Progress of Theoretical Physics, Vol. 37, No. 5, May 1967

The Unitarity Condition of the S-Matrix for High Energy Elastic Scattering

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(Recieved December 5, 1966))

The inelasticity of the intermediate states in the unitarity condition of the S-matrix is investigated for high energy elastic scattering of strongly interacting particles. The inelastic process is treated as the multiple production of mesons, for which we have assumed statistical independence. The energy-momentum conservation law in the intermediate states is derived in terms of the meson momentum distribution, and it is shown that the effective intermediate states are in general varied with the elastic scattering angles. The property of the intermediate states is discussed with existing experimental data on the multiple meson production. In high energy accelerator researches and analyses of cosmic ray jet, it has been revealed that the momentum distribution of the secondaries is sharply peaked in the longitudinal direction and the average transverse momentum is constant independent of the incident energies. From these experimental results, we show that the inelasticity of the intermediate states responsible for large angle elastic scattering is quite small compared with that for small angle elastic scattering.

§ 1. Introduction

The probability conservation in scattering processes leads to the unitarity condition for the S-matrix. For the scattering from the state $|a\rangle$ to the state $|a'\rangle$, this condition is written as

$$i[\langle a'|T^+|a\rangle - \langle a'|T|a\rangle] = \sum_n \langle a'|T^+|n\rangle\langle n|T|a\rangle, \qquad (1\cdot1)$$

where T is the transition matrix and T^+ its hermitian conjugate. The intermediates states $\{|n\rangle\}$ in Eq. (1·1) should include all possible states which satisfy the conservation laws of the system. Therefore, under this condition, high energy elastic scattering should be discussed together with inelastic processes.

Consider the unitarity condition for the elastic scattering of two nucleons. For the scattering into the same state as the initial state, the relevant intermediate states are obviously the states realized by the collision of the two nucleons, which will mainly be multiple meson production states at high energies. However, this is not always the case for the scattering into a different state from the initial state. It is still an open question what states they will be. The main purpose of the present paper is to investigate this problem for high energy elastic scattering of strongly interacting particles.

Some important features in multiple production phenomena have been revealed by recent accelerator researches[1)] and analyses of cosmic ray jets[2)]: (i) the average multiplicity is large compared with one (several to a few decades) and is increasing proportionally to $E^{1/2}$ or $\log E$ (E is the center-of-mass incident energy*)), and the eighty or more percent of produced particles are identified with the π meson, (ii) the inelasticity is widely distributed over 0.1 to 0.9 and its average value is around 0.4, (iii) the average longitudinal momentum is very large and seems to increase with the incident energies, while the average transverse momentum $\langle \boldsymbol{k}_\perp \rangle$ is constant of the order of 0.4 BeV*) over an extensive range 2.5×10 to 10^6 BeV of the incident energies, and its distribution is reproduced fairly well by the formula

$$f(\boldsymbol{k}_\perp)\,d\boldsymbol{k}_\perp \propto \exp\left[-\frac{2\boldsymbol{k}_\perp}{\langle \boldsymbol{k}_\perp \rangle}\right] d\boldsymbol{k}_\perp\,, \tag{1·2}$$

$\boldsymbol{k}_\perp$ being the transverse momentum and $\langle \boldsymbol{k}_\perp \rangle \simeq 0.4$ BeV.

We shall treat high energy inelastic processes as multiple meson production, which is the most typical and dominant phenomenon in high energy collisions. The multiple production will be realized through strong short range forces, and the produced mesons will exert the complicated, final interaction on themselves. However, as was stated above, the average transverse momentum of the mesons is found to be constant irrespective of incident energies. Owing to this fact, we assume the statistical independence of the meson.[3] Then the S-matrix elements for multiple meson production are factorizable into the S-matrix elements for the production of an individual meson:

$$\langle \boldsymbol{k}_1, \cdots \boldsymbol{k}_n, \boldsymbol{q}_1, \boldsymbol{q}_2; \mathbf{out}|\boldsymbol{p}; \mathbf{in}\rangle$$
$$= i\delta^4(Q+k_1+\cdots+k_n-P)\frac{1}{\sqrt{n!}}\varphi(\boldsymbol{k}_1)\cdots\varphi(\boldsymbol{k}_n)\,T(\boldsymbol{q}_1, \boldsymbol{q}_2; \boldsymbol{p}). \tag{1·3}$$

In Eq. (1·3), $\boldsymbol{p}$ and $\boldsymbol{q}_j\,(j=1, 2)$ are the incident and recoil nucleon momenta, P and Q their total values, T the nucleon recoil amplitude, and $\varphi(\boldsymbol{k}_j)$ the single particle wave function of the meson with the momentum $\boldsymbol{k}_j\,(j=1\cdots n)$. Several successes[4] in explaining the experimental data of multiple meson production based on the assumption have been reported. The interrelation between elastic and inelastic processes has been discussed by a number of authors[5] by making use of some main features of multiple production phenomena experimentally established.

In the next section, the multiple meson production process will be formulated on the above assumption. In § 3, we shall derive the energy-momentum conservation law in the intermediate states of the unitarity condition for high energy elastic scattering. In the last section, the inelasticity in the intermediate states will be investigated in the light of the experimental data on multiple meson production and our conclusions will be given.

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* Throughout this paper, the coordinate system of reference is always the center-of-mass (c.m.) system and the natural unit: $c=\hbar=1$ is used, unless otherwise stated.

§ 2. The formulation of multiple production

In this section, we want to formulate briefly the multiple meson production process in high energy collision of strongly interacting particles (nucleon-nucleon collision) based on the assumption stated in § 1. Although no essentially new feature is presented except for exactly taking account of energy-momentum conservation law, some formulas are necessary for the later discussions. The produced mesons are assumed to be of one kind (π meson). For simplicity, spin and charge which give rise to a very small effect at high energies are neglected.

We expand the incident state $|\boldsymbol{p};\ \mathbf{in}\rangle$ of two nucleons into the complete set of the final states in which the bound state is assumed to be absent:

$$|\boldsymbol{p};\ \mathbf{in}\rangle=\sum_n\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2 d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_n|\boldsymbol{k}_1\cdots\boldsymbol{k}_n,\ \boldsymbol{q}_1,\ \boldsymbol{q}_2:\ \mathbf{out}\rangle \times\langle\boldsymbol{k}_1\cdots\boldsymbol{k}_n,\ \boldsymbol{q}_1,\ \boldsymbol{q}_2;\ \mathbf{out}|\boldsymbol{p};\ \mathbf{in}\rangle. \tag{2·1}$$

Substituting the matrix elements (1·3) into Eq. (2·1), we can express the incident state in terms of the outgoing states as

$$|\boldsymbol{p};\ \mathbf{in}\rangle=|\boldsymbol{p};\ \mathbf{out}\rangle+ +\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2 T(\boldsymbol{q}_1,\ \boldsymbol{q}_2;\ \boldsymbol{p})\int\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}\varphi(\boldsymbol{k})e^{ik\rho}a_k^{\mathrm{out}}\right]|\boldsymbol{q}_1,\ \boldsymbol{q}_2;\ \mathbf{out}\rangle, \tag{2·2}$$

where a_k^{out} is the creation operator of the outgoing meson with the momentum $\boldsymbol{k}$, ρ a four-dimensional vector, and both $(Q-P)\rho$ and $k\rho$ the four-dimensional scalar products:*)

$$(Q-P)\rho=\sum_\mu(Q_\mu-P_\mu)\rho_\mu,\quad(\mu=0,\ x,\ y,\ z) \tag{2·3a}$$

and

$$k\rho=\sum_\mu k_\mu\rho_\mu\,. \tag{2·3b}$$

From Eq. (2·2), various cross sections are given as follows: the total elastic cross section

$$\sigma_{\mathrm{el}}(P)=(\pi\varepsilon_p)^2\int d\Omega_q|T(\boldsymbol{q}_1,\ \boldsymbol{q}_2;\ \boldsymbol{p})|^2,\quad(\varepsilon_p=\sqrt{|\boldsymbol{p}|^2+M^2}), \tag{2·4}$$

the total inelastic cross section

$$\sigma_{\mathrm{inel}}(P)=\frac{2\pi^2\varepsilon_p}{p}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2|T(\boldsymbol{q}_1,\ \boldsymbol{q}_2;\ \boldsymbol{p})|^2 \times\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}|\varphi(\boldsymbol{k})|^2e^{ik\rho}\right] \tag{2·5}$$

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* The scalar product of the four-vectors A and B is defined as
$$\sum_\mu A_\mu B_\mu=A_0B_0-A_xB_x-A_yB_y-A_zB_z\,.$$

and the total cross section

$$\sigma_{\text{tot}}(P)=\frac{2\pi^2\varepsilon_p}{P}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2|T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})|^2$$

$$\times\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}|\varphi(\boldsymbol{k})|^2 e^{ik\rho}\right] \tag{2·6a}$$

$$=\frac{4\pi^2\varepsilon_p}{p}\,\text{Im}\,T(\boldsymbol{p};\boldsymbol{p}). \tag{2·6b}$$

The momentum distribution $\overline{N}(\boldsymbol{k})$ of the produced mesons is

$$\overline{N}(\boldsymbol{k})=A(\boldsymbol{k})|\varphi(\boldsymbol{k})|^2, \tag{2·7a}$$

$$A(\boldsymbol{k})\equiv\frac{1}{\sigma_{\text{tot}}(P)}\frac{2\pi^2\varepsilon_p}{p}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2|T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})|^2$$

$$\times\int\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}|\varphi(\boldsymbol{k})|^2 e^{ik\rho}\right]. \tag{2·8a}$$

Similarly for the average multiplicity $\overline{N}$ of the mesons and the average four-momentum $\overline{K}_\mu$ transferred to the mesons are

$$\overline{N}=\int d^3\boldsymbol{k}\overline{N}(\boldsymbol{k}) \tag{2·9a}$$

and

$$\overline{K}_\mu=\int d^3\boldsymbol{k}\overline{N}(\boldsymbol{k})k_\mu. \tag{2·10a}$$

Since the mesons are experimentally found not so energetic, the nucleon recoil amplitude will not vary drastically with sufficiently high incident energies. Then from Eq. (2·6a), we can approximate $A(\boldsymbol{k})$ as

$$A(\boldsymbol{k})\simeq\frac{\sigma_{\text{tot}}(P-k)}{\sigma_{\text{tot}}(P)}$$

$$\simeq 1. \tag{2·8b}$$

Accordingly one gets

$$\overline{N}(\boldsymbol{k})\simeq|\varphi(\boldsymbol{k})|^2, \tag{2·7b}$$

$$\overline{N}\simeq\int d^3\boldsymbol{k}|\varphi(\boldsymbol{k})|^2 \tag{2·9b}$$

and

$$\overline{K}_\mu\simeq\int d^3\boldsymbol{k}|\varphi(\boldsymbol{k})|^2 k_\mu. \tag{2·10b}$$

Statistically independent mesons will have random phases, so that we shall consistently treat the meson wave function as real. Therefore one can determine the analytic expression for $\varphi(\boldsymbol{k})$ through the formulas (2·7), (2·9) and (2·10), if the sufficient data on multiple production are obtained. By making use of the experimental data stated in § 1, $\varphi(\boldsymbol{k})$ is given as follows:

$$\varphi(\boldsymbol{k})=\varphi_{\boldsymbol{p}}(\boldsymbol{k})\propto\exp\left[-\frac{k_{\perp}}{\langle k_{\perp}\rangle}\right]\left(\text{for } \boldsymbol{k}\perp\boldsymbol{p}\right), \tag{2·11}$$

with $\langle k_{\perp}\rangle\simeq 0.4$ BeV. Of course, it is normalized to the average multiplicity $\bar{N}$.

§ 3. The energy-momentum conservation law in the intermediate states appearing in the unitarity condition

From Eq. (2·2), the unitarity condition for the elastic scattering of the two nucleons is given by

$$\operatorname{Im} T(p,\theta)=\frac{1}{2}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2 T^*(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p}')T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})$$

$$\times\int\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})e^{ik\rho}\right], \tag{3·1}$$

where θ is the elastic scattering angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$. One rewrite Eq. (3·1) in the form

$$\operatorname{Im} T(p,\theta)=\frac{1}{2}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2 T^*(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p}')T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})$$

$$\times\exp\left[\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})\right]D(Q;p,\theta), \tag{3·2}$$

where

$$D(Q;p,\theta)=\exp\left[-\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})\right]$$

$$\times\int\frac{d^4\rho}{(2\pi)^4}\exp\left[i(Q-P)\rho+\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})e^{ik\rho}\right]. \tag{3·3}$$

The function D has the following properties:

(i) $D(Q;p,\theta)$ is positive.

(ii) $\int d^4Q D(Q;p,\theta)=1$.

Owing to the properties (i) and (ii), $D(Q;p,\theta)$ can be interpreted as a probability distribution with respect to the total energy-momentum Q of the recoil nucleons in the intermediate states of the unitarity condition. This may be understood when one considers the following example: let $\varphi_{\boldsymbol{p}}(\boldsymbol{k})$ be equal to zero in Eq. (3·3), which corresponds to neglecting all the inelastic intermediate states. Then D reduces to

$$D(Q;p,\theta)=\delta^4(Q-P). \tag{3·4}$$

This is nothing other than the well-known Q distribution for elastic unitarity.

The quantity D will therefore tell us about the inelasticity in the intermediate states. One should note that the inelasticity varies in general with the elastic scattering angles. The quantity $\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})\exp(ik\rho)$ in Eq. (3·3) is the characteristic function for the distribution $\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})$. It is difficult to integrate over $\boldsymbol{k}$ unless we know the analytic expression for $\varphi_{\boldsymbol{p}}(\boldsymbol{k})$, so that we calculate it by making use of some approximations.

The factor $\exp(ik\rho)$ can be expanded into the Maclaurin series as

$$e^{ik\rho} = \sum_{s=0}^{n-1}(ik\rho)^s + \alpha_n \frac{(ik\rho)^n}{n!}, \tag{3·5}$$

where α_n denotes a real or complex quantity of modulus not exceeding unity:

$$|\alpha_n| \leqq 1 . \tag{3·6}$$

First of all, let us deal with the case $n=1$ in the series (3·5):

$$e^{ik\rho} = 1 + i\alpha_1 \cdot k\rho . \tag{3·7}$$

Making use of Eq. (3·7), it follows that

$$\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})e^{ik\rho}$$
$$= \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k}) + i\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})\alpha_1\cdot k\rho . \tag{3·8}$$

$|\alpha_1|$ will have a complicated behavior though not exceeding unity. Therefore its average contribution to the integration (3·3) may be characterized by the factor $k\rho$ of definite linear behavior. Thus we replace the average contribution from the unknown α_1 in Eq. (3·8) by a real parameter $\bar{\alpha}_1$, which will be adjusted later:

$$\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})e^{ik\rho}$$
$$= \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k}) + i\sum_{\mu}[\bar{\alpha}_1\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu]\rho_\mu . \tag{3·9}$$

Substituting Eq. (3·9) into Eq. (3·3), we have

$$D(Q;\, p,\, \theta) = \delta^4(Q_\mu + \bar{\alpha}_1\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu - P_\mu). \tag{3·10}$$

A delta function-like distribution has resulted in the present approximation and it could be interpreted as giving the energy-momentum conservation law for a representative quantity of distribution. To see the meaning of the distribution (3·10) and the property of $\bar{\alpha}_1$, consider the forward elastic scattering. From Eq. (3·10), the total energy-momentum transferred to the mesons in the intermediate states is given by

$$\bar{\alpha}_1\int d^3\boldsymbol{k}[\varphi_{\boldsymbol{p}}(\boldsymbol{k})]^2 k_\mu . \tag{3·11}$$

On the other hand, since the intermediate state is nothing other than the inelastic states realized by the collision of the incident nucleons, the average value of the momentum transfer is also given by Eq. (2·9). Comparing Eq. (3·11) with Eq. (2·9), we find that Eq. (3·10) is the conservation law related to the mean value of the distribution, and that

$$\bar{\alpha}_1 \simeq 1 . \tag{3·12}$$

When the distribution is not essentially affected by the energy-momentum conservation law and the nucleon recoil effects, it is in general justified that the quantity (3·11) supplemented by the condition (3·12) gives the average energy-momentum transferred to the mesons. In fact, if the effects can be neglected, Eq. (2·2) reduces to

$$|p; \mathbf{in}\rangle = \exp[i \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}}(\boldsymbol{k})\, a_{\boldsymbol{k}}^{\text{out}}]\, |\boldsymbol{p}; \mathbf{out}\rangle. \tag{3·13}$$

Then K_μ is given exactly by

$$\bar{K}_\mu = \int d^3\boldsymbol{k}[\varphi_{\boldsymbol{p}}(\boldsymbol{k})]^2 k_\mu , \tag{3·14}$$

which corresponds to $\bar{\alpha}_1 = 1$ in Eq. (3·11). Though Eq. (3·12) has been derived by referring to the forward scattering, its validity for other scattering angles could be easily understood by regarding $[\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})]^{1/2}$ as the meson wave function anew.

Next, we treat the case $n=2$ in the series (3·5):

$$e^{ik\rho} = 1 + ik\rho - \frac{1}{2}\alpha_2 \cdot (k\rho)^2. \tag{3·15}$$

In the same way as in the previous case, we approximate the contribution from α_2 by a real parameter $\bar{\alpha}_2$:

$$\begin{aligned}\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})e^{ik\rho} &= \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k}) + i\sum_\mu [\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu]\rho_\mu \\ &\quad - \frac{1}{2}\sum_{\mu\nu}[\bar{\alpha}_2 \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu k_\nu]\rho_\mu\rho_\nu .\end{aligned} \tag{3·16}$$

It follows from Eq. (3·16) that

$$D(Q; p, \theta) = \frac{1}{(2\pi)^2\sqrt{\det(A_{\mu\nu})}} \exp\left[-\frac{1}{2}\sum_{\mu\nu}(Q_\mu + K_\mu - P_\mu)A_{\mu\nu}^{-1}(Q_\nu + K_\nu - P_\nu)\right], \tag{3·17}$$

where

$$K_\mu = K_\mu(p, \theta) = \int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu \tag{3·18}$$

and

$$A_{\mu\nu}=A_{\mu\nu}(p,\theta)=\bar{\alpha}_2\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu k_\nu\,. \tag{3·19}$$

The reciprocal matrix of $A_{\mu\nu}$ is denoted by $A_{\mu\nu}^{-1}$ and its determinant by $\det(A_{\mu\nu})$, which is assumed to be positive. Equation (3·17) is a normal distribution, in which the mode is nearly given by the mean value and the standard deviation by $A_{\mu\nu}^{-1}$. By making use of the central limit theorem in the probability theory, we can determine the parameter $A_{\mu\nu}$ in an approximation. The detailed discussion is presented in the Appendix.

Here we want to discuss briefly the normalization of the inelastic nucleon amplitude. Since D is a normalized distribution, we can write the inelastic nucleon amplitude T in Eq. (2·5) as

$$T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})=\mathcal{T}(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})\exp\{\int d^3\boldsymbol{k}[\varphi_{\boldsymbol{p}}(\boldsymbol{k})]^2\}, \tag{3·20}$$

where $\mathcal{T}$ is normalized as

$$\sigma_{\rm inel}(P)=\frac{2\pi^2\varepsilon_p}{p}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2|\mathcal{T}(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})|^2 D(Q;p,\theta). \tag{3·21}$$

Using Eqs. (2·9) and (3·18), one gets

$$|T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})|^2\propto\exp(-\bar{N}). \tag{3·22}$$

§ 4. Conclusions and discussions

In the preceding section, we have shown the inelasticity in the intermediate states of the unitarity condition for high energy elastic scattering is generally varied with elastic scattering angles. We want to investigate its qualitative features in the light of the experimental data. It is convenient to use the distribution of the mean value, Eq. (3·10), supplemented by the condition (3·12).

From the energy-momentum conservation law (3·10), the average energy-momentum transferred to the mesons is given by

$$K_\mu(p,\theta)=\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu\,. \tag{4·1}$$

According to the analyses of the existing data on multiple meson production, the meson wave function $\varphi_{\boldsymbol{p}}(\boldsymbol{k})$ is obtained as Eq. (2·11) normalized to the average multiplicity $\bar{N}$. Therefore

$$K_0\left(p,\theta\simeq\frac{\pi}{2}\right)\simeq\bar{N}\langle\boldsymbol{k}_\perp\rangle,$$

$$\simeq\bar{N}\times0.4\,(\text{BeV}), \tag{4·2a}$$

$$K_0(p,\theta\simeq0)\simeq\bar{N}\langle\boldsymbol{k}_{/\!/}\rangle \tag{4·2b}$$

and

$$K_0\left(p,\ \theta\simeq\frac{\pi}{2}\right)\ll K_0(p,\ \theta\simeq 0). \tag{4·3}$$

Here $\langle \boldsymbol{k}_{/\!/}\rangle$ and $\langle \boldsymbol{k}_{\perp}\rangle$ are, respectively, the average values of the longitudinal and transverse momenta of a meson, and the value 0.4 BeV is the experimental one for the latter. Equations (4·2) and (4·3) are also written in terms of inelasticity η as follows:

$$\eta\left(p,\ \theta\simeq\frac{\pi}{2}\right)=\frac{K_0(p,\ \theta\simeq\pi/2)}{2\varepsilon_p}\simeq\frac{\overline{N}\langle \boldsymbol{k}_{\perp}\rangle}{2\varepsilon_p}$$

$$\simeq\frac{1}{\varepsilon_p{}^{1/2}}\ \text{or}\ \frac{\log\varepsilon_p}{\varepsilon_p}, \tag{4·4a}$$

$$\eta(p,\ \theta\simeq 0)=\frac{K_0(p,\ \theta\simeq 0)}{2\varepsilon_p}\simeq\frac{\overline{N}\langle \boldsymbol{k}_{/\!/}\rangle}{2\varepsilon_p}$$

$$\simeq 0.4 \tag{4·4b}$$

and

$$\eta\left(p,\ \theta\simeq\frac{\pi}{2}\right)\ll\eta(p,\ \theta\simeq 0), \tag{4·5}$$

where we have used the experimental values for the average multiplicity and the average inelasticity.

From Eq. (4·5), we can obtain the following conclusions: the inleasticity in the intermediate states responsible for the large angle elastic scattering is quite small compared with that for small angle elastic scattering. Its value is of course maximum for the forward scattering.

Owing to this property of the intermediate states, the small angle elastic scattering is considered to be dominated by inelastic intermediate states, and elastic unitarity may, to a good approximation, be neglected in the diffraction region of the scattering angles. For the large angle elastic scattering, however, this is not the case and elastic unitarity (or the real part of the scattering amplitude) could have appreciable contribution. These are quite consistent with the results in the analyses of high energy elastic scattering obtained by a number of authors. In fact, the authors of references 3) and 5) have supposed the forward elastic scattering as a shadow process into inelastic scattering and succeeded in explaining its various behavior found in experiment. Cottingham and Peierls[6)] have derived the conclusion, in the analysis of the $p-p$ large angle elastic scattering, that the elastic unitarity is necessary in explaining the isotropic distribution at large angles. In view of the inelasticity in the intermediate states, one can understand self-consistently the validity of different treatments of the unitarity condition adopted according to the various scattering angles.

As is well known, the nucleon has an extended structure, and the multiple meson production is considered to be caused by breaking up the structure. The smallness of the inelasticity means that the nucleon will not be badly broken in the intermediate process. Therefore the strong damping of the large angle scattering amplitude with increasing energies is interpreted as causing the difficulty in giving the nucleon the large momentum transfer without breaking up its structure. On the basis of the inelasticity, one can give another reasoning to the speculation for the large angle elastic scattering proposed by Wu and Yang.[7)]

Acknowledgments

The author would like to thank Professor H. Fukuda and Professor C. Iso of Tokyo Institute of Technology for helpful discussions and their continual guidance throughout the course of the present work. He should express his sincere gratitude to Professor D. Ito, Professor Y. Murai, Professor K. Mori and Dr. E. Kageyama for their encouragement and their generous help. Discussions with Professor T. Kobayashi of Waseda University are deeply appreciated.

Appendix

We want to derive the parameters $A_{\mu\nu}$ in Eq. (3·18). We approximate the inelastic states as the multiple meson state with the average multiplicity $\bar{N}$:

$$|\text{inel};\ \mathbf{out}\rangle=\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2 d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_{\bar{N}}\delta^4(Q+k_1+\cdots+k_{\bar{N}}-P)\,T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})$$

$$\times\frac{1}{\sqrt{\bar{N}}}\varphi_{\boldsymbol{p}}(\boldsymbol{k}_1)\cdots\varphi_{\boldsymbol{p}}(\boldsymbol{k}_{\bar{N}})\,|\boldsymbol{k}_1\cdots\boldsymbol{k}_{\bar{N}},\boldsymbol{q}_1,\boldsymbol{q}_2;\mathbf{out}\rangle. \qquad (\text{A}\cdot 1)$$

The unitarity condition for the forward elastic scattering is written as

$$\text{Im}\,T(p,0)=\frac{1}{2}\int d^3\boldsymbol{q}_1 d^3\boldsymbol{q}_2|T(\boldsymbol{q}_1,\boldsymbol{q}_2;\boldsymbol{p})|^2\frac{N^{\bar{N}}}{\bar{N}!}D_{\bar{N}}(Q;p,0), \qquad (\text{A}\cdot 2)$$

where

$$N=\int d^3\boldsymbol{k}|\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2 \qquad (\text{A}\cdot 3)$$

and

$$D_{\bar{N}}(Q;p,0)=\int\frac{d^4\rho}{(2\pi)^4}e^{i(Q-P)\rho}\left[\int d^3\boldsymbol{k}f(\boldsymbol{k})e^{ik\rho}\right]^{\bar{N}}, \qquad (\text{A}\cdot 4)$$

$f(\boldsymbol{k})$ being the normalized distribution

$$f(\boldsymbol{k})=\frac{|\varphi(\boldsymbol{k})|^2}{N}. \qquad (\text{A}\cdot 5)$$

$\int d^3\boldsymbol{k} f(\boldsymbol{k}) \exp(ik\rho)$ is the characteristic function of the distribution f, and

$$[\int d^3\boldsymbol{k} f(\boldsymbol{k})\, e^{ik\rho}]^{\bar{N}} \tag{A·6}$$

is the total characteristic function of the $\bar{N}$ mesons with the same distrbution f. From the central limit theorem,*) for a sufficiently large $\bar{N}$, the quantity (A·6) is given by

$$[\int d^3\boldsymbol{k} f(\boldsymbol{k})\, e^{ik\rho}]^{\bar{N}} = \exp\left[i\sum_{\mu} K_\mu \rho_\mu - \frac{1}{2}\sum_{\mu\nu} A_{\mu\nu}\rho_\mu\rho_\nu\right] + O\left(\frac{1}{\sqrt{\bar{N}}}\right), \tag{A·7}$$

where

$$K_\mu = \bar{N}\int d^3\boldsymbol{k} f(\boldsymbol{k})\, k_\mu \tag{A·8}$$

and

$$A_{\mu\nu} = \bar{N}\int d^3\boldsymbol{k} f(\boldsymbol{k})\, (k_\mu - \langle k_\mu\rangle)\, (k_\nu - \langle k_\nu\rangle) \tag{A·9}$$

with

$$\langle k_\mu \rangle = \frac{\int d^3\boldsymbol{k} f(\boldsymbol{k})\, k_\mu}{N}. \tag{A·10}$$

Substituting Eq. (A·7) into Eq. (A·3),

$$D_{\bar{N}}(Q;\, p,\, 0) = \frac{1}{(2\pi)^2\sqrt{\det(A_{\mu\nu})}} \exp\left[-\frac{1}{2}(Q_\mu + K_\mu - P_\mu)\, A_{\mu\nu}^{-1}(Q_\nu + K_\nu - P_\nu)\right], \tag{A·11}$$

where we denotes the reciprocal matrix of $A_{\mu\nu}$ by $A_{\mu\nu}^{-1}$ and its determinant by det $(A_{\mu\nu})$ which is assumed to be positive.

Now remenbering Eqs. (2·9b) and (2·10b), we have

$$N = \int d^3\boldsymbol{k} |\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2 \simeq \bar{N}, \tag{A·12}$$

$$K_\mu = \int d^3\boldsymbol{k} |\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2 k_\mu \simeq \bar{K}_\mu \tag{A·13a}$$

and

$$A_{\mu\nu} = \int d^3\boldsymbol{k} |\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2 (k_\mu - \langle k_\mu\rangle)\, (k_\nu - \langle k_\nu\rangle) \tag{A·14a}$$

with

$$\langle k_\mu \rangle \equiv \frac{\int d^3\boldsymbol{k} |\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2 k_\mu}{\int d^3\boldsymbol{k} |\varphi_{\boldsymbol{p}}(\boldsymbol{k})|^2} \simeq \frac{\bar{K}_\mu}{\bar{N}}. \tag{A·15a}$$

*) See, for example, H. Cramer, *Mathematical Methods of Statistics* (Princeton, 1954), p. 213.

In the same way, K_μ and $A_{\mu\nu}$ for the non-forward scattering are given in the present approximation by

$$K_\mu(p,\theta)\simeq\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})k_\mu \tag{A·13b}$$

and

$$A_{\mu\nu}(p,\theta)\simeq\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})(k_\mu-\langle k_\mu\rangle_\theta)(k_\nu-\langle k_\nu\rangle_\theta) \tag{A·14b}$$

with

$$\langle k_\mu\rangle_\theta\equiv\frac{K_\mu(p,\theta)}{\int d^3\boldsymbol{k}\varphi_{\boldsymbol{p}'}(\boldsymbol{k})\varphi_{\boldsymbol{p}}(\boldsymbol{k})}\,. \tag{A·15b}$$

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