given by*

$$x_k + y_k\sqrt{d} = \varepsilon^k, \tag{1.3.10}$$

$k = 1, 2, 3, \cdots$, where $\varepsilon = x_0 + y_0\sqrt{d}$ is the fundamental solution of Equation (1.3.9)*. Define $\overline{\varepsilon} = x_0 - y_0\sqrt{d}$. Then we have $\varepsilon\overline{\varepsilon} = 1$ and*

$$x_k = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2}, \quad y_k = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}}, \quad k = 1, 2, 3, \cdots . \tag{1.3.11}$$

Lemma 1.3.22. *([56]) Suppose that the Pell equation*

$$x^2 - dy^2 = -1 \tag{1.3.12}$$

is solvable. Let $\rho = x_0 + y_0\sqrt{d}$ be the fundamental solution of Equation (1.3.12)*, where $d(> 1)$ is a positive integer but not a perfect square. Then we have the following results.*

(1) All positive integral solutions x_k, y_k of Equation (1.3.12) *are given by*

$$x_k + y_k\sqrt{d} = \rho^k, \quad k = 1, 3, 5, \cdots . \tag{1.3.13}$$

(2) All positive integral solutions x_k, y_k of Equation (1.3.9) *are given by relation* (1.3.13)*, $k = 2, 4, 6, \cdots$.*

(3) Let $\overline{\rho} = x_0 - y_0\sqrt{d}$, then $\rho\overline{\rho} = -1$, and the solutions x_k, y_k in (1) and (2) can be defined by

$$x_k = \frac{\rho^k + \overline{\rho}^k}{2}, \quad y_k = \frac{\rho^k - \overline{\rho}^k}{2\sqrt{d}}, \quad k = 1, 2, 3, \cdots . \tag{1.3.14}$$

Lemma 1.3.23. *([49, 90])*

(1) *Let d (> 1) be a positive integer with square-free divisors. If there exist $d_1 > 1$ and d_2 such that $d = d_1 d_2$ and the Diophantine equation*

$$d_1 x^2 - d_2 y^2 = 1 \tag{1.3.15}$$

has positive integral solutions, then d_1, d_2 are uniquely determined by d.

(2) *Let $\varepsilon_1 = x_1\sqrt{d_1}+y_1\sqrt{d_2}$ be the fundamental solution of Equation (1.3.15). Then all positive integral solutions x_n, y_n of Equation (1.3.15) are given by*

$$x_n\sqrt{d_1} + y_n\sqrt{d_2} = \varepsilon_1^n, \quad 2 \nmid n. \tag{1.3.16}$$

(3) *Let $\overline{\varepsilon}_1 = x_1\sqrt{d_1} - y_1\sqrt{d_2}$. Then $\varepsilon_1\overline{\varepsilon}_1 = 1$ and the solutions in (2) have the form*

$$x_n = \frac{\varepsilon_1^n + \overline{\varepsilon}_1^n}{2\sqrt{d_1}}, \quad y_n = \frac{\varepsilon_1^n - \overline{\varepsilon}_1^n}{2\sqrt{d_2}}, \quad 2 \nmid n. \tag{1.3.17}$$

Theorem 1.3.24. *([16])*

(1) *Let d (> 1) be a positive integer but not a perfect square, and let x_k, y_k be defined by (1.3.11). If $m = d(\frac{y_n - y_l}{2})^2$, $r = d(\frac{y_n+y_l}{2})^2$, and $t = (\frac{x^2_{\frac{n+l}{2}} - x^2_{\frac{n-l}{2}}}{4})^2$, where $n > l > 0$, n and l are even, then the trees $T^t[m, r]$ of diameter 5 are integral.*

(2) *Let $d > 1$ be such that there exist positive integral solutions for Equation (1.3.12), and let x_k, y_k be defined by (1.3.14). If $m = d(x_n y_n - x_l y_l)^2$, $r = d(x_n y_n + x_l y_l)^2$, and $t = (\frac{x^2_{n+l} - x^2_{n-l}}{4})^2$, where $n > l > 0$, then the trees $T^t[m, r]$ of diameter 5 are integral.*

The result on integral tree of diameter 5 of R.Y. Liu [50] is a special case of Theorem 1.3.24 (when $d = 2$ and $k = l + 1$ in (2)).

Theorem 1.3.25. *([49])*

(1) *Let d (> 1) be a positive integer with square-free divisors, $d = d_1 d_2$, $d_1 > 1$ such that Equation (1.3.15) has positive integral solutions, and let x_k, y_k be defined by (1.3.17). If $m = dx_k^2 y_l^2$, $r = dx_l^2 y_k^2$, and $t = d_1^2(\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $2 \nmid kl$, then the trees $T^t[m, r]$ of diameter 5 are integral.*

(2) *Let d (>1) be a positive integer with square-free divisors, and x_k, y_k be defined by* (1.3.11). *Let $m = dx_k^2 y_l^2$, $r = dx_l^2 y_k^2$, and $t = (\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $\varepsilon = x_0 + y_0\sqrt{d}$ is the fundamental solution of Equation* (1.3.9). *If $2 \nmid x_0$ or $2|x_0$, and $k \equiv l (mod\, 2)$, then the trees $T^t[m, r]$ of diameter 5 are integral.*

In fact, the results on integral trees of diameter 5 of Z. F. Cao [16] (*i.e.* (2) and (1) of Theorem 1.3.24) are special cases of Theorem 1.3.25 (1) and (2), respectively (take $d_1 = d$ and even k, l).

In Chapter 4 of this thesis, the structure of integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 was found for the first time. The complete results are described in Section 4.1 of Chapter 4 (see also [69]).

Theorem 1.3.26. *([38]) There is no balanced integral tree of diameter $4k+1$ $(k \in N)$.*

As for diameter $4k-1$, so far the following result is known.

Theorem 1.3.27. *([38]) There is no balanced integral tree of diameter seven.*

The question of finding an integral tree of diameter six was posed for the first time in [79]. It was the observation of C. Godsil that one can construct integral trees of diameter six by attaching t new end-vertices to each vertex of the tree $T(r, m)$ (in our notation $T^t(r, m)$). The parameters t, r, m must be chosen so that m, $m+r$, t, $m+4t$ and $m+r+4t$ are perfect squares, which can be achieved by taking

$$m = (a^2 - b^2)^2, r = (c^2 - d^2)^2 - (a^2 - b^2)^2, t = a^2b^2 = c^2d^2.$$

For example, $a = 3$, $b = 2$, $c = 6$, $d = 1$ yields an integral tree of diameter six with 1 123 236 vertices.

Balanced trees $T(r, m, t)$ of diameter 6 can be constructed as follows.

Theorem 1.3.28. *([47]) The tree $T(r, m, t)$ of diameter 6 (where $r, m > 1$) is integral if and only if t and $m+t$ are perfect squares, and $x^4 - (r+m+t)x^2 + rt$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are positive integers.*

For example, let $p, q \in N$, $p > q$, and put $t = 4p^2q^2$, $m = (p^2 - q^2)^2$, $r = (p^2 + q^2)^2$. Then if $2(p^2 + q^2)$ is a perfect square, the tree $T(r, m, t)$ is integral. Thus, for $p = 7$, $q = 1$ we have one such example and by Theorem 1.3.18 the number of integral trees $T(r, m, t)$ is infinite.

A somewhat different form of the same result can be found in [38].

Theorem 1.3.29. *([38]) The tree $T(r, m, t)$ of diameter 6 (where $r, m > 1$)*

is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $r = a^2b^2/k^2$, *where* a, b, k, n *are positive integers satisfying*

$$(k^2 - b^2)(a^2 - k^2) = k^2(n^2 + 2nk), \quad b < k < a. \tag{1.3.18}$$

In [75], we also give some different sufficient conditions for $T(r, m, t)$ to be an integral tree (see also the previous Theorems 1.3.10 and 1.3.12).

Theorem 1.3.30. *([75]) Let* m_1, t_1, r_1, a, b, c *and* d *be positive integers satisfying the following conditions*

$m_1 + t_1 + r_1 = a^2 + b^2 = c^2 + d^2$,

where $c > a > d$, $c > b > d$, $(a, b) = 1$, $(c, d) = 1$ *and* $a|cd$ *or* $b|cd$. *Let* $m = m_1n^2$, $t = t_1n^2$ *and* $r = r_1n^2$, *where* n *is a positive integer. Then the tree* $T(r, m, t)$ *of diameter 6 is integral if one of the following conditions holds.*

(1) $a|cd$, $m_1 = b^2 - (\frac{cd}{a})^2$, $t_1 = (\frac{cd}{a})^2$ *and* $r_1 = a^2$.

(2) $b|cd$, $m_1 = a^2 - (\frac{cd}{b})^2$, $t_1 = (\frac{cd}{b})^2$ *and* $r_1 = b^2$.

In [69], a generalization of the previous result is given.

Theorem 1.3.31. *([69]) The tree* $K_{1,s} \bullet T(r, m, t)$ *of diameter 6 (where* $r, m > 1$*) is integral if and only if* t *and* $m + t$ *are perfect squares, and* $x^4 - (m + t + r + s)x^2 + rt + s(m + t)$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$, *where* a *and* b *are integers.*

Particularly, if $s = t$ we have the following.

Corollary 1.3.32. *([69]) For* $s = t$ *the tree* $K_{1,s} \bullet T(r, m, t)$ *of diameter 6 is integral if and only if* t, $m + t$ *and* $m + t + r$ *are perfect squares.*

Theorem 1.3.33. *([69]) For the tree* $K_{1,r} \bullet T(s, m, t)$ *of diameter 6, let the numbers* m, t, s, m_1, t_1, s_1, a, b, c *and* d *be the same as those of (1) or (3) in Theorem 1.3.9, and let* $r = t$ *and* $m_1 + t_1 + s_1$ *be perfect squares. Then* $K_{1,t} \bullet T(s, m, t)$ *is an integral tree with diameter 6.*

We also presented a list of examples of such integral trees in [69]. In Chapter 3 (see also [77]), a theorem equivalent to the previous one is given.

Theorem 1.3.34. *([77]) The tree* $K_{1,s} \bullet T(r, m, t)$ *of diameter 6 (where* $r, m > 1$*) is integral if and only if* $t = k^2$, $m = n^2 + 2nk$, $s = k^2 + \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk} (\geq 1)$ *and* $r = a^2 + b^2 - (n + k)^2 - k^2 - \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk} (> 1)$, *where* a, b, k *and* n *are positive integers.*

An interesting result on such a type of trees is as follows:

Theorem 1.3.35. *([77]) Suppose the tree $K_{1,s} \bullet T(r,m,t)$ of diameter 6 is integral, and m (≥ 2) and r (≥ 2) are positive integers. Then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is an integral tree.*

The previous integral trees $T^t(r,m)$ of diameter 6 have also been studied by R.Y. Liu [51], Z.F. Cao [16], and L.G. Wang and X.L. Li [70].

Theorem 1.3.36. *(see [51, 70] Let a, b be positive integers satisfying $b < a < b^2$. If $m = (a^2 - b^2)^2$, $r = (ab^2 - a)^2 - (a^2 - b^2)^2$, $t = a^2b^2$, then for any positive integer n the tree $T^{tn^2}(rn^2, mn^2)$ of diameter 6 is integral.*

Theorem 1.3.37.

(1) Let d (> 1) be a positive integer but not a perfect square, and x_k, y_k be defined by (1.3.11). If $m = (dy_{k+l}y_{k-l})^2$, $r = x_{2k}x_{2l}$, and $t = (\frac{x_{k+l}^2 - x_{k-l}^2}{4})^2$, where $k > l > 0$, k and l are positive integers, then all trees $T^t(r,m)$ ([16]) and $T^{tn^2}(rn^2, mn^2)$ ([70]) of diameter 6 are integral for every $n \in N$.

(2) Let d (> 1) be a positive integer but not a perfect square, let Equation (1.3.12) have positive integral solutions, and let x_k, y_k be defined by (1.3.14). If

$$m = \begin{cases} (dy_{k+l}y_{k-l})^2, & \text{if } k \equiv l(mod2), \\ (x_{k+l}x_{k-l})^2, & \text{if } k \not\equiv l(mod2), \end{cases}$$

$r = x_{2k}x_{2l}$, and $t = (\frac{x_{k+l}^2 - x_{k-l}^2}{4})^2$, where $k > l > 0$, then all trees $T^t(r,m)$ ([16]) and $T^{tn^2}(rn^2, mn^2)$ ([70]) of diameter 6 are integral for every $n \in N$.

Besides these general facts in the previous theorems, we have described many particular results on integral trees of diameter six. In most of these cases we have a construction of a set of sufficient conditions for such a tree to be integral, combined with a computer search which provides examples. Various results on integral trees of diameter six can be found in [2, 15, 16, 37, 38, 39, 47, 48, 50, 51, 69, 70, 74, 75, 77, 79].

Integral trees $T^t(r,m)$, $T(r,m,t)$, and $K_{1,s} \bullet T(r,m,t)$ of diameter 6 were investigated in [2, 16, 48, 51, 70, 77, 79], [2, 15, 37, 38, 39, 47, 48, 50, 75, 76, 77] and [2, 48, 69, 75, 76, 77], respectively. In Chapter 4, the structures of integral trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 are found for the first time.

In 1998, P. Híc and R. Nedela firstly constructed infinitely many balanced integral trees with diameter 8 (see also [38]). We also constructed independently some families of integral trees of diameter 8 by using a different method in Chapter 3, [74] or [75]. The following theorem can be found in [38].

Theorem 1.3.38. *([38]) The tree $T(q,r,m,t)$ of diameter 8 (where $q,r,m > 1$) is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $r = \frac{a^2b^2}{k^2}$, $q = \frac{c^2d^2-a^2b^2}{(n+k)^2}$, where a, b, c, d, k, n are positive integers satisfying* (1.3.18) *and*

$$(c^2+d^2)(n+k)^2k^2 = (n+k)^4k^2+a^2b^2(n^2+2nk)+c^2d^2k^2, \quad a^2b^2 < c^2d^2. \tag{1.3.19}$$

In [74, 75], we obtained the following somewhat different form of the same result.

Theorem 1.3.39. *([74] or [75]) The tree $T(q,r,m,t)$ of diameter 8 (where $q,r,m > 1$) is integral if and only if both t and $m+t$ are perfect squares, $x^4 - (m+t+r)x^2 + rt$ can be factorized as $(x^2-a^2)(x^2-b^2)$, and $x^4 - (q+m+t+r)x^2 + rt + q(m+t)$ can be factorized as $(x^2-c^2)(x^2-d^2)$, where a, b, c and d are integers.*

Particularly, if $q = t$ we have the following.

Corollary 1.3.40. *([74] or [75]) If $q = t$, then the tree $T(q,r,m,t)$ of diameter 8 (where $q,r > 1$) is integral if and only if t, $m+t$ and $m+t+r$ are perfect squares, and $x^4 - (m+t+r)x^2 + rt$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are positive integers.*

Theorem 1.3.41. *([74] or [75]) Let the numbers m, t, r, m_1, t_1, r_1, a, b, c and d be as in Theorem 1.3.30, and let $q = t$ and $m_1 + t_1 + r_1$ be a perfect square. Then $T(q,r,m,t)$ is an integral tree of diameter 8, and $T(r,m,t)$ is an integral tree of diameter 6.*

Some examples and sufficient conditions for integral trees of diameter 8 are given in Chapter 3 and [39, 74, 76, 77]. In Chapter 4, we find the structure of integral trees $K_{1,s} \bullet T(q,r,m,t)$ of diameter 8 for the first time. In Chapters 3 and 4 and [76, 77], some results treat interrelations between integral trees of various diameters. Let us give an example.

Theorem 1.3.42. *([77]) If a balanced tree $T(s,r,m,t)$ of diameter 8 is integral, and $s,r,m > 1$, then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral too.*

In 2003, an infinite class of integral balanced rooted trees of diameter 10 was found in [39]. Let us give an example (for details see [39]).

Theorem 1.3.43. *([39]) For any positive integer n we have the following result: the tree* $T(3006756n^2,\ 1051960n^2,\ 751689n^2,\ 283360n^2,\ 133956n^2)$ *is an integral balanced rooted tree of diameter 10, and its spectrum is given by* $Spec(T) = \{0,\ \pm 289n;\ \pm 306n,\ \pm 366n,\ \pm 527n, \pm 646n,\ \pm 918n, \pm 1037n, \pm 1394n,\ \pm 2074n\}$.

In fact, for every k a system (S_k) of diophantine equations can be found such that every solution of (S_k) gives an integral tree $T(n_k, n_{k-1}, \ldots, n_1)$ and vice versa. However, at the moment no solutions of (S_k) is known for $k \geq 6$. Moreover, no integral tree of diameters 7, 9 and greater than 10 has been found so far.

1.3.2 Results on integral graphs

Secondly, we consider integral graphs. These are investigated in the third part of this thesis, Chapters 5 through 7. There are many results concerning some particular classes of integral graphs. Next we list some known main results on integral graphs.

(1) All such connected integral cubic graphs were obtained by D. Cvetković and F. C. Bussemaker [20, 14], and independently in 1976 by A. J. Schwenk [63]. There are exactly thirteen connected cubic integral graphs.

(2) An infinite family of integral complete tripartite graphs was constructed by M. Roitman [60] in 1984.

(3) All 13 connected nonregular nonbipartite integral graphs whose maximum degree equals 4 were determined by Z. Radosavljević and S. Simić [58, 65].

(4) All 24 connected 4-regular integral graphs avoiding ± 3 in the spectrum were determined by D. Stevanović [66].

(5) 1888 possible spectra of 4-regular bipartite integral graphs were found by D. Cvetković, S. Simić and D. Stevanović (see [24] or [68]).

(6) Nonexistence results for some 4-regular integral graphs were obtained by D. Stevanović [67].

(7) Some graphs with exact three distinct eigenvalues were investigated by W. G. Bridges and R. A. Mena in 1979 and 1981 (see [12, 13]).

(8) Nonregular or regular graphs with few distinct eigenvalues were further studied by E.R. van Dam *et al.* in [25, 26, 27, 28, 55].

(9) It was proved by K. Balińska *et al.* [1] that there are exactly 150 connected integral graphs up to 10 vertices. The results on all connected integral graphs with 11 and 12 vertices can be found in [3, 4, 23].

(10) Some results on nonregular, bipartite, integral graphs with maximum degree 4 were obtained by K. Balińska and S. Simić *et al.* [5, 6, 7].

(11) A family of integral complete split graphs was characterized by P. Hansen, H. Mélot and D. Stevanovic [34] (see also [33]).

(12) Some constructions on integral graphs were studied and integral graphs K_n^t, $K_{a,b}^t$, $K_{a,a,\ldots,a}^t$, etc. were obtained by L.G. Wang, X.L. Li and S.G. Zhang [76] in 2000.

(13) Some new integral graphs based on the study of bipartite semiregular graphs were obtained by D.L. Zhang and H.W. Zhou [88] in 2003.

(14) Results on integral graphs which belong to the class $\overline{\alpha K_{a,b}}$, $\overline{\alpha K_a \cup \beta K_b}$ or $\overline{\alpha K_a \cup \beta K_{b,b}}$ were presented by M. Lepović [41, 42, 43, 44] in 2003 and 2004.

In particular, K. T. Balińska *et al.* [2] presented a survey of results on integral graphs and on the corresponding proof techniques used until 2002. Note that a few errata of the article [2] appeared in [23] in 2004.

Next we introduce some main results on integral graphs derived in this thesis.

In Chapter 5 (see also [71]), we give a useful sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving certain Diophantine equations. The discovery of these integral complete r-partite graphs is a new contribution to the research on integral graphs. In fact, M. Roitman's result on the integral complete tripartite graphs is generalized (see also [60] MR0772296 (86a:05089)).

In Chapter 6, we shall construct fifteen classes of larger nonregular and bipartite integral graphs from the 21 known smaller integral graphs (see also [5]). Their spectra and characteristic polynomials are obtained from matrix theory. We obtain their integral property by number theory and computer search. All these classes are infinite. They are different from those in the literature. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving Diophantine equations. These results generalize some results of Balińska and Simić (see also [5], MR1830594 (2002a:05171)).

In Chapter 7, we determine the characteristic polynomials of four classes of graphs. We also obtain sufficient and necessary conditions for these graphs to be integral by using number theory and computer search. All these classes are infinite and different from those in the literature. We also prove that the problem of finding integral graphs is equivalent to the problem of solving Diophantine equations. At the same time, we also give some new cospectral graphs and cospectral integral graphs. Note that some results on cospectral graphs can be found in [21, 22, 27, 31, 45, 62, 76].

1.3.3 Further results on integral graphs

Thirdly, we consider Laplacian integral and integral regular graphs. In the fourth part of this thesis, Chapter 8, we mainly investigate two classes of Laplacian integral and integral regular graphs. These results generalize the following results of Harary and Schwenk [36] or Schwenk [63].

Theorem 1.3.44. *([14], [20] or [63]) There are exactly thirteen connected cubic integral graphs. They are: K_4, $K_{3,3}$, $C_3 + K_2$, $C_4 + K_2$, $C_6 + K_2$, the Petersen graph, $L(S(K4))$, Tutte's 8-cage, the graph on 10 vertices obtained from $K_{3,3}$ by specifying a pair of nonadjacent vertices and replacing each of them by a triangle, Desargues's graph and its cospectral-mate, the graph obtained from two (disjoint) copies of $K_{2,3}$ by adding three edges between vertices of degree two in different copies of $K_{2,3}$, and a bipartite graphs on 24 vertices (with girth 6).*

In Chapter 8 (see also [72]), we find the spectra and characteristic polynomials of two classes of regular graphs. We derive the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. These graphs are not only integral but also Laplacian integral. These results generalize some results of Harary and Schwenk in [36].

Chapter 2

Some facts in number theory and matrix theory

In this chapter, we shall give some facts in number theory and some notations on matrix theory. All other notations and terminology can be found in [17, 56, 57, 90] and [8, 18, 29, 52].

2.1 Some facts in number theory

2.1.1 Some specific useful results

The following Lemmas can be found in [57, 75].

Lemma 2.1.1. *([57]) If $x > 0$, $y > 0$, $z > 0$, $(x, y) = 1$ and $2|y$, then all positive integral solutions of the Diophantine equation $x^2 + y^2 = z^2$ are given by*

$$x = r^2 - s^2, \;\; y = 2rs, \;\; z = r^2 + s^2,$$

where $(r, s) = 1$, $r > s > 0$ and $2 \not| r + s$.

Lemma 2.1.2. *([75])*
There exist positive integers $n = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, with $l = 0$ or 1, $s \geq 2$, and primes p_i of the form $p_i \equiv 1 (mod\, 4)$, $i = 1, 2, \cdots, s$, such that n can be expressed as

$$n = a^2 + b^2 = c^2 + d^2 \tag{2.1.1}$$

satisfying $a|cd$ or $b|cd$, where a, b, c and d are positive integers with $c > a > d$, $c > b > d$, $(a, b) = 1$ and $(c, d) = 1$. In particular, there are such n's with

$n = (p_1 p_2 \cdots p_s)^2$.

Regarding Lemma 2.1.2, we simply list the following examples.

(i) For $n = 2^l p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$, we have

$$5 \times 13 = 7^2 + 4^2 = 8^2 + 1^2, \qquad 5 \times 17 = 7^2 + 6^2 = 9^2 + 2^2,$$
$$5 \times 41 = 13^2 + 6^2 = 14^2 + 3^2, \qquad 5 \times 53 = 12^2 + 11^2 = 16^2 + 3^2,$$
$$5 \times 101 = 19^2 + 12^2 = 21^2 + 8^2, \qquad 13 \times 17 = 11^2 + 10^2 = 14^2 + 5^2,$$
$$13 \times 37 = 16^2 + 15^2 = 20^2 + 9^2, \qquad 13 \times 53 = 20^2 + 17^2 = 25^2 + 8^2,$$
$$13 \times 97 = 30^2 + 19^2 = 35^2 + 6^2, \qquad 13 \times 113 = 37^2 + 10^2 = 38^2 + 5^2,$$
$$13 \times 181 = 47^2 + 12^2 = 48^2 + 7^2, \qquad 13 \times 313 = 62^2 + 15^2 = 63^2 + 10^2,$$
$$13 \times 317 = 61^2 + 20^2 = 64^2 + 5^2, \qquad 13 \times 337 = 59^2 + 30^2 = 66^2 + 5^2,$$
$$13 \times 613 = 87^2 + 20^2 = 88^2 + 15^2, \qquad 13 \times 733 = 77^2 + 60^2 = 85^2 + 48^2,$$
$$13 \times 757 = 79^2 + 60^2 = 96^2 + 25^2, \qquad 17 \times 37 = 23^2 + 10^2 = 25^2 + 2^2,$$
$$17 \times 53 = 26^2 + 15^2 = 30^2 + 1^2, \qquad 17 \times 257 = 63^2 + 20^2 = 65^2 + 12^2,$$
$$17 \times 73 = 29^2 + 20^2 = 35^2 + 4^2, \qquad 17 \times 137 = 40^2 + 27^2 = 48^2 + 5^2,$$
$$17 \times 193 = 41^2 + 40^2 = 55^2 + 16^2, \qquad 29 \times 37 = 28^2 + 17^2 = 32^2 + 7^2,$$
$$29 \times 41 = 30^2 + 17^2 = 33^2 + 10^2, \qquad 29 \times 61 = 37^2 + 20^2 = 40^2 + 13^2,$$
$$29 \times 89 = 41^2 + 30^2 = 50^2 + 9^2, \qquad 29 \times 281 = 57^2 + 70^2 = 90^2 + 7^2,$$
$$29 \times 389 = 84^2 + 65^2 = 105^2 + 16^2, \qquad 41 \times 61 = 49^2 + 10^2 = 50^2 + 1^2,$$
$$5 \times 13 \times 17 = 24^2 + 23^2 = 32^2 + 9^2, \qquad 5 \times 13 \times 17 = 31^2 + 12^2 = 32^2 + 9^2,$$
$$5 \times 13 \times 17 = 31^2 + 12^2 = 33^2 + 4^2,$$
$$5 \times 13 \times 17 \times 37 = 167^2 + 114^2 = 194^2 + 57^2,$$
$$257 \times 65537 = 4095^2 + 272^2 = 4097^2 + 240^2.$$

(ii) For $n = (p_1 p_2 \cdots p_s)^2$, we have

$$(5 \times 13)^2 = 56^2 + 33^2 = 63^2 + 16^2,$$
$$(5 \times 29)^2 = 143^2 + 24^2 = 144^2 + 17^2,$$
$$(13 \times 17)^2 = 171^2 + 140^2 = 220^2 + 21^2,$$
$$(17 \times 37)^2 = 460^2 + 429^2 = 621^2 + 100^2,$$
$$(41 \times 61)^2 = 2301^2 + 980^2 = 2499^2 + 100^2.$$

Remark 2.1.3. *We found the solutions above by checking* $5p_1$, $13p_2$, $17p_3$, $29p_4$, *with primes* $p_i \equiv 1 (mod\, 4)$, $i = 1, 2, 3, 4$ *such that* $13 \le p_1 \le 1009$, $17 \le p_2 \le 1009$, $29 \le p_3 \le 229$ *and* $37 \le p_4 \le 557$; *while other solutions are obtained from one by one checking. In addition, we note that some of the primes are Fermat primes* $F_m = 2^{2^m} + 1$, *for* $m = 1, 2, 3, 4$.

In connection with Lemma 2.1.2, the following problems arise, which are not only useful for the construction of integral trees but are also interesting purely as problems in number theory.

Problem 2.1.4. *Find all positive integral solutions for the Diophantine equation*

$$n = x^2 + y^2 = z^2 + w^2 \tag{2.1.2}$$

such that $x|zw$ *or* $y|zw$, *where* $z > x > w$, $z > y > w$, $(x, y) = 1$ *and* $(z, w) = 1$.

Problem 2.1.5. *For Problem 2.1.4, we conjecture that there are infinitely many solutions. Even infinitely many where* n *is a perfect square?*

Problem 2.1.6. *Find all solutions for Problem 2.1.4, for special* $n = (5p_1)^\alpha$, $(5 \times 13 \times p_1)^\alpha, (p_1 \times p_2)^\alpha, (p_1 \times p_2 \times p_3)^\alpha, \cdots$, *where* $\alpha = 1, 2, 3, 4, \cdots$, p_i *are primes with* $p_i \equiv 1(mod\, 4)$, $i = 1, 2, 3, \cdots$.

The motivation to raise the above problems is as follows.

(i) Construct the integral trees $T(r, m, t)$ and $K_{1,s} \bullet T(m, t)$ from any positive integer solution of the Diophantine equation (2.1.2) (see [15, 37, 38, 39, 47, 50, 69, 75, 82] or Theorems 1.3.28, 1.3.29, 1.3.30, 1.3.8, 1.3.9, 1.3.10 etc.).

(ii) Construct the integral trees $T(r, m, t)$, $K_{1,s} \bullet T(r, m, t)$ and $T(s, r, m, t)$ from any positive integer solution of the Diophantine equation (2.1.2) of the second kind in Problem 2.1.5 (see [38, 39, 69, 75] or Theorems 1.3.28, 1.3.29, 1.3.30, 1.3.33 (or Corollary 1.2.14), 1.3.41 etc.).

So, it is very important to find all solutions of the Diophantine equation (2.1.2). For Problem 2.1.4 and the first part of Problem 2.1.5, an affirmative answer has been given in [15, 69, 75].

2.1.2 Some results on Diophantine equations

Let d be a positive integer but not a perfect square, let $m \neq 0$ be an integer. We shall study the Diophantine equation

$$x^2 - dy^2 = m. \tag{2.1.3}$$

If x_1, y_1 is a solution of (2.1.3), for convenience, then $x_1 + y_1\sqrt{d}$ is also called a solution of the Diophantine equation (2.1.3). Let $s + t\sqrt{d}$ be any solution of the Pell equation

$$x^2 - dy^2 = 1. \tag{2.1.4}$$

Clearly,

$$(x_1 + y_1\sqrt{d})(s + t\sqrt{d}) = x_1 s + y_1 td + (y_1 s + x_1 t)\sqrt{d}$$

is also a solution of the Diophantine equation (2.1.3). This solution and $x_1 + y_1\sqrt{d}$ are called *associate*. If two solutions $x_1 + y_1\sqrt{d}$ and $x_2 + y_2\sqrt{d}$ of the Diophantine equation (2.1.3) are associate, then we denote them by $x_1 + y_1\sqrt{d} \sim x_2 + y_2\sqrt{d}$. It is easy to verify that the associate relation $\sim$ is an equivalence relation. Hence, if the Diophantine equation (2.1.3) has solutions, then all the solutions can be classified by the associate relation. Any two solutions in the same associate class are associate each other, any two solutions not in the same class are not associate.

The following Lemmas 2.1.7, 2.1.8 and 2.1.9 can be found in [17] or [90].

Lemma 2.1.7. *([17] or [90]) A necessary and sufficient condition for two solutions $x_1 + y_1\sqrt{d}$ and $x_2 + y_2\sqrt{d}$ of the Diophantine equation* (2.1.3) *(m fixed) to be in the same associate class K is that*

$$x_1x_2 - dy_1y_2 \equiv 0(mod|m|) \quad and \quad y_1x_2 - x_1y_2 \equiv 0(mod|m|).$$

Let $x_1 + y_1\sqrt{d}$ be any solution of the Diophantine equation (2.1.3). By Lemma 2.1.7, we see that $-(x_1 + y_1\sqrt{d}) \sim x_1 + y_1\sqrt{d}$, $-(x_1 - y_1\sqrt{d}) \sim x_1 - y_1\sqrt{d}$. Let K and $\overline{K}$ be two associate classes of solutions of the Diophantine equation (2.1.3) such that for any solution $x+y\sqrt{d} \in K$, it follows $x-y\sqrt{d} \in \overline{K}$. Then also the converse is true. Hence, K and $\overline{K}$ are called *conjugate classes*. If $K = \overline{K}$, then this class is called an *ambiguous class*. Let $u_0 + v_0\sqrt{d}$ be the *fundamental solution* of the associate class K, *i.e.* v_0 is positive and has the smallest value in the class K. If the class K is ambiguous, we can assume that $u_0 \geq 0$.

Lemma 2.1.8. *([17] or [90]) Let K be any associate class of solutions of the Diophantine equation* (2.1.3)*, and let $u_0 + v_0\sqrt{d}$ be the fundamental solution of the associate class K. Let $x_0 + y_0\sqrt{d}$ be the fundamental solution of the Pell equation* (2.1.4)*. Then*

$$0 \leq v_0 \leq \begin{cases} \frac{y_0\sqrt{m}}{\sqrt{2(x_0+1)}}, & \text{if } m > 0, \\ \frac{y_0\sqrt{-m}}{\sqrt{2(x_0-1)}}, & \text{if } m < 0. \end{cases} \tag{2.1.5}$$

$$0 \leq |u_0| \leq \begin{cases} \sqrt{\frac{1}{2}(x_0+1)m}, & \text{if } m > 0, \\ \sqrt{\frac{1}{2}(x_0-1)(-m)}, & \text{if } m < 0. \end{cases} \tag{2.1.6}$$

Lemma 2.1.9. *([17] or [90])*

(1) *Let d be a positive integer but not a perfect square, $m \neq 0$, and let m be an integer. Then there are only finitely many associate classes for the Diophantine equation* (2.1.3)*, and the fundamental solutions of all these classes can be found from* (2.1.5) *and* (2.1.6) *by a finite procedure.*

(2) *Let K be an associate class of solutions of the Diophantine equation* (2.1.3)*, and let $u_0 + v_0\sqrt{d}$ be the fundamental solution of the associate class K. Then all solutions of the class K are given by*

$$x + y\sqrt{d} = \pm(u_0 + v_0\sqrt{d})(x_0 + y_0\sqrt{d})^n,$$

where n is an integer, and $x_0 + y_0\sqrt{d}$ is the fundamental solution of the Pell equation (2.1.4)*.*

(3) *If u_0 and v_0 satisfy* (2.1.5) *and* (2.1.6) *but are not solutions of the Diophantine equation* (2.1.3)*, then there is no solution for the Diophantine equation* (2.1.3)*.*

The following Lemmas 2.1.10 and 2.1.11 can be found in [16] and [57], respectively.

Lemma 2.1.10. *([16]) Let d (> 1) be a positive integer that is not a perfect square. Then there exist solutions for the Pell equation* (2.1.4)*, and all the positive integral solutions x_k, y_k of Equation* (2.1.4) *are given by*

$$x_k + y_k\sqrt{d} = \varepsilon^k, \quad k = 1, 2, \cdots, \tag{2.1.7}$$

where $\varepsilon = x_0 + y_0\sqrt{d}$ is the fundamental solution of Equation (2.1.4)*. Put $\overline{\varepsilon} = x_0 - y_0\sqrt{d}$. Then we have $\varepsilon\overline{\varepsilon} = 1$ and*

$$x_k = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2}, \quad y_k = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}}, \quad k = 1, 2, \cdots. \tag{2.1.8}$$

Lemma 2.1.11. *([57]) Let u, v be the fundamental solution of the Pell equation* (2.1.4)*, where $d(> 1)$ is a positive integer but not a perfect square. Then the Pell equation*

$$x^2 - dy^2 = -1 \tag{2.1.9}$$

has solutions if and only if there exist positive integer solutions s and t for the equations

$$s^2 + dt^2 = u, \quad 2st = v,$$

such that moreover s and t are the fundamental solution of the Pell equation (2.1.9).

The following Lemmas 2.1.12 and 2.1.13 can be found in [56] and [89], respectively.

Lemma 2.1.12. *([56]) Suppose the Pell equation* (2.1.9) *is solvable. Let $\rho = x_0 + y_0\sqrt{d}$ be the fundamental solution of Equation* (2.1.9)*, where $d(> 1)$ is a positive integer but not a perfect square. Then the following holds.*

(1) All positive integral solutions x_k, y_k of Equation (2.1.9) *are given by*

$$x_k + y_k\sqrt{d} = \rho^k, \quad k = 1, 3, 5, \cdots . \tag{2.1.10}$$

(2) All positive integral solutions x_k, y_k of Equation (2.1.4) *are given by relation* (2.1.10)*, $k = 2, 4, 6, \cdots$.*

(3) Let $\overline{\rho} = x_0 - y_0\sqrt{d}$, then $\rho\overline{\rho} = -1$, and the solutions x_k, y_k in (1) and (2) can be given by

$$x_k = \frac{\rho^k + \overline{\rho}^k}{2}, \quad y_k = \frac{\rho^k - \overline{\rho}^k}{2\sqrt{d}}, \quad k = 1, 2, \cdots . \tag{2.1.11}$$

Lemma 2.1.13. *([89])*

(1) If there is a solution for the Diophantine equation (2.1.3)*, where $m \neq 0$ is integer and $d(> 1)$ is a positive integer but not a perfect square, then the Diophantine equation* (2.1.3) *has infinitely many solutions.*

(2) Let x_1, y_1 be the fundamental solution of the Diophantine equation

$$x^2 - dy^2 = 4, \tag{2.1.12}$$

where $d(> 1)$ is a positive integer but not a perfect square. Then all positive integral solutions x_k, y_k of the Diophantine equation (2.1.12) *are given by*

$$\frac{x_k + y_k\sqrt{d}}{2} = (\frac{x_1 + y_1\sqrt{d}}{2})^k, \quad k = 1, 2, \cdots . \tag{2.1.13}$$

The following Lemmas 2.1.14 and 2.1.15 can be found in [49, 90] and [57], respectively.

Lemma 2.1.14. *([49, 90])*

(1) Let d (>1) be a positive integer with square-free divisors. If there exist $d_1 > 1$ and d_2 such that $d = d_1 d_2$ and the Diophantine equation

$$d_1 x^2 - d_2 y^2 = 1 \tag{2.1.14}$$

has positive integral solutions, then d_1, d_2 are uniquely determined by d.

(2) Let $\varepsilon_1 = x_1\sqrt{d_1} + y_1\sqrt{d_2}$ be the fundamental solution of Equation (2.1.14). Then all positive integral solutions x_n, y_n of Equation (2.1.14) are given by

$$x_n\sqrt{d_1} + y_n\sqrt{d_2} = \varepsilon_1^n, \quad 2 \nmid n. \tag{2.1.15}$$

(3) Let $\overline{\varepsilon}_1 = x_1\sqrt{d_1} - y_1\sqrt{d_2}$. Then $\varepsilon_1\overline{\varepsilon}_1 = 1$ and the solutions in (2) have the form

$$x_n = \frac{\varepsilon_1^n + \overline{\varepsilon}_1^n}{2\sqrt{d_1}}, \quad y_n = \frac{\varepsilon_1^n - \overline{\varepsilon}_1^n}{2\sqrt{d_2}}, \quad 2 \nmid n. \tag{2.1.16}$$

Lemma 2.1.15. *([57]) Let m be a positive integer. If $2 \nmid m$ or $4|m$, then there exist positive integral solutions for the Diophantine equation*

$$x^2 - y^2 = m. \tag{2.1.17}$$

Remark 2.1.16. *We can give a method for finding the solutions of the Diophantine equation (2.1.17). Suppose that $m = m_1 m_2$. Let $x - y = m_1$, $x + y = m_2$ and $2|(m_1 + m_2)$. Then the solutions of the Diophantine equation (2.1.17) can be found easily (see [57]).*

The following Lemma 2.1.17 can be found in [61].

Lemma 2.1.17. *([61]) Let a, b and c be integers with $d = (a, b)$. Then we have*

(1) If $d \nmid c$, then the linear Diophantine equation in two variables

$$ax + by = c \tag{2.1.18}$$

does not have integral solutions.

(2) If $d|c$, then there are infinitely many integral solutions for Equation (2.1.18). *Moreover, if $x = x_0, y = y_0$ is a particular solution of Equation* (2.1.18), *then all its solutions are given by*

$$x = x_0 + (b/d)t, \quad y = y_0 - (a/d)t$$

where t is any integer.

2.2 Some notations from matrix theory

In this section, we give some definitions from matrix theory. For details we refer to [8, 29, 52].

(1) C and R denote the set of complex and real numbers, respectively.

(2) $C^{m\times n}$ and $R^{m\times n}$ denote the set of $m \times n$ matrices whose elements are in C and R, respectively.

(3) A^T denotes the transpose of the matrix A.

(4) A^* denotes the conjugate transpose of the matrix A.

(5) I_n denotes the $n \times n$ identity matrix.

(6) $J_{m\times n}$ and $0_{m\times n}$ denote the $m \times n$ matrix with all entries equal to 1 and the $m \times n$ matrix with all entries equal to 0, respectively.

Let $A \in \mathcal{BC}(m, r)$, *i.e.* A is a block circulant matrix given as follows:

$$A = \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix},$$

where $A_k \in R^{r\times r}$, $k = 0, 1, \cdots, m-1$.

The following Lemma 2.2.1 can also be found in [29] (see page 181)(or see also [18, 52]).

Lemma 2.2.1. *([29]) Let $A \in \mathcal{BC}(2, r)$ be symmetric, then $det(xI_{2r} - A) = det(xI_r - (A_0 + A_1)) \cdot det(xI_r - (A_0 - A_1))$. In particular, the eigenvalues of A are those of $A_0 + A_1$ together with those of $A_0 - A_1$.*

Chapter 3

Families of integral trees with diameters 4, 6 and 8

In this chapter we investigate the trees $K_{1,a_0} \bullet S(r;m_i)$, $S(r;m_i)$, $K_{1,s} \bullet T(m,t)$, $K_{1,s} \bullet S(m+q;t,r)$, $T(r,m,t)$, $K_{1,s} \bullet T(r,m,t)$ and $T(s,r,m,t)$ with diameters 4, 4, 4, 4, 6, 6 and 8, respectively. Some new families of integral trees with diameters 4, 6 and 8 are studied. Most of these classes are infinite. They are different from those of [15, 16, 19, 22, 36, 37, 38, 46, 47, 48, 49, 50, 51, 59, 69, 75, 76, 79, 80, 81, 82, 85, 86, 87]. Some results which treat interrelations between integral trees of various diameters are obtained for the first time. These results generalize some well-known results on integral trees. Finally, we propose several open problems on integral trees for further study.

3.1 Integral trees with diameter 4

In this section, we shall construct infinitely many new classes of integral trees with diameter 4. They are different from those of [15, 19, 22, 36, 38, 46, 47, 48, 51, 59, 69, 75, 76, 79, 80, 81, 82, 85, 86, 87].

Clearly the following two results in [46], [85] or [86] are corollaries of Lemma 1.2.8.

Corollary 3.1.1. *([46], [85] or [86]) For the tree $S(r;m_i)$=$S(a_1+a_2+\cdots+a_s;m_1,m_2,\cdots,m_s)$ of diameter 4, we have*

$$\begin{aligned} P[S(r;m_i),x] &= x^{1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s}(x^2-m_i)^{a_i-1}[\prod_{i=1}^{s}(x^2-m_i) \\ &\quad -\sum_{i=1}^{s} a_i \prod_{j=1,j\neq i}^{s}(x^2-m_j)]. \end{aligned}$$

Corollary 3.1.2. *([46], [85] or [86]) The tree* $S(r;m_i)$=$S(a_1+a_2+\cdots+a_s;$ $m_1,\ m_2,\ \cdots,m_s)$ *of diameter 4 is integral if and only if the equation*

$$\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}[\prod_{i=1}^{s}(x^2-m_i)-\sum_{i=1}^{s}a_i\prod_{j=1,j\neq i}^{s}(x^2-m_j)]=0$$

has only integral roots.

Theorem 3.1.3. *Let the tree* $K_{1,a_0}\bullet S(r;m_i)$=$K_{1,a_0}\bullet S(a_1+a_2+\cdots+a_s;m_1,m_2,\ \cdots,\ m_s)$ *of diameter 4 be obtained by identifying the center* w *of* K_{1,a_0} *and the center* v *of* $S(a_1+a_2+\cdots+a_s;m_1,m_2,\cdots,m_s)$*. Then*

$$P[K_{1,a_0}\bullet S(r;m_i),x]=x^{a_0-1+\sum_{i=1}^{s}a_i(m_i-1)}\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}$$
$$\times[(x^2-a_0)\prod_{i=1}^{s}(x^2-m_i)-x^2\sum_{i=1}^{s}a_i\prod_{j=1,j\neq i}^{s}(x^2-m_j)].$$

Proof. Because the vertex w is the center of K_{1,a_0} and the vertex v is the center of the tree $S(a_1+a_2+\cdots+a_s;m_1,m_2,\cdots,m_s)$, if we let $G=K_{1,a_0}$ and $H=S(a_1+a_2+\cdots+a_s;m_1,m_2,\cdots,m_s)$, then by Lemma 1.2.1 we know that

$$\begin{aligned}P[K_{1,a_0}\bullet S(r;m_i),x] &= P(K_{1,a_0},x)\prod_{i=1}^{s}P^{a_i}(K_{1,m_i},x)+x^{a_0}P[S(r;m_i),x]\\ &\quad -x^{a_0+1}\prod_{i=1}^{s}P^{a_i}(K_{1,m_i},x).\end{aligned}$$

By Lemma 1.2.9 and Corollary 3.1.1, we have

$$P[K_{1,a_0}\bullet S(r;m_i),x]=x^{a_0-1+\sum_{i=1}^{s}a_i(m_i-1)}\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}$$
$$\times[(x^2-a_0)\prod_{i=1}^{s}(x^2-m_i)-x^2\sum_{i=1}^{s}a_i\prod_{j=1,j\neq i}^{s}(x^2-m_j)].$$

The theorem is thus proved. ■

Remark 3.1.4. *For the tree* $S(a_1+a_2+\cdots+a_s;m_1,m_2,\cdots,m_s)$ *of diameter 4, with* $m_1=0$*, by Corollary 3.1.1 and Theorem 3.1.3, we have* $S(a_1+a_2+\cdots+a_s;0,m_2,m_3,\cdots,m_s)=K_{1,a_1}\bullet S(a_2+a_3+\cdots+a_s;m_2,m_3,\cdots,m_s)$*.*

The following result in [69] is a corollary of our Theorem 3.1.3.

Corollary 3.1.5. *([69])*

(1) $P[K_{1,s}\bullet T(m,t),x]=x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st]$.

(2) The tree $K_{1,s}\bullet T(m,t)$ *of diameter 4 is integral if and only if* t *is a perfect square, and* $x^4-(m+t+s)x^2+st$ *can be factorized as* $(x^2-a^2)(x^2-b^2)$*, where* a *and* b *are integers.*

Proof. (1) When $K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_k; m_1, m_2, \cdots, m_k)$=$K_{1,s} \bullet S(m+0+\cdots+0; t, 0, \cdots, 0) = K_{1,s} \bullet S(m;t) = K_{1,s} \bullet T(m,t)$, by Theorem 3.1.3 and Lemma 1.2.9, we get

$P[K_{1,s} \bullet T(m,t), x] = P[K_{1,s} \bullet S(m;t), x]$
$= P[K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_k; m_1, m_2, \cdots, m_k), x]$
$= P[K_{1,s} \bullet S(m+0+\cdots+0; t, 0, \cdots, 0), x]$
$= x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st]$.

Thus (1) is proved.

(2) Using (1), we find

$$P[K_{1,s} \bullet T(m,t), x] = x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st].$$

The zeroes of the characteristic polynomial are either 0, $\pm\sqrt{t}$ or solutions of $x^4-(m+t+s)x^2+st=0$. The conditions imply the integrality of the zeroes. From the integrality condition it follows that t must be a perfect square. If a is a zero of $x^4-(m+t+s)x^2+st$, then also $-a$ is a zero. This yields the possibility of a factorization $(x^2-a^2)(x^2-b^2)$. ■

As this is the first proof of its kind, we have written out the simple proof in full detail. From now on similar proofs will be presented in shorter form.

Theorem 3.1.6. *The tree* $K_{1,a_0} \bullet S(r;m_i)$= $K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ *of diameter 4 is integral if and only if the equation*

$$x^{a_0+1+\sum_{i=1}^{s} a_i(m_i-1)}[\prod_{i=1}^{s}(x^2-m_i)^{a_i}][1-\frac{a_0}{x^2}-\sum_{i=1}^{s}\frac{a_i}{x^2-m_i}]=0$$

has only integral roots.

Proof. By Theorem 3.1.3, we get

$$\begin{aligned}&P[K_{1,a_0} \bullet S(r;m_i), x] = x^{a_0-1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s}(x^2-m_i)^{a_i-1}\\ &\times[(x^2-a_0)\prod_{i=1}^{s}(x^2-m_i) - x^2\sum_{i=1}^{s} a_i \prod_{j=1, j\neq i}^{s}(x^2-m_j)]\\ &= x^{a_0+1+\sum_{i=1}^{s} a_i(m_i-1)}[\prod_{i=1}^{s}(x^2-m_i)^{a_i}][1-\frac{a_0}{x^2}-\sum_{i=1}^{s}\frac{a_i}{x^2-m_i}].\end{aligned}$$

Thus, the theorem is proved. ■

Theorem 3.1.7. *For any positive integer n, the following holds.*

(1) If the tree $S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ *of diameter 4 is integral, and* $m_1, m_2, \cdots, m_s$ *are perfect squares, then the tree* $S(a_1n^2+a_2n^2+\cdots+a_sn^2; m_1n^2, m_2n^2, \cdots, m_sn^2)$ *is integral too.*

(2) *If the tree* $K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ *of diameter 4 is integral , and* $m_1, m_2, \cdots, m_s$ *are perfect squares, then the tree* $K_{1,a_0n^2} \bullet S(a_1n^2+a_2n^2+ \cdots +a_sn^2; m_1n^2, m_2n^2, \cdots, m_sn^2)$ *is integral too.*

Proof. We firstly prove (1). By Corollary 3.1.1, we get

$P[S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s), x]$
$= x^{1+\sum_{i=1}^{s} a_i(m_i-1)} \prod_{i=1}^{s}(x^2-m_i)^{a_i-1}[\prod_{i=1}^{s}(x^2-m_i)$
$-\sum_{i=1}^{s} a_i \prod_{j=1,j\neq i}^{s}(x^2-m_j)]$,

and

$P[S(a_1n^2+a_2n^2+\cdots+a_sn^2; m_1n^2, m_2n^2, \cdots, m_sn^2), x]$
$= x^{1+\sum_{i=1}^{s} a_in^2(m_in^2-1)} \prod_{i=1}^{s}(x^2-m_in^2)^{a_in^2-1}[\prod_{i=1}^{s}(x^2-m_in^2)$
$-\sum_{i=1}^{s} a_in^2 \prod_{j=1,j\neq i}^{s}(x^2-m_jn^2)]$.

By putting $x = yn$, we find

$P[S(a_1n^2+a_2n^2+\cdots+a_sn^2; m_1n^2, m_2n^2, \cdots, m_sn^2), yn]$
$= (yn)^{1+\sum_{i=1}^{s} a_in^2(m_in^2-1)} \prod_{i=1}^{s}[(yn)^2-m_in^2]^{a_in^2-1}\{\prod_{i=1}^{s}[(yn)^2-m_in^2]$
$-\sum_{i=1}^{s} a_in^2 \prod_{j=1,j\neq i}^{s}[(yn)^2-m_jn^2]\}$.
$= (yn)^{1+\sum_{i=1}^{s} a_in^2(m_in^2-1)} \prod_{i=1}^{s}[(yn)^2-m_in^2]^{a_in^2-1}n^{2s}[\prod_{i=1}^{s}(y^2-m_i)$
$-\sum_{i=1}^{s} a_i \prod_{j=1,j\neq i}^{s}(y^2-m_j)]$.

Because the tree $S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 is integral, and $m_1, m_2, \cdots, m_s$ are perfect squares, by Corollary 3.1.2, we see that the equation

$\prod_{i=1}^{s}(x^2-m_i)^{a_i-1}[\prod_{i=1}^{s}(x^2-m_i) - \sum_{i=1}^{s} a_i \prod_{j=1,j\neq i}^{s}(x^2-m_j)] = 0$

has only integral roots.

Hence, the tree $S(a_1n^2+a_2n^2+\cdots+a_sn^2; m_1n^2, m_2n^2, \cdots, m_sn^2)$ is integral too.

From Theorems 3.1.3 and 3.1.6, it is easy to prove in a similar way that (2) is also true. ■

Theorem 3.1.8. *For the tree* $K_{1,s} \bullet S(m+q; t, r)$ *of diameter 4 the following results hold.*

(1) *If* $m = q = 1$*, then the tree* $K_{1,s} \bullet S(1+1; t, r)$ *is integral if and only if* $x^6-(s+r+t+2)x^4+[(r+t)(s+1)+rt]x^2-rts$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$*, where* a*,* b *and* c *are integers.*

(2) *If* $m = 1, q \geq 2$*, then the tree* $K_{1,s} \bullet S(1+q; t, r)$ *is integral if and only if* r *is a perfect square, and* $x^6-(q+s+r+t+1)x^4+[t(q+r+s)+r(s+1)]x^2-rts$ *can be factorized as* $(x^2-a^2)(x^2-b^2)(x^2-c^2)$*, where* a*,* b *and* c *are integers. For* $q = 1, m \geq 2$*, we have a similar result.*

(3) *If* $m, q \geq 2$*, then the tree* $K_{1,s} \bullet S(m+q; t, r)$ *is integral if and only if* t

and r are perfect squares, and $x^6-(m+q+s+r+t)x^4+[t(s+q+r)+r(m+s)]x^2-rts$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, where a, b and c are integers.

Proof. By Theorem 3.1.3, we have

$$P[K_{1,s}\bullet S(m+q;t,r),x]=x^{m(t-1)+q(r-1)+s-1}(x^2-t)^{m-1}(x^2-r)^{q-1}$$
$$\times\{x^6-(m+q+t+r+s)x^4+[t(q+r+s)+r(m+s)]x^2-rts\}.$$

The theorem thus follows by Theorem 3.1.6. ■

Corollary 3.1.9. *For the tree $K_{1,s}\bullet S(m+q;t,r)$ of diameter 4, let $t=r$. Then $K_{1,s}\bullet S(m+q;t,t)=K_{1,s}\bullet T(m+q,t)$ is integral if and only if t is a perfect square, and $x^4-(q+m+t+s)x^2+st$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are integers.*

Proof. From Theorems 3.1.3 and 3.1.8, it is easy to check the correctness by using a method similar to that used in (2) of Corollary 3.1.5. ■

Remark 3.1.10. *In [69], we obtained many integral trees $K_{1,s}\bullet T(m,t)$. The following Theorem 3.1.12 and Corollaries 3.1.11, 3.1.13 concern graphs different from the integral trees of [69].*

Corollary 3.1.11. *If the tree $K_{1,s}\bullet T(m,t)$ of diameter 4 is integral, and m (≥ 2) is a positive integer, then for any positive integer n the trees $K_{1,sn^2}\bullet T(mn^2,tn^2)$ and $K_{1,tn^2}\bullet T(mn^2,sn^2)$ are integral trees with diameter 4.*

Proof. Because the tree $K_{1,s}\bullet T(m,t)$ is integral, and m (≥ 2) is a positive integer, by Theorem 3.1.3 or Corollary 3.1.5, we deduce that t is a perfect square, and $x^4-(m+t+s)x^2+st$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are integers. So, we have $st=a^2b^2$, $m+t+s=a^2+b^2$ and assume that $t=t_1^2$. Hence,

$$P[K_{1,s}\bullet T(m,t),x]=x^{m(t-1)+(s-1)}(x^2-t)^{m-1}[x^4-(m+t+s)x^2+st]$$
$$=x^{m(t_1^2-1)+(s-1)}(x^2-t_1^2)^{m-1}(x^2-a^2)(x^2-b^2).$$

By Corollary 3.1.5, we get

$$P[K_{1,sn^2}\bullet T(mn^2,tn^2),x]$$
$$=x^{mn^2(tn^2-1)+(sn^2-1)}(x^2-tn^2)^{mn^2-1}[x^4-(m+t+s)n^2x^2+stn^4]$$
$$=x^{mn^2(t_1^2n^2-1)+(sn^2-1)}(x^2-t_1^2n^2)^{mn^2-1}(x^2-a^2n^2)(x^2-b^2n^2),$$

and

$$P[K_{1,tn^2}\bullet T(mn^2,sn^2),x]$$
$$=x^{mn^2(sn^2-1)+(tn^2-1)}(x^2-sn^2)^{mn^2-1}[x^4-(m+t+s)n^2x^2+stn^4]$$
$$=x^{mn^2(sn^2-1)+(t_1^2n^2-1)}(x^2-\tfrac{a^2b^2}{t_1^2}\cdot n^2)^{mn^2-1}(x^2-a^2n^2)(x^2-b^2n^2),$$

which proves the statement. ■

Theorem 3.1.12. *The tree $K_{1,s} \bullet T(m,t)$ of diameter 4 is integral if and only if $t = k^2$, $s = \frac{a^2b^2}{k^2}(\geq 1)$, and $m = a^2 + b^2 - k^2 - \frac{a^2b^2}{k^2}(> 1)$, where a, b and k are positive integers.*

Proof. By Corollary 3.1.5, we know that the tree $K_{1,s} \bullet T(m,t)$ with diameter 4 is integral if and only if t is a perfect square, and $x^4 - (m+t+s)x^2 + st$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are integers. Thus, we have that t is a perfect square, $m + t + s = a^2 + b^2$, $st = a^2b^2$.

Putting $t = k^2$, we get $t = k^2$, $s = \frac{a^2b^2}{k^2}(\geq 1)$, and $m = a^2 + b^2 - k^2 - \frac{a^2b^2}{k^2}$ (> 1), where a, b and k are positive integers. ■

The following result in [82] is a corollary of our Theorem 3.1.8.

Corollary 3.1.13. *Let $t = c^2d^2$, $s = a^2b^2$, and $m = a^2c^2 + b^2d^2 - c^2d^2 - a^2b^2(> 1)$, where a, b, c and d are positive integers. Then for any positive integer n the trees $K_{1,sn^2} \bullet T(mn^2, tn^2)$ of diameter 4 and $K_{1,tn^2} \bullet T(mn^2, sn^2)$ of diameter 4 are integral.*

Proof. From Corollaries 3.1.5 and 3.1.11, it is easy to check the correctness by using a method similar to that in Theorem 3.1.7. ■

For the tree $K_{1,s} \bullet S(m + q; t, r)$ of diameter 4, let $t \neq r$. The following Corollaries 3.1.14, 3.1.16 and 3.1.17 are obtained by computer search. They are solutions differing from those in the literature.

Corollary 3.1.14. *If $s = 25, m = q = 1, t = 18$ and $r = 32$, then the tree $K_{1,s} \bullet S(m + q; t, r) = K_{1,s} \bullet S(q + m; r, t)$ of diameter 4 is integral.*

Proof. By Theorem 3.1.8, we find

$$x^6 - 77x^4 + 1876x^2 - 14400 = (x^2 - 4^2)(x^2 - 5^2)(x^2 - 6^2),$$

and the corollary is proved. ■

The following result in [86] is a corollary of our Theorem 3.1.8.

Corollary 3.1.15. *For any positive integer n, the following holds.*

(1) Let $k = 2n^2$, let $s = n^2k(k+2), m = 1, q = \frac{1}{2}(k-1)(k+1)(k+2)k^2, t = k^2 + 2k$ and $r = k^2(k+1)^2$. Then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.

(2) Let $k = 2n(3n+2)$, let $s = k(3n+1)^2, m = 1, q = \frac{3}{4}(k+1)(k+2)(3k+2), t = k^2 + 2k$ and $r = 4(k+1)^2$. Then the tree $K_{1,s} \bullet S(m+q; t, r)$ of

diameter 4 is integral.

(3) Let $k = 2(n+1)(3n+1)$, let $s = \frac{1}{4}k(k+2)(3n+2)^2, m = 1, q = \frac{3}{8}k(k+2)(3k+2), t = k^2+2k$ and $r = 4(k+1)^2$. Then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.

Proof. (1) By (2) of Theorem 3.1.8, we know that the tree $K_{1,s} \bullet S(1+q; t, r)$ is integral if and only if r is a perfect square, and $x^6 - (q+s+r+t+1)x^4 + [t(q+r+s)+r(s+1)]x^2 - rts$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, where a, b and c are integers. By the conditions in (1), we get $r = k^2(k+1)^2$ and

$$x^6 - (q+s+r+t+1)x^4 + [t(q+r+s)+r(s+1)]x^2 - rts$$
$$= (x^2-k^2)[x^2-(k+1)^2][x^2-n^2k^2(k+2)^2],$$

where $k = 2n^2$, and n is a positive integer. Thus, the tree $K_{1,s} \bullet S(m+q; t, r)$ is integral.

From Theorem 3.1.8 or Theorem 3.1.6, (2) and (3) are proved in a similar way. ■

Corollary 3.1.16. *Let $m = 1$, $t < r$, let s, m, q, t, r, a, b, c be positive integers in Table 3.1, with a, b and c as in Theorem 3.1.8. Then the tree $K_{1,s} \bullet S(m+q; t, r)$ of diameter 4 is integral.*

s	m	q	t	r	a	b	c
8	1	24	8	36	2	3	8
14	1	48	14	144	3	4	14
22	1	80	22	400	4	5	22
20	1	300	80	1296	4	9	40
48	1	84	27	225	5	6	18
32	1	120	32	900	5	6	32
44	1	168	44	1764	6	7	44
180	1	140	80	144	8	9	20
···	···	···	···	···	···	···	···

Table 3.1: Integral trees $K_{1,s} \bullet S(m+q; t, r)$ with diameter 4.

Proof. From Theorem 3.1.8, the result is easy to check by using a method similar to that in Corollary 3.1.15. ■

Corollary 3.1.17. *Let $m, q \geq 2$, $t < r$, let s, m, q, t, r, a, b and c be positive integers in Table 3.2, and let a, b and c be as in Theorem 3.1.8. Then for* *any positive integer n the tree $K_{1,sn^2} \bullet S(mn^2 + qn^2; tn^2, rn^2)$ of diameter 4 is integral.*

s	m	q	t	r	a	b	c
49	51	63	9	64	2	6	14
25	24	63	9	144	2	6	15
16	33	110	25	81	2	6	15
36	75	105	16	64	2	6	16
36	35	140	9	144	2	6	18
9	15	135	25	256	2	6	20
25	45	210	16	144	2	6	20
16	33	48	16	196	2	7	16
81	40	208	16	100	3	6	20
36	56	125	36	144	3	8	18
100	63	150	16	144	3	8	20
144	96	117	16	100	3	8	20
36	65	128	36	225	3	9	20
64	90	105	25	225	3	10	20
64	156	72	36	225	3	12	20
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 3.2: Integral trees $K_{1,sn^2} \bullet S(mn^2+qn^2; tn^2, rn^2)$ with diameter 4, where n is a positive integer.

Proof. From Theorems 3.1.7 and 3.1.8, the result easily follows by using a method similar to that in Corollary 3.1.15. ■

3.2 Integral trees with diameters 6 and 8

In this section, we mainly consider families of integral trees with diameter 6. Some families of integral trees with diameter 8 and some families of integral trees with diameter 4 are also constructed. Moreover some results which treat interrelations between integral trees of various diameters are obtained for the first time.

Let the tree $K_{1,s} \bullet T(r, m, t)$ of diameter 6 be obtained by identifying the center w of $K_{1,s}$ and the center u of $T(r, m, t)$. The integral tree $K_{1,s} \bullet T(r, m, t)$ of diameter 6 was firstly studied in [69, 75]. Next we shall obtain some results on integral trees $K_{1,s} \bullet T(r, m, t)$ of diameter 6. These results differ from those in [69, 75].

Theorem 3.2.1. *If the tree $K_{1,s} \bullet T(r,m,t)$ of diameter 6 is integral, and m (≥ 2) and r (≥ 2) are positive integers, then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

Proof. From Lemmas 1.2.10 and 1.2.12, the theorem is proved by using a similar method as in Corollary 3.1.11. ∎

Theorem 3.2.2. *Let $s = t = l^2q^2$, $m = (k^2 - l^2)q^2(> 1)$ and $r = p^2 - k^2q^2(> 1)$, where l, k, p and q are positive integers. Then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

Proof. By Corollary 1.2.13 or Lemma 1.2.12, we see that the tree $K_{1,t} \bullet T(r,m,t)$ of diameter 6 is integral if and only if t, $m+t$ and $m+t+r$ are perfect squares.

Because $s = t = l^2q^2$, $m = (k^2 - l^2)q^2(> 1)$ and $r = p^2 - k^2q^2(> 1)$, where l, k, p and q are positive integers, we get $t = s = l^2q^2$, $m + t = k^2q^2$ and $m+t+r = p^2$. Consequently the tree $K_{1,t} \bullet T(r,m,t)$ is integral. By Theorem 3.2.1, also the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ is integral. ∎

The following result in [82] is a corollary of Theorem 3.2.2.

Corollary 3.2.3. *Let $s = t = q^2$, $m = 3q^2$ and $r = p^2 - 4q^2$, where p and q are positive integers, and $p > 2q$. Then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

If $q = 1, p = 3$, then $s = t = 1, m = 3, r = 5$ and we obtain the smallest integral tree $K_{1,1} \bullet T(5,3,1)$ with diameter 6 in this class. Its characteristic polynomial is $P(K_{1,1} \bullet T(5,3,1), x) = x^5(x^2-1)^{11}(x^2-4)^4(x^2-9)$ with order 37.

Theorem 3.2.4. *The tree $K_{1,s} \bullet T(r,m,t)$ of diameter 6 (with $r, m > 1$) is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $s = k^2 + \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}(\geq 1)$ and $r = a^2 + b^2 - (n+k)^2 - k^2 - \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}(> 1)$, where a, b, k and n are positive integers.*

Proof. By Lemma 1.2.12, the tree $K_{1,s} \bullet T(r,m,t)$ (where $r, m > 1$) is integral if and only if t and $m+t$ are perfect squares, and $x^4 - (m+t+r+s)x^2 + rt + s(m+t)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$. Hence $a^2 + b^2 = m+t+r+s$ and $a^2b^2 = rt + s(m+t)$.

Assume that $t = k^2$, $m = n^2 + 2nk$. Then it follows $s = k^2 + \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}(\geq$

1) and $r = a^2 + b^2 - (n+k)^2 - k^2 - \frac{(a^2-k^2)(b^2-k^2)}{n^2+2nk}(> 1)$, where a, b, k and n are positive integers.

Thus, by Lemma 1.2.12, the theorem is true. ■

Corollary 3.2.5. *Let $t = k^2$, $m = a^2b^2 - k^2(> 1)$, $r = a^2b^2$, and $s = a^2c^2 + b^2d^2 - 2a^2b^2 = c^2d^2 - k^2(\geq 1)$, where k, a, b, c and d are integers. Then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

Proof. From Lemma 1.2.12 and Theorem 3.2.1, the result follows by using a method similar to that in Theorem 3.2.2. ■

Remark 3.2.6. *The Diophantine equation $a^2c^2+b^2d^2-2a^2b^2 = c^2d^2-k^2(\geq 1)$ can be transformed into*

$$(ac + bd)^2 - a^2b^2 = (ab + cd)^2 - k^2. \tag{3.2.1}$$

There exist solutions $a = b = 2$, $c = k = 1$ and $d = 3$ for the Diophantine equation (3.2.1)*. We conjecture that there are infinitely many other solutions.*

By computer search, we can get the following corollary.

Corollary 3.2.7. *With $s \neq t$, let $s, r(> 1), m(> 1)$, t, a, b be positive integers in Table 3.3, and let a and b be as in Lemma 1.2.12. Then for any positive integer n the tree $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ of diameter 6 is integral.*

Proof. From Lemma 1.2.12 and Theorem 3.2.1, the result is proved similarly to Theorem 3.2.4. ■

Remark 3.2.8. *Consider the trees $K_{1,s} \bullet T(r, m, t)$ of diameter 6.*

(i) If $s = t$, we can construct infinitely many classes of such integral trees from Lemma 1.2.12, Corollaries 1.2.14 and 1.2.13 of Chapter 1 of this thesis, which are Theorems 4 and 5 and Corollary 5 of [69].

(ii) For $s \neq t$, we got some classes of such integral trees $K_{1,s} \bullet T(r, m, t)$ of diameter 6 and $T(s, r, m, t)$ of diameter 8 in [75, 38]. Here $T(s, r, m, t)$ is obtained by joining the centers of s copies of $T(r, m, t)$ to a new vertex y.

Remark 3.2.9. *By using a computer search, we have found 2694 positive in-*

a	b	m	r	s	t	a	b	m	r	s	t
2	5	8	10	10	1	2	6	15	16	8	1
2	7	8	25	19	1	2	7	24	21	7	1
2	9	8	45	31	1	2	9	15	52	17	1
2	9	24	49	11	1	2	9	48	30	6	1
2	11	8	70	46	1	2	11	15	84	25	1
2	11	24	84	16	1	2	11	84	24	1	16
3	7	24	16	17	1	3	8	12	28	29	4
3	8	24	26	22	1	3	10	12	49	44	4
3	10	24	50	34	1	3	10	32	54	19	4
3	10	60	33	12	4	3	11	15	49	65	1
3	11	24	64	41	1	3	11	45	64	17	4
3	11	48	60	21	1	3	11	80	36	13	1
3	13	24	96	57	1	3	13	96	56	1	25
3	13	48	100	29	1	4	7	24	9	31	1
4	8	45	11	20	4	4	8	35	16	28	1
4	9	24	44	4	25	4	9	21	24	48	4
4	9	24	21	51	1	4	9	48	22	26	1
4	10	32	40	40	4	4	11	16	54	58	9
4	11	24	36	76	1	4	12	21	51	84	4
4	12	27	80	44	9	4	12	60	64	32	4
4	13	16	81	79	9	4	13	80	40	1	64
4	13	24	54	106	1	4	13	40	99	37	9
4	13	35	76	73	1	4	13	45	88	48	4
4	13	63	80	41	1	4	13	60	84	37	4
4	13	112	45	19	9	4	13	120	42	22	1
4	14	32	100	76	4	4	14	96	84	28	4
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 3.3: Integral trees $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ with diameter 6, where n is a positive integer.

tegral solutions $a, b, n, t_1, t, m(> 1), r(> 1)$ *and* s *for the Diophantine equations*

$$\begin{cases} t = t_1^2 \\ m = n^2 + 2nt_1 \\ a^2b^2 = rt + s(m+t) \\ a^2 + b^2 = m + t + r + s, \end{cases}$$

where $s \neq t, 1 \leq a \leq 20$ *and* $a \leq b \leq a + 20$. *In Table 3.3, for* $a = 2, 3, 4$, $a \leq b \leq a + 10$, $s \neq t$, *we give these parameters* a, b, s, r, m *and* t. *We shall construct infinitely many classes of such integral trees* $K_{1,s} \bullet T(r, m, t)$ *from Corollaries 3.2.3 and 3.2.5 and Theorems 3.2.2 and 3.2.4. They are different from those in [15, 16, 37, 38, 47, 48, 50, 51, 69, 75, 76, 79, 82].*

Next we discuss the interrelations between integral trees of various diameters. Let $r * G$ be the graph formed by joining the roots of r copies of G to a new vertex w. Let $K_{1,r} \bullet G$ be the graph obtained by identifying the center z of $K_{1,r}$ and the root u of G.

The following Lemma 3.2.10 and Corollary 3.2.11 can be found in [76].

Lemma 3.2.10.

(1) $P(r * G, x) = P^{r-1}(G, x)[xP(G, x) - rP(G - u, x)]$.

(2) $P(K_{1,r} \bullet G, x) = x^{r-1}[xP(G, x) - rP(G - u, x)]$.

Corollary 3.2.11.

(1) If G *and* $K_{1,r} \bullet G$ *are integral graphs, then* $r * G$ *is integral.*

(2) If G *and* $r * G$ *are integral graphs, then* $K_{1,r} \bullet G$ *is integral.*

Theorem 3.2.12. *If* $T(s, r, m, t)$ *of diameter 8 is integral, and* $s, r, m > 1$, *then for any positive integer* n *the trees* $T(sn^2, rn^2, mn^2, tn^2)$ *of diameter 8,* $K_{1,sn^2} \bullet T(rn^2, mn^2, tn^2)$ *of diameter 6,* $T(rn^2, mn^2, tn^2)$ *of diameter 6,* $T(mn^2, tn^2)$ *of diameter 4 and* K_{1,tn^2} *of diameter 2 are integral, too.*

Proof. From Lemmas 1.2.9, 1.2.10 and 1.2.15, the statement is proved by using a method similar to that in Theorem 3.1.7. ∎

The following examples can be found in [37, 38, 69, 75, 79].

Example 3.2.13. *It is shown in [75] that the tree* $T(324, 3136, 765, 324)$ *of diameter 8 is integral. Thus for any positive integer* n *the trees* $T(324n^2, 3136n^2, 765n^2, 324n^2)$ *of diameter 8,* $K_{1,324n^2} \bullet T(3136n^2, 765n^2, 324n^2)$ *of diameter 6,* $T(3136n^2, 765n^2, 324n^2)$ *of diameter 6,* $T\ (765n^2, 324n^2)$ *of diameter 4 and* $K_{1,324n^2}$ *of diameter 2 are integral, too.*

Example 3.2.14. *It is shown in [38] that the tree* $T(616, 225, 672, 4)$ *of diameter 8 is integral. Thus for any positive integer* n *the trees* $T(616n^2, 225n^2, 672n^2, 4n^2)$ *of diameter 8,* $K_{1,616n^2} \bullet T(225n^2, 672n^2, 4n^2)$ *of diameter 6,* $T(225n^2, 672n^2, 4n^2)$ *of diameter 6,* $T(672n^2, 4n^2)$ *of diameter 4 and* $K_{1,4n^2}$ *of diameter 2 are integral, too.*

Theorem 3.2.15. *If the tree* $T(r, m, t)$ *of diameter 6 is integral, and* $r, m > 1$*, then for any positive integer* n *the trees* $T(rn^2, mn^2, tn^2)$ *of diameter 6,* $K_{1,rn^2} \bullet T(mn^2, tn^2)$ *of diameter 4,* $K_{1,tn^2} \bullet T(mn^2, rn^2)$ *of diameter 4,* $T(mn^2, tn^2)$ *of diameter 4,* K_{1,rn^2} *of diameter 2 and* K_{1,tn^2} *of diameter 2 are integral, too.*

Proof. From Lemma 1.2.9 and Corollary 3.1.5, the statement follows by using a method similar to that used in Theorem 3.1.7. ■

Example 3.2.16. *It is shown in [37, 75] that the tree* $T(16, 45, 4)$ *of diameter 6 is integral. Thus for any positive integer* n *the trees* $T(16n^2, 45n^2, 4n^2)$ *of diameter 6,* $K_{1,16n^2} \bullet T(45n^2, 4n^2)$ *of diameter 4,* $K_{1,4n^2} \bullet T(45n^2, 16n^2)$ *of diameter 4,* $T(45n^2, 4n^2)$ *of diameter 4,* $K_{1,16n^2}$ *of diameter 2 and* $K_{1,4n^2}$ *of diameter 2 are integral too.*

Theorem 3.2.17. *For any positive integer* n*, the following holds.*

(1) If $G = T(m, t)$ *of diameter 4 and* $K_{1,r} \bullet G = K_{1,r} \bullet T(m, t)$ *of diameter 4 are integral, then* $r * G = T(r, m, t)$ *of diameter 6 and* $T(rn^2, mn^2, tn^2)$ *of diameter 6 are integral too.*

(2) If $G = T(r, m, t)$ *of diameter 6 and* $K_{1,s} \bullet G = K_{1,s} \bullet T(r, m, t)$ *of diameter 6 are integral, then* $s * G = T(s, r, m, t)$ *of diameter 8 and* $T(sn^2, rn^2, mn^2, tn^2)$ *of diameter 8 are integral too.*

Proof. From Corollaries 3.2.11 and 3.1.5 and Lemmas 1.2.9 and 1.2.15, the statement is proved similarly to Theorem 3.1.7. ■

Example 3.2.18. *For any positive integer* n*, the following holds.*

(1) It is shown in [79, 69] that $G = T(280, 9)$ *of diameter 4 and* $K_{1,r} \bullet$

$G = K_{1,36} \bullet T(280,9)$ of diameter 4 are integral. So the trees $r * G = T(36,280,9)$ of diameter 6 and $T(36n^2, 280n^2, 9n^2)$ of diameter 6 are integral too.

(2) It is shown in [37, 38, 75] that $G = T(144,105,16)$ of diameter 6 and $K_{1,s} \bullet G = K_{1,676} \bullet T(144,105,16)$ of diameter 6 are integral. So the trees $r * G = T(676,144,105,16)$ of diameter 8 and $T(676n^2, 144n^2, 105n^2, 16n^2)$ of diameter 8 are integral too.

3.3 Further discussion

In this chapter, we have mainly investigated integral trees $K_{1,s} \bullet S(m+q;t,r)$ of diameter 4 and $K_{1,s} \bullet T(r,m,t)$ of diameter 6. We tried unsuccessfully to get some general results from Corollaries 3.1.14, 3.1.16, 3.1.17 and 3.2.7 by computer search. Thus, we raise the following question.

Question 3.3.1. *Can general results on integral trees $K_{1,s} \bullet S(m+q;t,r)$ of diameter 4 and $K_{1,s} \bullet T(r,m,t)$ of diameter 6 be derived from Corollaries 3.1.14, 3.1.16, 3.1.17 and 3.2.7?*

Results on integral trees of diameter 4 are given in [15, 19, 22, 36, 38, 46, 47, 48, 51, 59, 69, 75, 76, 79, 80, 81, 82, 85, 86, 87]. P. Yuan ([85]) gave a sufficient condition for graphs to be an integral tree $S(r;m_i)$ of diameter 4 and constructed many new classes of such integral trees based on [47]. The authors of [46] and [86] further give a useful sufficient and necessary condition for graphs to be such integral trees of diameter 4. For the integral trees $S(r;m_i) = S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$, when $s=2$, we can find such integral trees in [15, 19, 38, 47, 48, 51, 59, 75, 76, 79, 80, 81, 82, 85, 87]. In particular, H.Z. Ren obtained in [59] all parameter values such that $S(a_1+a_2; m_1, m_2)$ is an integral tree. When $s = 3,4,5$, we found such integral trees in [46, 85, 86]. Hence, we ask

Question 3.3.2. *Are there integral trees $S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 for arbitrarily large s?*

For integral trees $K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_s; m_1, m_2, \cdots, m_s)$ of diameter 4, some results can be found in [47, 48, 69, 75, 80, 81, 82, 85, 86] and in the present chapter. We have not found such integral trees for $s \geq 3$. Hence, we ask

Question 3.3.3. *Are there integral trees $K_{1,a_0} \bullet S(a_1+a_2+\cdots+a_s; m_1, m_2,$*

$\cdots, m_s)$ *of diameter 4 for arbitrarily large* s*?*

For the tree $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 and $K_{1,a_1} \bullet S(a_2 + a_3 + \cdots + a_s; m_2, m_3, \cdots, m_s)$ of diameter 4, when r is odd, there are such integral trees in [85]. When $r(> 2)$ is even and $s = 1, 2$, such integral trees can be found in [80, 81, 85, 87]. Thus, we ask

Question 3.3.4. *Are there any integral trees* $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ *of diameter 4 or* $K_{1,a_1} \bullet S(a_2 + a_3 + \cdots + a_s; m_2, m_3, \cdots, m_s)$ *of diameter 4 while* $r(> 2)$ *is even and* $s \geq 3$*?*

For the tree $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4 or $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ of diameter 4, such integral trees are constructed in [80, 81, 85, 86] and in the present chapter under the assumption that the number of nonsquares among $m_1, m_2, \cdots, m_s$ is at most 2. Thus, we ask

Question 3.3.5. *Let* $S(r; m_i) = S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ *or* $K_{1,a_0} \bullet S(a_1 + a_2 + \cdots + a_s; m_1, m_2, \cdots, m_s)$ *be an integral tree. Is the number of nonsquares among* $m_1, m_2, \cdots, m_s$ *limited?*

In [69] and in this chapter, we successfully constructed integral trees by identifying the centers of two trees. Let $G \bullet H$ denote the tree obtained by identifying the center z of the tree G with the center u of the tree H. Then we can pose

Question 3.3.6. *Are there any integral trees* $T(p, q) \bullet T(r, m, t)$*,* $T(p, q) \bullet T(s, r, m, t)$*,* $T(p, m, t) \bullet S(r; m_i)$*,* $T(r, m, t) \bullet T(s, p, q, l)$ *and so on ?*

Integral trees of diameters 1, 2, 3, 4, 5, 6 and 8 have been constructed in [15, 16, 19, 22, 36, 37, 38, 46, 47, 48, 49, 50, 51, 59, 69, 75, 76, 79, 80, 81, 82, 85, 86, 87] and in this chapter. Hence, we suggest the following question.

Question 3.3.7. *Are there integral trees of diameter 7?*

Chapter 4

Integral trees with diameters 5, 6 and 8

In this chapter, we determine the characteristic polynomials of the trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$, $T(p,q) \bullet T(r,m,t)$, $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(q,r,m,t)$ with diameters 5, 6, 6 and 8, respectively. We also obtain firstly sufficient and necessary conditions for these trees to be integral by using number theory and computer search. All these classes are infinite and different from those in the literature. We also prove that the problem of finding such integral trees is equivalent to the problem of solving some Diophantine equations. These results generalize results of [2, 15, 16, 36, 37, 38, 39, 47, 48, 49, 50, 51, 69, 75, 76, 77, 78, 79]. In particular integral trees of the type $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$, $T(p,q) \bullet T(r,m,t)$, $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(q,r,m,t)$ are obtained for the first time. We further present some new results on interrelations between integral trees of various diameters.

4.1 Integral trees of diameter 5

In this section, we shall construct infinitely many new integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5. They are different from those of [2, 15, 16, 38, 47, 48, 49, 50, 51].

Integral trees of diameter 5 were mentioned for the first time in [47], but the authors were not able to find any example. Infinitely many integral trees $T^t[m,r]$ of diameter 5 were first constructed by R.Y. Liu in [50]. Later Z.F. Cao obtained general results on these classes by using the solutions of some Pell equations in [16], and then Y. Li obtained more general results on these classes by using the solutions of more general quadratic Diophantine equations in [49]. The authors of [38] proved that there are no balanced integral trees $T(1; n_{2k}, n_{2k-1}, \ldots, n_1)$ of diameter $4k+1$ for $k \geq 1$. Here the structure of integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is found for the first time.

Let the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 be obtained by joining the center u of $K_{1,s} \bullet T(m,t)$ and the center v of $T(q,r)$ with a new edge.

Theorem 4.1.1. *The tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral if and only if the equation*

$$(x^2-t)^{m-1}(x^2-r)^{q-1}\{x^6-(m+t+s+q+r+1)x^4+[st+(q+r)(m+t+s)+r+t]x^2-t(sq+sr+r)\}=0$$

has only integral roots.

Proof. Note that the vertex u is the center of the tree $K_{1,s} \bullet T(m,t)$, and the vertex v is the center of the tree $T(q,r)$. Suppose that $G_1 = K_{1,s} \bullet T(m,t)$, $G_2 = T(q,r)$. Then, by Lemma 1.2.2 we know that

$$P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x)$$
$$= P(K_{1,s} \bullet T(m,t), x)P(T(q,r), x) - x^s P^m(K_{1,t}, x)P^q(K_{1,r}, x).$$

By Lemmas 1.2.9 and 1.2.10, we have

$$P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x)$$
$$= x^{m(t-1)+q(r-1)+s}(x^2-t)^{m-1}(x^2-r)^{q-1}\{[x^2-(q+r)][x^4-(m+t+s)x^2+st]-(x^2-t)(x^2-r)\}$$
$$= x^{m(t-1)+q(r-1)+s}(x^2-t)^{m-1}(x^2-r)^{q-1}\{x^6-(m+t+s+q+r+1)x^4+[st+(q+r)(m+t+s)+r+t]x^2-t(sq+sr+r)\}. \blacksquare$$

The following corollary can be found in [15].

Corollary 4.1.2. *([15]) If $s=0$, then the tree $[K_{1,0} \bullet T(m,t)] \ominus T(q,r) = T(m,\, t) \ominus T(q,r)$ of diameter 5 cannot be integral.*

Now we assume $s > 0$ throughout the whole section.

Corollary 4.1.3. *If $q+r=t$, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral if and only if $x^4-(m+t+s+1)x^2+st+r$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are positive integers, and one of the following two conditions holds: (i) $q=1$, t is a perfect square, or (ii) $q>1$, t and r are perfect squares.*

Proof. When $q+r=t$, by Theorem 4.1.1, we find

$$P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x) = x^{m(t-1)+q(r-1)+s}(x^2-t)^m(x^2-r)^{q-1}[x^4-(m+t+s+1)x^2+st+r].$$

Hence, when $q+r=t$, the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral if and only if one of the following holds: (i) $q = 1$, t is a perfect square, and $x^4-(m+t+s+1)x^2+st+r$ can be factorized as $(x^2-a^2)(x^2-b^2)$, (ii) $q > 1$, t and r are perfect squares, $x^4-(m+t+s+1)x^2+st+r$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are positive integers. ∎

Corollary 4.1.4. *For the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 satisfying $q+r=t$, the following holds:*

(1) For $q = 1$, let $d > 1$ be such that there exist positive integral solutions for Equation (2.1.9). *Then, all positive integral solutions x_{2k-1}, y_{2k-1} of Equation* (2.1.9) *are defined by* (2.1.11). *If $s = d-1$, $m = a^2+b^2-y_{2k-1}^2-d$, $t = y_{2k-1}^2$, $q = 1$, $r = y_{2k-1}^2-1$ and $ab = x_{2k-1}$, where k, a and b are positive integers, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral, and there are infinitely many such integral trees.*

(2) Suppose $q > 1$, $s = de^2$, $t = f^2y_k^2, q = f^2(y_k^2-e^2) > 0$, $r = e^2f^2$, $m = a^2+b^2-f^2y_k^2-de^2-1 > 0$, and $ab = efx_k$, where a, b, $d(> 1)$, e, f and k are positive integers, and d is not a perfect square, and x_k, y_k are positive integral solutions of Equation (2.1.4) *(i.e. given by* (2.1.8)*). Then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral, and there are infinitely many such integral trees.*

(3) If $q > 1$, t and r are perfect squares, $q = t-r > 1$, $s = \frac{a^2b^2-r}{t} > 0$, $m = a^2+b^2-\frac{a^2b^2-r}{t}-t-1 > 0$, where s, m, t, q, r, a and b are positive integers, then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral.

Proof. Using $q+r=t$, by Theorem 4.1.1 we get

$$\begin{aligned}&P(\{[K_{1,s} \bullet T(m,t)] \ominus T(q,r)\}, x)\\ &= x^{m(t-1)+q(r-1)+s}(x^2-t)^m(x^2-r)^{q-1}[x^4-(m+t+s+1)x^2+st+r].\end{aligned}$$

By Corollary 4.1.3, we know that the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ (where $q+r=t$) is integral if and only if there exist positive integral solutions for the Diophantine equations

$$\begin{cases} a^2b^2 = st+r, \\ a^2+b^2 = m+t+s+1, \end{cases} \tag{4.1.1}$$

satisfying one of the two conditions of Corollary 4.1.3.

(1) When $q = 1$, by Equation (4.1.1), Condition (i) of Corollary 4.1.3 and $q+r=t$, we obtain

$$a^2b^2-(s+1)t = -1. \tag{4.1.2}$$

Assume that $ab = x$, $s + 1 = d$, $t = t_1^2 = y^2$. Then Equation (4.1.2) can be transformed into Equation (2.1.9). Hence, by Lemmas 2.1.11 and 2.1.12, Equation (4.1.1) and Equation (4.1.2), we see that $s = d - 1$, $m = a^2 + b^2 - y_{2k-1}^2 - d$, $t = y_{2k-1}^2$, $q = 1$, $r = y_{2k-1}^2 - 1$ and $ab = x_{2k-1}$, where k, a and b are positive integers. Thus, by Corollary 4.1.3 and Lemma 2.1.12, the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ is integral, and there are infinitely many such integral trees.

(2) When $q > 1$, by Equation (4.1.1), Condition (ii) of Corollary 4.1.3 and $q + r = t = t_1^2$, we find

$$a^2b^2 - st = r \Rightarrow a^2b^2 - st_1^2 = r_1^2 \Rightarrow \frac{a^2b^2}{r_1^2} - \frac{st_1^2}{r_1^2} = 1 \tag{4.1.3}$$

Assume that $r = r_1^2 = e^2f^2$, $s = de^2$, $t = t_1^2 = f^2y^2$ and $ab = efx$, where a, b, $d(> 1)$, e, f and k are positive integers, and d is not a perfect square. Then Equation (4.1.3) simplifies to Equation (2.1.4). Thus, by Equation (4.1.1), Condition (ii) of Corollary 4.1.3 and $q + r = t$, and Lemmas 2.1.10 and 2.1.12, the correctness of *(2)* is easily seen.

(3) From Theorem 4.1.1 or Corollary 4.1.3, the statement is easily proved by using a method similar to that used in *(1)* or *(2)*. ■

Note that we obtain the smallest integral tree $[K_{1,2} \bullet T(3,4)] \ominus T(3,1)$ of diameter 5 in this class. Its characteristic polynomial is $P([K_{1,2} \bullet T(3,4)] \ominus T(3,1), x) = x^{11}(x^2-1)^3(x^2-4)^3(x^2-9)$ with order 25.

For Corollary 4.1.4 *(3)*, we simply list some examples of integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5.

Example 4.1.5. *Assume $q + r = t$, $q > 1$ and let s, m, t, q, r, a, b be the positive integers in Corollary 4.1.4* (3). *Then for any choice of positive integers a_1, b_1, k, k_1, k_2, l the following cases (1)-(10) yield integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5.*

(1) $s = k^2(l^2+2)$, $m = (l^2-k^2)(l^2-k^2+2) > 0$, $t = k^2l^2$, $q = k^2(l^2-k^2) > 1$, $r = k^4$, $a = k^2$ *and* $b = l^2 + 1$,

(2) $s = k^2(l^2 + 2)$, $m = k^2l^4 - 1 > 0$, $t = k^2l^2$, $q = k^2(l^2 - k^2) > 1$, $r = k^4$, $a = k$ *and* $b = k(l^2 + 1)$,

(3) $s = k^2(l^2 - 2) > 0$, $m = (l^2 - k^2)(l^2 - k^2 - 2) > 0$, $t = k^2l^2$, $q = k^2(l^2 - k^2) > 1$, $r = k^4$, $a = k^2$ *and* $b = l^2 - 1 > 0$,

(4) $s = k^2(l^2-2) > 0$, $m = k^2(l^2-2)^2-1 > 0$, $t = k^2l^2$, $q = k^2(l^2-k^2) > 1$, $r = k^4$, $a = k$ *and* $b = k(l^2 - 1) > 0$,

(5) $s = l^2 + 2$, $m = l^4 - k^2l^2 + l^2 + k^2 - 2 > 0$, $t = k^2l^2$, $q = k^2(l^2 - 1) > 1$, $r = k^2$, $a = k$ *and* $b = l^2 + 1$,

(6) $s = l^2 - 2 > 0$, $m = l^4 - k^2l^2 - 3l^2 + k^2 + 2 > 0$, $t = k^2l^2$, $q = k^2(l^2 - 1) > 1$, $r = k^2$, $a = k$ *and* $b = l^2 - 1 > 0$,

(7) $s = a_1^2b_1^2(l^2+2)$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2+1)^2 - a_1^2b_1^2(l^2+2) - k_1^2k_2^2a_1^2b_1^2l^2 - 1 > 0$, $t = k_1^2k_2^2a_1^2b_1^2l^2$, $q = k_1^2k_2^2a_1^2b_1^2(l^2 - a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2 + 1)$,

(8) $s = a_1^2b_1^2(l^2-2) > 0$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2-1)^2 - a_1^2b_1^2(l^2-2) - k_1^2k_2^2a_1^2b_1^2l^2 - 1 > 0$, $t = k_1^2k_2^2a_1^2b_1^2l^2$, $q = k_1^2k_2^2a_1^2b_1^2(l^2 - a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2 - 1) > 0$,

(9) $s = k_1^2k_2^2a_1^2b_1^2(l^2 + 2)$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2 + 1)^2 - k_1^2k_2^2a_1^2b_1^2(l^2 + 2) - a_1^2b_1^2l^2 - 1 > 0$, $t = a_1^2b_1^2l^2$, $q = a_1^2b_1^2(l^2 - k_1^2k_2^2a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2 + 1)$,

(10) $s = k_1^2k_2^2a_1^2b_1^2(l^2 - 2) > 0$, $m = k_1^2a_1^4 + k_2^2b_1^4(l^2 - 1)^2 - k_1^2k_2^2a_1^2b_1^2(l^2 - 2) - a_1^2b_1^2l^2 - 1 > 0$, $t = a_1^2b_1^2l^2$, $q = a_1^2b_1^2(l^2 - k_1^2k_2^2a_1^2b_1^2) > 1$, $r = k_1^2k_2^2a_1^4b_1^4$, $a = k_1a_1^2$ *and* $b = k_2b_1^2(l^2 - 1) > 0$.

Proof. We only prove (1). (2)-(10) are similarly proved.

Consider $s = k^2(l^2 + 2)$, $m = (l^2 - k^2)(l^2 - k^2 + 2) > 0$, $t = k^2l^2$, $q = k^2(l^2 - k^2) > 1$, $r = k^4$, where k and l are positive integers. Then, t and r are perfect squares, $x^4 - (m + t + s + 1)x^2 + st + r = (x^2 - a^2)(x^2 - b^2)$, where $a = k^2$ and $b = l^2 + 1$.

Thus, by Corollary 4.1.3, the tree $[K_{1,s} \bullet T(m, t)] \ominus T(q, r)$ is integral. ■

Corollary 4.1.6. *For* $q + r \neq t$, *the tree* $[K_{1,s} \bullet T(m, t)] \ominus T(q, r)$ *of diameter 5 is integral if and only if* $x^6 - (m + t + s + q + r + 1)x^4 + [st + (q + r)(m + t + s) + r + t]x^2 - t(sq + sr + r)$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)$, *where* a, b *and* c *are positive integers, and one of the following four conditions holds:*

(i) $m = 1$, $q = 1$,

(ii) $m > 1$, $q = 1$, t *is a perfect square,*

(iii) $m = 1$, $q > 1$, r *is a perfect square,*

(iv) $m > 1$, $q > 1$, t *and* r *are perfect squares.*

Proof. Using Theorem 4.1.1, this corollary is proved similarly to Corollary 4.1.3. ■

Corollary 4.1.7. *For $q+r \neq t$, $m=1$, $q>1$, let a, b, c, s, m, t, q and r be the positive integers in Corollary 4.1.6, given in Table 4.1. Then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral. (Note that a, b, c, s, m, t, q and r are obtained by computer search, and $1 \leq a \leq 16$, $a \leq b \leq a+8$, $b \leq c \leq b+10$, $q+r \neq t$, $m=1$ and $q>1$).*

a	b	c	s	m	t	q	r
2	9	10	4	1	80	90	9
8	9	10	71	1	75	72	25
8	9	10	71	1	96	72	4
15	16	18	239	1	243	240	81
15	16	18	239	1	320	240	4

Table 4.1: Integral tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5, where $q+r \neq t$, $m=1$ and $q>1$.

Proof. The result follows from Theorem 4.1.1 or Corollary 4.1.6 using a method similar to that used in Example 4.1.5. ■

Corollary 4.1.8. *Suppose that $q+r \neq t$, $m>1$, $q>1$, t and r are perfect squares. Let a, b, c, s, m, t, q and r be the positive integers of Corollary 4.1.6, as given in Table 4.2. Then the tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 is integral. (Note that a, b, c, s, m, t, q and r are obtained by computer search, and $1 \leq a \leq 7$, $a \leq b \leq 9$, $b \leq c \leq 20$, $q+r \neq t$, $m>1$ and $q>1$).*

Proof. From Theorem 4.1.1 or Corollary 4.1.6, the correctness is easy to check by using a method similar to that used in Example 4.1.5. ■

Remark 4.1.9. *In view of Theorem 4.1.1, it is important to find positive integral solutions of the following Diophantine equations* (4.1.4) *satisfying one of the four conditions of Corollary 4.1.6.*

$$\begin{cases} a^2+b^2+c^2 = m+t+s+q+r+1 \\ a^2b^2+b^2c^2+c^2a^2 = st+(q+r)(m+t+s)+t+r \\ a^2b^2c^2 = t(sq+sr+r) \end{cases} \tag{4.1.4}$$

By Theorem 4.1.1, Corollaries 4.1.3, 4.1.4, 4.1.6, 4.1.7 and 4.1.8, we know that there exist infinitely many such integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of

a	b	c	s	m	t	q	r	a	b	c	s	m	t	q	r
1	5	6	3	16	9	32	1	1	5	6	7	18	4	31	1
1	5	6	8	21	4	27	1	1	6	7	10	27	4	43	1
1	6	7	11	30	4	39	1	1	6	7	5	32	9	38	1
1	9	10	22	63	4	91	1	1	9	10	23	66	4	87	1
2	5	6	12	10	9	29	4	2	6	7	20	20	9	35	4
2	7	8	13	27	16	56	4	2	7	8	15	33	16	48	4
2	8	9	17	39	16	72	4	2	8	9	19	45	4	64	4
3	6	8	35	20	16	36	1	3	8	10	55	36	16	40	25
3	8	11	67	45	16	64	1	3	9	10	21	36	36	87	9
3	9	10	24	45	36	75	9	4	5	12	25	36	100	22	1
5	6	12	38	32	100	30	4	5	6	17	35	56	225	32	1
6	7	16	50	48	196	42	4	6	7	20	102	190	144	44	4
6	8	15	57	60	144	54	9	7	8	16	67	44	196	52	9

Table 4.2: Integral tree $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ with diameter 5, where $q+r \neq t$, $m > 1$ and $q > 1$.

diameter 5. However, we did not succeed in finding positive integral solutions of Equation (4.1.4) *by computer search for the cases: (i) when $m = 1$ and $q = 1$, $1 \leq a \leq 15$, $a \leq b \leq a + 15$, $b \leq c \leq b + 20$, (ii) when $m > 1$, $q = 1$, $q + r \neq t$, and t is a perfect square, $1 \leq a \leq 5$, $a \leq b \leq a + 8$, $b \leq c \leq b + 10$.*

Hence, we raise the following question.

Question 4.1.10. *Are there integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 with $m = 1$, $q = 1$ or $m > 1$, $q = 1$, $q + r \neq t$, t a perfect square?*

From Corollary 4.1.7, we raise the following question.

Question 4.1.11. *Are there infinitely many integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5 with $q + r \neq t$, $m = 1$ and $q > 1$?*

Remark 4.1.12. *For integral trees $[K_{1,s} \bullet T(m,t)] \ominus T(q,r)$ of diameter 5, by analyzing Table 4.2, we can see the following. If $q + r \neq t$, $m > 1$, $q > 1$, t and r are perfect squares, then the problem of finding such integral trees is equivalent to the problem of solving Equation* (4.1.4)*. In particular, we can also see that*

$$[x^2 - (q + r)][x^4 - (m + t + s)x^2 + st] - (x^2 - t)(x^2 - r)$$
$$= [x^2 - (q + r)](x^2 - r)(x^2 - \tfrac{st}{r}) - (x^2 - t)(x^2 - r)$$

$= (x^2 - r)[x^4 - (q + r + \frac{st}{r} + 1)x^2 + t + \frac{st(q+r)}{r})$
$= (x^2 - r)(x^2 - a^2)(x^2 - b^2).$

Thus, when $q + r \neq t$, $m > 1$ *and* $q > 1$, t *and* r *are perfect squares, that is,* $t = t_1^2$ *and* $r = r_1^2$, *we are interested to solve the following Diophantine equations* (4.1.5).

$$\begin{cases} a^2 + b^2 = q + r + \frac{st}{r} + 1 \\ a^2 b^2 = t + \frac{st(q+r)}{r} \\ m + t + s = r + \frac{st}{r} \end{cases} \tag{4.1.5}$$

Hence, we raise the following question.

Question 4.1.13. *When* $q + r \neq t$, $m > 1$ *and* $q > 1$, t *and* r *are perfect squares, can we prove that there are infinitely many positive integral solutions for Equation* (4.1.4) *or Equation* (4.1.5)*? Moreover, what are all positive integral solutions of Equation* (4.1.4) *or Equation* (4.1.5)*?*

Recall that the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 is obtained by joining the center u of $K_{1,s} \bullet T(m,t)$ and the center w of $K_{1,p} \bullet T(q,r)$ with a new edge.

Theorem 4.1.14. *The characteristic polynomial of the tree* $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ *of diameter 5 is*
$P(\{[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]\}, x) = x^{m(t-1)+q(r-1)+s+p-2}(x^2 - t)^{m-1} (x^2 - r)^{q-1}\{[x^4 - (m+t+s)x^2 + st][x^4 - (p+q+r)x^2 + pr] - x^2(x^2 - t)(x^2 - r)\}.$

Proof. The result follows from Lemmas 1.2.2, 1.2.9 and 1.2.10 by arguments similar to those used in Theorem 4.1.1. ∎

Clearly the following corollary follows directly from Theorem 4.1.14.

Corollary 4.1.15. *The tree* $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ *of diameter 5 is integral if and only if the equation*

$$(x^2 - t)^{m-1}(x^2 - r)^{q-1}\{x^8 - (m + t + s + p + q + r + 1)x^6 + [(m + t + s)(p + q + r) + st + pr + t + r]x^4 - [st(p + q + r) + pr(m + t + s) + rt]x^2 + prst\} = 0$$

has only integral roots.

For the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5, we obtain the following special cases:

(i) If $p = 0$, then $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)] = [K_{1,s} \bullet T(m,t)] \ominus T(q,r)$.

(ii) If $s = p = r = t$, then $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)] = T^t[m,q]$.

Integral trees $T^t[m,q]$ of diameter 5 were investigated in [16, 49, 50, 51]. We simply list some examples from [16, 49, 50, 51]. The following Example 4.1.16 can be found in [16, 49].

Example 4.1.16.

(1) Let d (>1) be a positive integer but not a perfect square, and let x_k, y_k be defined by (2.1.8)*. If $m = d(\frac{y_n - y_l}{2})^2$, $r = d(\frac{y_n + y_l}{2})^2$, $t = (\frac{x^2_{\frac{n+l}{2}} - x^2_{\frac{n-l}{2}}}{4})^2$, where $n > l > 0$, n and l are even, then the trees $T^t[m,r]$ of diameter 5 are integral.*

(2) Let $d > 1$ such that there exists positive integral solutions for Equation (2.1.9)*, and let x_k, y_k be defined by* (2.1.11)*. If $m = d(x_n y_n - x_l y_l)^2$, $r = d(x_n y_n + x_l y_l)^2$, and $t = (\frac{x^2_{n+l} - x^2_{n-l}}{4})^2$, where $n > l > 0$, then the trees $T^t[m,r]$ of diameter 5 are integral.*

(3) Let d (>1) be a positive integer with square-free divisors, $d = d_1 d_2$, $d_1 > 1$ such that Equation (2.1.14) *has positive integral solutions, and let x_k, y_k be defined by* (2.1.16)*. If $m = dx_k^2 y_l^2$, $r = dx_l^2 y_k^2$, and $t = d_1^2(\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $2 \nmid kl$, then the trees $T^t[m,r]$ of diameter 5 are integral.*

(4) Let d (>1) be a positive integer with square-free divisors, and let x_k, y_k be defined by (2.1.8)*. Let $m = dx_k^2 y_l^2$, $r = dx_l^2 y_k^2$, and $t = (\frac{x_k^2 - x_l^2}{4})^2$, where $k \neq l$, $\varepsilon = x_0 + y_0\sqrt{d}$ is the fundamental solution of Equation* (2.1.4)*. If $2 \nmid x_0$ or $2 | x_0$, and $k \equiv l (\mathrm{mod}\ 2)$, then the trees $T^t[m,r]$ of diameter 5 are integral.*

For the tree $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5, we have only found such integral trees for the case that $s = t = p = r$. Hence, we raise the following question.

Question 4.1.17. *Are there integral trees $[K_{1,s} \bullet T(m,t)] \ominus [K_{1,p} \bullet T(q,r)]$ of diameter 5 for s, t, p and r which are not all equal ?*

4.2 Two classes of integral trees of diameter 6

In this section, we determine the characteristic polynomials of two classes of trees with diameter 6. We also obtain for the first time sufficient and necessary conditions for two classes of trees to be integral. To do so, we again use number theory and apply a computer search. All these classes are infinite and different from those in the literature. We also prove that the problem of finding integral trees of diameter 6 is equivalent to solving some Diophantine equations. We give a positive answer to a question of Wang *et al.* [77].

Integral trees $T^t(r,m)$, $T(r,m,t)$, and $K_{1,s}\bullet T(r,m,t)$ of diameter 6 were investigated in [2, 16, 48, 51, 70, 77, 79], [2, 15, 37, 38, 39, 47, 48, 50, 75, 76, 77] and [2, 48, 69, 75, 76, 77], respectively. In the present thesis the structures of integral trees $T(p,q)\bullet T(r,m,t)$ of diameter 6 and $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6 are found for the first time.

4.2.1 The characteristic polynomials of two classes of trees

In this section, we shall determine the characteristic polynomials of the trees $T(p,q)\bullet T(r,m,t)$ and $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ with diameter 6.

Theorem 4.2.1. $P[T(p,q)\bullet T(r,m,t),x] = x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6-(p+q+m+t+r)x^4+[(p+q)(m+t)+r(q+t)]x^2-qrt\}$.

Proof. By using Lemmas 1.2.1 and 1.2.9, we find

$$\begin{aligned}&P[T(p,q)\bullet T(r,m,t),x]\\&=P[T(p,q),x]P^r[T(m,t),x]+P^p(K_{1,q},x)P[T(r,m,t),x]\\&-xP^p(K_{1,q},x)P^r[T(m,t),x]\\&=x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\\&\cdot\{x^6-(p+q+m+t+r)x^4+[(p+q)(m+t)+r(q+t)]x^2-qrt\},\end{aligned}$$

and the theorem is proved. ■

Corollary 4.2.2.

(1) For $r=1$ the characteristic polynomial of the tree $T(p,q)\bullet T(r,m,t)=T(p,q)\ominus T(m,t)$ of diameter 5 is $P[T(p,q)\bullet T(r,m,t),x]=P([T(p,q)\ominus T(m,t),x]=x^{m(t-1)+p(q-1)}(x^2-q)^{p-1}(x^2-t)^{m-1}\{x^6-(p+q+m+t+1)x^4+[(p+q)(m+t)+q+t]x^2-qt\}$.

(2) For $m+t=q$ the characteristic polynomial of the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 is $P[T(p,q)\bullet T(r,m,t),x]=x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p+r-1}(x^2-t)^{r(m-1)}[x^4-(p+q+r)x^2+rt]$.

(3) For $p+q=t$ the characteristic polynomial of the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 is $P[T(p,q)\bullet T(r,m,t),x]=x^{rm(t-1)+p(q-1)+r-1}(x^2-q)^{p-1}(x^2-t)^{r(m-1)+1}[x^2-(m+t)]^{r-1}[x^4-(m+t+r)x^2+qr]$.

(4) *For $p = m$, $q = t$ the characteristic polynomial of the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 is* $P[T(m,t) \bullet T(r,m,t), x] = x^{m(r+1)(t-1)+r-1} (x^2 - t)^{(r+1)(m-1)}[x^2 - (m+t)]^{r-1}\{x^6 - [2(m+t)+r]x^4 + [(m+t)^2 + 2rt]x^2 - rt^2\}$.

Proof. The proof follows directly from Theorem 4.2.1. ■

Theorem 4.2.3. $P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t), x] = x^{rm(t-1)+p(q-1)+r+s-1}(x^2 - q)^{p-1}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1}\{x^6 - (p+q+s+m+t+r)x^4 + [(p+q+s)(m+t) + r(q+t) + qs]x^2 - q[s(m+t) + rt]\}$.

Proof. By (2) of Lemma 1.2.5, Lemma 1.2.9 and Theorem 4.2.1, we obtain

$$
\begin{aligned}
&P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t), x]\\
&= x^{s-1}\{xP[T(p,q) \bullet T(r,m,t), x] - sP^p(K_{1,q}, x)P^r[T(m,t), x]\}\\
&= x^{rm(t-1)+p(q-1)+r+s-1}(x^2 - q)^{p-1}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1}\\
&\cdot\{x^6 - (p+q+s+m+t+r)x^4 + [(p+q+s)(m+t) + r(q+t) + qs]x^2\\
&-q[s(m+t) + rt]\}.
\end{aligned}
$$
■

As special cases we find

Corollary 4.2.4.

(1) *(See Theorem 4.1.1 of Chapter 4) If $r = 1$, then the characteristic polynomial of the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t) = K_{1,s} \bullet T(p,q) \ominus T(m,t)$ of diameter 5 is*

$P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t), x] = P[K_{1,s} \bullet T(p,q) \ominus T(m,t), x]$

$= x^{m(t-1)+p(q-1)+s}\ (x^2 - q)^{p-1}(x^2 - t)^{m-1}\{x^6 - (p+q+s+m+t+1)x^4 + [(p+q+s)(m+t) + q + t + qs]x^2 - q[s(m+t) + t]\}$.

(2) *If $m+t = q$, then the characteristic polynomial of the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 is*

$P[K_{1,s} \bullet T(p,q) \bullet T(r,m,t), x] = x^{rm(t-1)+p(q-1)+r+s-1}(x^2 - q)^{p+r-1}(x^2 - t)^{r(m-1)}[x^4 - (p+q+s+r)x^2 + qs + rt]$.

4.2.2 Integral trees of diameter 6

In this section, we derive sufficient and necessary conditions for the trees $T(p,q) \bullet T(r,m,t)$ and $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 to be integral. Some concrete sufficient conditions are also obtained by computer search. All these classes are infinite and different from those of [2, 15, 16, 37, 38, 39, 47, 48, 50, 51, 69, 75, 76, 77, 79]. Recall that the trees $T(p,q) \ominus T(m,t)$ and

$[K_{1,s} \bullet T(p,q)] \ominus T(m,t)$ with diameter 5 were studied in [15, 38, 47] and Section 4.1.

From the characteristic polynomial in Theorem 4.2.1 the following is trivial.

Theorem 4.2.5. *The tree $T(p,q) \bullet T(r,m,t)$ is integral if and only if the equation*

$(x^2 - q)^{p-1}(x^2 - t)^{r(m-1)}[x^2 - (m+t)]^{r-1}\{x^6 - (p+q+m+t+r)x^4 + [(p+q)(m+t) + r(q+t)]x^2 - qrt\} = 0$

has only integral roots, where p, q, r, m and t are positive integers.

Theorem 4.2.6. *(See [15]) If $r = 1$, then the tree $T(p,q) \bullet T(r,m,t) = T(p,q) \ominus T(m,t)$ of diameter 5 is not an integral tree.*

Theorem 4.2.7. *If $m + t = q$, then the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r > 1$) is integral if and only if $x^4 - (p+q+r)x^2 + rt$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are integers, and one of the following two conditions holds: (i) t and $q(= m+t)$ are perfect squares, (ii) $m = 1$, $q(= 1+t)$ is a perfect square.*

Proof. When $m + t = q$, using Theorem 4.2.5 or Corollary 4.2.2 (2), we find

$P[T(p,q) \bullet T(r,m,t), x] = x^{rm(t-1)+p(q-1)+r-1}(x^2 - q)^{p+r-1}(x^2 - t)^{r(m-1)}$ $[x^4 - (p+q+r)x^2 + rt]$.

From this representation the statement directly follows. ■

Corollary 4.2.8.

(1) *For $m + t = q$, $m > 1$ the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r > 1$) is integral if and only if $t = k^2$, $m = n^2 + 2nk$, $r = \frac{a^2b^2}{k^2}(> 1)$, $q = (n+k)^2$, and $p = a^2 + b^2 - (n+k)^2 - \frac{a^2b^2}{k^2}(\geq 1)$, where a, b, n and k are positive integers.*

(2) *For $m + t = q$, $m = 1$ the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r > 1$) is integral if and only if $t = n^2 + 2n$, $m = 1$, $r = \frac{a^2b^2}{n^2+2n}(> 1)$, $q = (n+1)^2$, and $p = a^2 + b^2 - (n+1)^2 - \frac{a^2b^2}{n^2+2n}(\geq 1)$, where a, b and n are positive integers.*

Proof. From Theorem 4.2.7, it follows that the tree $T(p,q) \bullet T(r,m,t)$ (where $r > 1$ and $m+t = q$) is integral if and only if a, b, p, q, r, t are positive integral solutions for the equations (4.2.1) and one of the following two conditions holds: (i) t and $q(= m+t)$ are perfect squares, (ii) $m = 1$, $q(= 1+t)$ is a perfect square.

$$\begin{cases} a^2 + b^2 = p + q + r, \\ a^2b^2 = rt. \end{cases} \tag{4.2.1}$$

(1) Assume that $t = k^2$, $q(= m + t) = (n + k)^2$, *i.e.* $m = n^2 + 2nk$. Then by Equation (4.2.1), we get $r = \frac{a^2b^2}{k^2}(> 1)$, $q = m + t = (n + k)^2$, and $p = a^2 + b^2 - (n + k)^2 - \frac{a^2b^2}{k^2}(\geq 1)$, where a, b and k are positive integers.

(2) Assume that $m = 1$, $t = n^2 + 2n$. Then by Equation (4.2.1), we find $r = \frac{a^2b^2}{n^2+2n}(> 1)$, $q = 1+t = (n+1)^2$, and $p = a^2 + b^2 - (n+1)^2 - \frac{a^2b^2}{n^2+2n}(\geq 1)$, where a, b and n are positive integers. ■

Example 4.2.9. *For the tree $T(p, q) \bullet T(r, m, t)$ of diameter 6, let a, b, p, q, $r(> 1)$, m and t be as in Theorem 4.2.7. Then the following holds.*

(1) Let $m+t = q$, $m > 1$, and let a, b, p, q, r, m, t be positive integers in Table 4.3. Then the tree $T(p, q) \bullet T(r, m, t)$ of diameter 6 is integral.(Table 4.3 is obtained by computer search, where $1 \leq a \leq 6$, $a \leq b \leq a + 6$).

(2) Let $m+t = q$, $m = 1$, and let a, b, p, q, r, m, t be positive integers in Table 4.4. Then the tree $T(p, q) \bullet T(r, m, t)$ of diameter 6 is integral.(Table 4.4 is obtained by computer search, where $1 \leq a \leq 6$, $a \leq b \leq a + 6$).

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	4	4	9	4	5	4	1	6	17	16	4	7	9
1	6	19	9	9	5	4	1	6	8	25	4	16	9
1	6	12	16	9	12	4	1	6	3	25	9	21	4
2	6	6	25	9	9	16	2	6	8	16	16	7	9
2	8	27	25	16	9	16	2	8	16	36	16	20	16
2	8	3	49	16	33	16	3	8	8	49	16	13	36
3	8	12	25	36	9	16	3	8	1	36	36	20	16
4	9	12	49	36	13	36	4	10	10	81	25	17	64
4	10	16	36	64	11	25	4	10	3	49	64	24	25
6	12	16	100	64	19	81	6	12	18	81	81	17	64

Table 4.3: Integral trees $T(p, q) \bullet T(r, m, t)$ of diameter 6, with $m + t = q$, $m > 1$.

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	3	3	4	3	1	3	1	4	6	9	2	1	8
1	6	21	4	12	1	3	2	4	3	9	8	1	8
2	6	13	9	18	1	8	2	6	9	25	6	1	24
2	8	27	9	32	1	8	3	5	3	16	15	1	15
3	8	24	25	24	1	24	3	8	12	49	12	1	48
4	6	3	25	24	1	24	4	9	18	25	54	1	24
4	9	21	49	27	1	48	4	10	15	81	20	1	80
5	7	3	36	35	1	35	6	8	3	49	48	1	48
6	10	12	49	75	1	48	6	10	10	81	45	1	80
6	11	13	100	44	1	99	6	12	23	49	108	1	48

Table 4.4: Integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t=q$, $m=1$.

Proof. From Theorem 4.2.7 or Corollary 4.2.8, the statement easily follows similarly to Corollary 4.2.8. ■

Theorem 4.2.10. *If $p+q=t$, then the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $x^4+(m+t+r)x^2+qr$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where a and b are integers, and one of the following two conditions holds: (i) t, q and $m+t$ are perfect squares, (ii) $p=1$, t and $m+t$ are perfect squares.*

Proof. The result follows from Theorem 4.2.5 or Corollary 4.2.2 (3) similarly to Theorem 4.2.7. ■

Corollary 4.2.11.

(1) *If $p+q=t$, t, q and $m+t$ are perfect squares, then the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $t=k^2$, $m=n^2+2nk$, $r=\frac{a^2b^2}{s^2}(>1)$, $q=s^2$, and $p=k^2-s^2(\geq 1)$, where a, b, k, n and s are positive integers satisfying*

$$s^2(a^2+b^2)=s^2(n+k)^2+a^2b^2. \tag{4.2.2}$$

(2) *If $p+q=t$, $p=1$, then the tree $T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $t=k^2$, $m=n^2+2nk$, $r=\frac{a^2b^2}{k^2-1}(>1)$, $q=k^2-1(\geq 1)$, and $p=1$, where a, b, k and n are positive integers*

satisfying

$$(k^2-1)(a^2+b^2)=(k^2-1)(n+k)^2+a^2b^2. \tag{4.2.3}$$

Proof. By using Theorem 4.2.5 or Theorem 4.2.10, the proof is similar to the proof of Corollary 4.2.8. ■

Example 4.2.12. *For the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6, with a, b, p, q, $r(>1)$, m and t be as in Theorem 4.2.10, the following holds.*

(1) If $p+q=t$, t, q and $m+t$ are perfect squares, a, b, p, q, r, m and t are positive integers in Table 4.5, then the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 is integral.(Table 4.5 is obtained by computer search, where $1\le a\le 5$, $a\le b\le 2a+10$).

(2) If $p+q=t$, $p=1$, a, b, p, q, r, m and t are positive integers in Table 4.6, then the tree $T(p,q)\bullet T(r,m,t)$ of diameter 6 is integral.(Table 4.6 is obtained by computer search, where $1\le a\le 5$, $a\le b\le a+20$).

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	8	32	4	16	13	36	1	8	21	4	16	24	25
1	8	12	4	16	33	16	1	8	5	4	16	40	9
2	9	27	9	36	13	36	2	9	16	9	36	24	25
2	9	7	9	36	33	16	3	14	95	49	36	25	144
3	14	72	49	36	48	121	3	14	51	49	36	69	100
3	14	32	49	36	88	81	3	14	15	49	36	105	64
3	16	84	16	144	21	100	3	16	65	16	144	40	81
3	16	48	16	144	57	64	3	16	33	16	144	72	49
3	16	20	16	144	85	36	3	16	9	16	144	96	25
4	18	133	36	144	27	169	4	18	108	36	144	52	144
4	18	85	36	144	75	121	4	18	64	36	144	96	100
4	18	45	36	144	115	81	4	18	28	36	144	132	64
4	18	13	36	144	147	49	5	14	51	49	100	21	100
5	14	32	49	100	40	81	5	14	15	49	100	57	64

Table 4.5: Integral trees $T(p,q)\bullet T(r,m,t)$, with $p+q=t$, t, q and $m+t$ are perfect squares.

a	b	p	q	r	m	t	a	b	p	q	r	m	t
1	6	1	3	12	21	4	1	16	1	8	32	216	9
2	8	1	8	32	27	9	2	15	1	15	60	153	16
3	8	1	24	24	24	25	3	10	1	15	60	33	16
3	16	1	24	96	144	25	3	20	1	15	240	153	16
4	12	1	24	96	39	25	4	15	1	80	45	115	81
4	24	1	48	192	351	49	5	14	1	35	140	45	36

Table 4.6: Integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6, where $p+q=t$, $p=1$.

Proof. The statement follows from Theorem 4.2.10 or Corollary 4.2.11 similarly to Corollary 4.2.8. ∎

Remark 4.2.13. *By using Theorem 4.2.10, we can construct integral trees $T(p,q) \bullet T(r,m,t)$ from any positive integral solution of the equations* (4.2.4) *satisfying one of the two conditions of Theorem 4.2.10.*

$$\begin{cases} a^2+b^2 = m+t+r, \\ a^2b^2 = qr. \end{cases} \tag{4.2.4}$$

For the case $p+q=t$, by Corollary 4.2.11 and Example 4.2.12, we know that there exist many such integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6 satisfying one of the two conditions of Theorem 4.2.10.

Hence, we raise the following question.

Question 4.2.14. *For $p+q=t$, are there infinitely many positive integral solutions for Equation* (4.2.4) *satisfying one of the two conditions of Theorem 4.2.10? If yes, how can they be found, i.e., what are all positive integral solutions for Equation* (4.2.2) *or Equation* (4.2.3)*?*

Theorem 4.2.15. *If $p=m$, $q=t$, then the tree $T(p,q) \bullet T(r,m,t) = T(m,t) \bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $x^6-[2(m+t)+r]x^4+[(m+t)^2+2rt]x^2-rt^2$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, where a, b and c are integers, and one of the following two conditions holds: (i) t and $m+t$ are perfect squares, (ii) $m=1$, $m+t$ is a perfect square.*

Proof. From Theorem 4.2.5 or Corollary 4.2.2 (4), the correctness follows similarly to Theorem 4.2.7. ∎

Corollary 4.2.16. *Let $p = m$, $q = t$, and suppose that the tree $T(m,t) \bullet T(r,m,t)$ of diameter 6 is integral, such that t and $m+t$ are perfect squares. Then for any positive integer n the tree $T(mn^2,tn^2) \bullet T(rn^2,mn^2,tn^2)$ of diameter 6 is integral, too.*

Proof. Because the tree $T(m,t) \bullet T(r,m,t)$ is integral, t and $m+t$ are perfect squares. By Theorem 4.2.15 or Corollary 4.2.2 (4), we find

$$\begin{aligned}
&P[T(m,t) \bullet T(r,m,t), x] \\
&= x^{m(r+1)(t-1)+r-1}(x^2-t)^{(r+1)(m-1)}[x^2-(m+t)]^{r-1} \\
&\cdot\{x^6-[2(m+t)+r]x^4+[(m+t)^2+2rt]x^2-rt^2\} \\
&= x^{m(r+1)(t-1)+r-1}(x^2-t)^{(r+1)(m-1)}[x^2-(m+t)]^{r-1} \\
&\cdot(x^2-a^2)(x^2-b^2)(x^2-c^2),
\end{aligned}$$

where a, b and c are integers, t and $m+t$ are perfect squares.

By using again Theorem 4.2.15 or Corollary 4.2.2 (4), we obtain

$$\begin{aligned}
&P[T(mn^2,tn^2) \bullet T(rn^2,mn^2,tn^2), x] \\
&= x^{mn^2(rn^2+1)(tn^2-1)+rn^2-1}(x^2-tn^2)^{(rn^2+1)(mn^2-1)}[x^2-(m+t)n^2]^{rn^2-1} \\
&\cdot\{x^6-[2(m+t)+r]n^2x^4+[(m+t)^2+2rt]n^4x^2-rt^2n^6\} \\
&= x^{mn^2(rn^2+1)(tn^2-1)+rn^2-1}(x^2-tn^2)^{(rn^2+1)(mn^2-1)}[x^2-(m+t)n^2]^{rn^2-1} \\
&\cdot(x^2-a^2n^2)(x^2-b^2n^2)(x^2-c^2n^2),
\end{aligned}$$

where a, b and c are integers, t and $m+t$ are perfect squares.

This proves the corollary. ■

Corollary 4.2.17. *Suppose $p = m$, $q = t$, t and $m+t$ are perfect squares. Let a, b, c, p, q, $r(>1)$, m, t be as in Theorem 4.2.15 and given in Table 4.7. Then for any positive integer n the tree $T(mn^2,tn^2) \bullet T(rn^2,mn^2,tn^2)$ of diameter 6 is integral. (Table 4.7 is obtained by computer search, where $1 \le a \le 10$, $a \le b \le 50$, $b \le c \le 200$),*

Proof. The result follows from Theorem 4.2.15 and Corollary 4.2.16 similarly to Corollary 4.2.8. ■

Remark 4.2.18. *In view of Theorem 4.2.15, we have to find positive integral solutions of the following Diophantine equations* (4.2.5) *satisfying one of the two conditions of Theorem 4.2.15.*

$$\begin{cases} a^2+b^2+c^2 = 2(m+t)+r, \\ a^2b^2+b^2c^2+c^2a^2 = (m+t)^2+2rt, \\ a^2b^2c^2 = rt^2. \end{cases} \tag{4.2.5}$$

For the tree $T(m,t) \bullet T(r,m,t)$ of diameter 6, by Corollaries 4.2.16 and 4.2.17, there exist infinitely many such integral trees satisfying Condition (i)

a	b	c	r	m	t	a	b	c	r	m	t
1	6	15	100	72	9	1	7	27	441	160	9
1	19	75	3249	1344	25	1	22	54	1089	1120	36
1	37	147	12321	5280	49	1	40	104	4225	4032	64
2	12	30	400	288	36	2	14	54	1764	640	36
2	38	150	12996	5376	100	2	44	108	4356	4480	144
3	7	25	441	96	25	3	10	42	1225	288	36
3	13	49	1521	480	49	3	18	45	900	648	81
3	21	81	3969	1440	81	3	31	121	8649	3360	121
3	35	60	784	1800	225	3	43	169	16641	6720	169
4	13	48	1521	420	64	4	24	60	1600	1152	144
4	28	108	7056	2560	144	4	49	192	21609	8580	256
5	30	75	2500	1800	225	5	35	135	11025	4000	225
6	14	50	1764	384	100	6	15	90	6561	800	100
6	20	84	4900	1152	144	6	26	98	6084	1920	196
6	33	49	484	1080	441	6	36	90	3600	2592	324
6	42	162	15876	5760	324	7	42	105	4900	3528	441
7	49	189	21609	7840	441	8	26	96	6084	1680	256
8	48	120	6400	4608	576	9	21	75	3969	864	225
9	30	126	11025	2592	324	9	39	147	13689	4320	441
10	21	165	23716	1800	225	10	49	65	676	1800	1225

Table 4.7: Integral trees $T(mn^2, tn^2) \bullet T(rn^2, mn^2, tn^2)$, where n is a positive integer.

of Theorem 4.2.15.

Hence, we raise the following

Question 4.2.19. *What are all positive integral solutions for Equation* (4.2.5) *satisfying Condition (i) of Theorem 4.2.15?*

However, we have to mention that for the case (ii), when $m = 1$, $m + t$ is a perfect square, $1 \leq a \leq 20$, $a \leq b \leq 5a + 10$, $b \leq c \leq 5b + 10$, we have not found positive integral solutions of Equation (4.2.5) by computer search. Hence, we raise the following question.

Question 4.2.20. *Are there positive integral solutions for Equation* (4.2.5) *satisfying Condition (ii) of Theorem 4.2.15? If the answer is yes how can we* *find all positive integral solutions of Equation* (4.2.5)*?*

Theorem 4.2.21. *If* $m + t \neq q$*,* $p + q \neq t$*, and* $(p, q) \neq (m, t)$*, then the tree* $T(p, q) \bullet T(r, m, t)$ *of diameter 6 (where* $r > 1$*) is integral if and only if* $x^6 - (p+q+m+t+r)x^4 + [(p+q)(m+t) + r(q+t)]x^2 - qrt$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)$*, where* a*,* b *and* c *are integers, and one of the following four conditions holds: (i)* q*,* t *and* $m+t$ *are perfect squares,(ii)* $p = 1$*,* $m > 1$*,* t *and* $m + t$ *are perfect squares, (iii)* $m = 1$*,* $p > 1$*,* q *and* $m + t$ *are perfect squares, (iv)* $p = m = 1$*,* $m + t$ *is a perfect square.*

Proof. The statement follows from Theorem 4.2.5 similarly to Theorem 4.2.7. ■

Theorem 4.2.22. *Suppose* $m + t \neq q$*,* $p + q \neq t$*, and* $(p, q) \neq (m, t)$*, and let* a*,* b*,* c*,* p*,* q*,* $r(> 1)$*,* m*,* t *be as in Theorem 4.2.21. Then the following hold.*

(1) If q*,* t *and* $m+t$ *are perfect squares,* a*,* b*,* c*,* p*,* q*,* r*,* m *and* t *are positive integers in Table 4.8, then the tree* $T(p, q) \bullet T(r, m, t)$ *of diameter 6 is integral. (Table 4.8 is obtained by computer search, where* $1 \leq a \leq 7$*,* $a \leq b \leq a + 9$*,* $b \leq c \leq 20$*).*

(2) If $p = 1$*,* $m > 1$*,* t *and* $m + t$ *are perfect squares,* a*,* b*,* c*,* p*,* q*,* r*,* m *and* t *are positive integers in Table 4.9, then the tree* $T(p, q) \bullet T(r, m, t)$ *of diameter 6 is integral. (Table 4.9 is obtained by computer search, where* $1 \leq a \leq 18$*,* $a \leq b \leq a + 10$*,* $b \leq c \leq b + 20$*).*

Proof. Using Theorem 4.2.5 or Theorem 4.2.21, the result follows in a way similar to that of Corollary 4.2.8. ■

Remark 4.2.23. *When* $m + t \neq q$*,* $p + q \neq t$*, and* $(p, q) \neq (m, t)$*, by Theorem 4.2.21, we have to find positive integral solutions of the following Diophantine equations* (4.2.6) *satisfying one of the four conditions of Theorem 4.2.21.*

$$\begin{cases} a^2 + b^2 + c^2 = p + q + m + t + r, \\ a^2b^2 + b^2c^2 + c^2a^2 = (p + q)(m + t) + r(q + t), \\ a^2b^2c^2 = qrt. \end{cases} \tag{4.2.6}$$

For the tree $T(p, q) \bullet T(r, m, t)$ of diameter 6, by Theorems 4.2.21 and 4.2.22, there exist many such integral trees satisfying Condition (i) or Condition (ii) of Theorem 4.2.21. Hence, we raise the following

a	b	c	p	q	r	m	t	a	b	c	p	q	r	m	t
1	4	9	33	4	36	16	9	1	4	9	28	9	36	21	4
1	4	15	17	4	100	112	9	1	4	15	12	9	100	117	4
1	7	15	48	9	49	144	25	1	7	15	32	25	49	160	9
1	7	20	81	4	196	144	25	1	7	20	60	25	196	165	4
1	9	20	60	25	36	325	36	1	9	20	49	36	36	336	25
1	10	14	135	16	25	72	49	1	10	14	102	49	25	105	16
2	5	18	70	9	225	33	16	2	5	18	63	16	225	40	9
2	6	20	24	16	144	231	25	2	6	20	15	25	144	240	16
2	8	18	132	16	144	64	36	2	8	18	112	36	144	84	16
2	11	20	210	25	121	105	64	2	11	20	171	64	121	144	25
3	8	15	68	81	100	33	16	3	12	20	220	64	100	88	81
3	12	20	203	81	100	105	64	4	10	18	88	144	144	39	25

Table 4.8: Integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t \neq q$, $p+q \neq t$, and $(p,q) \neq (m,t)$.

a	b	c	p	q	r	m	t	a	b	c	p	q	r	m	t
8	9	20	1	75	108	105	256	14	16	27	1	252	252	100	576

Table 4.9: Integral trees $T(p,q) \bullet T(r,m,t)$ of diameter 6, where $p=1$, $m+t \neq q$, $p+q \neq t$, and $(p,q) \neq (m,t)$.

Question 4.2.24. *What are all positive integral solutions for Equation* (4.2.6) *satisfying Condition (i) or Condition (ii) of Theorem 4.2.21?*

However we have to note that for $m+t \neq q$, $p+q \neq t$, and $(p,q) \neq (m,t)$, we have not found positive integral solutions of Equation (4.2.6) by computer search in the following cases: (iii) when $m=1$, $p>1$, q and $m+t$ are perfect squares, $1 \leq a \leq 10$, $a \leq b \leq a+10$, $b \leq c \leq b+20$, (iv) when $p=m=1$, $m+t$ is a perfect square, $1 \leq a \leq 10$, $a \leq b \leq a+10$, $b \leq c \leq b+20$.

Question 4.2.25. *For $m+t \neq q$, $p+q \neq t$, and $(p,q) \neq (m,t)$, are there positive integral solutions for Equation* (4.2.6) *satisfying Condition (iii) or Condition (iv) of Theorem 4.2.21? If yes how to find all such positive integral solutions of Equation* (4.2.6)*?*

Theorem 4.2.26. *The tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ is integral if and only if the equation*

$(x^2-q)^{p-1}(x^2-t)^{r(m-1)}[x^2-(m+t)]^{r-1}\{x^6-(p+q+s+m+t+r)x^4+[(p+q+s)(m+t)+r(q+t)+qs]x^2-q[s(m+t)+rt]\}=0$
has only integral roots, where s, p, q, r, m and t are positive integers.

Proof. Follows from Theorem 4.2.3 similarly to Theorem 4.2.7. ■

Theorem 4.2.27. *(See Theorem 4.1.1 of Chapter 4) If $r=1$, then the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)=[K_{1,s}\bullet T(p,q)]\ominus T(m,t)$ of diameter 5 is integral if and only if the equation*
$(x^2-q)^{p-1}(x^2-t)^{m-1}\{x^6-(p+q+s+m+t+1)x^4+[(p+q+s)(m+t)+q+t+qs]x^2-q[s(m+t)+t]\}=0$
has only integral roots, where s, p, q, m and t are positive integers.

Remark 4.2.28. *In Section 4.1 of Chapter 4, we have proven for the first time that there are infinitely many integral trees $K_{1,s}\bullet T(p,q)\bullet T(1,m,t)=[K_{1,s}\bullet T(p,q)]\ominus T(m,t)$ of diameter 5.*

Theorem 4.2.29.

(1) If $m+t=q$, $m>1$, then the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $t=k^2$, $m=n^2+2nk$, $q=(n+k)^2$, and s, p, $q(=(n+k)^2)$, r, $m(=n^2+2nk)$, $t(=k^2)$ are positive integers satisfying

$$\begin{cases} a^2+b^2=p+(n+k)^2+r+s, \\ a^2b^2=rk^2+s(n+k)^2. \end{cases} \tag{4.2.7}$$

(2) If $m+t=q$, $m=1$, then the tree $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $t=n^2+2n$, $m=1$, $q=(n+1)^2$, and s, p, $q(=(n+1)^2)$, r, $m(=1)$, $t(=n^2+2n)$ are positive integers satisfying

$$\begin{cases} a^2+b^2=p+(n+1)^2+r+s, \\ a^2b^2=r(n^2+2n)+s(n+1)^2. \end{cases} \tag{4.2.8}$$

Proof. The results follows from Theorem 4.2.26 or Corollary 4.2.4 (2) similarly to Theorem 4.2.7. ■

For the trees $K_{1,s}\bullet T(p,q)\bullet T(r,m,t)$ of diameter 6, by computer search, based on Theorem 4.2.29 we have found 4483 solutions of the equations (4.2.7) satisfying $m+t=q$, $m=n^2+2nk>1$, $t=k^2$, $r>1$, $q=(n+k)^2$, $1\le a\le 10$,

$a \le b \le a + 10$. We have also detected 438 solutions of the equations (4.2.8) satisfying $m + t = q$, $m = 1$, $t = n^2 + 2n$, $r > 1$, $q = (n + 1)^2$, $1 \le a \le 10$, $a \le b \le a + 10$. Some of them are given in Example 4.2.30.

Example 4.2.30. *For the tree $K_{1,s} \bullet T(p, q) \bullet T(r, m, t)$ of diameter 6, let a, b, s, p, q, $r(> 1)$, m and t be as in Theorem 4.2.29. Then the following holds.*

(1) If $m + t = q$, $m > 1$, a, b, s, p, q, r, m and t are positive integers in Table 4.10, then the tree $K_{1,s} \bullet T(p, q) \bullet T(r, m, t)$ is integral. (Table 4.10 is obtained by computer search, where $1 \le a \le 3$, $a \le b \le a + 4$).

(2) If $m + t = q$, $m = 1$, a, b, s, p, q, r, m and t are positive integers in Table 4.11, then the tree $K_{1,s} \bullet T(p, q) \bullet T(r, m, t)$ is integral. (Table 4.11 is obtained by computer search, where $1 \le a \le 4$, $a \le b \le a + 5$).

a	b	s	p	q	r	m	t	a	b	s	p	q	r	m	t
1	4	3	6	4	4	3	1	1	4	2	3	4	8	3	1
1	5	1	12	9	4	5	4	1	5	5	12	4	5	3	1
1	5	4	9	4	9	3	1	1	5	3	6	4	13	3	1
1	5	2	3	4	17	3	1	1	5	2	8	9	7	8	1
2	5	4	5	16	4	7	9	2	5	8	5	9	7	5	4
2	5	5	3	16	5	12	4	2	5	6	3	16	4	15	1
2	6	12	10	9	9	5	4	2	6	8	5	9	18	5	4
2	6	8	12	16	4	12	4	2	6	7	9	16	8	12	4
2	6	15	7	9	9	8	1	2	6	6	6	16	12	12	4
2	6	5	3	16	16	12	4	3	6	18	7	16	4	7	9
3	6	4	2	25	14	9	16	3	6	19	5	16	5	12	4
3	6	18	2	16	9	12	4	3	6	20	5	16	4	15	1
3	6	12	2	25	6	21	4	3	7	6	7	36	9	11	25
3	7	18	7	16	17	7	9	3	7	1	6	25	26	9	16
3	7	17	12	25	4	21	4	3	7	11	6	36	5	27	9
3	7	10	3	36	9	27	9	3	7	27	6	16	9	15	1

Table 4.10: Integral trees $K_{1,s} \bullet T(p, q) \bullet T(r, m, t)$ of diameter 6 , where $m + t = q$, $m > 1$, $r > 1$.

We raise the following

a	b	s	p	q	r	m	t	a	b	c	p	q	r	m	t
1	4	1	8	4	4	1	3	1	5	4	15	4	3	1	3
1	5	1	14	4	7	1	3	1	5	1	14	9	2	1	8
1	6	6	23	4	4	1	3	1	6	3	22	4	8	1	3
2	5	4	8	9	8	1	8	2	6	8	14	9	9	1	8
2	7	20	22	9	2	1	8	2	7	12	21	9	11	1	8
2	7	4	20	9	20	1	8	2	7	1	24	16	12	1	15
2	7	4	20	25	4	1	24	3	6	9	8	16	12	1	15
3	7	21	14	16	7	1	15	3	7	6	13	16	23	1	15
3	7	9	15	25	9	1	24	3	8	21	20	16	16	1	15
3	8	6	19	16	32	1	15	4	7	16	8	25	16	1	24
4	7	14	7	36	8	1	35	4	8	16	13	25	26	1	24
4	8	9	15	36	20	1	35	4	8	16	10	49	5	1	48
4	9	48	20	25	4	1	24	4	9	24	19	25	29	1	24
4	9	1	24	36	36	1	35	/	/	/	/	/	/	/	/

Table 4.11: Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t=q$, $m=1$, $r>1$.

Question 4.2.31. *For $m+t=q$, are there infinitely many positive integral solutions for Equation* (4.2.7) *or Equation* (4.2.8) *in Theorem 4.2.29? If yes, how to find them?*

Theorem 4.2.32. *If $m+t \neq q$, then the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6 (where $r>1$) is integral if and only if $x^6-(p+q+s+m+t+r)x^4+[(p+q+s)(m+t)+r(q+t)+qs]x^2-q[s(m+t)+rt]$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$, where a, b and c are integers, and one of the following four conditions holds: (i) q, t and $m+t$ are perfect squares, (ii) $p=1$, $m>1$, t and $m+t$ are perfect squares, (iii) $m=1$, $p>1$, q and $m+t$ are perfect squares, (iv) $m=p=1$, $m+t$ is a perfect square.*

Proof. From Theorem 4.2.3, the result follows as before. ∎

Based on Theorem 4.2.32 we obtain the following

Corollary 4.2.33. *For the tree $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, with $m+t \neq q$, and a, b, s, p, q, $r(>1)$, m, t as in Theorem 4.2.32, the following holds.*

(1) *If* $m+t \neq q$, q, t *and* $m+t$ *are perfect squares,* a, b, c, s, p, q, r, m *and* t *are positive integers in Table 4.12, then the tree* $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ *is integral. (Table 4.12 is obtained by computer search, where* $1 \leq a \leq 4$, $a \leq b \leq a+4$, $b \leq c \leq 15$*).*

(2) *If* $m+t \neq q$, $p = 1$, $m > 1$, t *and* $m+t$ *are perfect squares,* a, b, s, p, q, r, m *and* t *are positive integers in Table 4.13, then the tree* $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ *is integral. (Table 4.13 is obtained by computer search, where* $1 \leq a \leq 9$, $a \leq b \leq 10$, $b \leq c \leq 20$*).*

a	b	c	s	p	q	r	m	t
1	3	10	11	30	4	49	15	1
1	3	10	3	6	4	33	63	1
1	3	14	21	60	4	105	15	1
1	3	14	5	12	4	121	63	1
1	4	8	5	12	4	11	48	1
1	4	9	12	33	4	24	24	1
1	4	10	10	27	4	40	35	1
1	4	12	2	63	64	28	3	1
1	4	12	21	60	4	51	24	1
1	4	12	8	21	4	64	63	1
1	4	13	12	33	4	88	48	1
1	4	14	19	54	4	100	35	1
1	4	14	7	18	4	84	99	1
1	5	8	8	21	4	8	48	1
1	5	12	7	48	9	57	48	1
2	5	9	18	16	9	18	48	1
2	5	11	7	18	100	9	15	1
2	5	12	28	30	9	57	45	4
2	5	12	10	18	16	65	60	4
3	5	14	31	66	36	81	7	9
3	5	14	27	78	100	9	15	1
3	6	14	9	15	16	80	85	36

Table 4.12: Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t \neq q$.

a	b	c	s	p	q	r	m	t
2	3	8	5	1	8	27	32	4
2	3	10	8	1	5	18	77	4
3	4	10	18	1	10	32	55	9
3	4	10	16	1	15	44	45	4
3	4	14	8	1	14	54	128	16
3	5	8	12	1	24	12	48	1
4	5	12	18	1	24	78	48	16
4	5	18	18	1	18	72	220	36
4	5	18	24	1	18	66	240	16
5	6	13	50	1	26	72	56	25
5	6	13	66	1	26	56	72	9
5	6	17	18	1	34	128	133	36
5	6	18	32	1	27	100	189	36
6	7	16	39	1	48	153	64	36
6	7	16	90	1	48	102	96	4
6	7	19	14	1	38	168	144	81
6	7	19	54	1	38	128	189	36
6	7	19	70	1	38	112	216	9
6	7	19	74	1	38	108	224	1
6	9	20	55	1	80	125	252	4
7	8	12	78	1	63	34	56	25
8	9	20	68	1	80	252	80	64
8	9	20	176	1	80	144	140	4
8	10	13	128	1	65	18	57	64
9	10	16	75	1	96	69	96	100

Table 4.13: Integral trees $K_{1,s} \bullet T(p,q) \bullet T(r,m,t)$ of diameter 6, where $m+t \neq q$, $p = 1$.

Remark 4.2.34. *For $m + t \neq q$, in view of Theorem 4.2.32, we wish to find positive integral solutions of the following Diophantine equations* (4.2.9) *satisfying one of the four conditions of Theorem 4.2.32.*

$$\begin{cases} a^2 + b^2 + c^2 = p + q + s + m + t + r, \\ a^2b^2 + b^2c^2 + c^2a^2 = (p + q + s)(m + t) + r(q + t) + qs, \\ a^2b^2c^2 = q[s(m + t) + rt]. \end{cases} \tag{4.2.9}$$

So, the following question arises.

Question 4.2.35. *When $m+t \neq q$, what are all positive integral solutions for Equation* (4.2.9) *satisfying Condition (i) or Condition (ii) of Theorem 4.2.32?*

However again, when $m+t \neq q$, we have not found positive integral solutions of Equation (4.2.9) by computer search in the cases: (iii) when $m = 1$, $p > 1$, q and $m + t$ are perfect squares, $1 \leq a \leq 7$, $a \leq b \leq 10$, $b \leq c \leq 20$, (iv) when $m = p = 1$, $m + t$ is a perfect square, $1 \leq a \leq 10$, $a \leq b \leq 10$, $b \leq c \leq 20$. Hence, we raise the following

Question 4.2.36. *When $m + t \neq q$, are there positive integral solutions for Equation* (4.2.9) *satisfying Condition (iii) or Condition (iv) of Theorem 4.2.32? If yes, how to find them?*

4.3 Integral trees of diameter 8

In this section, we mainly give some new classes of integral trees $K_{1,s} \bullet T(q,r,m,t)$ with diameter 8, different from [2, 38, 39, 48, 75, 76, 77]. Integral trees $T(q,r,m,t)$ of diameter 8 have been investigated in [2, 38, 39, 48, 75, 76, 77]. In [76], we derived the characteristic polynomials of the trees $T(s,q,r,m,t)$ of diameter 10 and $K_{1,s} \bullet T(q,r,m,t)$ of diameter 8. But we did not find such integral trees. P. Híc and M. Pokorný found an infinite class of integral trees $T(s,q,r,m,t)$ of diameter 10 in [39] (see Corollary 4.3.6 or Theorem 1.3.43). Here we present some new results which treat interrelations between integral trees of various diameters. Integral trees $K_{1,s} \bullet T(q,r,m,t)$ of diameter 8 are found for the first time.

Theorem 4.3.1. *The tree $T(s,q,r,m,t)$ of diameter 10 (where $s,q,r,m > 1$) is integral if and only if t and $m+t$ are perfect squares, $x^4-(m+t+r)x^2+rt$ can be factorized as $(x^2-a^2)(x^2-b^2)$, $x^4-(q+m+t+r)x^2+rt+q(m+t)$ can be factorized as $(x^2-c^2)(x^2-d^2)$, and $x^6-(s+q+m+t+r)x^4+[rt+q(m+t)+s(m+t+r)]x^2-rst$ can be factorized as $(x^2-e^2)(x^2-f^2)(x^2-g^2)$, where a, b, c, d, e, f and g are integers.*

Proof. By Lemma 1.2.16 (1), we find
$P[T(s,q,r,m,t),x] = x^{sqrm(t-1)+sq(r-1)+s-1}(x^2-t)^{sqr(m-1)}[x^4-(m+t+r)x^2+rt]^{s(q-1)}[x^2-(m+t)]^{sq(r-1)}[x^4-(q+m+t+r)x^2+rt+q(m+t)]^{s-1}\{x^6-(s+q+m+t+r)x^4+[rt+q(m+t)+s(m+t+r)]x^2-rst\}$.

From this representation the statement directly follows. ■

Similarly, from Lemma 1.2.16 (2) we obtain

Theorem 4.3.2. *The tree* $K_{1,s} \bullet T(q, r, m, t)$ *of diameter 8 (where* $q, r, m > 1$*) is integral if and only if both* t *and* $m+t$ *are perfect squares,* $x^4-(m+t+r)x^2+rt$ *can be factorized as* $(x^2-a^2)(x^2-b^2)$*, and* $x^6-(s+q+m+t+r)x^4+[rt+q(m+t)+s(m+t+r)]x^2-rst$ *can be factorized as* $(x^2-c^2)(x^2-d^2)(x^2-e^2)$*, where* a*,* b*,* c*,* d *and* e *are integers.*

Corollary 4.3.3. *The tree* $K_{1,s} \bullet T(q, r, m, t)$ *of diameter 8 (where* $q, r, m > 1$*) is integral if and only if* $t = k^2$*,* $m = n^2+2nk$*,* $r = \frac{a^2b^2}{k^2}$*,* $q = c^2+d^2+e^2-a^2-b^2-\frac{c^2d^2e^2}{a^2b^2} > 1$*,* $s = \frac{c^2d^2e^2}{a^2b^2}$*, where* a*,* b*,* c*,* d*,* e*,* k *and* n *are positive integers satisfying* $(k^2-b^2)(a^2-k^2) = k^2(n^2+2nk)$ *and* $a^2b^2(c^2d^2+d^2e^2+e^2c^2-a^2b^2) = a^2b^2(n+k)^2(c^2+d^2+e^2-a^2-b^2)+c^2d^2e^2[a^2+b^2-(n+k)^2]$*, where* $a < k < b$*.*

Proof. The result follows easily from Theorem 4.3.2 ■

Theorem 4.3.4. *Let* s*,* q*,* r*,* m*,* t *be positive integers, then for any positive integer* n *the following holds.*

(1) If the tree $T(s, q, r, m, t)$ *of diameter 10 is integral, and* s*,* q*,* r*,* $m > 1$*, then the trees* $T(sn^2, qn^2, rn^2, mn^2, tn^2)$*,* $T(qn^2, rn^2, mn^2, tn^2)$*,* $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$*,* $T(rn^2, mn^2, tn^2)$*,* $K_{1,qn^2} \bullet T(rn^2, mn^2, tn^2)$*,* $K_{1,rn^2} \bullet T(mn^2, tn^2)$*,* $T(mn^2, tn^2)$ *and* $T(tn^2) = K_{1,tn^2}$ *are integral trees with diameters 10, 8, 8, 6, 6, 4, 4 and 2, respectively.*

(2) If the tree $K_{1,s} \bullet T(q, r, m, t)$ *of diameter 8 is integral, and* q*,* r*,* $m > 1$*, then the trees* $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$*,* $T(rn^2, mn^2, tn^2)$*,* $T(mn^2, tn^2)$*,* $K_{1,rn^2} \bullet T(mn^2, tn^2)$ *and* $T(tn^2) = K_{1,tn^2}$ *are integral trees with diameters 8, 6, 4, 4 and 2, respectively.*

(3) If the trees $T(q, r, m, t)$ *of diameter 8 and* $K_{1,s} \bullet T(q, r, m, t)$ *of diameter 8 are integral, and* q*,* r*,* $m > 1$*, then the tree* $T(sn^2, qn^2, rn^2, mn^2, tn^2)$ *of diameter 10 is an integral tree.*

(4) If the trees $T(q, r, m, t)$ *of diameter 8 and* $T(s, q, r, m, t)$ *of diameter 10 are integral, and* s*,* q*,* r*,* $m > 1$*, then the tree* $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$ *of diameter 8 is an integral tree.*

Proof. We only prove (1) for the tree $T(s, q, r, m, t)$. (2)-(4) are similarly proved.

Because $T(s, q, r, m, t)$ is integral, and s, q, r, $m > 1$, by Lemma 1.2.16 (1) or Theorem 4.3.1, we find

$P[T(s,q,r,m,t),x] = x^{sqrm(t-1)+sq(r-1)+s-1}(x^2-t)^{sqr(m-1)}[x^4-(m+t$
$+r)x^2+rt]^{s(q-1)}[x^2-(m+t)]^{sq(r-1)}[x^4-(q+m+t+r)x^2+rt+q(m$
$+t)]^{s-1}\{x^6-(s+q+m+t+r)x^4+[rt+q(m+t)+s(m+t+r)]x^2-rst\}$
$= x^{sqrm(t-1)+sq(r-1)+s-1}(x^2-t)^{sqr(m-1)}[x^2-(m+t)]^{sq(r-1)}[(x^2-a^2)$
$(x^2-b^2)]^{s(q-1)}[(x^2-c^2)(x^2-d^2)]^{s-1}(x^2-e^2)(x^2-f^2)(x^2-g^2),$
where t and $m+t$ are perfect squares, and a, b, c, d and e are integers.

Hence, by Lemma 1.2.16 (1) or Theorem 4.3.1, we get
$P[T(sn^2,qn^2,rn^2,mn^2,tn^2),x] = x^{sqrm(tn^2-1)n^8+sq(rn^2-1)n^4+sn^2-1}(x^2$
$-tn^2)^{sqr(mn^2-1)n^6}\cdot[x^2-(m+t)n^2]^{sq(rn^2-1)n^4}[x^4-(m+t+r)n^2x^2$
$+rtn^4]^{sn^2(qn^2-1)}[x^4-(q+m+t+r)n^2x^2+rtn^4+q(m+t)n^4]^{sn^2-1}$
$\cdot\{x^6-(s+q+m+t+r)n^2x^4+[rt+q(m+t)+s(m+t+r)]n^4x^2-rstn^6\}$
$= x^{sqrm(tn^2-1)n^8+sq(rn^2-1)n^4+sn^2-1}(x^2-tn^2)^{sqr(mn^2-1)n^6}$
$\cdot[x^2-(m+t)n^2]^{sq(rn^2-1)n^4}[(x^2-(an)^2)(x^2-(bn)^2)]^{sn^2(qn^2-1)}$
$\cdot[(x^2-(cn)^2)(x^2-(dn)^2)]^{sn^2-1}(x^2-(en)^2)(x^2-(fn)^2)(x^2-(gn)^2),$
where t and $m+t$ are perfect squares, and a, b, c, d and e are integers.

Thus the tree $T(sn^2,qn^2,rn^2,mn^2,tn^2)$ is integral. ■

Theorem 4.3.5. *Let n_i be positive integers, $i=1,2,\ldots,k$, where k is any positive integer. Then for any positive integer n the following holds.*

(1) If the tree $T(n_k,n_{k-1},\ldots,n_1)$ of diameter $2k$ is integral, and $n_i>1$, $i=2,3,\ldots,k$, then the trees $T(n_jn^2,\ n_{j-1}n^2,\ \ldots,n_1n^2)$ (where $1\le j\le k$, see [38]), $K_{1,n_jn^2}\bullet T(n_{j-1}n^2,n_{j-2}n^2,\ \ldots,\ n_1n^2)$ $(2\le j\le k)$ and $T(n_{j-1}n^2,n_{j-2}n^2,\ \ldots,\ n_1n^2)$ $(2\le j\le k)$ are integral trees with diameters $2j$, $2(j-1)$ and $2(j-1)$, respectively.

(2) If the trees $T(n_{k-1},n_{k-2},\ldots,n_1)$ of diameter $2(k-1)$ and $K_{1,n_k}\bullet T(n_{k-1}, n_{k-2},\ldots,n_1)$ of diameter $2(k-1)$ are integral, and $n_i>1$, $i=2,3,\ldots,k-1$, then the tree $T(n_k,\ n_{k-1},\ \ldots,n_1)$ of diameter $2k$ is integral.

(3) If the trees $T(n_{k-1},n_{k-2},\ldots,n_1)$ of diameter $2(k-1)$ and $T(n_k,n_{k-1}, \ldots,\ n_1) = n_k * T(n_{k-1},n_{k-2},\ldots,\ n_1)$ of diameter $2k$ are integral, and $n_i>1$, $i=2,3,\ldots,k$, then the tree $K_{1,n_k}\bullet T(n_{k-1},\ n_{k-2},\ \ldots,\ n_1)$ of diameter $2(k-1)$ is integral.

Proof. (1) Because $T(n_k,n_{k-1},\ldots,n_1)$ is integral, n_1 and n_i (>1) are positive integers for $i=2,3,\ldots,k$, by Theorems 1.3.19 and 1.3.18 (which are Corollary 3.4 and Theorem 3.5 of [38]), we find that $T(n_jn^2,n_{j-1}n^2,\ldots,n_1n^2)$ $(2\le j\le k)$ is integral for any positive integer n.

By Lemma 1.2.6, we know that for any positive integer n the graphs $G_1 = (n_jn^2-1)K_1\cup T(n_jn^2,n_{j-1}n^2,\ldots,n_1n^2)$ and $G_2 = K_{1,n_jn^2}\bullet T(n_{j-1}n^2,n_{j-2}n^2,$

$\ldots, n_1n^2) \cup (n_jn^2-1)T(n_{j-1}n^2, n_{j-2}n^2, \ldots, n_1n^2)$ $(2 \le j \le k)$ are cospectral forests. Since $T(n_jn^2, n_{j-1}n^2, \ldots, n_1n^2)$ $(2 \le j \le k)$ is integral, it follows that both $K_{1,n_jn^2} \bullet T(n_{j-1}n^2, n_{j-2}n^2, \ldots, n_1n^2)$ and $T(n_{j-1}n^2, n_{j-2}n^2, \ldots, n_1n^2)$ $(2 \le j \le k)$ are integral trees of diameter $2(j-1)$.

(2)-(3) are proven similarly using Lemmas 1.2.6 and 1.2.7. ■

Corollary 4.3.6. *(see [39]) Let* $s = 3006756$, $q = 1051960$, $r = 751689$, $m = 283360$ *and* $t = 133956$. *Then for any positive integer n the tree* $T(sn^2, qn^2, rn^2, mn^2, tn^2)$ *of diameter 10 is an integral balanced rooted tree.*

Proof. From Theorem 4.3.1, we can easily check the correctness by using a method similar to that used in (1) of Theorem 4.3.4. ■

Corollary 4.3.7. *Let* $s = 3006756$, $q = 1051960$, $r = 751689$, $m = 283360$ *and* $t = 133956$. *Then for any positive integer n the trees* $T(sn^2, qn^2, rn^2, mn^2, tn^2)$, $T(qn^2, rn^2, mn^2, tn^2)$, $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$, $T(rn^2, mn^2, tn^2)$, $K_{1,qn^2} \bullet T(rn^2, mn^2, tn^2)$, $K_{1,rn^2} \bullet T(mn^2, tn^2)$, $T(mn^2, tn^2)$ *and* $T(tn^2) = K_{1,tn^2}$ *are integral trees with diameters 10, 8, 8, 6, 6, 4, 4 and 2, respectively.*
Proof. The result follows by Theorems 4.3.1, 4.3.2 and 4.3.4 and Lemmas 1.2.9, 1.2.10, 1.2.15 and 1.2.16. ■

Based on Theorems 4.3.2 and 4.3.4, by computer search we obtain.

Corollary 4.3.8. *Let a, b, c, d, e, s,* $q(>1)$, $r(>1)$, $m(>1)$, *t be positive integers as in Theorem 4.3.2, given in Table 4.14. Then for any positive integer n the trees* $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$, $T(rn^2, mn^2, tn^2)$, $T(mn^2, tn^2)$, $K_{1,rn^2} \bullet T(mn^2, tn^2)$ *and* $T(tn^2) = K_{1,tn^2}$ *are integral trees with diameters 8, 6, 4, 4 and 2, respectively. (Table 4.14 is obtained by computer search, where* $1 \le a \le 7$, $a \le b \le a+20$,$1 \le c \le a$ $a \le d \le 5a+1$, $d \le e \le 10d+3$*).*

Analyzing Table 4.14, we can see that all its rows except the row $(s, q, r, m, t) = (9, 312, 100, 72, 49)$ have the following properties: $t = k^2$, $s = d^2 = m + t$, $a^2b^2 = c^2e^2 = rt$, where a, b, c, d, e, k, s, q, r, m and t are positive integers. This observation leads to.

Corollary 4.3.9. *Let* $t = k^2$, $m = d^2 - k^2 > 1$, $r = \frac{a^2b^2}{k^2}$, $q = c^2 + e^2 - a^2 - b^2 > 1$ *and* $s = d^2$, *where a, b, c, d, e and k are positive integers satisfying* $a^2b^2 = c^2e^2$ *and* $(a^2+b^2)k^2 = a^2b^2 + d^2k^2$. *Then for any positive integer n the trees* $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$, $T(rn^2, mn^2, tn^2)$, $K_{1,rn^2} \bullet T(mn^2, tn^2)$, $T(mn^2, tn^2)$ *and* $T(tn^2) = K_{1,tn^2}$ *are integral trees with diameters 8, 6, 4, 4 and 2, respectively.*

a	b	c	d	e	s	q	r	m	t
2	9	1	7	18	49	240	36	40	9
3	14	1	13	42	169	1560	36	120	49
3	14	2	13	21	169	240	36	120	49
3	16	1	11	48	121	2040	144	105	16
3	16	2	11	24	121	315	144	105	16
4	18	1	14	72	196	4845	144	160	36
4	18	2	14	36	196	960	144	160	36
4	18	3	14	24	196	245	144	160	36
5	14	1	10	21	9	312	100	72	49
5	14	1	11	70	121	4680	100	72	49
5	14	2	11	35	121	1008	100	72	49
7	26	1	23	182	529	32400	196	360	169
7	26	2	23	91	529	7560	196	360	169

Table 4.14: Integral trees $K_{1,sn^2} \bullet T(qn^2, rn^2, mn^2, tn^2)$, $T(rn^2, mn^2, tn^2)$, $T(mn^2, tn^2)$, $K_{1,rn^2} \bullet T(mn^2, tn^2)$ and $T(tn^2) = K_{1,tn^2}$, where $n \in \mathbb{N}$.

Proof. The result follows from Theorems 4.3.2 and 4.3.4. ■

In view of Theorems 4.3.2 and 4.3.4 and Lemmas 1.2.9, 1.2.10, 1.2.15 and 1.2.16, the following conjecture is true for $k = 3, 4, 5$, Hence we are led to

Conjecture 4.3.10. *For any positive integers k and n, if the tree $K_{1,n_k} \bullet T(n_{k-1}, n_{k-2}, \ldots, n_1)$ of diameter $2(k-1)$ is integral, and $n_i > 1$ for $i = 2, 3, \ldots, k-1$, then the trees $K_{1,n_k n^2} \bullet T(n_{k-1}n^2, n_{k-2}n^2, \ldots, n_1 n^2)$ of diameter $2(k-1)$, $K_{1,n_{j-2}n^2} \bullet T(n_{j-3}n^2, n_{j-4}n^2, \ldots, n_1 n^2)$ of diameter $2(j-3)$ (where $4 \le j \le k$) and $T(n_{j-2}n^2, n_{j-3}n^2, \ldots, n_1 n^2)$ of diameter $2(j-2)$ (where $3 \le j \le k$) are integral too.*

In [2, 15, 16, 36, 37, 38, 39, 47, 48, 50, 51, 69, 75, 76, 77, 79] and in the present chapter, we found integral trees $T(n_k, n_{k-1}, \ldots, n_1)$ of diameter $2k$ and integral trees $K_{1,n_k} \bullet T(n_{k-1}, n_{k-2}, \ldots, n_1)$ of diameter $2(k-1)$ for positive integer $k \le 5$.

Question 4.3.11. *Are there integral trees $K_{1,n_k} \bullet T(n_{k-1}, n_{k-2}, \ldots, n_1)$ of diameter $2(k-1)$ and integral trees $T(n_k, n_{k-1}, \ldots, n_1)$ of diameter $2k$ for arbitrary large integer k?*

Chapter 5

Integral complete r-partite graphs

An infinite family of integral complete tripartite graphs was constructed by M. Roitman in 1984 (see [60]), where he mentioned the general problem of finding integral complete multipartite graphs. He conjectured that for $r > 3$ there exist an infinite number of integral complete r-partite graphs. However, he did not find such integral graphs. Balińska and Simić [5] remarked in 2001 that the general problem seems to be intractable. In this chapter, we give a sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs. We will show that the problem of finding such integral graphs is equivalent to solving certain Diophantine equations. M. Roitman's result on the integral complete tripartite graphs is generalized in this chapter. We finally propose several basic open problems for further study.

5.1 A sufficient and necessary condition for complete r-partite graphs to be integral

In this section, we shall derive a sufficient and necessary condition for complete r-partite graphs to be integral. The following Lemma 5.1.1 can be found in [22].

Lemma 5.1.1. *([22]) For the complete r-partite graph $K_{p_1,\cdots,p_r}$ on n vertices, we have*

$$P(K_{p_1,p_2,\cdots,p_r},x) = x^{n-r}(1-\sum_{i=1}^{r}\frac{p_i}{x+p_i})\prod_{j=1}^{r}(x+p_j).$$

Assume that the number of distinct integers of $p_1, p_2, \cdots, p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < \cdots < p_s$. Suppose that a_i is the multiplicity of p_i for each $i = 1, 2, \cdots, s$. The complete r-partite graph $K_{p_1,p_2,\cdots,p_r}$=$K_{p_1,\cdots,p_1,p_2,\cdots,p_2,\cdots,p_s,\cdots,p_s}$ is also denoted by $K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_i p_i$.

Example 5.1.2. *([22]) The complete 2-partite graph K_{p_1,p_2} (i.e. $s = 2$ and $a_1 = a_2 = 1$) is integral if and only if $p_1 p_2$ is a perfect square.*

Corollary 5.1.3. *For the complete r-partite graph $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,\cdots,a_s\cdot p_s}$ on n vertices, we have*

$$P(K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s},x) = x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}[\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1,i\neq j}^{s}(x+p_i)].$$

The following theorem is immediate.

Theorem 5.1.4. *The complete r-partite graph $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ is integral if and only if*

$$\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1,i\neq j}^{s}(x+p_i) = 0. \tag{5.1.1}$$

has only integral roots.

We now discuss Equation (5.1.1) to get more information. Firstly, we divide both sides of Equation (5.1.1) by $\prod_{i=1}^{s}(x+p_i)$, and obtain

$$F(x) := \sum_{i=1}^{s}\frac{a_i p_i}{x+p_i} - 1 = 0. \tag{5.1.2}$$

Clearly, $-p_i$ is not a root of Equation (5.1.1), for $1 \le i \le s$. Hence, the solutions of Equation (5.1.1) are the same as those of Equation (5.1.2). Now we consider the roots of $F(x)$ over the set of real numbers. Note that $F(x)$ is discontinuous at each point $-p_i$. For $1 \le i \le s$, we have that $F(-p_i-0) = -\infty$, $F(-p_i+0) = +\infty$, $F(-\infty) = F(+\infty) = -1$, $F'(x) = -\sum_{i=1}^{s}\frac{a_i p_i}{(x+p_i)^2}$. We deduce that $F(x)$ is strictly monotone decreasing on each of the intervals where $F(x)$ is continuous. Using the Weierstrass Intermediate Value Theorem of Analysis, we deduce that $F(x)$ has s distinct real roots, $-\infty < u_s < u_{s-1} < \ldots < u_1 < +\infty$ of satisfying

$$-p_s < u_s < -p_{s-1} < u_{s-1} < \ldots < -p_2 < u_2 < -p_1 < 0 < u_1 < +\infty. \quad (5.1.3)$$

On the other hand, Equation (5.1.2) can be written as

$$\frac{a_1p_1}{x+p_1} + \frac{a_2p_2}{x+p_2} + \cdots + \frac{a_sp_s}{x+p_s} = 1. \quad (5.1.4)$$

From the above discussion, we have

Theorem 5.1.5. *The complete r-partite graph* $K_{p_1,p_2,\cdots,p_r}$=$K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ *is integral if and only if all the solutions of Equation* (5.1.4) *are integers, i.e. there exist integers* $u_1, u_2, \cdots, u_s$ *satisfying* (5.1.3) *such that the following linear system in* $a_1, a_2, \cdots, a_s$

$$\begin{cases} \frac{a_1p_1}{u_1+p_1} + \frac{a_2p_2}{u_1+p_2} + \cdots + \frac{a_sp_s}{u_1+p_s} = 1 \\ \cdots\cdots\cdots \quad \cdots\cdots\cdots \\ \frac{a_1p_1}{u_s+p_1} + \frac{a_2p_2}{u_s+p_2} + \cdots + \frac{a_sp_s}{u_s+p_s} = 1 \end{cases} \quad (5.1.5)$$

has positive integral solutions $(a_1, a_2, \cdots, a_s)$.

Theorem 5.1.6. *The complete r-partite graph* $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ *on n vertices is integral if and only if there exist integers* u_i *and positive integers* p_i $(i = 1, 2, \cdots, s)$ *such that* (5.1.3) *holds and*

$$a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i=1,i\neq k}^{s}(p_k - p_i)}, \quad (k = 1, 2, \cdots, s) \quad (5.1.6)$$

are positive integers.

Proof. From Cauchy's result on determinants in [11], we deduce

$$\begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \cdots & \frac{1}{\alpha_1+\beta_s} \\ \vdots & \ddots & \vdots \\ \frac{1}{\alpha_s+\beta_1} & \cdots & \frac{1}{\alpha_s+\beta_s} \end{vmatrix} = \frac{\prod_{1\leq i<j\leq s}(\alpha_j - \alpha_i)(\beta_j - \beta_i)}{\prod_{1\leq i,j\leq s}(\alpha_i + \beta_j)}.$$

The determinant of the coefficient matrix D of the linear system (5.1.5) is the following:

$$|D| = \begin{vmatrix} \frac{p_1}{u_1+p_1} & \cdots & \frac{p_s}{u_1+p_s} \\ \vdots & \ddots & \vdots \\ \frac{p_1}{u_s+p_1} & \cdots & \frac{p_s}{u_s+p_s} \end{vmatrix} = \prod_{i=1}^{s} p_i \begin{vmatrix} \frac{1}{u_1+p_1} & \cdots & \frac{1}{u_1+p_s} \\ \vdots & \ddots & \vdots \\ \frac{1}{u_s+p_1} & \cdots & \frac{1}{u_s+p_s} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{s} p_i \prod_{1\le i<j\le s} (u_j - u_i)(p_j - p_i)}{\prod_{1\le i,j\le s} (u_i + p_j)} \neq 0.$$

$$|D_k| = \begin{vmatrix} \frac{p_1}{u_1+p_1} & \frac{p_2}{u_1+p_2} & \cdots & \frac{p_{k-1}}{u_1+p_{k-1}} & 1 & \frac{p_{k+1}}{u_1+p_{k+1}} & \cdots & \frac{p_{s-1}}{u_1+p_{s-1}} & \frac{p_s}{u_1+p_s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{p_1}{u_s+p_1} & \frac{p_2}{u_s+p_2} & \cdots & \frac{p_{k-1}}{u_s+p_{k-1}} & 1 & \frac{p_{k+1}}{u_s+p_{k+1}} & \cdots & \frac{p_{s-1}}{u_s+p_{s-1}} & \frac{p_s}{u_s+p_s} \end{vmatrix}$$

$$= lim_{p_k\to+\infty}|D| = \frac{\prod_{i=1,i\neq k}^{s} p_i \prod_{1\le i<j\le s} (u_j - u_i)(p_j - p_i)}{\prod_{1\le i,j\le s,j\neq k} (u_i + p_j) \prod_{i=1,i\neq k}^{s} (p_k - p_i)},$$

for $k = 1, 2, \cdots, s$.

By using the well-known Cramer Rule to solve the linear system (5.1.5) in $a_1, a_2, \cdots, a_s$, we obtain

$$a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i=1,i\neq k}^{s}(p_k - p_i)}, \quad (k = 1, 2, \cdots, s).$$

Because $-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and $p_i \ge 1$ for $i = 1, 2, \cdots, s$, we can deduce that $a_k > 0$ $(k = 1, 2, \cdots, s)$ and $u_1 > 0$.

So the result follows from Theorem 5.1.5. ■

Corollary 5.1.7. *If the complete r-partite graph $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,\cdots,a_s\cdot p_s}$ on n vertices is integral, then the following holds.*

(1) $a_k > 0$ *for* $k = 1, 2, \cdots, s$, *and* $u_1 > 0$.

(2) $\sum_{i=1}^{s} u_i = \sum_{i=1}^{s}(a_i - 1)p_i$.

(3) $\prod_{i=1}^{s}(-u_i) = \prod_{i=1}^{s} p_i(1 - \sum_{i=1}^{s} a_i)$.

(4) $Spec(K_{a_1\cdot p_1, a_2\cdot p_2,\cdots,a_s\cdot p_s}) =$

$$\begin{pmatrix} -p_s & u_s & -p_{s-1} & u_{s-1} & \cdots & u_2 & -p_1 & 0 & u_1 \\ a_s-1 & 1 & a_{s-1}-1 & 1 & \cdots & 1 & a_1-1 & \sum\limits_{i=1}^{s} a_i(p_i-1) & 1 \end{pmatrix}.$$

Proof. The result in (1) has been shown before. By Corollary 5.1.3, we have

$P(K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}, x)$

$= x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}[\prod_{i=1}^{s}(x+p_i) - \sum_{j=1}^{s} a_j p_j \prod_{i=1,i\neq j}^{s}(x+p_i)]$
$= x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}\{x^s - [\sum_{i=1}^{s}(a_i-1)p_i]x^{s-1} + \ldots$
$+ \prod_{i=1}^{s} p_i(1-\sum_{i=1}^{s} a_i)\}$
$= x^{n-r}\prod_{i=1}^{s}(x+p_i)^{a_i-1}\prod_{i=1}^{s}(x-u_i).$

By using the relation between roots and coefficients of polynomials, and the inequality (5.1.3), we obtain the results in (2)-(4). ■

Lemma 5.1.8. *Define*

$$\Psi_{\vec{a},\vec{p}}(x) = \sum_{i=1}^{s}\frac{a_i p_i}{x+p_i}, \quad \Phi_{\vec{a},\vec{p}}(x) = (\prod_{i=1}^{s}(x+p_i))(1-\Psi_{\vec{a},\vec{p}}(x)).$$

with vectors $\vec{a} = (a_1, a_2, \ldots, a_s)$, $\vec{p} = (p_1, p_2, \ldots, p_s) \in \mathbb{Z}^s$. *Let q be a non-zero integer. Then u is an integral root of $\Phi_{\vec{a},q\vec{p}}(x)$ if and only if u/q is an integral root of $\Phi_{\vec{a},\vec{p}}(x)$.*

Proof. It is easy to see that v is a root of $\Phi_{\vec{a},\vec{p}}(x)$ if and only if qv is a root of $\Phi_{\vec{a},q\vec{p}}(x)$. Therefore if all the roots of $\Phi_{\vec{a},\vec{p}}(x)$ are integers, then the roots of $\Phi_{\vec{a},q\vec{p}}(x)$ are integers as well.

Assume now that v is an integral root of $\Phi_{\vec{a},q\vec{p}}(x)$. Then v/q is a rational root of $\Phi_{\vec{a},\vec{p}}(x)$. Since $\Phi_{\vec{a},\vec{p}}(x)$ is a monomial polynomial with integral coefficients, its rational roots should be integers. Therefore $v/q \in \mathbb{Z}$. ■

From the above lemma we can obtain the following result.

Theorem 5.1.9. *For any positive integer q, the complete r-partite graph $K_{a_1\cdot p_1 q,\ a_2\cdot p_2 q,\ \cdots,\ a_s\cdot p_s q}$ is integral if and only if the complete r-partite graph $K_{p_1,p_2,\ \cdots,\ p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ is integral.*

Remark 5.1.10. *The above Theorem 5.1.9 shows that we have to study Equation* (5.1.2) *only for the case $(p_1, p_2, \cdots, p_s) = 1$. Let us call such a vector $\vec{p}$ primitive.*

5.2 Integral complete r-partite graphs

In this section, we shall construct infinitely many new classes of integral complete r-partite graphs $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, differing from those of [21, 22, 60]. The idea is as follows:

First, we properly choose positive integers $p_1, p_2, \cdots, p_s$. Then, we try to find integers u_i $(i = 1, 2, \cdots, s)$ satisfying (5.1.3) such that there are positive integral solutions $(a_1, a_2, \cdots, a_s)$ for the linear system (5.1.5) (or such that all a_k's of (5.1.6) are positive integers). Finally, we obtain positive integers $a_1, a_2, \cdots, a_s$ such that all solutions of Equation (5.1.4) are integers. In this way we construct many new classes of integral graphs $K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$.

Example 5.2.1. *Let $p_1 = 1, p_2 = 9$ and $u_2 = -4$. If $u_1 = 72t - 9$ and t is a positive integer, then $K_{a_1\cdot p_1,a_2\cdot p_2}$ is integral for the infinitely many positive integers a_1, a_2 given by* (5.2.1) *and* (5.2.2).

Proof. From Theorem 5.1.6 we find

$$a_1 = \frac{(p_1 + u_1)(p_1 + u_2)}{p_1(p_1 - p_2)} = \frac{3}{8}(u_1 + 1) \tag{5.2.1}$$

and

$$a_2 = \frac{(p_2 + u_1)(p_2 + u_2)}{p_2(p_2 - p_1)} = \frac{5}{72}(u_1 + 9). \tag{5.2.2}$$

So, $K_{a_1\cdot p_1,a_2\cdot p_2}$ is integral if and only if a_1 and a_2 are positive integers. From (5.2.1) and (5.2.2), we get the Diophantine equation

$$27a_2 - 5a_1 = 15. \tag{5.2.3}$$

A result in elementary number theory yields that all positive integral solutions of Equation (5.2.3) are given by $a_1 = 27t - 3, a_2 = 5t$, implying $u_1 = 72t - 9$, where t is a positive integer.

Hence, $K_{p_1,p_2,p_3,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2}$ is integral for the above infinitely many integers a_1 and a_2. ■

Theorem 5.2.2. *Let $p_1 < p_2$. Then $K_{a_1\cdot p_1,a_2\cdot p_2}$ is integral if and only if one of the following two conditions holds:*

(1) For $(m, k) = 1$, let $p_1 = m, p_2 = m + k$, $m \geq 1$, $k \geq 2$, $1 \leq q < k$, where m, k and q are positive integers. Then, a_1 and a_2 must be positive integral solutions for the Diophantine equation

$$q(m + k)a_2 - m(k - q)a_1 = q(k - q). \tag{5.2.4}$$

(2) *For* $(m,k) = d \geq 2$, *let* $p_1 = m, p_2 = m+k$, $m = m_1 d$, $k = k_1 d$, $(m_1, k_1) = 1$, $q = q_1 d$, $1 \leq q_1 < k_1$, m_1, k_1, *where* q_1 *and* d *are positive integers. Then,* a_1 *and* a_2 *must be positive integral solutions for the Diophantine equation*

$$q_1(m_1 + k_1)a_2 - m_1(k_1 - q_1)a_1 = q_1(k_1 - q_1). \tag{5.2.5}$$

Proof. Because $p_1 < p_2$, from Theorem 5.1.6, we know that $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is integral if and only if there exist integers u_1, u_2 and positive integers p_1, p_2 satisfying $-p_2 < u_2 < -p_1 < u_1 < +\infty$ such that a_1 and a_2 are positive integers, where $a_k = \frac{\prod_{i=1}^{2}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{2}(p_k - p_i)}$ for $k = 1, 2$. Hence, we choose $p_1 = m, p_2 = m+k$, $u_2 = -(m+q)$, $m \geq 1$, $k \geq 2$, $1 \leq q < k$, where m, k and q are positive integers, and we obtain

$$a_1 = \frac{q(m+u_1)}{mk}, \quad a_2 = \frac{(m+k+u_1)(k-q)}{k(m+k)}.$$

Hence, we get Equation (5.2.4). From Lemma 2.1.17, we know that there are solutions for Equation (5.2.4) if and only if $d_1 | q(k-q)$, where $d_1 = (q(m+k), m(k-q))$.

Now, we discuss the following two cases.

Case 1. When $(m,k) = 1$, we have $(m+k, m) = 1$, and $d_1 | q(k-q)$. From Lemma 2.1.17 and the condition $(m,k) = 1$, we know that there are infinitely many integral solutions for Equation (5.2.4). Therefore, there are infinitely many positive integral solutions (a_1, a_2) for Equation (5.2.4).

Case 2. When $(m,k) = d \geq 2$, let $m = m_1 d$, $k = k_1 d$, $(m_1, k_1) = 1$, where m_1, k_1 and d are positive integers. We have $(m_1 + k_1, m_1) = 1$, $d_1 = (qd(m_1 + k_1), m_1 d(k_1 d - q))$. If $d_1 | q(k-q) = q(k_1 d - q)$, then $d | q$. Thus, let $q = q_1 d$, $1 \leq q_1 < k_1$, where q_1 is a positive integer. We can reduce Equation (5.2.4) to Equation (5.2.5). Hence, from Lemma 2.1.17 and the condition $(m_1, k_1) = 1$, we know that there are infinitely many integral solutions for Equation (5.2.5). Therefore, there are infinitely many positive integral solutions (a_1, a_2) for Equation (5.2.5). ■

Example 5.2.3.

(1) *For* $s = 3$, *let* $p_1 = 1$, $p_2 = 5$, $p_3 = 9$, $u_2 = -3$ *and* $u_3 = -7$. *If* $u_1 = 120t - 105$, t *is a positive integer, then* $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ *is integral for the infinitely many positive integers* a_1, a_2 *and* a_3 *given by* (5.2.6), (5.2.7) *and* (5.2.8).

(2) *For any positive integer q, let $p'_i = p_i q$ and $u'_i = u_i q$ for $i = 1, 2, 3$, where p_i, u_i and a_i $(i = 1, 2, 3)$ are the same as in (1). Then $K_{a_1 \cdot p'_1, a_2 \cdot p'_2, a_3 \cdot p'_3} = K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q}$ is integral too.*

Proof. (1). From Theorem 5.1.6, we have

$$a_1 = \frac{3}{8}(u_1 + 1), \tag{5.2.6}$$

$$a_2 = \frac{1}{20}(u_1 + 5), \tag{5.2.7}$$

$$a_3 = \frac{1}{24}(u_1 + 9). \tag{5.2.8}$$

So, $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral if and only if a_1, a_2 and a_3 are positive integers. By (5.2.7) and (5.2.8), we get the Diophantine equation

$$6a_3 - 5a_2 = 1. \tag{5.2.9}$$

From elementary number theory we learn that all positive integral solutions of Equation (5.2.9) are given by $a_2 = 6t - 5, a_3 = 5t - 4$, where t is a positive integer. From (5.2.6) and (5.2.7), we have $u_1 = 120t - 105$, $a_1 = 45t - 39$.

Hence, when $p_1 = 1, p_2 = 5, p_3 = 9, a_1 = 45t - 39$, $a_2 = 6t - 5, a_3 = 5t - 4$, where t is a positive integer, then the graph $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral.

(2). From Theorem 5.1.9 and Example 5.2.3 (1), it is easy to prove that for any positive integer q the graph $K_{a_1 \cdot p'_1, a_2 \cdot p'_2, a_3 \cdot p'_3} = K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q}$ is integral too. ■

Example 5.2.4.

(1) *(See [60]) For $s = 3$, let $a_1 = a_2 = a_3 = 1$, $p_1 = 4u^2(u^2 + v^2)^3$, $p_2 = 3u^2 v^2 (u^2 + 6uv + v^2)(-u^2 + 6uv - v^2)$, $p_3 = 4v^2(u^2 + v^2)^3$, where u, v are positive integers such that $(3 - \sqrt{8})v < u < v$. Then the graph $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral.*

(2) *For any positive integer q, if $s = 3$, and a_i, p_i $(i = 1, 2, 3)$ are as in (1), then the graph $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q} = K_{p_1 q, p_2 q, p_3 q}$ is integral too.*

Proof. (1). (See [60]) We show first that p_1, p_2 and p_3 are distinct positive integers. In fact, the condition $0 < (3 - \sqrt{8})v < u < v$, ensures $-u^2 + 6uv - v^2 > 0$, $p_3 > 0$ and $p_1 < p_3$. Assume that $p_1 = p_2$ and let $u = du_0$, $v = dv_0$, where $(u_0, v_0) = 1$. Then,

$$4u_0^2(u_0^2+v_0^2)^3 = 3u_0^2v_0^2(34u_0^2v_0^2 - u_0^4 - v_0^4)$$

and so

$$u_0^2+v_0^2 \equiv 0(mod3) \Rightarrow u_0 \equiv v_0 \equiv 0(mod3),$$

contradicting $(u_0, v_0) = 1$. Therefore $p_1 \neq p_2$. Similarly $p_3 \neq p_2$.

By Corollary 5.1.3, we have
$P(K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3}, x)$

$$= x^{p_1+p_2+p_3-3}[\prod_{i=1}^{3}(x+p_i) - \sum_{j=1}^{3} p_j \prod_{i=1, i\neq j}^{3}(x+p_i)]$$
$$= x^{-3+4(u^2+v^2)^4+3u^2v^2(-u^2+6uv-v^2)(u^2+6uv+v^2)}(x-u_1)(x-u_2)(x-u_3),$$

where $u_1 = 24u^2v^2(u^2+v^2)^2$, $u_2 = -2uv(u^2+v^2)^2(-u^2+6uv-v^2)$, $u_3 = -2uv(u^2+v^2)^2(u^2+6uv+v^2)$. From Theorem 5.1.4, we conclude that $K_{a_1\cdot p_1, a_2\cdot p_2,\ a_3\cdot p_3} = K_{p_1,p_2,p_3}$ is integral.

(2). From Theorem 5.1.9 and (1), it is easy to see that the graph $K_{a_1\cdot p_1 q, a_2\cdot p_2 q,\ a_3\cdot p_3 q} = K_{p_1 q, p_2 q, p_3 q}$ is integral too. ■

Theorem 5.2.5. *For $s = 3$, let p_i $(i = 1, 2, 3)$ be positive integers as in Table 5.1, $a_1 = a_2 = a_3 = 1$, and let u_i $(i = 1, 2, 3)$ be as in Theorem 5.1.6. Then for any positive integer q the graph $K_{a_1\cdot p_1 q, a_2\cdot p_2 q, a_3\cdot p_3 q} = K_{p_1 q, p_2 q, p_3 q}$ is integral.*

Proof. Using a method similar to that in Example 5.2.4, the result follows from Theorems 5.1.4, 5.1.5 or 5.1.6 and 5.1.9. ■

Remark 5.2.6. *An infinite family of integral complete tripartite graphs K_{p_1,p_2,p_3} was constructed in [60]. In Table 5.1, by using a computer search, we have found 34 solutions $(p_1,\ p_2,\ p_3)$, where $p_1 < p_2 < p_3$, $1 \leq p_1 \leq 50$, $p_1+1 \leq p_2 \leq p_1+50$, and $p_2+1 \leq p_3 \leq p_2+100$. We shall construct infinitely many classes of such integral graphs from Theorems 5.1.4, 5.1.5, 5.1.6 and 5.1.9. They are different from those in the literature (see [21, 22, 60]). For any positive integer q, the complete tripartite graph $K_{5q,8q,12q}$ is integral and*

$$Spec(K_{5q,8q,12q}) = \begin{pmatrix} -10q & -6q & 0 & 16q \\ 1 & 1 & 25q-3 & 1 \end{pmatrix}.$$

According to Table 5.1, the complete tripartite graph $K_{5,8,12}$ is integral and of order 25, which is much smaller than the order of the graphs given in [21, 22, 60].

p_1	p_2	p_3	u_1	u_2	u_3	p_1	p_2	p_3	u_1	u_2	u_3
3	17	65	39	-5	-34	4	13	48	32	-6	-26
5	8	12	16	-6	-10	5	12	77	40	-7	-33
6	34	130	78	-10	-68	7	13	45	35	-9	-26
8	26	96	64	-12	-52	9	25	91	63	-13	-50
10	16	24	32	-12	-20	12	17	56	48	-14	-34
12	25	88	66	-16	-50	13	24	28	42	-16	-26
14	26	90	70	-18	-52	15	24	36	48	-18	-30
15	37	133	95	-21	-74	16	37	132	96	-22	-74
17	33	35	55	-21	-34	20	32	48	64	-24	-40
21	39	135	105	-27	-78	24	34	112	96	-28	-68
24	41	140	112	-30	-82	25	40	60	80	-30	-50
26	48	56	84	-32	-52	29	36	80	90	-32	-58
29	39	77	91	-33	-58	30	48	72	96	-36	-60
34	66	70	110	-42	-68	35	56	84	112	-42	-70
37	63	85	119	-45	-74	39	72	84	126	-48	-78
40	64	96	128	-48	-80	41	60	104	130	-48	-82
45	72	108	144	-54	-90	50	80	120	160	-60	-100

Table 5.1: Integral complete tripartite graphs K_{p_1q,p_2q,p_3q}, where q is any positive integer.

Remark 5.2.7. *Theorem 5.2.5 generates an infinite set of vectors* $\vec{p} = (p_1, p_2, p_3)$ *for which Equation* (5.1.2) *has only integral solutions. But there is only a finite number of primitive vectors in this infinite set (in general, the primitive vectors are the only ones which are of interest). The infinite series built in [60] gives an infinite series of primitive solutions. Thus Theorem 5.2.5 is much weaker than the result of [60]. However, by analyzing Table 5.1 one can see that all its rows except the row* $\vec{p} = (5, 12, 77)$ *have the following property:* $u_3/p_i = -2$ *for a suitable* $i \in \{1, 2, 3\}$*. This observation gives a hint to the construction of a new infinite series of primitive triples* $\vec{p}$*, as will be seen in the following.*

Let $u_3 < u_2 < u_1$ be the roots of $F(x) = F_{\vec{p}}(x) = 0$ (*cf.* (5.1.2)). Set $v_3 = -u_3$, $v_2 = -u_2$, $v_1 = u_1$. Then the v_i's are positive integers which satisfy the following conditions (see Corollary 5.1.7, with $a_i = 1$, $i = 1, 2, 3$):

$$\begin{aligned} v_1 &= v_2 + v_3, \\ v_2^2 + v_2 v_3 + v_3^2 &= p_1 p_2 + p_1 p_3 + p_2 p_3, \\ (v_2 + v_3) v_2 v_3 &= 2 p_1 p_2 p_3. \end{aligned} \tag{5.2.10}$$

Let us look for solutions of (5.2.10) such that $v_3 = 2p_i$, for some i=1,2,3. Changing the ordering of the p_i's we may assume $v_3 = 2p_3$. Then (5.2.10) is equivalent to

$$\begin{aligned} p_1 + p_2 &= 2v_3, \\ p_1 p_2 &= (v_2 + v_3) v_2. \end{aligned} \tag{5.2.11}$$

These equations have integral solutions for p_1, p_2 if and only if $v_3^2 - (v_2 + v_3)v_2$ is a perfect square, say m^2. Then $p_1 = v_3 - m$, $p_2 = v_3 + m$, $p_3 = v_3/2$.

$$v_3^2 - (v_2 + v_3)v_2 = m^2 \Longleftrightarrow$$

$$\Longleftrightarrow x^2 - xy - y^2 = 1, \quad \text{where } x = \frac{v_3}{m}, \quad y = \frac{v_2}{m}, \Longleftrightarrow$$

$$\Longleftrightarrow x = \frac{t^2 + 1}{t^2 + t - 1}, \quad y = t(x - 1), \quad t \in \mathbb{Q}.$$

We may assume that $m > 0$. It follows from $p_1 = m(x-1) > 0$ and $v_2 > 0$ that $x > 1$ and $t > 0$. The first inequality is equivalent to

$$\frac{2 - t}{t^2 + t - 1} > 0 \Longleftrightarrow \frac{\sqrt{5} - 1}{2} = 0.618.. < t < 2 \text{ or } t < \frac{-\sqrt{5} - 1}{2} = -1.618..$$

Thus $p_1 = v_3 - m = m(x - 1)$, $p_2 = v_3 + m = m(x + 1)$, $p_3 = v_3/2 = mx/2$ are non-negative.

Writing $t = a/b$ where $(a, b) = 1$ and $a > 0$, $b > 0$ we obtain

$$\frac{v_3}{m} = x = \frac{a^2 + b^2}{a^2 + ab - b^2}, \quad \frac{v_2}{m} = y = \frac{2ab - a^2}{a^2 + ab - b^2}.$$

After some routine transformations we find

$$\begin{aligned} p_1 &= \frac{2b(2b - a)}{d}, \quad p_2 = \frac{2a(2a + b)}{d}, \quad p_3 = \frac{a^2 + b^2}{d}, \\ u_1 &= \frac{2b(b + 2a)}{d}, \quad u_2 = -\frac{2a(2b - a)}{d}, \quad u_3 = -\frac{2(a^2 + b^2)}{d}, \end{aligned} \tag{5.2.12}$$

where $d = (2b(2b - a), 2a(2a + b), a^2 + b^2)$. Note that $d \in \{1, 2, 5, 10\}$.

Take for example $a = 2$, $b = 3$. Then $d = 1$ and

$p_1 = 24, \quad p_2 = 28, \quad p_3 = 13, \quad u_1 = 42, \quad u_2 = -16, \quad u_3 = -26.$

This is one of the triples given in Table 5.1. All triples given in this table (up to numbering) except $(5, 12, 77)$ may be obtained from (5.2.12).

We come to the following result.

Theorem 5.2.8. *For $s = 3$, let p_i and u_i $(i = 1, 2, 3)$ be positive integers given by formulae* (5.2.12), *and let $a_1 = a_2 = a_3 = 1$. Then for any positive integer q the graph $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q} = K_{p_1 q, p_2 q, p_3 q}$ is integral.*

Example 5.2.9. *For any positive integer q, if $s = 3$, let $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$, then there do not exist positive integers a_1, a_2, a_3 such that $K_{p_1, p_2, \cdots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral.*

Proof. Let $s = 3$, $p_1 = q$, $p_2 = 3q$, $p_3 = 5q$, $u_2 = -2q$, $u_3 = -4q$. From Theorem 5.1.6, we know that $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ is integral if and only if there exist integers u_i and positive integers p_i $(i = 1, 2, 3)$ satisfying $-p_3 < u_3 < -p_2 < u_2 < -p_1 < u_1 < +\infty$ such that $a_k = \frac{\prod_{i=1}^{3}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{3}(p_k - p_i)}$ $(k = 1, 2, 3)$ are positive integers.

Hence, we obtain

$$a_1 = \frac{3}{8q}(u_1 + q), \tag{5.2.13}$$

$$a_2 = \frac{1}{12q}(u_1 + 3q), \tag{5.2.14}$$

$$a_3 = \frac{3}{40q}(u_1 + 5q). \tag{5.2.15}$$

By (5.2.14) and (5.2.15), it follows

$$20a_3 - 18a_2 = 3. \tag{5.2.16}$$

By Lemma 2.1.17, there are no integral solutions for Equation (5.2.16) and so the graph $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3}$ cannot be integral. ■

Theorem 5.2.10. *For the complete r-partite graph $K_{p_1, p_2, \cdots, p_r} = K_{a_1 \cdot p_1,\ a_2 \cdot p_2,\ \cdots,\ a_s \cdot p_s}$ on n vertices, let m, s and q be positive integers, and $s \geq 3$, then we have*

(1) If $p_i = m + 2(i-1)$ for $i = 1, 2, \cdots, s$, then no integers a_i $(i = 1, 2, \cdots, s)$ exist such that $K_{a_1 \cdot p_1, a_2 \cdot p_2, \cdots, a_s \cdot p_s}$ is an integral graph.

(2) If $p'_i = p_i q = [m + 2(i-1)]q$ for $i = 1, 2, \cdots, s$, let $u'_j = -(m + 2j - 3)q$ for $j = 2, 3, \cdots, s$, then no integers a'_i $(i = 1, 2, \cdots, s)$ exist such that $K_{a'_1 \cdot p'_1, a'_2 \cdot p'_2, \cdots, a'_s \cdot p'_s}$ is an integral graph.

Proof. (1). From Theorem 5.1.6, we know that $K_{a_1 \cdot p_1, a_2 \cdot p_2, \cdots, a_s \cdot p_s}$ is integral if and only if there exist integers u_i and positive integers p_i $(i = 1, 2, \cdots, s)$ satisfying $-p_s < u_s < -p_{s-1} < u_{s-1} < \cdots < u_2 < -p_1 < u_1 < +\infty$ such that all a_k $(k = 1, 2, \cdots, s)$ are positive integers, where $a_k = \frac{\prod_{i=1}^{s}(p_k + u_i)}{p_k \prod_{i=1, i \neq k}^{s}(p_k - p_i)}$. Hence, we can only choose

$u_j = -(m + 2j - 3), \quad j = 2, 3, \cdots, s.$

We obtain

$$a_{s-1} = \frac{(m + 2s - 4 + u_1) \cdot (2s-5)!!}{2(m + 2s - 4) \cdot (2s-4)!!}, \tag{5.2.17}$$

$$a_s = \frac{(m + 2s - 2 + u_1) \cdot (2s-3)!!}{(m + 2s - 2) \cdot (2s-2)!!}, \tag{5.2.18}$$

From (5.2.17) and (5.2.18), it follows

$$(m+2s-2) \cdot (2s-2)!! \cdot a_s - 2(m+2s-4)(2s-3) \cdot (2s-4)!! \cdot a_{s-1} = 2 \cdot (2s-3)!!. \tag{5.2.19}$$

Since $s \geq 3$, let $d = ((m+2s-2+u_1) \cdot (2s-2)!!, 2(m+2s-4)(2s-3) \cdot (2s-4)!!)$, then $d = 2 \cdot (2s-4)!! \cdot ((m + 2s - 2 + u_1)(s-1), (m + 2s - 4)(2s-3))$. Thus, $d \not| [2 \cdot (2s-3)!!]$. From Lemma 2.1.17, we find that there are no integral solutions (a_{s-1}, a_s) for Equation (5.2.19). Hence, $K_{a_1 \cdot p_1, a_2 \cdot p_2, \cdots, a_s \cdot p_s}$ cannot be an integral graph.

(2). From Theorem 5.1.6, we similarly obtain

$$a'_{s-1} = \frac{[q(m + 2s - 4) + u'_1] \cdot (2s-5)!!}{2q(m + 2s - 4) \cdot (2s-4)!!}, \tag{5.2.20}$$

$$a'_s = \frac{[q(m + 2s - 2) + u'_1] \cdot (2s-3)!!}{q(m + 2s - 2) \cdot (2s-2)!!}. \tag{5.2.21}$$

From (5.2.20) and (5.2.21), we find

$$(m+2s-2) \cdot (2s-2)!! \cdot a'_s - 2(m+2s-4)(2s-3) \cdot (2s-4)!! \cdot a'_{s-1} = 2 \cdot (2s-3)!!. \tag{5.2.22}$$

Lemma 2.1.17 implies that there are no integral solutions (a'_{s-1}, a'_s) for Equation (5.2.22). Hence, $K_{a'_1 \cdot p'_1 q, a'_2 \cdot p'_2, \cdots, a'_s \cdot p'_s}$ cannot be an integral graph. ■

5.3 Further discussion

In this chapter, we have mainly investigated integral complete r-partite graphs $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,\ a_2\cdot p_2,\cdots,\ a_s\cdot p_s}$ on n vertices. For $s = 1, 2, 3$, some results can be found in [21, 22, 60] and in the present chapter. For $s \geq 4$, we have not found such integral graphs. Thus, we raise the following questions for further study.

Question 5.3.1. *Are there integral complete r-partite graphs $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ with arbitrarily large s?*

For complete r-partite graphs $K_{p_1,p_2,\cdots,p_r}=K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, when $s = 1, 2, 3$, and $a_1=a_2 = \cdots = a_s = 1$, then some results for such integral graphs can be found in [21, 22, 60] and in the present chapter. However, for $s \geq 4$, $a_1 = a_2 = \cdots = a_s = 1$, we have not found such integral graphs. Hence, we ask

Question 5.3.2. *Are there integral complete r-partite graphs $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, with $a_1= a_2 = \cdots = a_s = 1$, and $s \geq 4$?*

For complete r-partite graphs $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$, we gave a sufficient and necessary condition for $K_{p_1,p_2,\cdots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ to be integral. In particular, when $s = 1, 2$, we have found all parameters a_i, p_i for $K_{a_1\cdot p_1,a_2\cdot p_2\cdots,a_s\cdot p_s}$ to be an integral graph in [21, 22] and in the present chapter. When $s \geq 3$, we have not obtained such general results. Hence, we ask

Question 5.3.3. *When $s = 3, 4, 5, \cdots$, can we give a better sufficient and necessary condition for $K_{a_1\cdot p_1,a_2\cdot p_2,\cdots,a_s\cdot p_s}$ to be integral ?*

Chapter 6

Integral nonregular bipartite graphs

In this chapter, we shall construct fifteen classes of integral graphs from the known 21 integral graphs (see also [5]). These classes consist of nonregular bipartite graphs. Their spectra and characteristic polynomials are obtained from matrix theory. Their integral property is analyzed by using number theory and computer search. All these classes are infinite and different from those in the literature. It is proved that the problem of finding such integral graphs is equivalent to solving Diophantine equations. These results generalize results of Balińska and Simić (see also [5], MR1830594 (2002a:05171)). Finally, we propose several open problems for further study.

6.1 The characteristic polynomials of some classes of graphs

In this section, we investigate the structures of the nonregular bipartite integral graphs in [5]. Fifteen new classes of larger graphs are constructed based on the structures of the 21 smaller integral graphs in [5].

The following Theorem 6.1.1 can be found in [5].

Theorem 6.1.1. *([5]) The graphs in Figures 6.1 and 6.2 are nonregular bipartite integral graphs with maximum degree four. (The graphs in Figure 6.1 are integral graphs with number of vertices up to 16.)*

We can generalize the result of Theorem 6.1.1 and construct fifteen types of graphs. The following Theorem 6.1.2 on their characteristic polynomials is

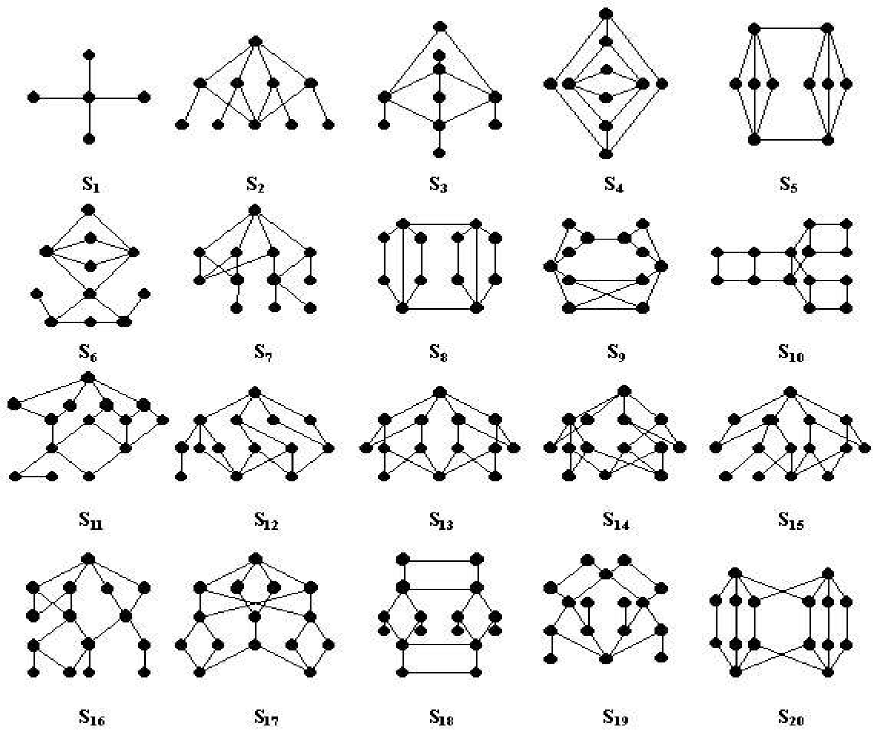


Figure 6.1: Nonregular bipartite integral graphs with maximum degree four and at most 16 vertices.

obtained from matrix theory.

Theorem 6.1.2. *Let m, n, p, q and t be nonnegative integers. Then the characteristic polynomials of the fifteen types of graphs in Figures 6.3 and 6.4 are as follows:*

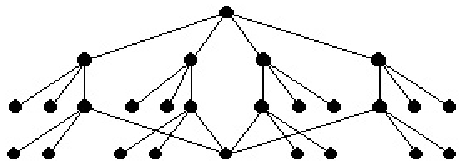
(1) *(see [22] or [47])* $P(K_{1,t}, x) = x^{t-1}(x^2 - t)$, $(t \geq 0)$.

(2) $P(S_2(n,t), x) = x^{n(t-1)+2}(x^2 - t)^{n-1}[x^2 - (2n + t)]$, $(n \geq 1,\ t \geq 0)$.

(3) $P(S_3(m,n,t), x) = x^{m+n+4(t-1)}(x^2 - t)^2[x^4 - 2(m+n+t+2)x^2 + (2m + t)(2n + t)]$, $(m \geq 0,\ n \geq 1,\ t \geq 0$ *or* $m \geq 1,\ n \geq 0,\ t \geq 0)$.

(3.1) $P(S_3(n,n,t), x) = x^{2n+4(t-1)}(x^2 - t)^2(x^2 + 2x - 2n - t)(x^2 + 2x - 2n - t)$, $(n \geq 1,\ t \geq 0)$.

(3.2) $P(S_3(m,n,0), x) = x^{m+n}[x^4 - 2(m + n + 2)x^2 + 4mn]$, $(m \geq 1,\ n \geq 1)$.



S_{21}

Figure 6.2: A nonregular bipartite integral graph with maximum degree four and 26 vertices.

(4) $P(S_4(m,n,p,q),x) = x^{mp+n+2q-2}(x^2-2m)^{p-1}(x^2-pq)[x^4-(2m+2n+4q+pq)x^2+4mn+8mq+2npq]$, $(m \geq 0,\ n \geq 0,\ p \geq 1,\ q \geq 1)$.

(4.1) $P(S_4(n,n,p,q),x) = x^{n(p+1)+2q-2}(x^2-2n)^p(x^2-pq)\ [x^2-(2n+4q+pq)]$, $(n \geq 0,\ p \geq 1,\ q \geq 1)$.

(4.2) $P(S_4(n,n,p,p),x) = x^{n(p+1)+2p-2}(x^2-2n)^p(x+p)(x-p)\ [x^2-(2n+p^2+4p)]$, $(n \geq 1,\ p \geq 1)$.

(4.3) $P(S_4(2,2,p,p),x) = x^{4p}(x+p+2)(x+p)(x+2)^p(x-2)^p(x-p)(x-p-2)$, $(p \geq 1)$.

(4.4) $P(S_4(0,n,p,q),x) = x^{n+2p+2q-4}(x^2-pq)[x^4-(2n+4q+pq)x^2+2npq]$, $(n \geq 0,\ p \geq 1,\ q \geq 1)$.

(4.5) $P(S_4(m,0,p,q),x) = x^{mp+2q-2}(x^2-2m)^{p-1}(x^2-pq)[x^4-(2m+4q+pq)x^2+8mq]$, $(m \geq 0,\ p \geq 1,\ q \geq 1)$.

(5) $P(S_5(m,n),x) = x^{m+n-2}(x+1)(x-1)[x^4-(2m+2n+1)x^2-4mn]$, $(m \geq 0,\ n \geq 0)$.

(5.1) $P(S_5(n,n),x) = x^{2n-2}(x+1)(x-1)(x^2+x-2n)(x^2-x-2n)$, $(n \geq 0)$.

(5.2) $P(S_5(0,n),x) = P(S_5(n,0),x) = x^n(x+1)(x-1)[x^2-(2n+1)]$, $(n \geq 0)$.

(6) $P(S_6(m,n,t),x) = x^{n(t-1)+m+2}(x^2-t)^{n-1}[x^4-(2m+2n+t+2)x^2+2n(2m+1)+2t(m+1)]$, $(m \geq 0,\ n \geq 1,\ t \geq 0)$ *or* $(m \geq 0,\ n = t = 0)$.

(6.1) $P(S_6(0,n,t),x) = x^{n(t-1)+2}(x^2-t)^{n-1}[x^4-(2n+t+2)x^2+2n+2t]$, $(n \geq 1,\ t \geq 0)$.

(6.2) $P(S_6(m,0,0),x) = P(K_{2,m+1} \cup K_1,x) = x^{m+2}[x^2-(2m+2)]$, $(m \geq 1)$.

(6.3) $P(S_6(m,n,0),x) = x^{n+m}[x^4 - (2m+2n+2)x^2 + 2n(2m+1)]$, $(m \geq 0, n \geq 0)$.

(6.4) $P(S_6(m,1,t),x) = x^{m+t+1}[x^4 - (2m+t+4)x^2 + 2(2m+1) + 2t(m+1)]$, $(m \geq 0,\ t \geq 0)$.

(6.5) $P(S_6(n-1,n,1),x) = x^{n+1}(x+1)^{n-1}(x-1)^{n-1}(x^2+x-2n)(x^2-x-2n)$, $(n \geq 1)$.

(6.6) $P(S_6(n+1,n,1),x) = x^{n+3}(x+1)^{n-1}(x-1)^{n-1}(x^2+x-2n-2)(x^2-x-2n-2)$, $(n \geq 0)$.

(6.7) $P(S_6(n+1,n,9),x) = x^{9n+3}(x+3)^{n-1}(x-3)^{n-1}(x^2+x-2n-6)(x^2-x-2n-6)$, $(n \geq 1)$.

(7) $P(S_8(m,n),x) = (x+1)^{m+n-2}(x-1)^{m+n-2}[x^4 - 4x^3 - (m+n-5)x^2 + (2m+2n-2)x + mn - m - n][x^4 + 4x^3 - (m+n-5)x^2 - (2m+2n-2)x + mn - m - n]$, $(m \geq 0,\ n \geq 0)$.

(7.1) $P(S_8(n,n),x) = (x+1)^{2n-2}(x-1)^{2n-2}(x^2+x-n)(x^2-x-n)(x^2+3x-n+2)(x^2-3x-n+2)$, $(n \geq 0)$.

(7.2) $P(S_8(0,n),x) = P(S_8(n,0),x) = (x+1)^n(x-1)^n(x^2+2x-n)(x^2-2x-n)$, $(n \geq 0)$.

(8) $P(S_9(m,n,p,q),x) = x^{m+n+p+q-2}[x^6 - (2m+n+2p+q+nq+1)x^4 + (m+n+mn+p+4mp+2np+q+2mq+2nq+2mnq+pq+2npq)x^2 - (mp+np+2mnp+mq+nq+2mnq+2mpq+2npq+4mpq)]$, $(m \geq 1, n \geq 1\ p \geq 1,\ q \geq 1)$.

(8.1) $P(S_9(n,n,n,n),x) = x^{4n-2}(x^2-2n)^2(x+n+1)(x-n-1)$, $(n \geq 1)$.

(9) $P(S_{10}(n),x) = x^{2(n-1)}(x+2)^{n-1}(x+1)(x-1)(x-2)^{n-1}(x^2+2x-n)(x^2-2x-n)$, $(n \geq 0)$.

(10) $P(S_{13}(m,n),x) = x^2(x+1)^{n(m-1)}(x-1)^{n(m-1)}(x^2+x-m)^{n-1}[x^2+x-m(n+1)](x^2-x-m)^{n-1}[x^2-x-m(n+1)]$, $(m \geq 1,\ n \geq 1)$.

(11) $P(S_{17}(m,n,p,q),x) = x^{mq+p+n-1}(x^2-2m)^{q-1}\{x^6 - (2m+2n+p+q+pq+1)x^4 + [m(2+4n+2p+q+pq)+n+p+np+2nq+2pq+2npq+pq^2]x^2 - [2m(n+p+np+nq+pq+npq)+2npq(q+1)]\}$, $(m \geq 1,\ n \geq 1,\ p \geq 1,\ q \geq 1)$.

(11.1) $P(S_{17}(n,n,n,n),x) = x^{n^2+2n-1}(x+n+1)(x-n-1)(x^2-2n)^{n+1}$, $(n \geq 1)$.

(11.2) $P(S_{17}(n,n,p,q),x) = x^{nq+n+p-1}(x^2-2n)^q\{x^4-[2n+(p+1)(q+1)]x^2+(q+1)[n(p+1)+p(q+1)]\}$, $(n \geq 1,\ p \geq 1,\ q \geq 1)$.

(11.3) $P(S_{17}(n,n,1,q),x) = x^{nq+n}(x^2-2n)^q(x^2-q-1)(x^2-2n-q-1)$, $(n \geq 1,\ q \geq 1)$.

(11.4) $P(S_{17}(m,n,m,n),x) = x^{mn+m+n-1}(x^2-2m)^{n-1}(x^2-m-n)[x^4-(2m+2n+mn+1))+2m(n+1)^2]$, $(m \geq 1,\ n \geq 1)$.

(11.5) $P(S_{17}(2,n,2,n),x) = x^{3n+1}(x+2)^{n-1}(x-2)^{n-1}(x^2-n-2)(x^2+x-2n-2)(x^2-x-2n-2)$, $(n \geq 1)$.

(12) $P(S_{18}(n,p,q,t),x) = x^{2n(t-1)}(x+1)^{p+q-2}(x-1)^{p+q-2}(x^2-t)^{2(n-1)}[x^6-4x^5-(2n+p+q+t-6)x^4+(6n+2p+2q+4t-4)x^3-(6n+p-np+q-nq-pq+6t-pt-qt-1)x^2+(2n-np-nq+4t-2pt-2qt)x-t+pt+qt-pqt][x^6+4x^5-(2n+p+q+t-6)x^4-(6n+2p+2q+4t-4)x^3-(6n+p-np+q-nq-pq+6t-pt-qt-1)x^2-(2n-np-nq+4t-2pt-2qt)x-t+pt+qt-pqt]$, $(n \geq 1,\ p \geq 0,\ q \geq 0,\ t \geq 0)$.

(12.1) $P(S_{18}(n,p,p,t),x) = x^{2n(t-1)}(x+1)^{2p-2}(x-1)^{2p-2}(x^2-t)^{2(n-1)}[(x+1)^2-p][(x-1)^2-p][x^4-2x^3-(p+t+2n-1)x^2+2(n+t)x+t(p-1)][x^4+2x^3-(p+t+2n-1)x^2-2(n+t)x+t(p-1)]$, $(n \geq 1,\ p \geq 0,\ t \geq 0)$.

(12.2) $P(S_{18}(n,t,t,t),x) = x^{2n(t-1)}(x+1)^{2t-2}(x-1)^{2t-2}(x^2-t)^{2(n-1)}[(x+1)^2-t][(x-1)^2-t][x^4-2x^3-(2t+2n-1)x^2+2(n+t)x+t(t-1)][x^4+2x^3-(2t+2n-1)x^2-2(n+t)x+t(t-1)]$, $(n \geq 1,\ t \geq 0)$.

(12.3) $P(S_{18}(n,1,1,1),x) = x^4(x+2)(x+1)^{2n-1}(x-1)^{2n-1}(x-2)(x^2+x-2n-2)(x^2-x-2n-2)$, $(n \geq 1)$.

(12.4) $P(S_{18}(2k^2,k^2,k^2,k^2),x) = x^{4k^2(k^2-1)}(x+k+1)(x+k)^{2(2k^2-1)}(x+k-1)(x+1)^{2k^2-2}(x-1)^{2k^2-2}(x-k+1)(x-k)^{2(2k^2-1)}(x-k-1)[x^2+(2k+1)x-k(k-1)][x^2-(2k+1)x-k(k-1)][x^2+(2k-1)x-k(k+1)][x^2-(2k-1)x-k(k+1)]$, $(k \geq 1)$.

(13) $P(S_{19}(m,n,p,t),x) = x^{mn(t-1)+n}(x+1)^{n(p-1)}(x-1)^{n(p-1)}[x^4-(m+t+p+1)x^2+m+t+pt]^{n-1}(x^2-t)^{n(m-1)}\{x^6-(m+n+mn+p+np+t+1)x^4+[m+n+mn+mn^2-2np+2mnp+mn^2p+np^2+t(1+n+p+np)]x^2-n(p-1)^2(mn+t)\}$, $(m \geq 1,\ n \geq 1,\ p \geq 1,\ t \geq 0)$.

(13.1) $P(S_{19}(m,n,1,t),x) = x^{mn(t-1)+n+2}(x^2-t)^{n(m-1)}[x^4-(m+t+2)x^2+m+2t]^{n-1}[x^4-(m+2n+mn+t+2)x^2+(n+1)(m+2mn+2t)$, $(m \geq 1,\ n \geq 1,\ t \geq 0)$.

(13.2) $P(S_{19}(2,n,1,1),x) = x^{n+2}(x+1)^{2n-1}(x-1)^{2n-1}(x+2)^{n-1}(x-2)^{n-1}(x^2+x-2n-2)(x^2-x-2n-2)$, $(n \geq 1)$.

(14) $P(S_{20}(n,p,q),x) = (x+1)^{p(n-1)+q}(x-1)^{p(n-1)+q}(x^2-x-n)^{p-1}(x^2+x-n)^{p-1}(x^2-x-pq-n)(x^2+x-pq-n)$, $(n \geq 1,\ p \geq 1,\ q \geq 1)$.

(15) $P(S_{21}(m,t),x) = x^{2m(t-1)+2}(x^2-x-t)^{m-1}(x^2+x-t)^{m-1}(x^2-x-m-t)(x^2+x-m-t)$, $(m \geq 1,\ t \geq 0)$.

Proof. We only prove (2) and (10). The characteristic polynomials of the other 13 types can be obtained similarly.

(2). By properly ordering the vertices of the graph $S_2(n,t)$, the adjacency matrix $A = A(S_2(n,t))$ of $S_2(n,t)$ can be written as the $(nt+t+2)\times(nt+t+2)$ matrix such that

$$A = A(S_2(n,t)) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} & B_1 & 0_{t\times 2} \\ A_{21} & A_{22} & \cdots & A_{2n} & B_2 & 0_{t\times 2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} & B_n & 0_{t\times 2} \\ B_1^T & B_2^T & \cdots & B_n^T & 0_{n\times n} & J_{n\times 2} \\ 0_{2\times t} & 0_{2\times t} & \cdots & 0_{2\times t} & J_{2\times n} & 0_{2\times 2} \end{bmatrix},$$

where $A_{ij} = 0_{t\times t}$ for $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,n$, and

$$B_k = [a_{ij}^{(k)}] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}, \quad B_k \in R^{t\times n}, \quad \text{for } k = 1,2,\cdots,n.$$

Then we have

$$P[S_2(n,t),x] = |xI_{nt+t+2} - A(S_2(n,t))|$$
$$= \begin{vmatrix} xI_t & 0_{t\times t} & \cdots & 0_{t\times t} & -B_1 & 0_{t\times 2} \\ 0_{t\times t} & xI_t & \cdots & 0_{t\times t} & -B_2 & 0_{t\times 2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{t\times t} & 0_{t\times t} & \cdots & xI_t & -B_n & 0_{t\times 2} \\ -B_1^T & -B_2^T & \cdots & -B_n^T & xI_n & -J_{n\times 2} \\ 0_{2\times t} & 0_{2\times t} & \cdots & 0_{2\times t} & -J_{2\times n} & xI_2 \end{vmatrix}.$$

By careful calculation, we can prove that the characteristic polynomial of $S_2(n,t)$ is

$$P(S_2(n,t),x) = x^{n(t-1)+2}(x^2-t)^{n-1}[x^2-(2n+t)].$$

(10). By properly ordering the vertices of the graph $S_{13}(m,n)$, the adjacency matrix $A = A(S_{13}(m,n))$ of $S_{13}(m,n)$ can be written as the $(2mn+2n+2)\times(2mn+2n+2)$ symmetric block circulant matrix such that $A =$

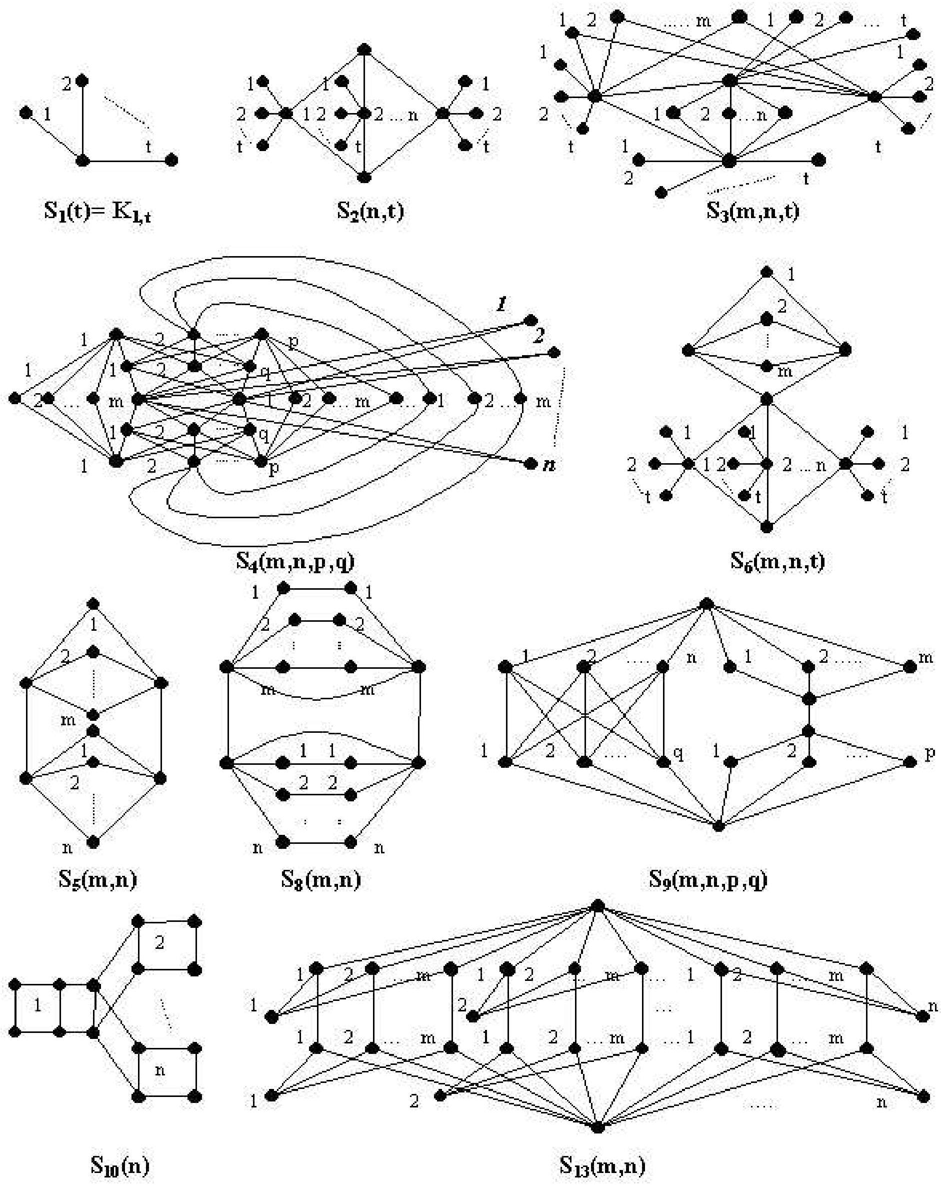


Figure 6.3: Nonregular bipartite graphs.

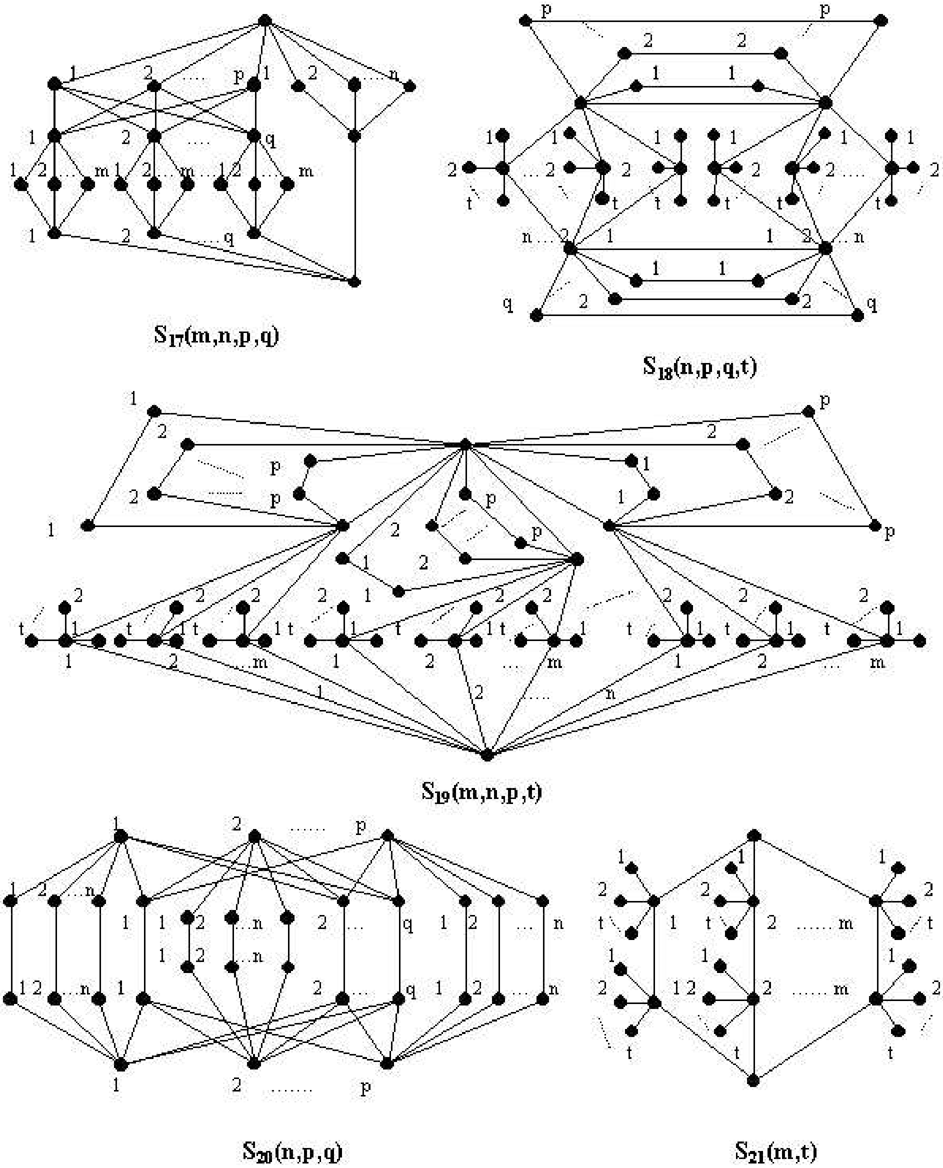


Figure 6.4: Nonregular bipartite graphs.

$A(S_{13}(m,n)) \in \mathcal{BC}(2, mn+n+1)$ and

$$A = A(S_{13}(m,n)) = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix},$$

where

$$A_0 = \begin{bmatrix} 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & B_1 & J_{m\times 1} \\ 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & B_2 & J_{m\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} & B_n & J_{m\times 1} \\ B_1^T & B_2^T & \cdots & B_n^T & 0_{n\times n} & 0_{n\times 1} \\ J_{1\times m} & J_{1\times m} & \cdots & J_{1\times m} & 0_{1\times n} & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} I_m & 0_{m\times m} & \cdots & 0_{m\times m} & 0_{m\times n} & 0_{m\times 1} \\ 0_{m\times m} & I_m & \cdots & 0_{m\times m} & 0_{m\times n} & 0_{m\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \cdots & I_m & 0_{m\times n} & 0_{m\times 1} \\ 0_{n\times m} & 0_{n\times m} & \cdots & 0_{n\times m} & 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times m} & 0_{1\times m} & \cdots & 0_{1\times m} & 0_{1\times n} & 0 \end{bmatrix},$$

and

$$B_k = [a_{ij}^{(k)}] = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}, \quad B_k \in R^{m\times n}, \quad \text{for } k = 1, 2, \cdots, n.$$

In view of Lemma 2.2.1, we distinguish between the following two cases.

Case 1. Let $b_0 = |xI_{mn+n+1} - (A_0 + A_1)|$. Then we have

$$b_0 = \begin{vmatrix} (x-1)I_m & 0_{m\times m} & \cdots & 0_{m\times m} & -B_1 & -J_{m\times 1} \\ 0_{m\times m} & (x-1)I_m & \cdots & 0_{m\times m} & -B_2 & J_{m\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \cdots & (x-1)I_m & -B_n & -J_{m\times 1} \\ -B_1^T & -B_2^T & \cdots & -B_n^T & xI_n & 0_{n\times 1} \\ -J_{1\times m} & -J_{1\times m} & \cdots & -J_{1\times m} & 0_{1\times n} & x \end{vmatrix}$$

By careful calculation, we can find

$$b_0 = x(x-1)^{n(m-1)}(x^2 - x - m)^{n-1}[x^2 - x - m(n+1)].$$

Case 2. Let $b_1 = |xI_{mn+n+1} - (A_0 - A_1)|$. Then we have

$$b_1 = \begin{vmatrix} (x+1)I_m & 0_{m\times m} & \cdots & 0_{m\times m} & -B_1 & -J_{m\times 1} \\ 0_{m\times m} & (x+1)I_m & \cdots & 0_{m\times m} & -B_2 & J_{m\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{m\times m} & 0_{m\times m} & \cdots & (x+1)I_m & -B_n & -J_{m\times 1} \\ -B_1^T & -B_2^T & \cdots & -B_n^T & xI_n & 0_{n\times 1} \\ -J_{1\times m} & -J_{1\times m} & \cdots & -J_{1\times m} & 0_{1\times n} & x \end{vmatrix}$$

By careful calculation, we can find

$$b_1 = x(x+1)^{n(m-1)}(x^2+x-m)^{n-1}[x^2+x-m(n+1)].$$

Hence, the characteristic polynomial of $S_{13}(m,n)$ is

$$P(S_{13}(m,n),x) = x^2(x+1)^{n(m-1)}(x-1)^{n(m-1)}(x^2+x-m)^{n-1}[x^2+x \\ -m(n+1)](x^2-x-m)^{n-1}[x^2-x-m(n+1)].$$

The proof is now complete. ■

We note that these classes of graphs in Figures 6.3 and 6.4 are constructed from the smaller graphs in Figures 6.1 and 6.2 (or Figures 4 and 5 of [5]). We believe that it is useful to construct new classes of integral graphs by using this method.

6.2 Integral nonregular bipartite graphs

In this section, by using number theory and computer search, we shall obtain some new classes of integral graphs from Theorem 6.1.2. All these classes are infinite and consist of connected graphs except for several unconnected graphs for which one or more of their parameters are taken zero.

Theorem 6.2.1. *(see [22, 36] or [47]) The tree $K_{1,t}$ is integral if and only if t is a perfect square.*

Theorem 6.2.2. *The graph $S_2(n,t)$ is integral if and only if one of the following holds: (i) t and $2n+t$ are perfect squares, or (ii) $n=1$ and $t+2$ is a perfect square, where t (≥ 0) and n (≥ 1) are integers.*

Proof. By (2) of Theorem 6.1.2, we know

$$P(S_2(n,t),x) = x^{n(t-1)+2}(x^2-t)^{n-1}[x^2-(2n+t)],$$

where n (≥ 1) and $t(\geq 0)$ are integers. Hence, a sufficient and necessary condition for the graph $S_2(n,t)$ to be integral is the following: (i) when $n \geq 2$, t and $2n+t$ are perfect squares, or (ii) when $n=1$, $t+2$ is a perfect square, where t (≥ 0) and n (≥ 1) are integers.

Thus, this theorem is proved. ■

Corollary 6.2.3.

(1) Let n (≥ 2), t (≥ 0) and k (≥ 1) be integers. If the graph $S_2(n,t)$ is integral, then the graph $S_2(nk^2, tk^2)$ is integral too.

(2) Let t and k be positive integers. If the graph $S_2(1, t-2) = K_{1,t}$ is integral, then the graph $S_2(1, tk^2-2) = K_{1,tk^2}$ is integral too.

Proof. By (2) of Theorem 6.1.2, we get

$$P(S_2(n,t),x) = x^{n(t-1)+2}(x^2-t)^{n-1}[x^2-(2n+t)]$$

and

$$P(S_2(nk^2,tk^2),x) = x^{nk^2(tk^2-1)+2}(x^2-tk^2)^{n-1}[x^2-(2n+t)k^2],$$

where $n \geq 1$, $t \geq 0$.

(1) Because n (≥ 2), t (≥ 0) and k (≥ 1) are integers, and the graph $S_2(n,t)$ is integral, we get that t and $2n+t$ are perfect squares. By Theorem 6.2.2, the graph $S_2(nk^2,tk^2)$ is integral.

(2) Because t and k are positive integers, and the graph $S_2(1,t-2) = K_{1,t}$ is integral, t must be a perfect square. By Theorem 6.2.2, also the graph $S_2(1, tk^2-2) = K_{1,tk^2}$ is integral. ■

Corollary 6.2.4. *If $t = a^2 \geq 0$, $n = \frac{b^2-a^2}{2} \geq 1$, $b > a$, and a, b, n (≥ 1), t (≥ 0) are integers, then for any positive integer k the graph $S_2(nk^2,tk^2)$ is integral too.*

Proof. Because $t = a^2 \geq 0$, $n = \frac{b^2-a^2}{2} \geq 1$, $b > a$, and a, b, n (≥ 1), t (≥ 0), k (≥ 1) are integers, by (2) of Theorem 6.1.2, it follows

$$P(S_2(\tfrac{b^2-a^2}{2}, a^2), x) = x^{\frac{b^2-a^2}{2}(a^2-1)+2}(x^2-a^2)^{\frac{b^2-a^2}{2}-1}(x^2-b^2).$$

So, the graph $S_2(n,t) = S_2(\frac{b^2-a^2}{2}, a^2)$ is integral. By Theorem 6.2.2, also the graph $S_2(nk^2,tk^2) = S_2(\frac{b^2-a^2}{2}\cdot k^2, a^2k^2)$ is integral. ■

Theorem 6.2.5. *The graph $S_3(m,n,t)$ is integral if and only if t is a perfect square, and $x^4 - 2(m+n+t+2)x^2 + (2m+t)(2n+t)$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where m (≥ 0), n (≥ 0), t (≥ 0), a and b are integers, and $m^2+n^2 \neq 0$.*

Proof. By (3) of Theorem 6.1.2, we see
$P(S_3(m,n,t),x)$
$= x^{m+n+4(t-1)}(x^2-t)^2[x^4-2(m+n+t+2)x^2+(2m+t)(2n+t)]$,
where m (≥ 0), n (≥ 1), t (≥ 0) are integers or m (≥ 1), n (≥ 0), t (≥ 0) are integers. Hence, this theorem is true. ∎

Corollary 6.2.6.

(1) The graph $S_3(n,n,t)$ is integral if and only if t is a perfect square, and $2n+t=k(k+2)$, where n (≥ 1), k (≥ 1) and t (≥ 0) are integers.

(2) If $m=0$, then $S_3(0,n,t)=S_3(n,0,t)$ is integral if and only if t is a perfect square, and $x^4-2(n+t+2)x^2+t(2n+t)$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where n (≥ 1), t (≥ 0), a and b are integers.

(3) If $m=t=0$, then the graph $S_3(0,n,0)=S_3(n,0,0)=S_2(n+2,0)$ is integral if and only if $n=2k^2-2$, where n and k are positive integers.

Proof. We only prove (1). (2) and (3) are similarly proved. By (3) of Theorem 6.1.2, we get
$P(S_3(n,n,t),x)=x^{2n+4(t-1)}(x^2-t)^2(x^2+2x-2n-t)(x^2+2x-2n-t)$,
where n (≥ 1) and t (≥ 0) are integers. Hence, the graph $S_3(n,n,t)$ is integral if and only if t is a perfect square, and $2n+t=k(k+2)$, where n (≥ 1), k (≥ 1) and t (≥ 0) are integers. ∎

Corollary 6.2.7. *Let m (≥ 0), n (≥ 0), t (≥ 0), l (≥ 0), k (≥ 0) be integers, and $m^2+n^2\neq 0$. Then we have*

(1) If $m=n, t=n^2$, then the graph $S_3(n,n,n^2)$ is integral.

(2) If $m=n=2l(l+1)-2k^2\geq 1$, $t=4k^2\geq 0$, then the graph $S_3(n,n,t)$ is integral.

(3) If $m=n=2l(l+2)+1-2k(k+1)\geq 1$, $t=(2k+1)^2$, then the graph $S_3(n,n,t)$ is integral.

(4) If $m\neq n$, let a, b, m (≥ 1), n (≥ 1), t (≥ 0) be integers in Table 6.1, and let a and b be as in Theorem 6.2.5. Then the graph $S_3(m,n,t)=S_3(n,m,t)$ is integral.(Table 6.1 is obtained by computer search, where $1\leq a\leq 50$, $a\leq b\leq a+10$, $m<n$, $0\leq t\leq 2500$).

a	b	m	n	t	a	b	m	n	t
5	9	13	37	1	5	9	9	33	9
5	9	1	25	25	14	20	100	196	0
14	20	98	194	4	14	20	92	188	16
14	20	82	178	36	14	20	68	164	64
14	20	50	146	100	14	20	28	124	144
14	20	2	98	196	24	28	294	384	0
24	28	292	382	4	24	28	286	376	16
24	28	276	366	36	24	28	262	352	64
24	28	244	334	100	24	28	222	312	144
24	28	196	286	196	24	28	166	256	256
24	28	132	222	324	24	28	94	184	400
24	28	52	142	484	24	28	6	96	576
27	35	367	607	1	27	35	363	603	9
27	35	355	595	25	27	35	343	583	49
27	35	327	567	81	27	35	307	547	121
27	35	283	523	169	27	35	255	495	225
27	35	223	463	289	27	35	187	427	361
27	35	147	387	441	27	35	103	343	529
27	35	55	295	625	27	35	3	243	729
44	54	972	1452	0	44	54	970	1450	4
44	54	964	1444	16	44	54	954	1434	36
44	54	940	1420	64	44	54	922	1402	100
44	54	900	1380	144	44	54	874	1354	196
44	54	844	1324	256	44	54	810	1290	324
44	54	772	1252	400	44	54	730	1210	484
44	54	684	1164	576	44	54	634	1114	676
44	54	580	1060	784	44	54	522	1002	900
44	54	460	940	1024	44	54	394	874	1156
44	54	324	804	1296	44	54	250	730	1444
44	54	172	652	1600	44	54	90	570	1764
44	54	4	484	1936	/	/	/	/	/

Table 6.1: Integral graphs $S_3(m,n,t) = S_3(n,m,t)$.

Proof. (1). Because $m = n$, $t = n^2$, it follows $x^4 - 2(m+n+t+2)x^2 + (2m+t)(2n+t) = (x^2 - n^2)[x^2 - (n+2)^2]$, where n (≥ 1) is an integer. Thus by Theorem 6.2.5, the graph $S_3(n, n, n^2)$ is integral.

Similarly to the proof of (1), we can prove (2), (3) and (4). For the case (4), we have found many integral graphs $S_3(m, n, t)$ by using computer search. 61 of them are shown in Table 6.1. They are found as positive integral solutions a, b, m, n, t and t_1 for the Diophantine equations

$$\begin{cases} t = t_1^2, \\ a^2 + b^2 = 2m + 2n + 2t + 4, \\ a^2b^2 = (2m+t)(2n+t), \end{cases} \tag{6.2.1}$$

where $m < n$, $1 \leq a \leq 50$, $a \leq b \leq a + 10$, $0 \leq t_1 \leq 50$ and $0 \leq t \leq 2500$. ∎

Theorem 6.2.8. *The graph $S_4(m, n, p, q)$ is integral if and only if $x^4 - (2m + 2n + 4q + pq)x^2 + 4mn + 8mq + 2npq$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, and one of the following holds: (i) $2m$ and pq ($= c^2$) are perfect squares, (ii) $p = 1$, q ($= c^2$) is a perfect square, where m (≥ 0), n (≥ 0) p (≥ 1), q (≥ 1), a and b are integers.*

Proof. By using Theorem 6.1.2 (4), this theorem is shown similarly to Theorem 6.2.2. ∎

Corollary 6.2.9.

(1) If $m = n$, then the graph $S_4(n, n, p, q)$ is integral if and only if $2n$, pq and $2n + 4q + pq$ are perfect squares, where n (≥ 0), p (≥ 1) and q (≥ 1) are integers.

(2) If $m = n$, $p = q$, then the graph $S_4(n, n, p, p)$ is integral if and only if $2n$ and $2n + p^2 + 4p$ are perfect squares, where n and p are positive integers.

Proof. From (4.1) and (4.2) of Theorem 6.1.2, this theorem is proved similarly to Theorem 6.2.2. ∎

Corollary 6.2.10. *Let a, b, c, m, n, p, q be as in Theorem 6.2.8, and let m (≥ 1), n (≥ 1), p (≥ 1), q (≥ 1), l (≥ 1), k (≥ 1), r (≥ 1), b_1, a, b, c be integers. Then we have*

(1) If $m = n = 2$, $p = q$, then the graph $S_4(2, 2, p, p)$ is integral.

(2) Let $m = n = 2l^2r^2$, $q = pk^2r^2$, $a = 2lr$, $b = b_1r$, $c = pkr$, and let b_1, p, k, l be positive integers satisfying the Diophantine equation

$$b_1^2 - (p^2 + 4p)k^2 = 4l^2. \tag{6.2.2}$$

Then the graph $S_4(m,n,p,q)$ *is integral.*

(3) Let $m=n=2l^2r^2$, $p=qk^2r^2$, $a=2lr$, $b=b_1r$, $c=qkr$, *and let* b_1, q, k, l *be positive integers satisfying the Diophantine equation*

$$b_1^2-(q^2+4q)k^2=4l^2. \tag{6.2.3}$$

Then the graph $S_4(m,n,p,q)$ *is integral.*

(4) Let $m=n=2l^2r^2$, $p=p_1^2$, $q=q_1^2r^2$, $a=2lr$,$b=b_1r$, $c=p_1q_1r$, *and let* b_1, p_1, q_1, l *be positive integers satisfying the Diophantine equation*

$$b_1^2-(p_1^2+4)q_1^2=4l^2. \tag{6.2.4}$$

Then the graph $S_4(m,n,p,q)$ *is integral.*

(5) Let $m=n=2l^2$, $p=q$, $a=2l$, $c=p$, *and let* b, p, l *be positive integers satisfying the Diophantine equation*

$$b^2-4l^2=p(p+4). \tag{6.2.5}$$

Then the graph $S_4(m,n,p,q)$ *is integral.*

(6) Let $m=n$, a, b, c, m, n, p *and* q *be given as in Table 6.2. Then the graph* $S_4(m,n,p,q)$ *is integral.(Table 6.2 is obtained by computer search, where* $0\le a\le 10$, $a\le b\le a+10$, $m=n$ *and* m, n, p *and* q *are not as in (1), but represent solutions of (2)-(5)).*

Proof. (1) Because $m=n=2$, $p=q$, we have by (4) of Theorem 6.1.2,

$P(S_4(2,2,p,p),x)=x^{4p}(x+p+2)(x+p)(x+2)^p(x-2)^p(x-p)(x-p-2)$, where p is a positive integer. Hence, the graph $S_4(2,2,p,p)$ is integral.

(2)-(5) When $m=n$, by Theorem 6.2.8, the graph $S_4(m,n,p,q)$ is integral if and only if $2n$ and pq $(=c^2)$ are perfect squares, and $x^4-(2m+2n+4q+pq)x^2+4mn+8mq+2npq=(x^2-2n)[x^2-(2n+4q+pq)]$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where $m=n$ (≥ 1), p (≥ 1), q (≥ 1), a , b and c are integers. Without loss of generality, assume that $a^2=2n$, $b^2=2n+4q+pq$. Hence, the graph $S_4(m,n,p,q)$ is integral if and only if the equations

$$\begin{cases} a^2=2n, \\ b^2=2n+4q+pq, \\ pq=c^2. \end{cases} \tag{6.2.6}$$

have only integral roots. We distinguish between the following four cases:

a	b	c	m	n	p	q	a	b	c	m	n	p	q
2	7	3	2	2	1	9	4	6	2	8	8	1	4
4	6	4	8	8	16	1	4	8	4	8	8	2	8
4	8	6	8	8	12	3	4	10	6	8	8	3	12
4	12	8	8	8	4	16	4	14	6	8	8	1	36
4	14	10	8	8	5	20	4	14	12	8	8	16	9
6	7	3	18	18	9	1	6	9	3	18	18	1	9
6	9	5	18	18	5	5	6	11	9	18	18	81	1
6	12	6	18	18	2	18	6	12	10	18	18	50	2
6	14	12	18	18	36	4	6	15	9	18	18	3	27
8	12	4	32	32	1	16	8	12	8	32	32	16	4
8	16	8	32	32	2	32	8	16	12	32	32	12	12
8	18	16	32	32	256	1	10	11	3	50	50	3	3
10	12	6	50	50	18	2	10	14	8	50	50	8	8
10	15	5	50	50	1	25	10	15	11	50	50	121	1
10	16	12	50	50	48	3	10	17	9	50	50	3	27
10	18	14	50	50	28	7	10	19	15	50	50	25	9
10	20	10	50	50	2	50	/	/	/	/	/	/	/

Table 6.2: Integral graphs $S_4(m, n, p, q)$.

Case 1. Suppose that $m = n = 2l^2r^2$, $q = pk^2r^2$, $a = 2lr$, $b = b_1r$, $c = pkr$, where l (≥ 1), r (≥ 1), b_1 and $k(\geq 1)$ are integers. By the equations (6.2.6), we get the Diophantine equation (6.2.2). From Corollary 6.2.9, we see that (2) of Corollary 6.2.10 is true.

Case 2. Suppose that $m = n = 2l^2r^2$, $p = qk^2r^2$, $a = 2lr$, $b = b_1r$, $c = qkr$, where l (≥ 1), r (≥ 1), b_1 and $k(\geq 1)$ are integers. By the equations (6.2.6), we get the Diophantine equation (6.2.3). From Corollary 6.2.9, the result in (3) follows.

Case 3. Suppose that $m = n = 2l^2r^2$, $p = p_1^2$, $a = 2lr$, $b = b_1r$, $c = p_1q_1r$ and $q = q_1^2r^2$, where l (≥ 1), r (≥ 1), b_1, $p_1(\geq 1)$ and $q_1(\geq 1)$ are integers. Equations (6.2.6) yields the Diophantine equation (6.2.4), which proves Corollary 6.2.10 (4).

Case 4. Suppose that $m = n = 2l^2$, $q = p$, $a = 2l$ and $c = p$, where l (≥ 1) and $p(\geq 1)$ are integers. Equations (6.2.6) leads to the Diophantine equation (6.2.5). This shows Corollary 6.2.10 (5).

(6) The result can be shown similarly to (1) by using Theorem 6.1.2 (4).■

Remark 6.2.11. *For the Diophantine equations* (6.2.2)-(6.2.4) *and* (6.2.5), *all positive integral solutions can be found from Lemmas 2.1.7-2.1.13 and Lemma 2.1.15, respectively. This shows that there are infinitely many integral graphs* $S_4(m, n,\ p,\ q)$.

Corollary 6.2.12. *Let* a, b, c, m, n, p, q *be as in Theorem 6.2.8, and let* m (≥ 1), n (≥ 1), p (≥ 1), q (≥ 1), a , b *and* c *be integers. Then we have*

(1) Suppose $m \neq n$, $p = 1$, *and let (i)* $a = 4$, $b = 16$, $c = 6$, $m = 2$, $n = 44$, $p = 1$ *and* $q = 36$, *or (ii)* $a = 6$, $b = 19$, $c = 3$, $m = 14$, $n = 162$, $p = 1$, $q = 9$ *hold. Then the graph* $S_4(m, n, p, q)$ *is integral. (Here* a, b, c, m, n, p *and* q *are obtained by computer search, and* $0 \leq a \leq 15$, $a \leq b \leq a + 30$, $m \neq n$, $p = 1$*).*

(2) If $m \neq n$, *and* a, b, c, m, n, p, q *are given in Table 6.3, then the graph* $S_4(m, n, p, q)$ *is integral.(Table 6.3 is obtained by computer search, where* $1 \leq a \leq 10$, $a \leq b \leq a + 20$ *and* $m \neq n$*).*

a	b	c	m	n	p	q	a	b	c	m	n	p	q
4	6	4	2	14	16	1	4	8	6	2	12	9	4
4	16	6	2	44	1	36	6	10	8	2	26	16	4
6	14	8	2	50	4	16	6	14	8	50	2	4	16
6	16	6	2	114	6	6	6	18	8	2	82	2	32
6	22	12	162	2	12	12	7	24	15	162	8	15	15
8	12	10	2	44	25	4	8	12	8	8	56	16	4
8	12	6	18	60	9	4	8	12	6	50	28	9	4
8	14	8	2	92	32	2	8	14	12	8	38	24	6
8	16	12	8	48	9	16	8	18	8	2	152	16	4
8	18	8	128	26	16	4	9	12	9	18	48	27	3
10	14	12	2	66	36	4	10	16	12	2	86	16	9
10	16	12	32	56	16	9	10	20	12	32	74	4	36
10	30	24	18	66	9	64	/	/	/	/	/	/	/

Table 6.3: Integral graphs $S_4(m, n, p, q)$.

Proof. As in the proof of Corollary 6.2.10 (1), we easily check the correctness by using (4) of Theorem 6.1.2 or Theorem 6.2.8. ■

Similarly we find

Corollary 6.2.13. *Let* $m = 0$ *or* $n = 0$, *and let* a, b, c, m, n, $p(\geq 1)$, $q(\geq 1)$ *be as in Theorem 6.2.8, and given in Table 6.4, then the graph* $S_4(m, n, p, q)$

is integral.(Table 6.4 is obtained by computer search, where $1 \le a \le 10$, $a \le b \le a+30$ *and* $m = 0$ *or* $n = 0$*).*

a	b	c	m	n	p	q	a	b	c	m	n	p	q
1	4	3	2	0	9	1	2	8	6	8	0	9	4
2	28	24	98	0	144	4	3	12	9	18	0	9	9
4	16	12	32	0	9	1	4	21	15	98	0	25	9
5	20	15	50	0	9	25	6	20	12	0	50	3	48
6	24	18	72	0	9	36	7	28	21	98	0	9	49
8	30	24	0	50	8	72	8	32	24	128	0	9	64
9	36	27	162	0	9	81	10	40	30	200	0	9	100

Table 6.4: Integral graphs $S_4(m, n, p, q)$.

Theorem 6.2.14. *The graph* $S_5(m, n)$ *is integral if and only if* $x^4 - (2m + 2n + 1)x^2 + 4mn$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, where* m (≥ 0)*,* n (≥ 0)*,* a *and* b *are integers.*

Proof. As in the proof of Theorem 6.2.2, the result follows by Theorem 6.1.2 (5). ■

Corollary 6.2.15. *The graph* $S_5(n, n)$ *is integral if and only if* $n = \frac{1}{2}k(k+1)$*, where* k (≥ 0) *is an integer.*

Proof. The result is a direct consequence of Theorem 6.1.2 (5.1) or Theorem 6.2.14. ■

From Theorem 6.1.2 (5.2) we obtain

Corollary 6.2.16. *The graph* $S_5(0, n)$=$S_5(n, 0)$ *is integral if and only if* $n = 2k(k+1)$*, where* k (≥ 0) *is an integer.*

Based on Theorem 6.1.2 (5) by computer search we find

Corollary 6.2.17. *Let* $m \ne n$*, and let* a*,* b*,* m *and* n *be as in Theorem 6.2.14, and given in Table 6.5, where* $n > m$ (≥ 1)*,* $a \le b$*. Then the graph* $S_5(m, n)$ *is integral. (Table 6.5 is obtained by computer search, where* $1 \le a \le 155$*,* $a \le b \le a + 80$ *and* $n > m \ge 1$*).*

From Corollary 6.2.17 and Theorem 6.2.14, we can deduce the following corollary.

a	b	m	n	a	b	m	n	a	b	m	n
7	10	25	49	22	27	243	363	41	58	841	1681
76	85	2890	3610	115	126	6615	7935	/	/	/	/

Table 6.5: Integral graphs $S_5(m, n)$.

Corollary 6.2.18. *Let $(m, n) = d$, $1 \leq m < n$. Then the following holds.*

(1) If $2d$ is a perfect square, then $S_5(m, n)$ is not an integral graph.

(2) If $2d$ is a positive integer but not a perfect square, then all integral graphs $S_5(m, n)$ (where $1 \leq m < n$) are given by

$$m = 2d(\frac{y_k - y_l}{2})^2, \quad n = 2d(\frac{y_k + y_l}{2})^2, \quad k > l > 0,$$

where y_k, y_l are odd or even, $y_k, y_l \in \{y_n | y_0 = 0, y_1 = b_1, y_{n+2} = 2a_1 y_{n+1} - y_n, (n \geq 0)\}$, and $a_1 + b_1\sqrt{2d}$ is the fundamental solution of the Diophantine equation

$$x^2 - 2dy^2 = 1. \tag{6.2.7}$$

Proof. By Theorem 6.2.14, the necessary and sufficient condition for $S_5(m, n)$ to be an integral graph is that there are positive integers a and b such that

$$\begin{cases} a^2 + b^2 &= 2m + 2n + 1, \\ a^2 b^2 &= 4mn. \end{cases} \tag{6.2.8}$$

Let $(m, n) = d$, $1 \leq m < n$. By (6.2.8), we find

$$m = dm_1^2, \quad n = dn_1^2, \quad ab = 2dm_1 n_1, \tag{6.2.9}$$

where m_1 and n_1 are positive integers, and $(m_1, n_1) = 1$. By (6.2.8) and (6.2.9), we obtain

$$(a + b)^2 - 2d(m_1 + n_1)^2 = 1. \tag{6.2.10}$$

Clearly, if $2d$ is a perfect square, then the diophantine equation (6.2.10) has no integral solutions.

Let $2d$ be a positive integer but not a perfect square. Then the equation (6.2.10) is a Pell equation (see [40]). Let $\varepsilon = a_1 + b_1\sqrt{2d}$ be the fundamental solution of the equation (6.2.7). From (6.2.10), we deduce that

$$a + b = \frac{\varepsilon^k + \bar{\varepsilon}^k}{2}, \quad m_1 + n_1 = \frac{\varepsilon^k - \bar{\varepsilon}^k}{2\sqrt{2d}}, \quad k > 0, \tag{6.2.11}$$

where $\overline{\varepsilon} = a_1 - b_1\sqrt{2d}$ and $\varepsilon\overline{\varepsilon} = 1$ (see Lemma 2.1.10).

By using (6.2.11) and $ab = 2dm_1n_1$ (see (6.2.9)), we get

$$(2b - \frac{\varepsilon^k + \overline{\varepsilon}^k}{2})^2 - 2d(2n_1 - \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{2d}})^2 = 1.$$

Thus, we have

$$2b = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2} + \frac{\varepsilon^l + \overline{\varepsilon}^l}{2}, \quad 2n_1 = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{2d}} + \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{2d}}, \quad l > 0.$$

Hence,

$$m_1 = (\frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{2d}} - \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{2d}})/2, \quad n_1 = (\frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{2d}} + \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{2d}})/2, \quad k > l > 0.$$

Let

$$y_n = \frac{\varepsilon^n - \overline{\varepsilon}^n}{2\sqrt{2d}}, \quad n = 0, 1, 2, \cdots.$$

Then we get the Pell sequence (see [15] or [90])

$$y_0 = 0, \quad y_1 = b_1, \quad y_{n+2} = 2a_1 y_{n+1} - y_n, \quad (n \geq 0).$$

Hence, all integral graphs $S_5(m,n)$ (where $1 \leq m < n$) are given by

$$m = 2d(\frac{y_k - y_l}{2})^2, \quad n = 2d(\frac{y_k + y_l}{2})^2, \quad k > l > 0.$$

The proof is now complete. ■

In a similar way the next results can be derived by using Theorem 6.1.2 (6).

Theorem 6.2.19. *The graph $S_6(m,n,t)$ is integral if and only if $x^4 - (2m + 2n + t + 2)x^2 + 2n(2m+1) + 2t(m+1)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, with integers a, b, and one of the following two conditions holds: (i) t is a perfect square, (ii) $n = 1$, where m (≥ 0), n (≥ 1), t (≥ 0) or m (≥ 0), $n = t = 0$.*

Corollary 6.2.20. *The graph $S_6(m,0,0) = K_{2,m+1} \cup K_1$ is integral if and only if $m = 2k^2 - 1$, where k is a positive integer.*

Corollary 6.2.21. *For the graph $S_6(m,n,t)$, we have*

(1) *If* $m = 0$*, then the graph* $S_6(0, n, t)$ *is integral if and only if (i)* t *is a perfect square, and* $x^4 - (2n + t + 2)x^2 + 2n + 2t$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, or (ii)* $n = 1$*, and* $x^4 - (t + 4)x^2 + 2t + 2$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, where* n (≥ 1)*,* t (≥ 0)*,* a *and* b *are integers.*

(2) *If* $m = 0$*,* $t = 1$*, then the graph* $S_6(0, n, 1)$ *is integral if and only if* $n = 2k^2 - 1$*, where* k *is a positive integer.*

Corollary 6.2.22. *For the graph* $S_6(m, n, t)$*, we have*

(1) *For* $t = 0$*, the graph* $S_6(m, n, 0)$ *is integral if and only if* $x^4 - (2m+2n+2)x^2 + 2n(2m+1)$ *can be factorized as* $(x^2 - a^2)(x^2 - b^2)$*, where* m (≥ 0)*,* n (≥ 0)*, and* a *and* b *are integers.*

(2) *For* $t = 0$*, let* a*,* b*,* m*,* n *and* t *be as in Theorem 6.2.19. If* $a = 12$*,* $b = 14$*,* $m = 73$*,* $n = 96$*, or* $a = 2520$*,* $b = 2522$*,* $m = 3175537$*,* $n = 3179904$*, then the graph* $S_6(m, n, t)$ *is integral.(The first solution is obtained by computer search, where* $0 \leq a \leq 100$*,* $a \leq b \leq a + 30$ *and* $m \geq 0$*,* $n \geq 0$*)*

(3) *For* $t = 0$*, let* $(2m + 1, 2n) = d$*. We have the following results.*

(i) If d *is a perfect square, then* $S_6(m, n, 0)$ *is not an integral graph.*

(ii) If d *is a positive integer but not a perfect square, then all integral graphs* $S_6(m, n, 0)$ *(where* $1 \leq m < n$*) are given via*

$$2m + 1 = d(\frac{y_k - y_l}{2})^2, \quad 2n = d(\frac{y_k + y_l}{2})^2, \quad k > l > 0,$$

where y_k, y_l *are odd or even,* $y_k, y_l \in \{y_n | y_0 = 0, y_1 = b_1, y_{n+2} = 2a_1 y_{n+1} - y_n, (n \geq 0)\}$*, and* $a_1 + b_1\sqrt{d}$ *is the fundamental solution of the Pell equation* (2.1.4)*.*

Proof. (1)-(2). By using Theorem 6.1.2 (6) or Theorem 6.2.19, the result follows similarly to Theorem 6.2.2 and Corollary 6.2.10 (1).

For (3), by Theorem 6.2.19 or Corollary 6.2.22 (1), the necessary and sufficient condition for $S_6(m, n, 0)$ to be an integral graph is that there are positive integers a and b satisfying

$$\begin{cases} a^2 + b^2 &= 2m + 2n + 2, \\ a^2 b^2 &= 2n(2m + 1). \end{cases} \tag{6.2.12}$$

Let $(2m+1, 2n) = d$. By (6.2.12) we have

$$2m+1 = dm_1^2, \quad 2n = dn_1^2, \quad ab = dm_1n_1, \tag{6.2.13}$$

where m_1 and n_1 are positive integers, and $(m_1, n_1) = 1$. By using (6.2.12) and (6.2.13), we get

$$(a+b)^2 - d(m_1+n_1)^2 = 1. \tag{6.2.14}$$

Clearly, if d is a perfect square, then the Diophantine equation (6.2.14) has no integral solutions.

Let d be a positive integer but not a perfect square. Then the equation (6.2.14) is a Pell equation (see [40]). Let $\varepsilon = a_1 + b_1\sqrt{d}$ be the fundamental solution of the equation (2.1.4). By (6.2.14), we deduce as before that

$$a+b = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2}, \quad m_1+n_1 = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}}, \quad k > 0, \tag{6.2.15}$$

where $\overline{\varepsilon} = a_1 - b_1\sqrt{d}$ and $\varepsilon\overline{\varepsilon} = 1$.

In view of (6.2.15) and $ab = 2dm_1n_1$ (see (6.2.13)), we get

$$(2b - \frac{\varepsilon^k + \overline{\varepsilon}^k}{2})^2 - d(2n_1 - \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}})^2 = 1.$$

Thus, we have

$$2b = \frac{\varepsilon^k + \overline{\varepsilon}^k}{2} + \frac{\varepsilon^l + \overline{\varepsilon}^l}{2}, \quad 2n_1 = \frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}} + \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{d}}, \quad l > 0.$$

Hence,

$$m_1 = (\frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}} - \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{d}})/2, \quad n_1 = (\frac{\varepsilon^k - \overline{\varepsilon}^k}{2\sqrt{d}} + \frac{\varepsilon^l - \overline{\varepsilon}^l}{2\sqrt{d}})/2, \quad k > l > 0.$$

Letting

$$y_n = \frac{\varepsilon^n - \overline{\varepsilon}^n}{2\sqrt{d}}, \quad n = 0, 1, 2, \cdots,$$

we obtain the Pell sequence (see [15] or [90])

$$y_0 = 0, \quad y_1 = b_1, \quad y_{n+2} = 2a_1y_{n+1} - y_n, \quad (n \geq 0).$$

Hence, all integral graphs $S_6(m, n, 0)$ (where $1 \leq m < n$) are given via

$$2m+1 = d(\frac{y_k - y_l}{2})^2, \quad 2n = d(\frac{y_k + y_l}{2})^2, \quad k > l > 0,$$

where m and n are positive integers. ■

For $S_6(m,n,t)$, when $t=0$, $1 \le m < n$, we have obtained all integral graphs $S_6(m,n,0)$. However, when $t=0$, $1 \le n < m$, we have not found any such integral graph. So we raise the following question.

Question 6.2.23. *Are there integral graphs $S_6(m,n,0)$ with $1 \le n < m$? Can we give a sufficient and necessary condition for $S_6(m,n,0)$ ($1 \le n < m$) to be an integral graph?*

Corollary 6.2.24. *For the graph $S_6(m,n,t)$, we have*

(1) For $n=1$, the graph $S_6(m,1,t)$ is integral if and only if $x^4-(2m+t+4)x^2+2(2m+1)+2t(m+1)$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where m (≥ 0), t (≥ 0), a and b are integers.

(2) For $n=1$, let a, b, m, n, t be as in Theorem 6.2.19 (or Corollary 6.2.24 (1)), and $a=1$, $b=2$, $m=0$ and $n=t=1$. Then the graph $S_6(0,1,1)$ is integral.(This solution is obtained by computer search, where $0 \le a \le 25$, $a \le b \le a+20$, $m \ge 0$, $t \ge 0$).

Proof. Similar to the proof of Theorem 6.2.2 and Corollary 6.2.10 (1), the statement follows from Theorem 6.1.2 (6) or Theorem 6.2.19. ■

The following is a consequence of the formulae (6.5)-(6.7) of Theorem 6.1.2.

Corollary 6.2.25. *For the graph $S_6(m,n,t)$, we have*

(1) If $m=n-1$, $t=1$, then the graph $S_6(n-1,n,1)$ is integral if and only if $n=\frac{1}{2}k(k+1)$, where k is a positive integer.

(2) If $m=n+1$, $t=1$, then the graph $S_6(n+1,n,1)$ is integral if and only if $n=\frac{1}{2}k(k+1)-1$, where k is a positive integer.

(3) If $m=n+1$, $t=9$, then the graph $S_6(n+1,n,9)$ is integral if and only if $n=\frac{1}{2}k(k+1)-3$, where k (≥ 2) is a positive integer.

Based on Theorem 6.1.2 (6) by computer search we found

Corollary 6.2.26. *Let a, b, m, n and t be as in Theorem 6.2.19, and given in Table 6.6, then the graph $S_6(m,n,t)$ is integral.(Table 6.6 is obtained by computer search, where $1 \le a \le 10$, $a \le b \le a+20$). (Note that m, n and t are different from those in Corollaries 6.2.20, 6.2.21, 6.2.22, 6.2.24 and 6.2.25).*

a	b	m	n	t	a	b	m	n	t	a	b	m	n	t
3	6	4	13	9	3	8	4	27	9	3	10	4	45	9
3	12	4	67	9	3	14	4	93	9	3	16	4	123	9
3	18	4	157	9	3	20	4	195	9	3	22	4	237	9
4	5	11	4	9	5	6	12	5	25	5	6	16	9	9
5	8	12	19	25	5	10	12	37	25	5	12	12	59	25
5	14	12	85	25	5	16	12	115	25	5	18	12	149	25
5	20	12	187	25	5	22	12	229	25	5	24	12	275	25
6	7	18	11	25	6	7	22	15	9	6	7	23	6	25
7	8	24	7	49	7	8	25	18	25	7	8	29	22	9
7	8	30	13	25	7	10	24	25	49	7	12	24	47	49
7	14	24	73	49	7	16	24	103	49	7	18	24	137	49
7	20	24	175	49	7	22	24	217	49	7	24	24	263	49
7	26	24	313	49	8	9	32	15	49	8	9	33	26	25
8	9	37	30	9	8	9	38	21	25	8	9	39	8	49
8	11	32	55	9	8	11	59	28	9	9	10	40	9	81
9	10	41	24	49	9	10	42	35	25	9	10	46	39	9
9	10	47	30	25	9	10	48	17	49	9	12	40	31	81
9	14	40	57	81	9	16	40	87	81	9	18	40	121	81
9	20	40	159	81	9	22	40	201	81	9	24	40	247	81
9	26	40	297	81	9	28	40	351	81	10	11	50	19	81
10	11	51	34	49	10	11	52	45	25	10	11	56	49	9
10	11	57	40	25	10	11	58	27	49	10	11	59	10	81

Table 6.6: Integral graphs $S_6(m, n, t)$.

With similar arguments as before the following results are obtained by using Theorem 6.1.2 (7).

Theorem 6.2.27. *The graph $S_8(m, n)$ ($m \geq 0$, $n \geq 0$) is integral if and only if $x^4 - 4x^3 - (m+n-5)x^2 + (2m+2n-2)x + mn - m - n$ and $x^4 + 4x^3 - (m+n-5)x^2 - (2m+2n-2)x + mn - m - n$ can be factorized as $(x+a)(x-b)(x+c)(x-d)$ and $(x-a)(x+b)(x-c)(x+d)$, respectively, where m, n, a, b, c and d are nonnegative integers.*

Corollary 6.2.28. *The graph $S_8(0, n) = S_8(n, 0)$ is integral if and only if $n = k(k+2)$, where k is a nonnegative integer.*

Corollary 6.2.29. *For the graph $S_8(m, n)$, we have*

(1) *If $m = n$, then the graph $S_8(n, n)$ is integral if and only if $n = k(k + 1)$, where k is a nonnegative integer.*

(2) *If $m \neq n$, let a, b, c, d, m, n, t be as in Theorem 6.2.27, and given in Table 6.7, then the graph $S_8(m, n) = S_8(n, m)$ is integral.(Table 6.7 is obtained by computer search, where $0 \leq a \leq 100$, $0 \leq b \leq a + 30$, $a \leq c \leq a + 30$, $b \leq d \leq b + 30$ and $0 \leq m < n$).*

a	b	c	d	m	n	a	b	c	d	m	n
6	8	9	11	50	98	11	13	13	15	147	192
21	23	26	28	486	726	40	42	57	59	1682	3362
44	46	51	53	2028	2700	47	49	50	52	2312	2592
75	77	84	86	5780	7220	/	/	/	/	/	/

Table 6.7: Integral graphs $S_8(m, n) = S_8(n, m)$.

The next statements are derived from Theorem 6.1.2 (8) -(11).

Theorem 6.2.30. *The graph $S_9(m, n, p, q)$ $(m, n, p, q \geq 1)$ is integral if and only if $x^6 - (2m+n+2p+q+nq+1)x^4 + (m+n+mn+p+4mp+2np+q+2mq+2nq+2mnq+pq+2npq)x^2 - (mp+np+2mnp+mq+nq+2mnq+2mpq+2npq+4mpq)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)$, where a, b and c are integers.*

Corollary 6.2.31. *For the graph $S_9(m, n, p, q)$, we have*

(1) *If $m = n = p = q$, then the graph $S_9(n, n, n, n)$ is integral if and only if $n = 2k^2$, where k is a positive integer.*

(2) *If m, n, p, q are not as in (1), and a, b, c, m, n, p, q are as in Theorem 6.2.30, and given in Table 6.8 , then the graph $S_9(m, n, p, q) = S_9(p, q, m, n)$ is integral.(Table 6.8 is obtained by computer search, where $1 \leq a \leq 7$, $a \leq b \leq a + 5$, $b \leq c \leq b + 5$, and m, n, p, q are not as in (1)).*

a	b	c	m	n	p	q	a	b	c	m	n	p	q
3	4	5	8	1	8	8	6	10	12	51	3	21	33

Table 6.8: Integral graph $S_9(m, n, p, q) = S_9(p, q, m, n)$.

Theorem 6.2.32. *The graph $S_{10}(n)$ is integral if and only if $n = k(k+2)$, where k (≥ 0) is an integer.*

Theorem 6.2.33.

(1) If $m \geq 1$, $n \geq 1$, then the graph $S_{13}(m,n)$ is integral if and only if (i) $m = k(k+1)$ and $m(n+1) = l(l+1)$, where k and l are positive integers, or (ii) $n = 1$, and $m = \frac{1}{2}k(k+1)$, where k is a positive integer.

(2) For $m = n = k(k+1)$, k a positive integer, the graph $S_{13}(n,n)$ is integral.

Theorem 6.2.34. *The graph $S_{17}(m,n,p,q)$ is integral if and only if $x^6 - (2m+2n+p+q+pq+1)x^4 + [m(2+4n+2p+q+pq)+n+p+np+2nq+2pq+2npq+pq^2]x^2 - [2m(n+p+np+nq+pq+npq)+2npq(q+1)]$ can be factorized as $(x^2-a^2)(x^2-b^2)(x^2-c^2)$ with integers a, b, c, and one of the following two conditions holds: (i) $2m$ is a perfect square, (ii) $q = 1$, where m, n, p, q are positive integers.*

Corollary 6.2.35. *For the graph $S_{17}(m,n,p,q)$, we have*

(1) If $m = n$, then the graph $S_{17}(n,n,p,q)$ is integral if and only if $n = 2k^2$, and $x^4 - [2n+(p+1)(q+1)]x^2 + (q+1)[n(p+1)+p(q+1)]$ can be factorized as $(x^2-a^2)(x^2-b^2)$, where n, p, q, k are positive integers, and a, b, c are integers.

(2) If $m = n = p = q$, then the graph $S_{17}(n,n,n,n)$ is integral if and only if $n = 2k^2$, where k is a positive integer.

(3) If $m = n$, $p = 1$, then the graph $S_{17}(n,n,1,q)$ is integral if and only if $n = 2r^2s^2h^2$, $q = (r^2-s^2)^2h^2-1$, where $(r,s) = 1$, $r > s$, $2 \nmid r+s$, and n, q, r, s, h are positive integers.

(4) If $m = n = 2k^2l^2$, $p = 2l^2$ and $q = k^2(2l^2+1)-1$, then the graph $S_{17}(n,n,p,q)$ is integral.

(5) For $m = n$, let a, b, m, n, p and q be as in (1), and not as in (2)-(4), and given by Table 6.9. Then the graph $S_{17}(n,n,p,q)$ is integral.(Table 6.9 is obtained by computer search, where $1 \leq a \leq 40$, $a \leq b \leq a+20$, and m, n, p and q are not as in (2)-(4)).

Proof. (i) Similar to the proof of Theorem 6.2.2 and Corollary 6.2.10, the statements in (1), (2), (4) and (5) are proven by Theorem 6.1.2 (11) or Theorem 6.2.34.

a	b	m	n	p	q	a	b	m	n	p	q
8	18	50	50	11	23	14	34	288	288	193	3
16	36	200	200	11	95	/	/	/	/	/	/

Table 6.9: Integral graph $S_{17}(n, n, p, q)$.

(ii) Next we shall prove (3). By (11) of Theorem 6.1.2, the graph $S_{17}(n, n, 1, q)$ is integral if and only if $2n$, $q+1$ and $2n+q+1$ are perfect squares. Assume that $n = 2k^2h^2$, $q+1 = l^2h^2$ and $2n+q+1 = t^2h^2$, where k, l, t,h are positive integers, and $(l, 2k) = 1$. Then we get

$$l^2 + (2k)^2 = t^2.$$

Lemma 2.1.1 yields $l = r^2 - s^2$, $2k = 2rs$, $t = r^2 + s^2$, $n = 2r^2s^2h^2$, $q = (r^2 - s^2)^2h^2 - 1$, where $(r, s) = 1$, $r > s > 0$, $2 \not| r + s$, and n, q, r, s, h are positive integers. ■

Corollary 6.2.36. *For the graph $S_{17}(m, n, p, q)$, we have*

(1) If $m \neq n$, $p = m$, $q = n$, then the graph $S_{17}(m, n, m, n)$ is integral if and only if $x^4 - (2m + 2n + mn + 1)x^2 + 2m(n + 1)^2$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$ with integers a, b, and one of the following two conditions holds: (i) $2m$ and $n+m$ are perfect squares, (ii) $n = 1$, $m+1$ is a perfect square, where m, n are positive integers.

(2) If $m = p = 2$, $q = n$, then the graph $S_{17}(2, n, 2, n)$ is integral if and only if $n = l^2-2$ and $2n+2 = k(k+1)$, where n, l and k are positive integers.

(3) For $m = p = 2$, $q = n$, let n, l and k be as in (2) of Corollary 6.2.36, and given in Table 6.10. Then the graph $S_{17}(2, n, 2, n)$ is integral.(Table 6.10 is obtained by computer search, where $1 \leq k \leq 10000$).

n	l	k	n	l	k	n	l	k
2	2	2	14	4	5	119	11	15
527	23	32	4094	64	90	17954	134	189
139127	373	527	609959	781	1104	4726274	2174	3074
20720702	4552	6437	/	/	/	/	/	/

Table 6.10: Integral graphs $S_{17}(2, n, 2, n)$.

By (2) of Corollary 6.2.36, we see that the graph $S_{17}(2,n,2,n)$ is integral if and only if $n = l^2 - 2$ and $2l^2 - 2 = k(k+1)$, where n, l and k are positive integers. Hence, we raise the following question.

Question 6.2.37. *What are all positive integral solutions of the Diophantine equation $2l^2 - 2 = k(k+1)$?*

Corollary 6.2.38. *For the graph $S_{17}(m,n,p,q)$, with $m \neq n$, let a, b, c, m, n, p, q be as in Theorem 6.2.34, and given in Table 6.11. Then the graph $S_{17}(m,n,p,q)$ is integral.(Table 6.11 is obtained by computer search, where $1 \leq a \leq 13$, $a \leq b \leq a+5$, $b \leq c \leq b+10$, and $m \neq n$).*

a	b	c	m	n	p	q	a	b	c	m	n	p	q
4	5	6	2	14	2	14	5	5	6	2	16	1	24
7	9	10	18	48	1	48	9	10	11	18	52	1	80
11	11	12	8	64	1	120	11	14	15	50	100	1	120
11	15	16	2	119	2	119	13	14	15	18	108	1	168
13	15	16	32	124	1	168	/	/	/	/	/	/	/

Table 6.11: Integral graphs $S_{17}(m,n,p,q)$.

We finally list the results obtained from Theorem 6.1.2 (12)-(15).

Theorem 6.2.39. *The graph $S_{18}(n,p,q,t)$ is integral if and only if $x^6 - 4x^5 - (2n+p+q+t-6)x^4 + (6n+2p+2q+4t-4)x^3 - (6n+p-np+q-nq-pq+6t-pt-qt-1)x^2 + (2n-np-nq+4t-2pt-2qt)x - t(p-1)(q-1)$ can be factorized as $(x+a)(x+b)(x+c)(x-d)(x-e)(x-f)$, and one of the following two conditions holds: (i) t is a perfect square, (ii) $n=1$, where $n \geq 1$, p, q, t, a, b, c, d, e and f are nonnegative integers.*

Corollary 6.2.40. *For the graph $S_{18}(n,p,q,t)$, we have*

(1) If $p=q$, then the graph $S_{18}(n,p,p,t)$ is integral if and only if $x^4 - 2x^3 - (p+t+2n-1)x^2 + 2(n+t)x + t(p-1)$ can be factorized as $(x+a)(x-b)(x+c)(x-d)$, and one of the following two conditions holds: (i) p and t are perfect squares, (ii) $n=1$, and p is a perfect square, where n is a positive integer, p, t, a, b, c and d are nonnegative integers.

(2) If $p=q=t=1$, then the graph $S_{18}(n,1,1,1)$ is integral if and only if $n = \frac{1}{2}k(k+1) - 1$, where k (≥ 2) is a positive integer.

(3) *If $p = q = t = 0$, then the graph $S_{18}(n, 0, 0, 0) = S_5(n, n)$ is integral if and only if $n = \frac{1}{2}k(k + 1)$, where k is a positive integer.*

(4) *If $n = 2t$, $p = q = t = k^2$, then the graph $S_{18}(2k^2, k^2, k^2, k^2)$ is integral if and only if k is a positive integer satisfying the Pell equation*

$$l^2 - 8k^2 = 1. \tag{6.2.16}$$

Proof. Similar to the proof of Theorem 6.2.2 and Corollary 6.2.10, we easily check the correctness of the results in (1), (2) and (3) by using Theorem 6.1.2 (12) or Theorem 6.2.39.

Next we shall prove (4). By Theorem 6.1.2 (12.4), we see that the graph $S_{18}(2k^2, k^2, k^2, k^2)$ is integral if and only if there are positive integers k, r and s such that

$$\begin{cases} k(k-1) = r(r + 2k + 1), \\ k(k+1) = s(s + 2k - 1). \end{cases} \tag{6.2.17}$$

This relation yields

$$(2k + r + s)(s - r - 1) = 0 \quad \text{and} \quad s^2 + (2k - 1)s - k(k + 1) = 0.$$

Then $s = r + 1$, and $s = \frac{-(2k-1)\pm\sqrt{8k^2+1}}{2}$.

Hence, s is a positive integer if and only if $8k^2 + 1$ is a perfect square. Let $8k^2 + 1 = l^2$, then k is a positive integer satisfying the Pell equation (6.2.16). All positive integral solutions of (6.2.16) are given by

$$l + k\sqrt{8} = u_n + v_n\sqrt{8} = (3 + \sqrt{8})^n,$$

where $n = 1, 2, \cdots$.

Thus, the proof is complete. ■

Theorem 6.2.41. *The graph $S_{19}(m, n, p, t)$ is integral if and only if $(x^2 - t)^{n(m-1)}$ $[x^4 - (m + t + p + 1)x^2 + m + t + pt]^{n-1}\{x^6 - (m + n + mn + p + np + t + 1)x^4 + [m + n + mn + mn^2 - 2np + 2mnp + mn^2p + np^2 + t(1 + n + p + np)]x^2 - n(p - 1)^2(mn + t)\} = 0$ has only integral roots, where m (≥ 1), n (≥ 1), p (≥ 1), t (≥ 0) are integers.*

Corollary 6.2.42. *For the graph $S_{19}(m, n, p, t)$, we have*

(1) *If $p = 1$, then the graph $S_{19}(m, n, 1, t)$ is integral if and only if $(x^2 - t)^{n(m-1)}[x^4 - (m + t + 2)x^2 + m + 2t]^{n-1}[x^4 - (m + 2n + mn + t + 2)x^2 + (n + 1)(m + 2mn + 2t)] = 0$ has only integral roots, where $m(\geq 1)$, n (≥ 1) and t (≥ 0) are integers.*

(2) If $m = 2$, $p = t = 1$, then the graph $S_{19}(2, n, 1, 1)$ is integral if and only if $n = \frac{1}{2}k(k+1) - 1$, where k (≥ 2) is a positive integer.

Theorem 6.2.43. *The graph $S_{20}(n, p, q)$ is integral if and only if (i) $n = k(k+1)$ and $pq = (l+k+1)(l-k)$, or (ii) $p = 1$, $n = r$ and $q = s(s+1)-r \geq 1$, where n, p, q, l, k, r and s are positive integers, and $l > k$.*

Corollary 6.2.44. *If the graph $S_{20}(k(k+1), p, q)$ is integral, then the graph $S_{20}(k(k+1), q, p)$ is integral too.*

Theorem 6.2.45. *The graph $S_{21}(m, t)$ is integral if and only if (i) $t = k(k+1)$ and $m = (l+k+1)(l-k)$, or (ii) $m = 1$ and $t = r(r+1)-1$, where m (≥ 1), t (≥ 0) , l (≥ 1) , k (≥ 0) and r (≥ 1) are integers, and $l > k$.*

6.3 Further discussion

The search for integral graphs becomes easier if we use graph operations, such as graph sum, graph product, strong graph sum, etc. (see [22, 24, 36]). However, in general, the problem of characterizing integral graphs seems to be very difficult. In the present chapter, we have mainly investigated the 21 nonregular bipartite integral graphs of [5]. Fifteen classes of larger integral graphs were constructed based on the structures of the integral graphs of [5]. These classes are connected nonregular and bipartite graphs except for several unconnected graphs for which one or several of the parameters are zero. However, we have not found appropriate methods to construct new integral graphs from the graphs S_7, S_{11}, S_{12}, S_{14}, S_{15}, S_{16} of [5] or Theorem 6.1.1. Thus, we raise the following question.

Question 6.3.1. *Can we construct new integral graphs from the graphs S_7, S_{11}, S_{12}, S_{14}, S_{15}, S_{16} of [5] or Theorem 6.1.1?*

Although we obtained fifteen new classes of integral graphs from the graphs S_1, S_2, S_3, S_4, S_5, S_6, S_8, S_9, S_{10}, S_{13}, S_{17}, S_{18}, S_{19}, S_{20}, S_{21} in Theorem 6.1.1 or [5], we think that other methods can be found to construct new integral graphs. For example, let K_n^t be obtained by joining t new end vertices to each vertex of K_n, then the graph $K_{n(n+1)}^{n(n+1)}$ is integral (see [76]). We note that the graph K_n^t can be constructed from $S_1 = K_{1,4}$. Hence, we raise the following question.

Question 6.3.2. *How to construct new integral graphs from the graphs S_1, S_2, S_3, S_4, S_5, S_6, S_8, S_9, S_{10}, S_{13}, S_{17}, S_{18}, S_{19}, S_{20}, S_{21} in Theorem 6.1.1 or [5]?*

For the graphs $S_1(t) = K_{1,t}$, $S_2(n,t)$, $S_5(m,n)$, $S_{10}(n)$, $S_{13}(m,n)$, $S_{20}(n, p, q)$, $S_{21}(m,t)$, in fact, we have given a necessary and sufficient integrality condition. However, it is very difficult to find all integral graphs of the type $S_3(m,n,t)$, $S_4(m,n,p,q)$, $S_6(m,n,t)$, $S_8(m,n)$, $S_9(m,n,p,q)$, $S_{17}(m, n,p,q)$, $S_{18}(n,p,q,t)$, $S_{19}(m,n, p, t)$. Hence, we come to

Question 6.3.3. *Can we give a better necessary and sufficient condition for the above 8 classes of graphs to be integral?*

Note that in connection with Question 6.3.3, in the present chapter, we found some results for the above 8 classes of graphs by computer search and number theory. It was proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. Finally we ask

Question 6.3.4. *What are all positive integral solutions for these Diophantine equations, for example for the Diophantine equations* (6.2.1)-(6.2.5), (6.2.8), *and so on?*

Chapter 7

Families of integral graphs

In this chapter, the graphs $K_{1,r} \bullet K_n$, $r * K_n$, $K_{1,r} \bullet K_{m,n}$ and $r * K_{m,n}$ are studied. We determine the characteristic polynomials of these graphs and also obtain sufficient and necessary conditions for these graphs to be integral. Some sufficient conditions are found by using number theory and computer search. All these classes are infinite, and different from those in the literature. We also give some new cospectral integral graphs.

7.1 Integral graphs $K_{1,r} \bullet K_n$ and $r * K_n$

In this section, we shall determine the characteristic polynomials of the graphs $K_{1,r} \bullet K_n$ and $r * K_n$. We obtain sufficient and necessary conditions for these graphs to be integral by using number theory and computer search. Furthermore cospectral graphs and cospectral integral graphs are investigated. Note that some results on cospectral graphs can be found in [21, 22, 27, 31, 45, 62, 76].

Let $K_{1,r} \bullet K_n$ be the graph obtained by identifying the center w of $K_{1,r}$ with one vertex v of K_n.

Theorem 7.1.1. *The graph $K_{1,r} \bullet K_n$ is integral if and only if $x^3-(n-2)x^2-(r+n-1)x+r(n-2)$ can be factorized as $(x-a)(x-b)(x+c)$, where a, b and c are nonnegative integers.*

Proof. By (2) of Lemma 1.2.5 and (1) of Lemma 1.2.17, we get

$$\begin{aligned} &P(K_{1,r} \bullet K_n, x) = x^{r-1}[xP(K_n, x) - rP(K_{n-1}, x)] \\ &= x^{r-1}(x+1)^{n-2}[x^3-(n-2)x^2-(n+r-1)x+r(n-2)], \end{aligned}$$

proving the theorem. ■

Let $u \in V(K_n)$ be the root of K_n, and let $r * K_n$ be the graph obtained by joining the roots of r copies of K_n to a new vertex w.

Theorem 7.1.2. *The graph $r * K_n$ is integral if and only if $x^3 - (n-2)x^2 - (r+n-1)x + r(n-2)$ can be factorized as $(x-a)(x-b)(x+c)$, where a, b and c are nonnegative integers.*

Proof. By (1) of Lemma 1.2.5 and (1) of Lemma 1.2.17, we find

$$\begin{aligned} &P(r * K_n, x) = P^{r-1}(K_n, x)[xP(K_n, x) - rP(K_{n-1}, x)] \\ &= (x+1)^{r(n-1)-1}(x-n+1)^{r-1}[x^3 - (n-2)x^2 - (n+r-1)x + r(n-2)]. \end{aligned}$$

Thus this theorem is proved. ■

Corollary 7.1.3.

(1) Let $G_1 = (r-1)K_1 \cup r * K_n$, $G_2 = (r-1)K_n \cup [K_{1,r} \bullet K_n]$. Then G_1 and G_2 are cospectral.

(2) If $K_{1,r} \bullet K_n$ is integral, then $r * K_n$ is integral too.

(3) If $r * K_n$ is integral, then $K_{1,r} \bullet K_n$ is integral too.

(4) If $r * K_n$ or $K_{1,r} \bullet K_n$ is integral, then $G_1 = (r-1)K_1 \cup r * K_n$ and $G_2 = (r-1)K_n \cup [K_{1,r} \bullet K_n]$ are cospectral integral graphs.

Proof. (1) By Lemma 1.2.6 (1), Lemma 1.2.17 (1), and Theorems 7.1.1 and 7.1.2, we find

$$\begin{aligned} &P(G_1, x) = P(G_2, x) = x^{r-1}(x+1)^{r(n-1)-1}(x-n+1)^{r-1} \\ &\cdot[x^3 - (n-2)x^2 - (n+r-1)x + r(n-2)]. \end{aligned}$$

Thus (1) is proved.

(2) By Lemma 1.2.17 (1), we know that K_n is integral. When $K_{1,r} \bullet K_n$ is integral, by (1) of Lemma 1.2.7 and Theorems 7.1.1 and 7.1.2, we can prove that $r * K_n$ is integral too.

(3) By (1) of Lemma 1.2.17, the graph K_n is integral. When $r * K_n$ is integral, by (2) of Lemma 1.2.7 and Theorems 7.1.1 and 7.1.2, we see that also $K_{1,r} \bullet K_n$ is integral.

(4) The result follows from Lemmas 1.2.6, 1.2.7 and 1.2.17, and Theorems 7.1.1 and 7.1.2. ■

Corollary 7.1.4. *For any positive integer k, we have.*

(1) If $r = (k^2-1)(4k^2-1)$, $n = 2k^2+1$ and $k > 1$, then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

(2) If $r = k^2(4k^2-1)$, $n = 2k^2$, then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

Proof. (1) By Theorems 7.1.1 and 7.1.2, we get

$$P(K_{1,r} \bullet K_n, x) = x^{(k^2-1)(4k^2-1)-1}(x+1)^{2k^2-1}(x-a)(x-b)(x+c)$$

and

$$P(r * K_n, x) = (x+1)^{2k^2(k^2-1)(4k^2-1)-1}(x-2k^2)^{(k^2-1)(4k^2-1)-1} \cdot (x-a)(x-b)(x+c),$$

where k (> 1) is a positive integer, $a = (k-1)(2k+1)$, $b = (k+1)(2k-1)$ and $c = 2k^2 - 1$.

(2) From Theorems 7.1.1 and 7.1.2, we find

$$P(K_{1,r} \bullet K_n, x) = x^{k^2(4k^2-1)-1}(x+1)^{2k^2-2}(x-a)(x-b)(x+c)$$

and

$$P(r * K_n, x) = (x+1)^{k^2(4k^2-1)(2k^2-1)-1}(x-2k^2+1)^{k^2(4k^2-1)-1} \cdot (x-a)(x-b)(x+c),$$

where k is a positive integer, $a = (k-1)(2k+1)$, $b = (k+1)(2k-1)$ and $c = 2k^2$.

Thus this corollary is proved. ■

Corollary 7.1.5. *For positive integers p and q, we have.*

(1) If $r = q^2(p^2+2)$, $n = 2q^2$, and p, q is a positive integral solution of the Diophantine equation

$$p^2 - 2q^2 = -1, \tag{7.1.1}$$

then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

(2) If $r = 2pq(pq-1) > 0$, $n = pq+2$, and p, q is a positive integral solution of the Diophantine equation (7.1.1), then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

Proof. By Theorem 7.1.1 or 7.1.2, we know that $K_{1,r} \bullet K_n$ or $r * K_n$ is integral if and only if $x^3 - (n-2)x^2 - (r+n-1)x + r(n-2)$ can be factorized as $(x-a)(x-b)(x+c)$, where a, b and c are nonnegative integers. Hence $K_{1,r} \bullet K_n$ or $r * K_n$ is integral if and only if the equations

$$\begin{cases} a+b-c = n-2, \\ (a+b)c - ab = r+n-1, \\ abc = r(n-2). \end{cases} \tag{7.1.2}$$

have only integral solutions.

(1) Assume that $r = q^2(p^2+2)$, $n = b = 2q^2$, $a = pq - 1$ $c = a+2 = pq+1$. Thus, when $p^2 - 2q^2 = -1$, we obtain positive integral solutions of Equations (7.1.2). Hence $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

(2) Assume that $r = 2pq(pq-1) > 0$, $n = pq+2$, $a = pq-1$, $b = 2q^2$, $c = 2q^2 - 1$. Thus, when $p^2 - 2q^2 = -1$, we have given positive integral solutions of Equations (7.1.2). Hence, by Theorems 7.1.1 and 7.1.2, $K_{1,r} \bullet K_n$ and $r * K_n$ are integral. ■

Corollary 7.1.6. *Let all positive integral solutions x_k, y_k of Equation* (2.1.9) *be defined by* (2.1.11), $k = 1, 3, 5, \cdots$. *When $d = 2$, $\rho = 1 + \sqrt{2}$, $\overline{\rho} = 1 - \sqrt{2}$, we have.*

(1) For $r = y_{2k-1}^2(x_{2k-1}^2 + 2)$, $n = 2y_{2k-1}^2$, $k = 1, 2, \ldots$, the graphs $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

(2) For $r = 2x_{2k-1}y_{2k-1}(x_{2k-1}y_{2k-1} - 1)$, $n = x_{2k-1} + 2$, $k = 1, 2, \ldots$, the graphs $K_{1,r} \bullet K_n$ and $r * K_n$ are integral.

Proof. (1) By Corollary 7.1.5, we see that if $r = q^2(p^2+2)$, $n = 2q^2$, and p, q is a positive integral solution of Equation (7.1.1), then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral. By Lemma 2.1.12, we know that all positive integral solutions $p = x_{2k-1}$, $q = y_{2k-1}$ of Equation (7.1.1) are given by (2.1.11), for $k = 1, 2, 3, \cdots$, where $d = 2$, $\rho = 1 + \sqrt{2}$, $\overline{\rho} = 1 - \sqrt{2}$. Thus this proof is complete.

(2) Similar to the proof of (1), we can check the correctness by using Corollary 7.1.5 and Lemma 2.1.12. ■

Based on Theorems 7.1.1 and 7.1.2 we obtain

Corollary 7.1.7. *Let a, b, c, r, n be positive integers as in Theorem 7.1.1 or Theorem 7.1.2, and given in Table 7.1. Then $K_{1,r} \bullet K_n$ and $r * K_n$ are integral. (Here a, b, c, r and n are obtained by computer search, and $1 \le a \le 120$,*

$a \le b \le a + 40$, $1 \le c \le a + b$, and they are different from those of Corollaries 7.1.4-7.1.6).

a	b	c	r	n	a	b	c	r	n
63	80	66	4320	79	63	80	78	6048	67

Table 7.1: Integral graphs $K_{1,r} \bullet K_n$ and $r * K_n$.

In view of Theorems 7.1.1 and 7.1.2 and Corollaries 7.1.4-7.1.7, we raise the following question.

Question 7.1.8. *What are all the nonnegative integral solutions for Equations* (7.1.2)*?*

7.2 Integral graphs $K_{1,r} \bullet K_{m,n}$ and $r * K_{m,n}$

In this section, we shall determine the characteristic polynomials of the graphs $K_{1,r} \bullet K_{m,n}$ and $r * K_{m,n}$. We also obtain sufficient and necessary conditions for these graphs to be integral. Some new cospectral graphs and cospectral integral graphs are discussed.

A complete bipartite graph K_{p_1,p_2} is a graph with vertex set V such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where the two vertex classes V_1, V_2 are nonempty disjoint sets, $|V_i| = p_i$ for $i = 1, 2$, and such that two vertices in V are adjacent if and only if they belong to different classes.

Theorem 7.2.1. *Let $K_{m,n}$ be a complete bipartite graph with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, m\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n\}$. Let $K_{1,r} \bullet K_{m,n}$ be the graph obtained by identifying the center w of $K_{1,r}$ with the vertex u_1 of $K_{m,n}$. Then $K_{1,r} \bullet K_{m,n}$ is integral if and only if $x^4 - (mn + r)x^2 + rn(m - 1)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are integers.*

Proof. By (2) of Lemma 1.2.5 and (2) of Lemma 1.2.17, we get

$$\begin{aligned} &P(K_{1,r} \bullet K_{m,n}, x) = x^{r-1}[xP(K_{m,n}, x) - rP(K_{m-1,n}, x)] \\ &= x^{m+n+r-4}[x^4 - (mn + r)x^2 + rn(m - 1)], \end{aligned}$$

proving the theorem. ∎

Corollary 7.2.2. *If $K_{1,r} \bullet K_{m,n}$ is integral, then for any positive integer s the graph $K_{1,rs^2} \bullet K_{m,ns^2}$ is integral too.*

Proof. Because $K_{1,r} \bullet K_{m,n}$ is integral, by Theorem 7.2.1, we find

$$P(K_{1,r} \bullet K_{m,n}, x) = x^{m+n+r-4}[x^4 - (mn+r)x^2 + rn(m-1)]$$
$$= x^{m+n+r-4}(x^2 - a^2)(x^2 - b^2),$$

where a and b are integers. Hence, again by Theorem 7.2.1, we get

$$P(K_{1,rs^2} \bullet K_{m,ns^2}, x) = x^{m+ns^2+rs^2-4}[x^4 - (mn+r)s^2x^2 + rn(m-1)s^4]$$
$$= x^{m+ns^2+rs^2-4}[x^2 - (as)^2][x^2 - (bs)^2].$$

Thus this corollary is proved. ■

Theorem 7.2.3. *Let $K_{m,n}$ be a complete bipartite graph with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, m\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n\}$, let $u_1 \in V_1$ be the root of $K_{m,n}$. Let $r * K_{m,n}$ be the graph obtained by joining the roots of r copies of $K_{m,n}$ to a new vertex w. Then $r * K_{m,n}$ is integral if and only if mn is a perfect square, and $x^4 - (mn+r)x^2 + rn(m-1)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are integers.*

Proof. By Lemma 1.2.5 (1) and Lemma 1.2.17 (2),

$$P(r * K_{m,n}, x) = P^{r-1}(K_{m,n}, x)[xP(K_{m,n}, x) - rP(K_{m-1,n}, x)]$$
$$= x^{r(m+n-2)-1}(x^2 - mn)^{r-1}[x^4 - (mn+r)x^2 + rn(m-1)],$$

and the theorem is proved. ■

Corollary 7.2.4. *If $r * K_{m,n}$ is integral, and $r > 1$, then for any positive integer s the graph $(rs^2) * K_{m,ns^2}$ is integral too.*

Proof. Because $r * K_{m,n}$ is integral, by Theorem 7.2.3, we must have

$$P(r * K_{m,n}, x) = x^{r(m+n-2)-1}(x^2 - mn)^{r-1}[x^4 - (mn+r)x^2 + rn(m-1)]$$
$$= x^{r(m+n-2)-1}(x^2 - mn)^{r-1}(x^2 - a^2)(x^2 - b^2).$$

where mn is a perfect square, and a and b are integers. Hence, Theorem 7.2.3 yields

$$P[(rs^2) * K_{m,ns^2}, x] = x^{rs^2(m+ns^2-2)-1}(x^2 - mns^2)^{rs^2-1}$$
$$\times[x^4 - (mn+r)s^2x^2 + rn(m-1)s^4]$$
$$= x^{rs^2(m+ns^2-2)-1}(x^2 - mns^2)^{rs^2-1}[x^2 - (as)^2][x^2 - (bs)^2],$$

where mn is a perfect square, and a and b are integers. ■

Corollary 7.2.5.

(1) Let $G_1 = (r-1)K_1 \cup r * K_{m,n}$, $G_2 = (r-1)K_{m,n} \cup [K_{1,r} \bullet K_{m,n}]$. Then G_1 and G_2 are cospectral.

(2) If $K_{1,r} \bullet K_{m,n}$ and $K_{m,n}$ are integral, then $r * K_{m,n}$ is integral too.

(3) If $r * K_{m,n}$ and $K_{m,n}$ are integral, then $K_{1,r} \bullet K_{m,n}$ is integral too.

(4) If $r * K_{m,n}$ and $K_{m,n}$ are integral or $K_{1,r} \bullet K_{m,n}$ and $K_{m,n}$ are integral, then $G_1 = (r-1)K_1 \cup r * K_{m,n}$ and $G_2 = (r-1)K_{m,n} \cup [K_{1,r} \bullet K_{m,n}]$ are cospectral integral graphs.

Proof. Follows by using Lemmas 1.2.6, 1.2.7, 1.2.17 as well as Theorems 7.2.1 and 7.2.3. ■

Corollary 7.2.6. *Let a, b, r, m and n be positive integers as in Theorem 7.2.1 or Theorem 7.2.3. Then for any integer s the graph $K_{1,rs^2} \bullet K_{m,ns^2}$ is integral if a, b, r, m, t are given by one of the following cases where k, p, q are positive integers:*

In particular, if mn is a perfect square, then in these cases the graph $(rs^2) * K_{m,ns^2}$ is integral too.

(1) $a = k$, $b = k^2+1$, $r = k^2+1$, $m = k^2+2$ and $n = k^2$,

(2) $a = k^2$, $b = k^2+k$, $r = k^3(k+1)$, $m = k^2+k+1$ and $n = k^2$,

(3) $a = k^2-k$, $b = k^2$, $r = k^3(k-1)$, $m = k^2-k+1$ and $n = k^2$, where $k > 1$,

(4) $a = k$, $b = 2k$, $r = 2k^2$, $m = 3$ and $n = k^2$,

(5) $a = pq$, $b = (p^2+1)q$, $r = (p^2+1)q^2$, $m = p^2+2$ and $n = p^2q^2$,

(6) $a = p(p^2+2)$, $b = (p^2+2)(p^2+1)$, $r = p^2(p^2+2)^3$, $m = p^2+2$ and $n = (p^2+2)(p^2+1)$,

(7) $a = 2q$, $b = 2p$, $r = 2p^2$, $m = 2p^2-1$ and $n = 2$, where $p^2-2q^2 = 1$,

(8) $a = 2q$, $b = 2p$, $r = 8q^2$, $m = p^2+1$ and $n = 2$, where $p^2-2q^2 = 1$,

(9) $a = p$, $b = 2q$, $r = 2q^2$, $m = 4q^2-1$ and $n = 1$, where $p^2-2q^2 = -1$,

(10) $a = p$, $b = 2q$, $r = 2p^2$, $m = 2q^2+1$ and $n = 1$, where $p^2-2q^2 = -1$,

(11) $a = 3q$, $b = 2p$, $r = 3p^2$, $m = 12q^2 + 1$ *and* $n = 1$, *where* $p^2 - 3q^2 = 1$,

(12) $a = 3q$, $b = 2p$, $r = 12q^2$, $m = 3p^2 + 1$ *and* $n = 1$, *where* $p^2 - 3q^2 = 1$,

(13) $a = 2q$, $b = 3p$, $r = 6p^2$, $m = 6q^2 + 1$ *and* $n = 1$, *where* $3p^2 - 2q^2 = 1$,

(14) $a = 2q$, $b = 3p$, $r = 6q^2$, $m = 6p^2 + 1$ *and* $n = 1$, *where* $3p^2 - 2q^2 = 1$,

(15) $a = q^2 - 1$, $b = pq$, $r = (q^2 - 1)q^2$, $m = p^2 + 1$ *and* $n = q^2 - 1$, *where* $p^2 - 2q^2 = -2$,

(16) $a = q^2 - 1$, $b = pq$, $r = (q^2 - 1)p^2$, $m = q^2 + 1$ *and* $n = q^2 - 1$, *where* $p^2 - 2q^2 = -2$,

(17) $a = (k^2 + 1)(k^2 + 2)$, $b = k(k^2 + 2)$, $r = k^2(k^2 + 2)^3$, $m = k^2 + 2$ *and* $n = (k^2 + 2)(k^2 + 1)$,

(18) $a = q(q^2+1)$, $b = p(q^2+1)$, $r = (q^2+1)^3$, $m = 2(q^2+1)$ *and* $n = q^2(q^2+1)$, *where* $p^2 - 2q^2 = 1$,

(19) $a = p$, $b = 2q$, $r = 2q^2$, $m = 2q^2 - 1$ *and* $n = 2$, *where* $p^2 - 2q^2 = -2$,

(20) $a = k(k^2+k+1)$, $b = (k+1)(k^2+k+1)$, $r = (k^2+k+1)^3$, $m = k^2+k+1$ *and* $n = k(k + 1)(k^2 + k + 1)$.

Proof. We only prove (2). The other cases can be proved similarly. By Theorem 7.2.1, the graph $K_{1,r} \bullet K_{m,n}$ is integral if and only if $x^4 - (mn + r)x^2 + rn(m - 1)$ can be factorized as $(x^2 - a^2)(x^2 - b^2)$, where a and b are integers. Hence $K_{1,r} \bullet K_{m,n}$ is integral if and only if the equations

$$\begin{cases} a^2 + b^2 & = mn + r, \\ a^2b^2 & = rn(m - 1) \end{cases} \tag{7.2.1}$$

have only integral roots. Choosing $a = k^2$, $b = k^2 + k$, $r = k^3(k + 1)$, $m = k^2 + k + 1$ and $n = k^2$, where k is a positive integers, we can check that they are positive integral solutions of Equations (7.2.1). Hence, by Theorem 7.2.1 and Corollary 7.2.2, $K_{1,rs^2} \bullet K_{m,ns^2}$ is integral. By Theorem 7.2.3 and Corollary 7.2.4, if mn is a perfect square, then the graph $(rs^2) * K_{m,ns^2}$ is integral too. Thus this corollary is proved. ■

Remark 7.2.7. *For the Diophantine equations* $p^2 - 2q^2 = 1$, $p^2 - 2q^2 = -1$, $p^2 - 3q^2 = 1$, $3p^2 - 2q^2 = 1$, $p^2 - 2q^2 = -2$ *in Corollary 7.2.6, we can find all positive integral solutions from Lemmas 2.1.8-2.1.12 and 2.1.14.*

Let the tree $T[m,r]$ of diameter 3 be formed by joining the centers of $K_{1,m}$ and $K_{1,r}$ with a new edge.

Corollary 7.2.8. *(See [15]) For any positive integer t, we have.*

(1) (see also [79]) When $r = m-1 = t(t+1)$ and $n = 1$, then the tree $K_{1,r} \bullet K_{m,n} = T[r, m-1]$ of diameter 3 is integral.

(2) When $1 \le r < m-1$, $n = 1$, and $(r, m-1) = d$ the following holds.

(i) If d is a perfect square, then $K_{1,r} \bullet K_{m,n} = T[r, m-1] = K_{1,m-1} \bullet K_{r+1,n} = T[m-1, r]$ is not an integral tree.

(ii) If d is a positive integer but not a perfect square, then all integral trees $K_{1,r} \bullet K_{m,n} = T[r, m-1] = K_{1,m-1} \bullet K_{r+1,n} = T[m-1, r]$ (where $1 \le r < m-1$, $n = 1$) are given by

$$r = d(\frac{y_k - y_l}{2})^2, \quad m = 1 + d(\frac{y_k + y_l}{2})^2, \quad n = 1, \quad k > l > 0,$$

where y_k, y_l are odd or even, $y_k, y_l \in \{y_p | y_0 = 0, y_1 = b_1, y_{p+2} = 2a_1 y_{p+1} - y_p, (p \ge 0)\}$, and $a_1 + b_1\sqrt{d}$ is the fundamental solution of Equation (2.1.4).

Corollary 7.2.9. *For positive integers t and s, we have.*

(1) When $n = 1$, $r = m-1 = t(t+1)$, then $K_{1,rs^2} \bullet K_{m,ns^2}$ is an integral graph.

(2) When $n = 1$, $1 \le r < m-1$, $(r, m-1) = d$, then we have the following results.

(i) If d is a perfect square, then $K_{1,rs^2} \bullet K_{m,ns^2}$ and $K_{1,(m-1)s^2} \bullet K_{r+1,ns^2}$ are not integral graphs.

(ii) If d is a positive integer but not a perfect square, let $r = d(\frac{y_k - y_l}{2})^2$, $m = 1 + d(\frac{y_k + y_l}{2})^2$, $k > l > 0$, where y_k, y_l are odd or even, $y_k, y_l \in \{y_p | y_0 = 0, y_1 = b_1, y_{p+2} = 2a_1 y_{p+1} - y_p, (p \ge 0)\}$, and $a_1 + b_1\sqrt{d}$ is the fundamental solution of the Equation (2.1.4). Then $K_{1,rs^2} \bullet K_{m,ns^2}$ and $K_{1,(m-1)s^2} \bullet K_{r+1,ns^2}$ are integral graphs.

Proof. The result follows easily by using Corollaries 7.2.2 and 7.2.8. ■

Based on Theorem 7.2.1 and Corollary 7.2.2 we obtain

Corollary 7.2.10. *Let a, b, r, m and n be positive integers as in Theorem 7.2.1, and given in Table 7.2, then the graph $K_{1,rs^2} \bullet K_{m,ns^2}$ is an integral graph. (Table 7.2 is obtained by computer search, where $1 \le a \le 30$, $a \le b \le a+20$, and they are not those of Corollaries 7.2.6, 7.2.8 and 7.2.9).*

a	b	r	m	n	a	b	r	m	n
4	15	25	3	72	6	10	40	16	6
7	20	50	57	7	9	14	189	22	4
12	15	150	73	3	12	15	216	51	3
12	17	153	35	8	12	20	384	16	10
12	21	147	73	6	14	30	200	64	14
14	33	1078	23	9	15	18	243	51	6
15	20	250	25	15	15	20	375	25	10
20	26	416	66	10	20	27	405	181	4
21	24	567	225	2	21	26	637	40	12
21	32	448	113	9	22	25	605	126	4
22	30	880	56	9	24	35	1176	25	25
25	42	1750	71	9	28	30	800	442	2
28	30	882	401	2	28	33	847	57	18
28	33	1078	265	3	30	35	1125	50	20
30	37	925	112	12	/	/	/	/	/

Table 7.2: Integral graphs $K_{1,rs^2} \bullet K_{m,ns^2}$, where s is a positive integer.

From Theorems 7.2.1 and 7.2.3 and Corollaries 7.2.2 and 7.2.4 we easily deduce

Corollary 7.2.11. *Let $a = 24$, $b = 35$, $r = 1176$, $m = 25$ and $n = 25$, (see Theorem 7.2.1 or Theorem 7.2.3), then for any integer s both graphs $K_{1,rs^2} \bullet K_{m,ns^2}$ and $(rs^2) * K_{m,ns^2}$ are integral.*

In view of Theorems 7.2.1 and 7.2.3 and Corollaries 7.2.6, 7.2.8-7.2.11, we raise the following question.

Question 7.2.12. *What are all the positive integral solutions for Equation* (7.2.1)*?*

Chapter 8

Two classes of Laplacian integral and integral regular graphs

In this chapter, we derive the spectra and characteristic polynomials of the two classes of regular graphs $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$, as well as the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. When $p = n^2 + n + 1$, these graphs are not only integral but also Laplacian integral. These results generalize some results of Harary and Schwenk in [36].

8.1 The characteristic polynomials of two classes of regular graphs

In this section, we determine the characteristic polynomials of $K_{n,n+1} \equiv K_{n+1,n}$ and $K_{1,p}[(p-1)K_p]$.

The $(n+1)$-regular graph $K_{n,n+1} \equiv K_{n+1,n}$ on $4n+2$ vertices is obtained from two disjoint copies of $K_{n,n+1}$ with vertex classes $V_1 = \{u_i | i = 1, 2, \cdots, n\}$, $V_2 = \{v_i | i = 1, 2, \cdots, n+1\}$ and $U_1 = \{z_i | i = 1, 2, \cdots, n\}$, $U_2 = \{w_i | i = 1, 2, \cdots, n+1\}$, respectively, by adding the edges $\{v_i w_i | i = 1, 2, \cdots, n+1\}$. Let $K_{1,p}$ be a graph with vertex classes $V_1 = \{u_1\}$ and $V_2 = \{v_i | i = 1, 2, \cdots, p\}$. The i-th graph K_p of $(p-1)K_p$ has the vertex set $\{w_{i_j} | j = 1, 2, \cdots, p\}$, where $i = 1, 2, \cdots, p-1$. Then the p-regular graph $K_{1,p}[(p-1)K_p]$ on p^2+1 vertices is obtained by adding the edges $\{v_i w_{j_i} | j = 1, 2, \cdots, p-1\}$, $i = 1, 2, \cdots, p$, between the graph $K_{1,p}$ and the graph $(p-1)K_p$.

Theorem 8.1.1. *For the regular graph $K_{n,n+1} \equiv K_{n+1,n}$ of degree $(n+1)$ with $4n+2$ vertices, the characteristic polynomial is*

$$P(K_{n,n+1} \equiv K_{n+1,n}, x) = (x+n+1)(x+n)(x+1)^n x^{2n-2}(x-1)^n(x-n)(x-n-1).$$

Proof. By properly ordering the vertices of the graph $K_{n,n+1} \equiv K_{n+1,n}$, the adjacency matrix $A = A(K_{n,n+1} \equiv K_{n+1,n})$ can be written as the $(4n+2) \times (4n+2)$ symmetric block circulant matrix such that $A = A(K_{n,n+1} \equiv K_{n+1,n}) \in \mathcal{BC}(2, 2n+1)$ and

$$A = A(K_{n,n+1} \equiv K_{n+1,n}) = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix},$$

where $A_0 = \begin{bmatrix} 0_{n\times n} & J_{n\times(n+1)} \\ J_{(n+1)\times n} & 0_{(n+1)\times(n+1)} \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0_{n\times n} & 0_{n\times(n+1)} \\ 0_{(n+1)\times n} & I_{n+1} \end{bmatrix}$.

In view of Lemma 2.2.1, we distinguish between the following two cases.

Case 1. Let $b_0 = |xI_{2n+1} - (A_0 + A_1)|$, *i.e.*,

$$b_0 = \begin{vmatrix} xI_n & -J_{n\times(n+1)} \\ -J_{(n+1)\times n} & (x-1)I_{n+1} \end{vmatrix}.$$

After some careful calculation, we obtain

$$b_0 = x^{n-1}(x-1)^n(x+n)(x-n-1).$$

Case 2. Let $b_1 = |xI_{2n+1} - (A_0 - A_1)|$, *i.e.*,

$$b_1 = \begin{vmatrix} xI_n & -J_{n\times(n+1)} \\ -J_{(n+1)\times n} & (x+1)I_{n+1} \end{vmatrix}.$$

Here, we find

$$b_1 = x^{n-1}(x+1)^n(x-n)(x+n+1).$$

Hence, the characteristic polynomial of $K_{n,n+1} \equiv K_{n+1,n}$ is (see Lemma 2.2.1)

$$P(K_{n,n+1} \equiv K_{n+1,n}, x) = (x+n+1)(x+n)(x+1)^n x^{2n-2}(x-1)^n(x-n)(x-n-1),$$

and the proof is complete. ■

We note that the graph $K_{1,2} \equiv K_{2,1}$ coincides with the cycle C_6 and the graph $K_{2,3} \equiv K_{3,2}$ with the graph 3.20 of [22] (see page 293) or G_{10} of [63].

Theorem 8.1.2. *For the regular graph $K_{1,p}[(p-1)K_p]$ of degree p with p^2+1 vertices, the characteristic polynomial is*

$$P(K_{1,p}[(p-1)K_p],x) = (x+1)^{(p-1)(p-2)}(x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p.$$

Proof. By properly ordering the vertices of the graph $K_{1,p}[(p-1)K_p]$, the adjacency matrix $A = A(K_{1,p}[(p-1)K_p])$ can be written as the $(p^2+1)\times(p^2+1)$ matrix

$$\begin{aligned} A &= A(K_{1,p}[(p-1)K_p]) \\ &= \begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_{p-2} & A_{p-1} & I_p & 0_{p\times 1} \\ A_{p-1} & A_1 & A_2 & \cdots & A_{p-3} & A_{p-2} & I_p & 0_{p\times 1} \\ A_{p-2} & A_{p-1} & A_1 & \cdots & A_{p-4} & A_{p-3} & I_p & 0_{p\times 1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ A_3 & A_4 & A_5 & \cdots & A_1 & A_2 & I_p & 0_{p\times 1} \\ A_2 & A_3 & A_4 & \cdots & A_{p-1} & A_1 & I_p & 0_{p\times 1} \\ I_p & I_p & I_p & \cdots & I_p & I_p & 0_{p\times p} & J_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 0_{1\times p} & \cdots & 0_{1\times p} & 0_{1\times p} & J_{1\times p} & 0 \end{bmatrix}, \end{aligned}$$

where $A_1 = J_{p\times p} - I_p$ and $A_2 = A_3 = \cdots = A_{p-1} = 0_{p\times p}$. Then we have

$$\begin{aligned} P(K_{1,p}[(p-1)K_p],x) &= |xI_{p^2+1} - A(K_{1,p}[(p-1)K_p])| \\ &= \begin{vmatrix} X_{p\times p} & 0_{p\times p} & \cdots & 0_{p\times p} & -I_p & 0_{p\times 1} \\ 0_{p\times p} & X_{p\times p} & \cdots & 0_{p\times p} & -I_p & 0_{p\times 1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{p\times p} & 0_{p\times p} & \cdots & X_{p\times p} & I_p & 0_{p\times 1} \\ -I_p & -I_p & \cdots & -I_p & xI_p & -J_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & \cdots & 0_{1\times p} & -J_{1\times p} & x \end{vmatrix}, \end{aligned}$$

where $X_{p\times p} = (x+1)I_p - J_{p\times p}$.

After some careful calculation, we can prove that the characteristic polynomial of $K_{1,p}[(p-1)K_p]$ is

$$P(K_{1,p}[(p-1)K_p],x) = (x+1)^{(p-1)(p-2)}(x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p.$$

■

We note that the graph $K_{1,3}[2K_3]$ coincides with the graph 3.16 of [22] (see page 293) or G_{11} of [63].

8.2 Other results

In this section, we shall give the characteristic polynomials for $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$, $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$, $L(K_{1,p}[(p-1)K_p])$, $\overline{K_{1,p}[(p-1)K_p]}$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$. For integers $n \geq 0$ and $m \geq 0$, if a regular graph G is integral, then as we will show the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are not only integral but also Laplacian integral. We also mention some interesting results on characteristic polynomials of integral graphs, see [5, 14, 15, 16, 19, 21, 22, 36, 37, 38, 46, 47, 48, 50, 55, 60, 63, 69, 71, 73, 75, 76, 79, 85].

Theorem 8.2.1. *For the complement $\overline{K_{n,n+1} \equiv K_{n+1,n}}$ of the regular graph $K_{n,n+1} \equiv K_{n+1,n}$, the characteristic polynomial is*
$P(\overline{K_{n,n+1} \equiv K_{n+1,n}}, x) = (x+n+1)(x+2)^n(x+1)^{2n-2}x^n(x-n+1)(x-n)(x-3n)$.

Proof. Because $K_{n,n+1} \equiv K_{n+1,n}$ is a regular graph of degree $(n+1)$ with $4n+2$ vertices, by Theorem 8.1.1 and Lemma 1.2.19, we get

$$\begin{aligned}&P(\overline{K_{n,n+1} \equiv K_{n+1,n}}, x)\\ &= (-1)^{4n+2}\frac{x-4n-2+n+2}{x+n+2}P(K_{n,n+1} \equiv K_{n+1,n}, -x-1)\\ &= (x+n+1)(x+2)^n(x+1)^{2n-2}x^n(x-n+1)(x-n)(x-3n).\end{aligned}$$

Thus, this theorem is proved. ■

Theorem 8.2.2. *For the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{n,n+1} \equiv K_{n+1,n}$, we have the following results.*

(1) The characteristic polynomial of $L(K_{n,n+1} \equiv K_{n+1,n})$ is

$$P(L(K_{n,n+1} \equiv K_{n+1,n}), x) = (x+2)^{n(2n-1)}(x+1)(x-n+2)^n(x-n+1)^{2n-2}(x-n)^n(x-2n+1)(x-2n).$$

(2) The characteristic polynomial of $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ is

$$P(\overline{L(K_{n,n+1} \equiv K_{n+1,n})}, x) = (x+2n)(x+n+1)^n(x+n)^{2n-2}(x+n-1)^n x(x-1)^{n(2n-1)}(x-2n^2-n).$$

(3) The characteristic polynomial of $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ is

$$P(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}), x) = (x+2)^{(2n+1)(3n-2)}(x-2n+3)(x-3n+4)^n(x-3n+3)^{2n-2}(x-3n+2)^n(x-4n+3)(x-4n+2)(x-6n+2).$$

Proof. (1) Because $K_{n,n+1} \equiv K_{n+1,n}$ is a regular graph of degree $(n+1)$ with $4n+2$ vertices, by Theorem 8.1.1 and Lemma 1.2.20, we get

$$\begin{aligned}
&P(L(K_{n,n+1} \equiv K_{n+1,n}), x)\\
&= (x+2)^{(2n+1)(n-1)} P(K_{n,n+1} \equiv K_{n+1,n}, x+2-n-1)\\
&= (x+2)^{n(2n-1)}(x+1)(x-n+2)^n(x-n+1)^{2n-2}(x-n)^n\\
&\cdot(x-2n+1)(x-2n).
\end{aligned}$$

(2) Because $K_{n,n+1} \equiv K_{n+1,n}$ is a regular graph of degree $(n+1)$ with $4n+2$ vertices, by Lemma 1.2.18, we find that $L(K_{n,n+1} \equiv K_{n+1,n})$ is a regular graph of degree $2n$ with $(2n+1)(n+1)$ vertices. By (1) of Theorem 8.2.2 and Lemma 1.2.19, we get

$$\begin{aligned}
&P(\overline{L(K_{n,n+1} \equiv K_{n+1,n})}, x)\\
&= (-1)^{(2n+1)(n+1)} \frac{x-(2n+1)(n+1)+2n+1}{x+2n+1} P(L(K_{n,n+1} \equiv K_{n+1,n}), -x-1)\\
&= (x+2n)(x+n+1)^n(x+n)^{2n-2}(x+n-1)^n x(x-1)^{n(2n-1)}\\
&\cdot(x-2n^2-n).
\end{aligned}$$

(3) Because $K_{n,n+1} \equiv K_{n+1,n}$ is a regular graph of degree $(n+1)$ with $4n+2$ vertices, it follows that $\overline{K_{n,n+1} \equiv K_{n+1,n}}$ is a regular graph of degree $3n$ with $(4n+2)$ vertices. By Theorem 8.2.1 and Lemma 1.2.20, we get

$$\begin{aligned}
&P(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}), x)\\
&= (-1)^{(2n+1)(3n-2)} P(\overline{K_{n,n+1} \equiv K_{n+1,n}}, x+2-3n)\\
&= (x+2)^{(2n+1)(3n-2)}(x-2n+3)(x-3n+4)^n(x-3n+3)^{2n-2}\\
&\cdot(x-3n+2)^n(x-4n+3)(x-4n+2)(x-6n+2).
\end{aligned}$$

Thus, this theorem is proved. ■

Corollary 8.2.3. *The graphs* $K_{n,n+1} \equiv K_{n+1,n}$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ *and* $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ *are integral.*

Proof. The result follows directly from Theorems 8.1.1, 8.2.1 and 8.2.2. ■

Theorem 8.2.4. *For the complement of the regular graph* $K_{1,p}[(p-1)K_p]$, *the characteristic polynomial reads*

$$P(\overline{K_{1,p}[(p-1)K_p]}, x) = x^{(p-1)(p-2)}(x+p)^{p-2}(x-p^2+p)(x^2+x-p+1)^p.$$

Proof. Because $K_{1,p}[(p-1)K_p]$ is a regular graph of degree p with p^2+1 vertices, by Theorem 8.1.2 and Lemma 1.2.19, we get

$$\begin{aligned}
&P(\overline{K_{1,p}[(p-1)K_p]}, x)\\
&= (-1)^{p^2+1} \frac{x-p^2+p}{x+p+1} P(K_{1,p}[(p-1)K_p], -x-1)\\
&= x^{(p-1)(p-2)}(x+p)^{p-2}(x-p^2+p)(x^2+x-p+1)^p.
\end{aligned}$$

Thus , this theorem is proved ■

Theorem 8.2.5. *For the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{1,p}[(p-1)K_p]$, we have the following results.*

(1) The characteristic polynomial of $L(K_{1,p}[(p-1)K_p])$ is

$$P(L(K_{1,p}[(p-1)K_p]),x) = (x+2)^{\frac{1}{2}(p-2)(p^2+1)}(x-p+3)^{(p-1)(p-2)} \cdot(x-2p+3)^{p-2}(x-2p+2)[(x-p+2)^2+(x-p+2)-(p-1)]^p.$$

(2) The characteristic polynomial of $\overline{L(K_{1,p}[(p-1)K_p])}$ is

$$P(\overline{L(K_{1,p}[(p-1)K_p])},x) = (x-1)^{\frac{1}{2}(p-2)(p^2+1)}(x+p-2)^{(p-1)(p-2)}(x-2+2p)^{p-2}[x-\tfrac{1}{2}(p-1)^2(p+2)][(x+p-1)^2-(x+p-1)-(p-1)]^p.$$

(3) The characteristic polynomial of $L(\overline{K_{1,p}[(p-1)K_p]})$ is

$$P(L(\overline{K_{1,p}[(p-1)K_p]}),x) = [(x-p^2+p+2)^2+(x-p^2+p+2)-(p-1)]^p(x-p^2+2p+2)^{p-2}(x-2p^2+2p+2)(x+2)^{\frac{1}{2}(p+1)(p-2)(p^2+1)} \cdot(x-p^2+p+2)^{(p-1)(p-2)}.$$

Proof.(1) Because $K_{1,p}[(p-1)K_p]$ is a regular graph of degree p with p^2+1 vertices, by Theorem 8.1.2 and Lemma 1.2.20, we obtain

$$\begin{aligned}
&P(L(K_{1,p}[(p-1)K_p]),x)\\
&= (x+2)^{\frac{1}{2}(p^2+1)(p-2)}P(K_{1,p}[(p-1)K_p],x+2-p)\\
&= (x+2)^{\frac{1}{2}(p-2)(p^2+1)}(x-p+3)^{(p-1)(p-2)}(x-2p+3)^{p-2}\\
&\cdot(x-2p+2)[(x-p+2)^2+(x-p+2)-(p-1)]^p.
\end{aligned}$$

(2) Because $K_{1,p}[(p-1)K_p]$ is a regular graph of degree p with p^2+1 vertices, by Lemma 1.2.18, the graph $L(K_{1,p}[(p-1)K_p])$ is regular of degree $2(p-1))$ with $\frac{1}{2}p(p^2+1)$ vertices. Theorem 8.2.5 (1) and Lemma 1.2.19 yields

$$\begin{aligned}
&P(\overline{L(K_{1,p}[(p-1)K_p])},x)\\
&= (-1)^{\frac{1}{2}p(p^2+1)}\frac{x-\frac{1}{2}p(p^2+1)+2p-1}{x+2p-1}P(L(K_{1,p}[(p-1)K_p]),-x-1)\\
&= (x-1)^{\frac{1}{2}(p-2)(p^2+1)}(x+p-2)^{(p-1)(p-2)}(x+2p-2)^{p-2}\\
&\cdot[x-\tfrac{1}{2}(p-1)^2(p+2)][(x+p-1)^2-(x+p-1)-(p-1)]^p.
\end{aligned}$$

(3) Because $K_{1,p}[(p-1)K_p]$ is a regular graph of degree p with p^2+1 vertices, the graph $\overline{K_{1,p}[(p-1)K_p]}$ is regular of degree $p(p-1)$ with (p^2+1) vertices. By Theorem 8.2.4 and Lemma 1.2.20, we get

$$\begin{aligned}
&P(L(\overline{K_{1,p}[(p-1)K_p]}), x)\\
&= (-1)^{\frac{1}{2}(p^2+1)(p^2-p-2)} P(\overline{K_{1,p}[(p-1)K_p]}, x+2-p^2+p)\\
&= [(x-p^2+p+2)^2 + (x-p^2+p+2) - (p-1)]^p (x-p^2+2p+2)^{p-2}\\
&\cdot (x-2p^2+2p+2)(x+2)^{\frac{1}{2}(p+1)(p-2)(p^2+1)} (x-p^2+p+2)^{(p-1)(p-2)}.
\end{aligned}$$

■

Corollary 8.2.6. *The graphs $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ and $L(\overline{K_{1,p}[(p-1)K_p]})$ are integral if and only if $p = n^2+n+1$, where n is any positive integer.*

Proof. By Theorem 8.1.2, we get $P(K_{1,p}[(p-1)K_p], x) = (x+1)^{(p-1)(p-2)}(x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p$. Hence, the regular graph $K_{1,p}[(p-1)K_p]$ is integral if and only if $p-1 = n(n+1)$, *i.e.*, $p = n^2+n+1$, where n is a positive integer.

Similarly, we can prove the other results by using Theorems 8.2.4 and 8.2.5. ■

Corollary 8.2.7. *For the regular graphs $K_{1,p}[(p-1)K_p]$, the line graph, the complement of the line graph and the line graph of the complement of the regular graph $K_{1,p}[(p-1)K_p]$, let $p = n^2+n+1$ and n be any positive integer. Then we have the following results.*

(1) $P(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}], x) = (x+1)^{n(n+1)(n^2+n-1)}(x-n^2-n-1)[x-n(n+1)]^{n^2+n-1}(x+n+1)^{n^2+n+1}(x-n)^{n^2+n+1}$.

(2) $P(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}, x) = x^{n(n+1)(n^2+n-1)}(x+n^2+n+1)^{n^2+n-1}[x-n(n+1)(n^2+n+1)](x+n+1)^{n^2+n+1}(x-n)^{n^2+n+1}$.

(3) $P(L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]), x) = (x+2)^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}[x-(n+2)(n-1)]^{n(n+1)(n^2+n-1)}(x-2n^2-2n+1)^{n^2+n-1}[x-2n(n+1)](x-n^2+2)^{n^2+n+1}(x-n^2-2n+1)^{n^2+n+1}$.

(4) $P(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])}, x) = (x-1)^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}(x+n^2-1)^{n^2+n+1}(x+n^2+n-1)^{n(n+1)(n^2+n-1)}(x+2n^2+2n)^{n^2+n-1}[x-\frac{1}{2}n^2(n+1)^2(n^2+n+3)](x+n^2+2n)^{n^2+n+1}$.

(5) $P(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}), x) = (x-2n^4-4n^3-4n^2-2n+2)[x-(n^2+n-1)(n^2+n+2)]^{n(n+1)(n^2+n-1)}(x-n^4-2n^3-n^2+3)^{n^2+n-1}(x-$

$n^4 - 2n^3 - 2n^2 + 3)^{n^2+n+1}(x - n^4 - 2n^3 - 2n^2 - 2n + 2)^{n^2+n+1}(x + 2)^{\frac{1}{2}(n^2+n-1)(n^2+n+2)[(n^2+n+1)^2+1]}$.

Proof. (1) By Theorem 8.1.2, we get $P(K_{1,p}[(p-1)K_p], x) = (x+1)^{(p-1)(p-2)} (x-p+1)^{p-2}(x-p)(x^2+x-p+1)^p$. Hence, when $p = n^2+n+1$, we have

$P(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}], x) = (x+1)^{n(n+1)(n^2+n-1)}(x - n^2 - n - 1)[x - n(n+1)]^{n^2+n-1}(x+n+1)^{n^2+n+1}(x-n)^{n^2+n+1}$.

From Theorems 8.2.4 and 8.2.5, we can prove (2)-(5) in the same way. ■

Theorem 8.2.8. *If a regular graph G is integral, then for any integers $n \geq 0$ and $m \geq 0$ the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are integral.*

Proof. This theorem follows by using Lemmas 1.2.18, 1.2.21 and 1.2.22. ■

Corollary 8.2.9. *With integer $n \geq 1$, let $G = K_{n,n+1} \equiv K_{n+1,n}$ and $H = K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]$. Then for integers $p \geq 0$, $m \geq 0$ the graphs $L^m(G)$, $L^p(\overline{L^m(G)})$, $L^m(H)$ and $L^p(\overline{L^m(H)})$ are integral.*

Proof. The result follows by using Lemma 1.2.18, Corollaries 8.2.3, 8.2.6, 8.2.7 and Theorems 8.1.1, 8.1.2, 8.2.1, 8.2.2, 8.2.4, 8.2.5 and 8.2.8. ■

In the remainder of the chapter, we shall consider Laplacian integral graphs.

Theorem 8.2.10. *For the σ-polynomials of the regular graphs G, $\overline{G}$, $L(G)$, $\overline{L(G)}$ and $L(\overline{G})$, where $G = K_{n,n+1} \equiv K_{n+1,n}$, we have the following*

(1) The characteristic polynomial of $Lap(K_{n,n+1} \equiv K_{n+1,n})$ is

$\sigma(K_{n,n+1} \equiv K_{n+1,n}, x) = x(x-1)(x-n)^n(x-n-1)^{2n-2}(x-n-2)^n (x-2n-1)(x-2n-2)$.

(2) The characteristic polynomial of $Lap(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ is

$\sigma(\overline{K_{n,n+1} \equiv K_{n+1,n}}, x) = x(x-2n)(x-2n-1)(x-3n)^n(x-3n-1)^{2n-2}(x-3n-2)^n(x-4n-1)$.

(3) The characteristic polynomial of $Lap(L(K_{n,n+1} \equiv K_{n+1,n}))$ is

$\sigma(L(K_{n,n+1} \equiv K_{n+1,n}), x) = x(x-1)(x-n)^n(x-n-1)^{2n-2}(x-n-2)^n(x-2n-1)(x-2n-2)^{n(2n-1)}$.

(4) The characteristic polynomial of $Lap(\overline{L(K_{n,n+1} \equiv K_{n+1,n})})$ is

$\sigma(\overline{L(K_{n,n+1} \equiv K_{n+1,n})}, x) = x(x-2n^2-n+1)^{n(2n-1)}(x-2n^2-n)(x-2n^2-2n-1)^n(x-2n^2-2n)^{2n-2}(x-2n^2-2n+1)^n(x-2n^2-3n)$.

(5) The characteristic polynomial of $Lap(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}))$ *is*

$$\sigma(L(\overline{K_{n,n+1} \equiv K_{n+1,n}}), x) = x(x-2n)(x-2n-1)(x-3n)^n(x-3n-1)^{2n-2}(x-3n-2)^n(x-4n-1)(x-6n)^{(2n+1)(3n-2)}.$$

Proof. Because $K_{n,n+1} \equiv K_{n+1,n}$ is a regular graph of degree $n+1$ with $(4n+2)$ vertices, the graph $\overline{K_{n,n+1} \equiv K_{n+1,n}}$ is regular of degree $3n$ with $(4n+2)$ vertices. By Lemma 1.2.18, we find that $L(K_{n,n+1} \equiv K_{n+1,n})$ is a regular graph of degree $2n$ with $(2n+1)(n+1)$ vertices, that $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ is a regular graph of degree $(6n-2)$ with $3n(2n+1)$ vertices, and that $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ is a regular graph of degree $n(2n+1)$ with $(2n+1)(n+1)$ vertices.

By Lemma 1.2.23 and Theorem 8.1.1,

$$\sigma(K_{n,n+1} \equiv K_{n+1,n}, x) = (-1)^{4n+2}P(K_{n,n+1} \equiv K_{n+1,n}, n+1-x)$$
$$= x(x-1)(x-n)^n(x-n-1)^{2n-2}(x-n-2)^n(x-2n-1)(x-2n-2).$$

Thus, (1) is proved.

The formulae (2)-(5) are proved by Theorems 8.2.1, 8.2.2 and Lemmas 1.2.23, 1.2.24. ■

The next result is a direct consequence of Theorem 8.2.10 and Lemmas 1.2.23, 1.2.24, 1.2.25.

Corollary 8.2.11. *The graphs* $K_{n,n+1} \equiv K_{n+1,n}$, $\overline{K_{n,n+1} \equiv K_{n+1,n}}$, $L(K_{n,n+1} \equiv K_{n+1,n})$, $\overline{L(K_{n,n+1} \equiv K_{n+1,n})}$ *and* $L(\overline{K_{n,n+1} \equiv K_{n+1,n}})$ *are Laplacian integral.*

Theorem 8.2.12. *For the σ-polynomials of the regular graphs* $K_{1,p}\,[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $\overline{L(K_{1,p}[(p-1)K_p])}$, $L(\overline{K_{1,p}[(p-1)K_p]})$ *and* $L(K_{1,p}[(p-1)K_p])$ *the following holds.*

(1) The characteristic polynomial of $Lap(K_{1,p}[(p-1)K_p])$ *is*

$$\sigma(K_{1,p}[(p-1)K_p], x) = x(x-1)^{p-2}(x-p-1)^{(p-1)(p-2)}[(x-p)^2-(x-p)-(p-1)]^p.$$

(2) The characteristic polynomial of $Lap(\overline{K_{1,p}[(p-1)K_p]})$ *is*

$$\sigma(\overline{K_{1,p}[(p-1)K_p]}, x) = x(x-p^2+p)^{(p-1)(p-2)}(x-p^2)^{p-2}\cdot[(x-p^2+p)^2-(x-p^2+p)-(p-1)]^p.$$

(3) The characteristic polynomial of $Lap(L(K_{1,p}[(p-1)K_p]))$ *is*

$$\sigma(L(K_{1,p}[(p-1)K_p]), x) = x(x-p-1)^{(p-1)(p-2)}(x-1)^{p-2}\cdot(x-2p)^{\frac{1}{2}(p-2)(p^2+1)}[(x-p)^2-(x-p)-(p-1)]^p.$$

(4) The characteristic polynomial of $Lap(\overline{L(K_{1,p}[(p-1)K_p]})$ *is*

$$\sigma(\overline{L(K_{1,p}[(p-1)K_p])}, x) = x[x-\tfrac{1}{2}p(p^2-3)]^{\frac{1}{2}(p-2)(p^2+1)}[x-\tfrac{1}{2}(p^3-p-2)]^{(p-1)(p-2)} \cdot \{[x-\tfrac{1}{2}p(p-1)(p+1)]^2-[x-\tfrac{1}{2}p(p-1)(p+1)]-(p-1)\}^p \cdot [x-\tfrac{1}{2}(p-1)(p^2+p+2)]^{p-2}$$

(5) The characteristic polynomial of $Lap(L(\overline{K_{1,p}[(p-1)K_p]}))$ *is*

$$\sigma(L(\overline{K_{1,p}[(p-1)K_p]}), x) = x(x-p^2+p)^{(p-1)(p-2)}(x-p^2)^{p-2} \cdot (x-2p^2+2p)^{\frac{1}{2}(p+1)(p-2)(p^2+1)}[(x-p^2+p)^2-(x-p^2+p)-(p-1)]^p.$$

Proof. The result follows similarly to Theorem 8.2.10 (1) by using Theorems 8.1.2, 8.2.4, 8.2.5 and Lemmas 1.2.23, 1.2.24. ■

Corollary 8.2.13. *Any one of the graphs* $K_{1,p}[(p-1)K_p]$, $\overline{K_{1,p}[(p-1)K_p]}$, $L(K_{1,p}[(p-1)K_p])$, $\overline{L(K_{1,p}[(p-1)K_p])}$ *and* $L(\overline{K_{1,p}[(p-1)K_p]})$ *is Laplacian integral if and only if* $p = n^2+n+1$, *where* n *is a positive integer.*

Proof. By Theorem 8.2.12, we have $\sigma[K_{1,p}[(p-1)K_p], x] = x(x-1)^{p-2}(x-p-1)^{(p-1)(p-2)}[(x-p)^2-(x-p)-(p-1)]^p$. Hence, $K_{1,p}[(p-1)K_p]$ is Laplacian integral if and only if $p-1 = n(n+1)$, *i.e.*, $p = n^2+n+1$, where n is a positive integer.

Similarly, we can prove the other results of this corollary if we use Theorem 8.2.12 and Lemmas 1.2.23, 1.2.24, 1.2.25. ■

Corollary 8.2.14. *For the* σ*-polynomials of the graphs* $G = K_{1,p}[(p-1)K_p]$, $\overline{G}$, $L(G)$, $\overline{L(G)}$ *and* $L(\overline{G})$, *the following is true if we choose* $p = n^2+n+1$, *where* n *is any positive integer.*

(1) The characteristic polynomial of $Lap(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])$ *is*

$$\sigma(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}], x) = x(x-1)^{n^2+n-1}(x-n^2-n-2)^{n(n+1)(n^2+n-1)}(x-n^2-1)^{n^2+n+1}(x-n^2-2n-2)^{n^2+n+1}.$$

(2) The characteristic polynomial of $Lap(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]})$ *is*

$$\sigma(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}, x) = x[x-n(n+1)(n^2+n+1)]^{n(n+1)(n^2+n-1)}[x-(n^2+n+1)^2]^{n^2+n-1}[x-(n+1)^2(n^2+1)]^{n^2+n+1} \cdot [x-n^2(n^2+2n+2)]^{n^2+n+1}.$$

(3) The characteristic polynomial of $Lap(L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]))$ *is*

$$\sigma(L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]),x) = x(x-n^2-n-2)^{n(n+1)(n^2+n-1)} \cdot(x-1)^{n^2+n-1}[x-2(n^2+n+1)]^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}(x-n^2-2n -2)^{n^2+n+1}(x-n^2-1)^{n^2+n+1}.$$

(4) The characteristic polynomial of $Lap(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])})$ *is*

$$\sigma(\overline{L(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])},x) = x\{x-\tfrac{1}{2}[(n^2+n+1)^3 -(n^2+n+3)]\}^{n(n+1)(n^2+n+1)} \cdot [x-\tfrac{1}{2}n(n+1)(n^4+2n^3+4n^2+3n +3)]^{n^2+n-1}\{x-\tfrac{1}{2}(n+1)[n(n^2+n+1)(n^2+n+2)-2]\}^{n^2+n+1} \cdot\{x-\tfrac{1}{2}n[(n+1)(n^2+n+1)(n^2+n+2)+2]\}^{n^2+n+1} \cdot\{x-\tfrac{1}{2}(n^2+n+1)[(n^2+n+1)^2-3]\}^{\frac{1}{2}(n^2+n-1)[(n^2+n+1)^2+1]}.$$

(5) The characteristic polynomial of $Lap(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}))$ *is*

$$\sigma(L(\overline{K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}]}),x) = x[x-n(n+1)(n^2+n+ 1)]^{n(n+1)(n^2+n-1)} \cdot [x-(n^2+n+1)^2]^{n^2+n-1}[x-(n+1)^2(n^2+1)]^{n^2+n+1} \cdot[x-2n(n+1)(n^2+n+1)]^{\frac{1}{2}(n^2+n-1)(n^2+n+2)[(n^2+n+1)^2+1]} \cdot[x-n^2(n^2+2n+2)]^{n^2+n+1}.$$

Proof. (1) By Theorem 8.2.12, we have $\sigma[K_{1,p}[(p-1)K_p],x] = x(x-1)^{p-2}(x-p-1)^{(p-1)(p-2)}[(x-p)^2-(x-p)\ -(p-1)]^p$. Hence, if $p = n^2+n+1$, where n is a positive integer, we get

$\sigma(K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}],x) = x(x-1)^{n^2+n-1}(x-n^2-1)^{n^2+n+1} (x-n^2-n-2)^{n(n+1)(n^2+n-1)}(x-n^2-2n-2)^{n^2+n+1}$.

The other results of this corollary are similarly proven from Theorem 8.2.12. ■

Theorem 8.2.15. *If a regular graph G is integral, then for integers $n \geq 0$ and $m \geq 0$ the graphs $L^m(G)$ and $L^n(\overline{L^m(G)})$ are Laplacian integral.*

Proof. The result follows by Lemmas 1.2.18, 1.2.23, 1.2.24 and 1.2.25 and Theorem 8.2.8. ■

The next result is a consequence of Lemmas 1.2.18 1.2.23, 1.2.24, 1.2.25 and Corollary 8.2.14 as well as Theorems 8.1.1, 8.1.2 and 8.2.15.

Corollary 8.2.16. *Let n be a positive integer, and let $G = K_{n,n+1} \equiv K_{n+1,n}$ and $H = K_{1,n^2+n+1}[n(n+1)K_{n^2+n+1}])$. Then for integers $p \geq 0$ and $m \geq 0$*

the graphs $L^m(G)$, $L^p(\overline{L^m(G)})$, $L^m(H)$ *and* $L^p(\overline{L^m(H)})$ *are Laplacian integral graphs.*

Theorem 8.2.17. *If* G *is a regular graph of degree* k *with* n *vertices and* $m = \frac{1}{2}nk$ *edges, then for any integer* $t \geq 2$ *the characteristic polynomial of the* $(2^t k - 2^{t+1} + 2)$*-regular graph* $L^t(G)$ *with* $n\prod_{i=0}^{t-1}(2^{i-1}k - 2^i + 1)$ *vertices and* $n\prod_{i=0}^{t}(2^{i-1}k - 2^i + 1)$ *edges is*

$$P(L^t(G), x) = P(G, x + (2-k)(2^t - 1))(x+2)^{2^{t-2}n(k-2)\prod_{i=0}^{t-2}(2^{i-1}k-2^i+1)}$$
$$\cdot \prod_{j=2}^{t-1}[x + 2 + (2-k)\sum_{i=j}^{t-1} 2^i]^{2^{j-2}n(k-2)\prod_{i=0}^{j-2}(2^{i-1}k-2^i+1)}[x + 2 + (2-k)(2^t - 2)]^{\frac{1}{2}n(k-2)}.$$

Proof. G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges. By Lemmas 1.2.18 and 1.2.20, we therefore find that $L(G)$ is the $(2k-2)$-regular graph with $\frac{1}{2}nk$ vertices and $\frac{1}{2}nk(k-1)$ edges. Further, we deduce that $L^t(G)$ is a $(2^t k - 2^{t+1} + 2)$-regular graph with $n\prod_{i=0}^{t-1}(2^{i-1}k - 2^i + 1)$ vertices and $n\prod_{i=0}^{t}(2^{i-1}k - 2^i + 1)$ edges.

We prove the formula for the characteristic polynomial by induction on $t \geq 2$. When $t = 2$, from Lemmas 1.2.18 and 1.2.20, we obtain

$$\begin{aligned}
&P(L^2(G), x) = P(L(L(G)), x) \\
&= (x+2)^{\frac{1}{2}\cdot\frac{1}{2}nk\cdot(2k-4)}P(L(G), x + 2 - (2k-2)) \\
&= P(G, x + (2-k)(2^2 - 1))(x+2)^{2^{2-2}n(k-2)\prod_{i=0}^{2-2}(2^{i-1}k-2^i+1)} \\
&\cdot[x + 2 + (2-k)(2^2 - 2)]^{\frac{1}{2}n(k-2)}.
\end{aligned}$$

We now assume that for $t = r - 1 \geq 2$, the result on the characteristic polynomial of the $(2^{r-1}k - 2^r + 2)$-regular graph $L^{r-1}(G)$ with $n\prod_{i=0}^{r-2}(2^{i-1}k - 2^i + 1)$ vertices and $n\prod_{i=0}^{r-1}(2^{i-1}k - 2^i + 1)$ edges holds. When $t = r$, by Lemmas 1.2.18 and 1.2.20, we find

$P(L^r(G), x) = P(L(L^{r-1}(G)), x) = P\{L^{r-1}(G), x + 2 - [2^{r-1}k - 2^r + 2]\}(x + 2)^{\frac{1}{2}n\prod_{i=0}^{r-2}(2^{i-1}k-2^i+1)\cdot[(2^{r-1}k-2^r+2-2]}$.

Using the induction hypothesis, the characteristic polynomial of the $(2^r k - 2^{r+1} + 2)$-regular graph $L^r(G)$ with $n\prod_{i=0}^{r-1}(2^{i-1}k - 2^i + 1)$ vertices and $n\prod_{i=0}^{r}(2^{i-1}k - 2^i + 1)$ edges becomes

$$P(L^r(G), x) = P(G, x + (2-k)(2^r - 1))(x+2)^{2^{r-2}n(k-2)\prod_{i=0}^{r-2}(2^{i-1}k-2^i+1)}$$
$$\cdot \prod_{j=2}^{r-1}[x + 2 + (2-k)\sum_{i=j}^{r-1} 2^i]^{2^{j-2}n(k-2)\prod_{i=0}^{j-2}(2^{i-1}k-2^i+1)}[x + 2 + (2-k)(2^r - 2)]^{\frac{1}{2}n(k-2)}.$$

Hence, this theorem is proved. ∎

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Index

Summary

This monograph deals with integral graphs, Laplacian integral regular graphs, cospectral graphs and cospectral integral graphs. The organization of this work, which consists of eight chapters, is as follows.

In Chapter 1, we present a general introduction to the topics of the thesis and a survey of the main results on integral graphs against a background of related results, together with the relevant terminology and notations.

In Chapter 2, we shall give some facts in number theory and some notations on matrices.

In Chapter 3, some new families of integral trees with diameters 4, 6 and 8 are given. Most of these classes are infinite. They are different from those in the literature. We believe that this new contribution to the research of integral trees is useful for constructing other integral trees. At the same time, some results on the interrelations between integral trees of various diameters are obtained for the first time. These results generalize some of the well-known results or theorems on integral trees. Finally, we propose several basic open problems on integral trees for further study.

In Chapter 4, we determine the characteristic polynomials of some classes of trees with diameters 5, 6 and 8. We also obtain sufficient and necessary conditions for these trees to be integral by using number theory and computer search. All these classes are infinite and different from those in the literature. The results generalize some of the well-known theorems on integral trees. In particular, we firstly find some special structures of integral trees of diameters 5, 6 and 8. Furthermore, some new results which treat interrelations between integral trees of various diameters are found.

In Chapter 5, we give a useful sufficient and necessary condition for complete r-partite graphs to be integral, from which we can construct infinitely many new classes of such integral graphs. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving some Diophantine equations. The discovery of these integral complete r-partite graphs is a new contribution to the research on integral graphs. In fact, M. Roit-

man's result on integral complete tripartite graphs is generalized. Finally, we propose several basic open problems for further study.

In Chapter 6, we construct fifteen classes of larger integral graphs from known smaller ones. These classes consist of nonregular and bipartite graphs. Their spectra and characteristic polynomials are obtained from matrix theory. Their integral property is established by using number theory and computer search. All these classes are infinite and different from those in the literature. It is proved that the problem of finding such integral graphs is equivalent to the problem of solving Diophantine equations. We believe that these results are useful for constructing other integral graphs. They generalize some results of Balińska and Simić. Finally, we propose several open problems for further study.

In Chapter 7, we determine the characteristic polynomials of four classes of graphs and obtain sufficient and necessary conditions for these graphs to be integral by using number theory and computer search. All these classes are infinite. We also give some new cospectral graphs and cospectral integral graphs.

In Chapter 8, we study the spectra and characteristic polynomials of two classes of regular graphs. We derive the characteristic polynomials for their complement graphs, their line graphs, the complement graphs of their line graphs and the line graphs of their complement graphs. These graphs are not only integral but also Laplacian integral. The discovery of these integral graphs is a new contribution to the research of integral graphs. These results generalize some results of Harary and Schwenk.

Curriculum vitae

Ligong Wang was born on September 14, 1968, in Xining City of Qinghai Province, P.R. China. From 1975 until 1986 he attended primary and middle school in Xining City. In September 1986, he started to study pure mathematics at Shaanxi Normal University in Xi'an. After receiving his Bachelor's degree in July 1990, he worked at the fifth middle school in Xining City of Qinghai Province from July 1990 to August 1995. In September 1995, he became a graduate student at Qinghai Normal University. He specialized in Pure Mathematics, and completed his Master's degree thesis, entitled "Some Studies on Chromatic Uniqueness of Graphs", under the supervision of Professor Ruying Liu. In July 1998, he graduated with his Master's degree and is employed as a teacher at the Northwestern Polytechnical University. From 1998 on he is teaching mathematics and doing research on graph theory.

In September 2001 he started as a Ph.D. student under the supervision of Prof. Dr. Cornelis Hoede and Dr. Georg Still from the University of Twente, and Prof. Dr Xueliang Li from Northwestern Polytechnical University. One of the main results of the study during the past four years is the thesis that is now in the hand of the reader.

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